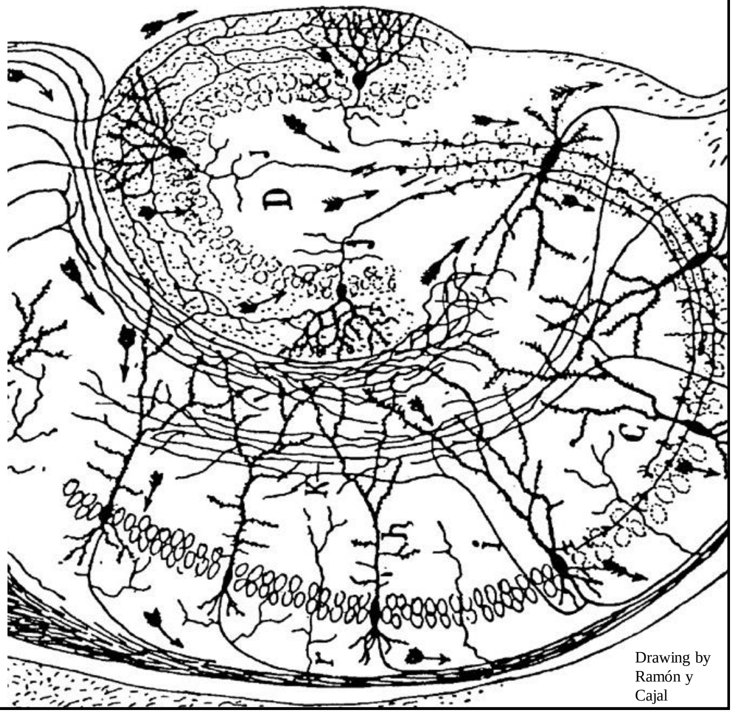
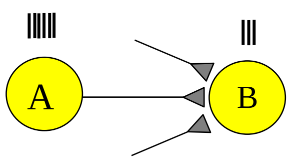
Synaptic Plasticity and Learning



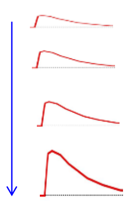
Drawing by Ramón y Cajal

Long Term Potentiation (LTP)

LTP = Experimentally observed *increase* in synaptic strength that lasts for hours or days



Increase in EPSP size for same input over time



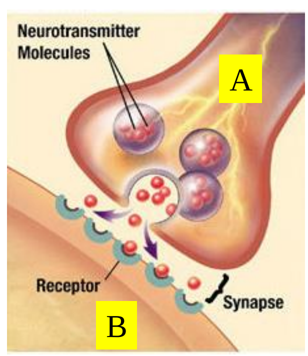
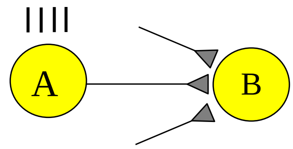


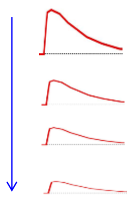
Image Source: Wikimedia Commons

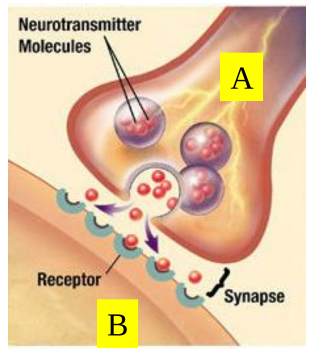
Long Term Depression (LTD)

LTD = Experimentally observed *decrease* in synaptic strength that lasts for hours or days



Decrease in EPSP size for same input over time





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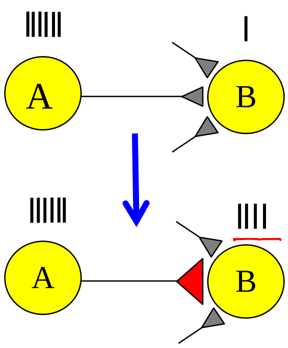
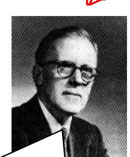
Hebb's Learning Rule





If neuron A repeatedly takes part in firing neuron B, then the synapse from A to B is strengthened

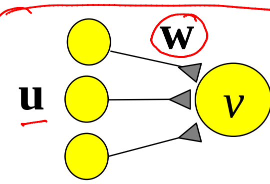
"Neurons that fire together wire together!"

Image Source: Wikimedia Commons

Formalizing Hebb's Rule

- Consider a single linear neuron with steady state output:

  $$v = \mathbf{w} \cdot \mathbf{u} = \mathbf{w}^T \mathbf{u} = \mathbf{u}^T \mathbf{w}$$



- Basic Hebb Rule: $\boxed{\tau_w \dfrac{d\mathbf{w}}{dt} = \mathbf{u}v}$

  Discrete Implementation:

  $$\tau_w \frac{\mathbf{w}(t+\Delta t) - \mathbf{w}(t)}{\Delta t} = \mathbf{u}v \quad \left(\text{or } \mathbf{w}(t+\Delta t) = \mathbf{w}(t) + \frac{\Delta t}{\tau_w}\mathbf{u}v\right)$$

  $$\mathbf{w}_{i+1} = \mathbf{w}_i + \varepsilon \cdot \mathbf{u}v \quad (\text{or } \Delta\mathbf{w} = \varepsilon \cdot \mathbf{u}v)$$

What is the average effect of the Hebb rule?

- Hebb Rule: $\tau_w \dfrac{d\mathbf{w}}{dt} = \mathbf{u}v$

- Average effect of the rule:

  $$\tau_w \frac{d\mathbf{w}}{dt} = \langle \mathbf{u}v \rangle_{\mathbf{u}} = \langle \mathbf{u}\mathbf{u}^T \mathbf{w} \rangle_{\mathbf{u}} = \langle \mathbf{u}\mathbf{u}^T \rangle_{\mathbf{u}} \mathbf{w} = Q\mathbf{w}$$

- Q is the input correlation matrix: $Q = \langle \mathbf{u}\mathbf{u}^T \rangle_{\mathbf{u}}$

Covariance Rule

- Hebb rule only increases synaptic weights (LTP)
  - What about LTD?
- Covariance rule:

$v > \langle v \rangle \Rightarrow LTP$

$v < \langle v \rangle \Rightarrow LTD$

$$\tau_w \frac{d\mathbf{w}}{dt} = \mathbf{u}(v - \langle v \rangle)$$

(Note: LTD for low or no output given some input)

- Average effect of the rule:

$$\tau_w \frac{d\mathbf{w}}{dt} = \langle \mathbf{u}(v - \langle v \rangle) \rangle_{\mathbf{u}} = \langle \mathbf{u}(\mathbf{u}^T - \langle \mathbf{u} \rangle^T)\mathbf{w} \rangle_{\mathbf{u}} = \left(\langle \mathbf{u}\mathbf{u}^T \rangle - \langle \mathbf{u} \rangle \langle \mathbf{u} \rangle^T \right)\mathbf{w}$$

$$= C\mathbf{w} \quad (C \text{ is the input covariance matrix } \langle \mathbf{u}\mathbf{u}^T \rangle - \langle \mathbf{u} \rangle \langle \mathbf{u} \rangle^T)$$

Are these learning rules stable?

- Does **w** converge to a stable value or explode?
  - Look at what happens to the length of **w** over time
- Hebb rule: $\tau_w \frac{d\mathbf{w}}{dt} = \mathbf{u}v$

$$\frac{d\|\mathbf{w}\|^2}{dt} = 2\mathbf{w}^T \frac{d\mathbf{w}}{dt} = 2\mathbf{w}^T(\mathbf{u}v / \tau_w) = \frac{2}{\tau_w} v^2 > 0$$

v

w grows without bound!

- Covariance rule: $\tau_w \frac{d\mathbf{w}}{dt} = \mathbf{u}(v - \langle v \rangle)$

$$\frac{d\|\mathbf{w}\|^2}{dt} = 2\mathbf{w}^T \frac{d\mathbf{w}}{dt} = 2\mathbf{w}^T(\mathbf{u}(v - \langle v \rangle) / \tau_w) = \frac{2}{\tau_w}(v^2 - v\langle v \rangle)$$

CONSTRAINT

$\|\omega\| = 1$

$\frac{\omega}{\|\omega\|}$

Averaging RHS, $\frac{d\|\mathbf{w}\|^2}{dt} = \frac{2}{\tau_w}(\langle v^2 \rangle - \langle v \rangle^2) = \frac{2}{\tau_w}\sigma_v^2 > 0$

w grows without bound!

Oja's Rule for Hebbian Learning

- Oja's rule: $\tau_w \frac{d\mathbf{w}}{dt} = \mathbf{u}v - \alpha v^2 \mathbf{w}$ $\quad (\alpha > 0)$

- Stable?

$$\frac{d\|\mathbf{w}\|^2}{dt} = 2\mathbf{w}^T \frac{d\mathbf{w}}{dt} = \frac{2}{\tau_w}\mathbf{w}^T(\mathbf{u}v - \alpha v^2 \mathbf{w}) = \frac{2}{\tau_w}(v^2 - \alpha v^2 \mathbf{w}^T\mathbf{w})$$

i.e., $\tau_w \frac{d\|\mathbf{w}\|^2}{dt} = 2v^2(1 - \alpha\|\mathbf{w}\|^2)$

At steady state : $\|\mathbf{w}\|^2 = \frac{1}{\alpha}$. $\quad (\|\mathbf{w}\| = \frac{1}{\sqrt{\alpha}})$

w does not grow without bound, i.e.,
Oja's rule is stable!

Summary: Hebbian Learning

- Hebb rule:

$\tau_w \frac{d\mathbf{w}}{dt} = \mathbf{u}v$ — Unstable (unless constraint on $\|\mathbf{w}\|$ is imposed)

- Covariance rule:

$\tau_w \frac{d\mathbf{w}}{dt} = \mathbf{u}(v - \langle v \rangle)$ — Unstable (unless constraint on $\|\mathbf{w}\|$ is imposed)

- Oja's rule:

$\tau_w \frac{d\mathbf{w}}{dt} = \mathbf{u}v - \alpha v^2 \mathbf{w}$ — Stable $\quad \|\mathbf{w}\| \to \frac{1}{\sqrt{\alpha}}$

What does Hebbian Learning do anyway?

- ✦ Start with the averaged Hebb rule: $\tau_w \frac{d\mathbf{w}}{dt} = Q\mathbf{w}$
- ✦ How do we solve this equation to find $\mathbf{w}(t)$?
  - ➪ Eigenvectors to the rescue (again)!
- ✦ Write $\mathbf{w}(t)$ in terms of *eigenvectors* of Q: $\mathbf{w}(t) = \sum_i c_i(t)\mathbf{e}_i$
- ✦ Substitute in Hebb rule diff. eq. and simplify as before:

$\tau_w \frac{dc_i}{dt} = \lambda_i c_i$ i.e., $c_i(t) = c_i(0)\exp(\lambda_i t/\tau_w)$

$$\mathbf{w}(t) = \sum_i c_i(t)\mathbf{e}_i = \sum_i c_i(0)\exp(\lambda_i t/\tau_w)\mathbf{e}_i$$

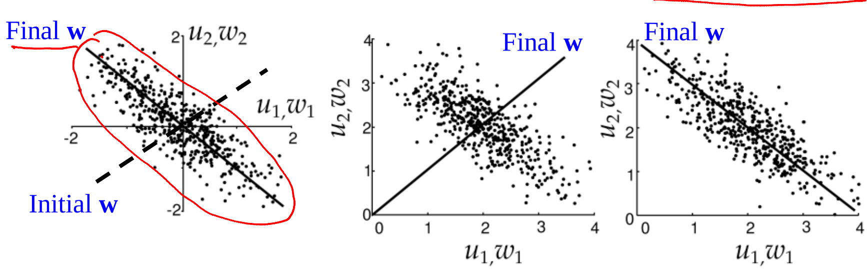
For large t, largest eigenvalue term dominates: $\mathbf{w}(t) \propto \mathbf{e}_1$

(For Oja's rule: $\mathbf{w}(t) = \frac{\mathbf{e}_1}{\sqrt{\alpha}}$)

The Brain can do Statistics!*

Hebbian Learning implements *Principal Component Analysis* (PCA)

Hebb Rule	Hebb Rule	Covariance Rule
Input mean = (0,0)	Input mean = (2,2)	Input mean = (2,2)

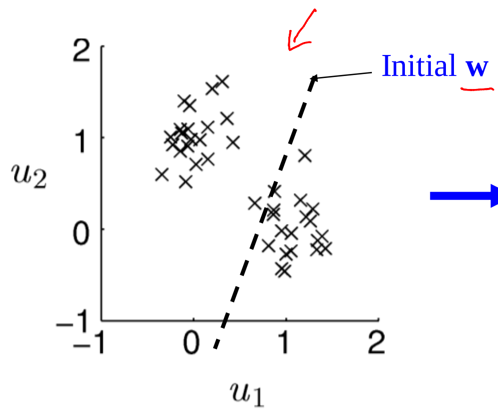


Hebbian learning learns a weight vector aligned with the principal eigenvector of input correlation/covariance matrix (i.e., direction of maximum variance)

*See last week's lecture for "The Brain can do Calculus!"

Image Source: Dayan & Abbott textbook

What about this input data?



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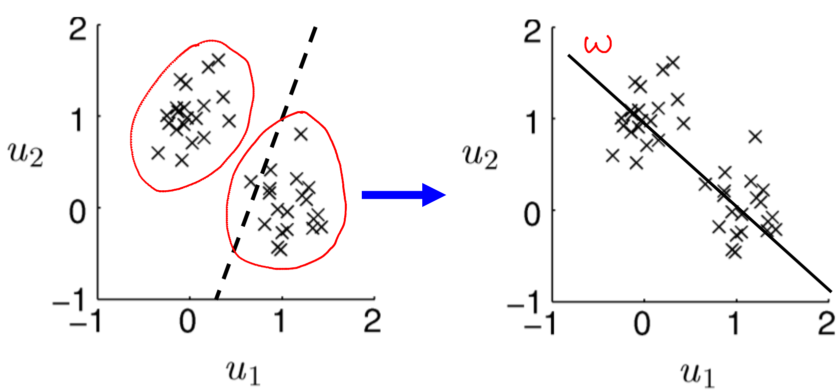
What does the covariance rule learn?

Image Source: Dayan & Abbott textbook

PCA does not correctly describe the data



What should a network of neurons learn from such data?

Next Lecture: Competitive Learning, Generative Models, and Unsupervised Learning

Image Source: Dayan & Abbott textbook