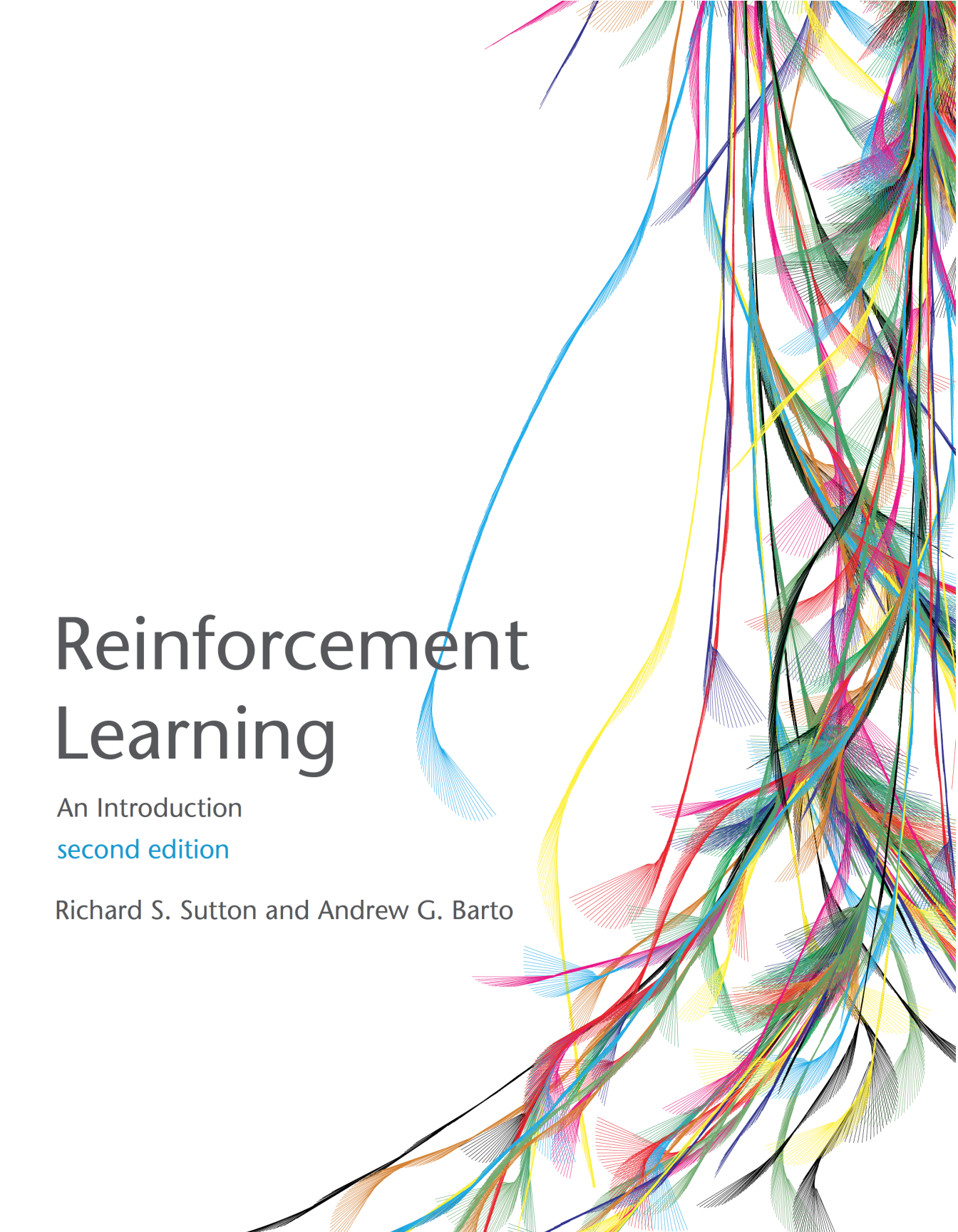

Reinforcement Learning

An Introduction

second edition

Richard S. Sutton and Andrew G. Barto

Adaptive Computation and Machine Learning

Francis Bach

A complete list of books published in the Adaptive Computation and Machine Learning series appears at the back of this book.

The cover design is based on the trajectories of a simulated bicycle controlled by a reinforcement learning system developed by Jette Randløv.

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In memory of A. Harry Klopf

Contents

Preface to the Second Edition

The twenty years since the publication of the first edition of this book have seen tremendous progress in artificial intelligence, propelled in large part by advances in machine learning, including advances in reinforcement learning. Although the impressive computational power that became available is responsible for some of these advances, new developments in theory and algorithms have been driving forces as well. In the face of this progress, a second edition of our 1998 book was long overdue, and we finally began the project in 2012. Our goal for the second edition was the same as our goal for the first: to provide a clear and simple account of the key ideas and algorithms of reinforcement learning that is accessible to readers in all the related disciplines. The edition remains an introduction, and we retain a focus on core, online learning algorithms. This edition includes some new topics that rose to importance over the intervening years, and we expanded coverage of topics that we now understand better. But we made no attempt to provide comprehensive coverage of the field, which has exploded in many different directions. We apologize for having to leave out all but a handful of these contributions.

As in the first edition, we chose not to produce a rigorous formal treatment of reinforcement learning, or to formulate it in the most general terms. However, our deeper understanding of some topics since the first edition required a bit more mathematics to explain; we have set off the more mathematical parts in shaded boxes that the non-mathematically-inclined may choose to skip. We also use a slightly different notation than was used in the first edition. In teaching, we have found that the new notation helps to address some common points of confusion. It emphasizes the difference between random variables, denoted with capital letters, and their instantiations, denoted in lower case. For example, the state, action, and reward at time step t are denoted S_t, A_t, and R_t, while their possible values might be denoted s, a, and r. Along with this, it is natural to use lower case for value functions (e.g., v_π) and restrict capitals to their tabular estimates (e.g., $Q_t(s,a)$). Approximate value functions are deterministic functions of random parameters and are thus also in lower case (e.g., $\hat{v}(s,\mathbf{w}_t) \approx v_\pi(s)$). Vectors, such as the weight vector $\mathbf{w}_t$ (formerly $\boldsymbol{\theta}_t$) and the feature vector $\mathbf{x}_t$ (formerly $\boldsymbol{\phi}_t$), are bold and written in lowercase even if they are random variables. Uppercase bold is reserved for matrices. In the first edition we used special notations, $\mathcal{P}^a_{ss'}$ and $\mathcal{R}^a_{ss'}$, for the transition probabilities and expected rewards. One weakness of that notation is that it still did not fully characterize the dynamics of the rewards, giving only their expectations, which is sufficient for dynamic programming but not for reinforcement learning. Another weakness

is the excess of subscripts and superscripts. In this edition we use the explicit notation of $p(s', r \mid s, a)$ for the joint probability for the next state and reward given the current state and action. All the changes in notation are summarized in a table on page xix.

The second edition is significantly expanded, and its top-level organization has been changed. After the introductory first chapter, the second edition is divided into three new parts. The first part (Chapters 2–8) treats as much of reinforcement learning as possible without going beyond the tabular case for which exact solutions can be found. We cover both learning and planning methods for the tabular case, as well as their unification in n-step methods and in Dyna. Many algorithms presented in this part are new to the second edition, including UCB, Expected Sarsa, Double learning, tree-backup, $Q(\sigma)$, RTDP, and MCTS. Doing the tabular case first, and thoroughly, enables core ideas to be developed in the simplest possible setting. The second part of the book (Chapters 9–13) is then devoted to extending the ideas to function approximation. It has new sections on artificial neural networks, the fourier basis, LSTD, kernel-based methods, Gradient-TD and Emphatic-TD methods, average-reward methods, true online TD(λ), and policy-gradient methods. The second edition significantly expands the treatment of off-policy learning, first for the tabular case in Chapters 5–7, then with function approximation in Chapters 11 and 12. Another change is that the second edition separates the forward-view idea of n-step bootstrapping (now treated more fully in Chapter 7) from the backward-view idea of eligibility traces (now treated independently in Chapter 12). The third part of the book has large new chapters on reinforcement learning's relationships to psychology (Chapter 14) and neuroscience (Chapter 15), as well as an updated case-studies chapter including Atari game playing, Watson's wagering strategy, and the Go playing programs AlphaGo and AlphaGo Zero (Chapter 16). Still, out of necessity we have included only a small subset of all that has been done in the field. Our choices reflect our long-standing interests in inexpensive model-free methods that should scale well to large applications. The final chapter now includes a discussion of the future societal impacts of reinforcement learning. For better or worse, the second edition is about twice as large as the first.

This book is designed to be used as the primary text for a one- or two-semester course on reinforcement learning. For a one-semester course, the first ten chapters should be covered in order and form a good core, to which can be added material from the other chapters, from other books such as Bertsekas and Tsitsiklis (1996), Wiering and van Otterlo (2012), and Szepesvári (2010), or from the literature, according to taste. Depending of the students' background, some additional material on online supervised learning may be helpful. The ideas of options and option models are a natural addition (Sutton, Precup and Singh, 1999). A two-semester course can cover all the chapters as well as supplementary material. The book can also be used as part of broader courses on machine learning, artificial intelligence, or neural networks. In this case, it may be desirable to cover only a subset of the material. We recommend covering Chapter 1 for a brief overview, Chapter 2 through Section 2.4, Chapter 3, and then selecting sections from the remaining chapters according to time and interests. Chapter 6 is the most important for the subject and for the rest of the book. A course focusing on machine learning or neural networks should cover Chapters 9 and 10, and a course focusing on artificial intelligence or planning should cover Chapter 8. Throughout the book, sections and chapters that are more difficult and not essential to the rest of the book are marked

with a $*$. These can be omitted on first reading without creating problems later on. Some exercises are also marked with a $*$ to indicate that they are more advanced and not essential to understanding the basic material of the chapter.

Most chapters end with a section entitled "Bibliographical and Historical Remarks," wherein we credit the sources of the ideas presented in that chapter, provide pointers to further reading and ongoing research, and describe relevant historical background. Despite our attempts to make these sections authoritative and complete, we have undoubtedly left out some important prior work. For that we again apologize, and we welcome corrections and extensions for incorporation into the electronic version of the book.

Like the first edition, this edition of the book is dedicated to the memory of A. Harry Klopf. It was Harry who introduced us to each other, and it was his ideas about the brain and artificial intelligence that launched our long excursion into reinforcement learning. Trained in neurophysiology and long interested in machine intelligence, Harry was a senior scientist affiliated with the Avionics Directorate of the Air Force Office of Scientific Research (AFOSR) at Wright-Patterson Air Force Base, Ohio. He was dissatisfied with the great importance attributed to equilibrium-seeking processes, including homeostasis and error-correcting pattern classification methods, in explaining natural intelligence and in providing a basis for machine intelligence. He noted that systems that try to maximize something (whatever that might be) are qualitatively different from equilibrium-seeking systems, and he argued that maximizing systems hold the key to understanding important aspects of natural intelligence and for building artificial intelligences. Harry was instrumental in obtaining funding from AFOSR for a project to assess the scientific merit of these and related ideas. This project was conducted in the late 1970s at the University of Massachusetts Amherst (UMass Amherst), initially under the direction of Michael Arbib, William Kilmer, and Nico Spinelli, professors in the Department of Computer and Information Science at UMass Amherst, and founding members of the Cybernetics Center for Systems Neuroscience at the University, a farsighted group focusing on the intersection of neuroscience and artificial intelligence. Barto, a recent Ph.D. from the University of Michigan, was hired as post doctoral researcher on the project. Meanwhile, Sutton, an undergraduate studying computer science and psychology at Stanford, had been corresponding with Harry regarding their mutual interest in the role of stimulus timing in classical conditioning. Harry suggested to the UMass group that Sutton would be a great addition to the project. Thus, Sutton became a UMass graduate student, whose Ph.D. was directed by Barto, who had become an Associate Professor. The study of reinforcement learning as presented in this book is rightfully an outcome of that project instigated by Harry and inspired by his ideas. Further, Harry was responsible for bringing us, the authors, together in what has been a long and enjoyable interaction. By dedicating this book to Harry we honor his essential contributions, not only to the field of reinforcement learning, but also to our collaboration. We also thank Professors Arbib, Kilmer, and Spinelli for the opportunity they provided to us to begin exploring these ideas. Finally, we thank AFOSR for generous support over the early years of our research, and the NSF for its generous support over many of the following years.

We have very many people to thank for their inspiration and help with this second edition. Everyone we acknowledged for their inspiration and help with the first edition

deserve our deepest gratitude for this edition as well, which would not exist were it not for their contributions to edition number one. To that long list we must add many others who contributed specifically to the second edition. Our students over the many years that we have taught this material contributed in countless ways: exposing errors, offering fixes, and—not the least—being confused in places where we could have explained things better. We especially thank Martha Steenstrup for reading and providing detailed comments throughout. The chapters on psychology and neuroscience could not have been written without the help of many experts in those fields. We thank John Moore for his patient tutoring over many many years on animal learning experiments, theory, and neuroscience, and for his careful reading of multiple drafts of Chapters 14 and 15. We also thank Matt Botvinick, Nathaniel Daw, Peter Dayan, and Yael Niv for their penetrating comments on drafts of these chapter, their essential guidance through the massive literature, and their interception of many of our errors in early drafts. Of course, the remaining errors in these chapters—and there must still be some—are totally our own. We thank Phil Thomas for helping us make these chapters accessible to non-psychologists and non-neuroscientists, and we thank Peter Sterling for helping us improve the exposition. We are grateful to Jim Houk for introducing us to the subject of information processing in the basal ganglia and for alerting us to other relevant aspects of neuroscience. José Martínez, Terry Sejnowski, David Silver, Gerry Tesauro, Georgios Theocharous, and Phil Thomas generously helped us understand details of their reinforcement learning applications for inclusion in the case-studies chapter, and they provided helpful comments on drafts of these sections. Special thanks are owed to David Silver for helping us better understand Monte Carlo Tree Search and the DeepMind Go-playing programs. We thank George Konidaris for his help with the section on the Fourier basis. Emilio Cartoni, Thomas Cederborg, Stefan Dernbach, Clemens Rosenbaum, Patrick Taylor, Thomas Colin, and Pierre-Luc Bacon helped us in a number important ways for which we are most grateful.

Sutton would also like to thank the members of the Reinforcement Learning and Artificial Intelligence laboratory at the University of Alberta for contributions to the second edition. He owes a particular debt to Rupam Mahmood for essential contributions to the treatment of off-policy Monte Carlo methods in Chapter 5, to Hamid Maei for helping develop the perspective on off-policy learning presented in Chapter 11, to Eric Graves for conducting the experiments in Chapter 13, to Shangtong Zhang for replicating and thus verifying almost all the experimental results, to Kris De Asis for improving the new technical content of Chapters 7 and 12, and to Harm van Seijen for insights that led to the separation of n-step methods from eligibility traces and (along with Hado van Hasselt) for the ideas involving exact equivalence of forward and backward views of eligibility traces presented in Chapter 12. Sutton also gratefully acknowledges the support and freedom he was granted by the Government of Alberta and the National Science and Engineering Research Council of Canada throughout the period during which the second edition was conceived and written. In particular, he would like to thank Randy Goebel for creating a supportive and far-sighted environment for research in Alberta. He would also like to thank DeepMind their support in the last six months of writing the book.

Finally, we owe thanks to the many careful readers of drafts of the second edition that we posted on the internet. They found many errors that we had missed and alerted us to potential points of confusion.

Preface to the First Edition

We first came to focus on what is now known as reinforcement learning in late 1979. We were both at the University of Massachusetts, working on one of the earliest projects to revive the idea that networks of neuronlike adaptive elements might prove to be a promising approach to artificial adaptive intelligence. The project explored the "heterostatic theory of adaptive systems" developed by A. Harry Klopf. Harry's work was a rich source of ideas, and we were permitted to explore them critically and compare them with the long history of prior work in adaptive systems. Our task became one of teasing the ideas apart and understanding their relationships and relative importance. This continues today, but in 1979 we came to realize that perhaps the simplest of the ideas, which had long been taken for granted, had received surprisingly little attention from a computational perspective. This was simply the idea of a learning system that *wants* something, that adapts its behavior in order to maximize a special signal from its environment. This was the idea of a "hedonistic" learning system, or, as we would say now, the idea of reinforcement learning.

Like others, we had a sense that reinforcement learning had been thoroughly explored in the early days of cybernetics and artificial intelligence. On closer inspection, though, we found that it had been explored only slightly. While reinforcement learning had clearly motivated some of the earliest computational studies of learning, most of these researchers had gone on to other things, such as pattern classification, supervised learning, and adaptive control, or they had abandoned the study of learning altogether. As a result, the special issues involved in learning how to get something from the environment received relatively little attention. In retrospect, focusing on this idea was the critical step that set this branch of research in motion. Little progress could be made in the computational study of reinforcement learning until it was recognized that such a fundamental idea had not yet been thoroughly explored.

The field has come a long way since then, evolving and maturing in several directions. Reinforcement learning has gradually become one of the most active research areas in machine learning, artificial intelligence, and neural network research. The field has developed strong mathematical foundations and impressive applications. The computational study of reinforcement learning is now a large field, with hundreds of active researchers around the world in diverse disciplines such as psychology, control theory, artificial intelligence, and neuroscience. Particularly important have been the contributions establishing and developing the relationships to the theory of optimal control and dynamic programming.

The overall problem of learning from interaction to achieve goals is still far from being solved, but our understanding of it has improved significantly. We can now place component ideas, such as temporal-difference learning, dynamic programming, and function approximation, within a coherent perspective with respect to the overall problem.

Our goal in writing this book was to provide a clear and simple account of the key ideas and algorithms of reinforcement learning. We wanted our treatment to be accessible to readers in all of the related disciplines, but we could not cover all of these perspectives in detail. For the most part, our treatment takes the point of view of artificial intelligence and engineering. Coverage of connections to other fields we leave to others or to another time. We also chose not to produce a rigorous formal treatment of reinforcement learning. We did not reach for the highest possible level of mathematical abstraction and did not rely on a theorem–proof format. We tried to choose a level of mathematical detail that points the mathematically inclined in the right directions without distracting from the simplicity and potential generality of the underlying ideas.

...

In some sense we have been working toward this book for thirty years, and we have lots of people to thank. First, we thank those who have personally helped us develop the overall view presented in this book: Harry Klopf, for helping us recognize that reinforcement learning needed to be revived; Chris Watkins, Dimitri Bertsekas, John Tsitsiklis, and Paul Werbos, for helping us see the value of the relationships to dynamic programming; John Moore and Jim Kehoe, for insights and inspirations from animal learning theory; Oliver Selfridge, for emphasizing the breadth and importance of adaptation; and, more generally, our colleagues and students who have contributed in countless ways: Ron Williams, Charles Anderson, Satinder Singh, Sridhar Mahadevan, Steve Bradtke, Bob Crites, Peter Dayan, and Leemon Baird. Our view of reinforcement learning has been significantly enriched by discussions with Paul Cohen, Paul Utgoff, Martha Steenstrup, Gerry Tesauro, Mike Jordan, Leslie Kaelbling, Andrew Moore, Chris Atkeson, Tom Mitchell, Nils Nilsson, Stuart Russell, Tom Dietterich, Tom Dean, and Bob Narendra. We thank Michael Littman, Gerry Tesauro, Bob Crites, Satinder Singh, and Wei Zhang for providing specifics of Sections 4.7, 15.1, 15.4, 15.4, and 15.6 respectively. We thank the Air Force Office of Scientific Research, the National Science Foundation, and GTE Laboratories for their long and farsighted support.

We also wish to thank the many people who have read drafts of this book and provided valuable comments, including Tom Kalt, John Tsitsiklis, Pawel Cichosz, Olle Gällmo, Chuck Anderson, Stuart Russell, Ben Van Roy, Paul Steenstrup, Paul Cohen, Sridhar Mahadevan, Jette Randlov, Brian Sheppard, Thomas O'Connell, Richard Coggins, Cristina Versino, John H. Hiett, Andreas Badelt, Jay Ponte, Joe Beck, Justus Piater, Martha Steenstrup, Satinder Singh, Tommi Jaakkola, Dimitri Bertsekas, Torbjörn Ekman, Christina Björkman, Jakob Carlström, and Olle Palmgren. Finally, we thank Gwyn Mitchell for helping in many ways, and Harry Stanton and Bob Prior for being our champions at MIT Press.

Summary of Notation

Capital letters are used for random variables, whereas lower case letters are used for the values of random variables and for scalar functions. Quantities that are required to be real-valued vectors are written in bold and in lower case (even if random variables). Matrices are bold capitals.

$\doteq$	equality relationship that is true by definition
$\approx$	approximately equal
$\propto$	proportional to
$\Pr\{X=x\}$	probability that a random variable X takes on the value x
$X \sim p$	random variable X selected from distribution $p(x) \doteq \Pr\{X=x\}$
$\mathbb{E}[X]$	expectation of a random variable X, i.e., $\mathbb{E}[X] \doteq \sum_x p(x)x$
$\arg\max_a f(a)$	a value of a at which $f(a)$ takes its maximal value
$\ln x$	natural logarithm of x
e^x	the base of the natural logarithm, $e \approx 2.71828$, carried to power x; $e^{\ln x} = x$
$\mathbb{R}$	set of real numbers
$f : \mathcal{X} \to \mathcal{Y}$	function f from elements of set $\mathcal{X}$ to elements of set $\mathcal{Y}$
$\leftarrow$	assignment
$(a, b]$	the real interval between a and b including b but not including a

ε	probability of taking a random action in an ε-greedy policy
α, β	step-size parameters
γ	discount-rate parameter
λ	decay-rate parameter for eligibility traces
$\mathbb{1}_{predicate}$	indicator function ($\mathbb{1}_{predicate} \doteq 1$ if the *predicate* is true, else 0)

In a multi-arm bandit problem:

k	number of actions (arms)
t	discrete time step or play number
$q_*(a)$	true value (expected reward) of action a
$Q_t(a)$	estimate at time t of $q_*(a)$
$N_t(a)$	number of times action a has been selected up prior to time t
$H_t(a)$	learned preference for selecting action a at time t
$\pi_t(a)$	probability of selecting action a at time t
$\bar{R}_t$	estimate at time t of the expected reward given π_t

In a Markov Decision Process:

s, s'	states
a	an action
r	a reward
$\mathcal{S}$	set of all nonterminal states
$\mathcal{S}^+$	set of all states, including the terminal state
$\mathcal{A}(s)$	set of all actions available in state s
$\mathcal{R}$	set of all possible rewards, a finite subset of $\mathbb{R}$
$\subset$	subset of; e.g., $\mathcal{R} \subset \mathbb{R}$
$\in$	is an element of; e.g., $s \in \mathcal{S}$, $r \in \mathcal{R}$
$\lvert\mathcal{S}\rvert$	number of elements in set $\mathcal{S}$
t	discrete time step
$T, T(t)$	final time step of an episode, or of the episode including time step t
A_t	action at time t
S_t	state at time t, typically due, stochastically, to S_{t-1} and A_{t-1}
R_t	reward at time t, typically due, stochastically, to S_{t-1} and A_{t-1}
π	policy (decision-making rule)
$\pi(s)$	action taken in state s under *deterministic* policy π
$\pi(a\lvert s)$	probability of taking action a in state s under *stochastic* policy π
G_t	return following time t
h	horizon, the time step one looks up to in a forward view
$G_{t:t+n}, G_{t:h}$	n-step return from $t+1$ to $t+n$, or to h (discounted and corrected)
$\bar{G}_{t:h}$	flat return (undiscounted and uncorrected) from $t+1$ to h (Section 5.8)
G_t^λ	λ-return (Section 12.1)
$G_{t:h}^\lambda$	truncated, corrected λ-return (Section 12.3)
$G_t^{\lambda s}, G_t^{\lambda a}$	λ-return, corrected by estimated state, or action, values (Section 12.8)
$p(s', r \,\vert\, s, a)$	probability of transition to state s' with reward r, from state s and action a
$p(s' \,\vert\, s, a)$	probability of transition to state s', from state s taking action a
$r(s, a)$	expected immediate reward from state s after action a
$r(s, a, s')$	expected immediate reward on transition from s to s' under action a
$v_\pi(s)$	value of state s under policy π (expected return)
$v_*(s)$	value of state s under the optimal policy
$q_\pi(s, a)$	value of taking action a in state s under policy π
$q_*(s, a)$	value of taking action a in state s under the optimal policy
V, V_t	array estimates of state-value function v_π or v_*
Q, Q_t	array estimates of action-value function q_π or q_*
$\bar{V}_t(s)$	expected approximate action value, e.g., $\bar{V}_t(s) \doteq \sum_a \pi(a\lvert s) Q_t(s, a)$
U_t	target for estimate at time t

δ_t	temporal-difference (TD) error at t (a random variable) (Section 6.1)
δ_t^s, δ_t^a	state- and action-specific forms of the TD error (Section 12.9)
n	in n-step methods, n is the number of steps of bootstrapping
d	dimensionality—the number of components of $\mathbf{w}$
d'	alternate dimensionality—the number of components of $\boldsymbol{\theta}$
$\mathbf{w}, \mathbf{w}_t$	d-vector of weights underlying an approximate value function
$w_i, w_{t,i}$	ith component of learnable weight vector
$\hat{v}(s,\mathbf{w})$	approximate value of state s given weight vector $\mathbf{w}$
$v_{\mathbf{w}}(s)$	alternate notation for $\hat{v}(s,\mathbf{w})$
$\hat{q}(s,a,\mathbf{w})$	approximate value of state–action pair s, a given weight vector $\mathbf{w}$
$\nabla\hat{v}(s,\mathbf{w})$	column vector of partial derivatives of $\hat{v}(s,\mathbf{w})$ with respect to $\mathbf{w}$
$\nabla\hat{q}(s,a,\mathbf{w})$	column vector of partial derivatives of $\hat{q}(s,a,\mathbf{w})$ with respect to $\mathbf{w}$
$\mathbf{x}(s)$	vector of features visible when in state s
$\mathbf{x}(s,a)$	vector of features visible when in state s taking action a
$x_i(s), x_i(s,a)$	ith component of vector $\mathbf{x}(s)$ or $\mathbf{x}(s,a)$
$\mathbf{x}_t$	shorthand for $\mathbf{x}(S_t)$ or $\mathbf{x}(S_t, A_t)$
$\mathbf{w}^\top\mathbf{x}$	inner product of vectors, $\mathbf{w}^\top\mathbf{x} \doteq \sum_i w_i x_i$; e.g., $\hat{v}(s,\mathbf{w}) \doteq \mathbf{w}^\top\mathbf{x}(s)$
$\mathbf{v}, \mathbf{v}_t$	secondary d-vector of weights, used to learn $\mathbf{w}$ (Chapter 11)
$\mathbf{z}_t$	d-vector of eligibility traces at time t (Chapter 12)
$\boldsymbol{\theta}, \boldsymbol{\theta}_t$	parameter vector of target policy (Chapter 13)
$\pi(a\|s,\boldsymbol{\theta})$	probability of taking action a in state s given parameter vector $\boldsymbol{\theta}$
$\pi_{\boldsymbol{\theta}}$	policy corresponding to parameter $\boldsymbol{\theta}$
$\nabla\pi(a\|s,\boldsymbol{\theta})$	column vector of partial derivatives of $\pi(a\|s,\boldsymbol{\theta})$ with respect to $\boldsymbol{\theta}$
$J(\boldsymbol{\theta})$	performance measure for the policy $\pi_{\boldsymbol{\theta}}$
$\nabla J(\boldsymbol{\theta})$	column vector of partial derivatives of $J(\boldsymbol{\theta})$ with respect to $\boldsymbol{\theta}$
$h(s,a,\boldsymbol{\theta})$	preference for selecting action a in state s based on $\boldsymbol{\theta}$
$b(a\|s)$	behavior policy used to select actions while learning about target policy π
$b(s)$	a baseline function $b : \mathcal{S} \mapsto \mathbb{R}$ for policy-gradient methods
b	branching factor for an MDP or search tree
$\rho_{t:h}$	importance sampling ratio for time t through time h (Section 5.5)
ρ_t	importance sampling ratio for time t alone, $\rho_t \doteq \rho_{t:t}$
$r(\pi)$	average reward (reward rate) for policy π (Section 10.3)
$\bar{R}_t$	estimate of $r(\pi)$ at time t
$\mu(s)$	on-policy distribution over states (Section 9.2)
$\boldsymbol{\mu}$	$\|\mathcal{S}\|$-vector of the $\mu(s)$ for all $s \in \mathcal{S}$
$\|v\|_\mu^2$	μ-weighted squared norm of value function v, i.e., $\|v\|_\mu^2 \doteq \sum_{s\in\mathcal{S}} \mu(s)v(s)^2$
$\eta(s)$	expected number of visits to state s per episode (page 199)
Π	projection operator for value functions (page 268)
B_π	Bellman operator for value functions (Section 11.4)

$\mathbf{A}$	$d \times d$ matrix $\mathbf{A} \doteq \mathbb{E}\Big[\mathbf{x}_t\big(\mathbf{x}_t - \gamma\mathbf{x}_{t+1}\big)^\top\Big]$
$\mathbf{b}$	d-dimensional vector $\mathbf{b} \doteq \mathbb{E}[R_{t+1}\mathbf{x}_t]$
$\mathbf{w}_{\text{TD}}$	TD fixed point $\mathbf{w}_{\text{TD}} \doteq \mathbf{A}^{-1}\mathbf{b}$ (a d-vector, Section 9.4)
$\mathbf{I}$	identity matrix
$\mathbf{P}$	$\lvert\mathcal{S}\rvert \times \lvert\mathcal{S}\rvert$ matrix of state-transition probabilities under π
$\mathbf{D}$	$\lvert\mathcal{S}\rvert \times \lvert\mathcal{S}\rvert$ diagonal matrix with $\boldsymbol{\mu}$ on its diagonal
$\mathbf{X}$	$\lvert\mathcal{S}\rvert \times d$ matrix with the $\mathbf{x}(s)$ as its rows
$\bar{\delta}_{\mathbf{w}}(s)$	Bellman error (expected TD error) for $v_{\mathbf{w}}$ at state s (Section 11.4)
$\bar{\delta}_{\mathbf{w}}$, BE	Bellman error vector, with components $\bar{\delta}_{\mathbf{w}}(s)$
$\overline{\text{VE}}(\mathbf{w})$	mean square value error $\overline{\text{VE}}(\mathbf{w}) \doteq \lVert v_{\mathbf{w}} - v_\pi\rVert^2_\mu$ (Section 9.2)
$\overline{\text{BE}}(\mathbf{w})$	mean square Bellman error $\overline{\text{BE}}(\mathbf{w}) \doteq \big\lVert\bar{\delta}_{\mathbf{w}}\big\rVert^2_\mu$
$\overline{\text{PBE}}(\mathbf{w})$	mean square projected Bellman error $\overline{\text{PBE}}(\mathbf{w}) \doteq \big\lVert\Pi\bar{\delta}_{\mathbf{w}}\big\rVert^2_\mu$
$\overline{\text{TDE}}(\mathbf{w})$	mean square temporal-difference error $\overline{\text{TDE}}(\mathbf{w}) \doteq \mathbb{E}_b\big[\rho_t\delta_t^2\big]$ (Section 11.5)
$\overline{\text{RE}}(\mathbf{w})$	mean square return error (Section 11.6)

Chapter 1

Introduction

The idea that we learn by interacting with our environment is probably the first to occur to us when we think about the nature of learning. When an infant plays, waves its arms, or looks about, it has no explicit teacher, but it does have a direct sensorimotor connection to its environment. Exercising this connection produces a wealth of information about cause and effect, about the consequences of actions, and about what to do in order to achieve goals. Throughout our lives, such interactions are undoubtedly a major source of knowledge about our environment and ourselves. Whether we are learning to drive a car or to hold a conversation, we are acutely aware of how our environment responds to what we do, and we seek to influence what happens through our behavior. Learning from interaction is a foundational idea underlying nearly all theories of learning and intelligence.

In this book we explore a *computational* approach to learning from interaction. Rather than directly theorizing about how people or animals learn, we primarily explore idealized learning situations and evaluate the effectiveness of various learning methods.[1] That is, we adopt the perspective of an artificial intelligence researcher or engineer. We explore designs for machines that are effective in solving learning problems of scientific or economic interest, evaluating the designs through mathematical analysis or computational experiments. The approach we explore, called *reinforcement learning*, is much more focused on goal-directed learning from interaction than are other approaches to machine learning.

1.1 Reinforcement Learning

Reinforcement learning is learning what to do—how to map situations to actions—so as to maximize a numerical reward signal. The learner is not told which actions to take, but instead must discover which actions yield the most reward by trying them. In the most interesting and challenging cases, actions may affect not only the immediate

[1] The relationships to psychology and neuroscience are summarized in Chapters 14 and 15.

reward but also the next situation and, through that, all subsequent rewards. These two characteristics—trial-and-error search and delayed reward—are the two most important distinguishing features of reinforcement learning.

Reinforcement learning, like many topics whose names end with "ing," such as machine learning and mountaineering, is simultaneously a problem, a class of solution methods that work well on the problem, and the field that studies this problem and its solution methods. It is convenient to use a single name for all three things, but at the same time essential to keep the three conceptually separate. In particular, the distinction between problems and solution methods is very important in reinforcement learning; failing to make this distinction is the source of many confusions.

We formalize the problem of reinforcement learning using ideas from dynamical systems theory, specifically, as the optimal control of incompletely-known Markov decision processes. The details of this formalization must wait until Chapter 3, but the basic idea is simply to capture the most important aspects of the real problem facing a learning agent interacting over time with its environment to achieve a goal. A learning agent must be able to sense the state of its environment to some extent and must be able to take actions that affect the state. The agent also must have a goal or goals relating to the state of the environment. Markov decision processes are intended to include just these three aspects—sensation, action, and goal—in their simplest possible forms without trivializing any of them. Any method that is well suited to solving such problems we consider to be a reinforcement learning method.

Reinforcement learning is different from *supervised learning*, the kind of learning studied in most current research in the field of machine learning. Supervised learning is learning from a training set of labeled examples provided by a knowledgable external supervisor. Each example is a description of a situation together with a specification—the label—of the correct action the system should take to that situation, which is often to identify a category to which the situation belongs. The object of this kind of learning is for the system to extrapolate, or generalize, its responses so that it acts correctly in situations not present in the training set. This is an important kind of learning, but alone it is not adequate for learning from interaction. In interactive problems it is often impractical to obtain examples of desired behavior that are both correct and representative of all the situations in which the agent has to act. In uncharted territory—where one would expect learning to be most beneficial—an agent must be able to learn from its own experience.

Reinforcement learning is also different from what machine learning researchers call *unsupervised learning*, which is typically about finding structure hidden in collections of unlabeled data. The terms supervised learning and unsupervised learning would seem to exhaustively classify machine learning paradigms, but they do not. Although one might be tempted to think of reinforcement learning as a kind of unsupervised learning because it does not rely on examples of correct behavior, reinforcement learning is trying to maximize a reward signal instead of trying to find hidden structure. Uncovering structure in an agent's experience can certainly be useful in reinforcement learning, but by itself does not address the reinforcement learning problem of maximizing a reward signal. We therefore consider reinforcement learning to be a third machine learning paradigm, alongside supervised learning and unsupervised learning and perhaps other paradigms.

One of the challenges that arise in reinforcement learning, and not in other kinds of learning, is the trade-off between exploration and exploitation. To obtain a lot of reward, a reinforcement learning agent must prefer actions that it has tried in the past and found to be effective in producing reward. But to discover such actions, it has to try actions that it has not selected before. The agent has to *exploit* what it has already experienced in order to obtain reward, but it also has to *explore* in order to make better action selections in the future. The dilemma is that neither exploration nor exploitation can be pursued exclusively without failing at the task. The agent must try a variety of actions *and* progressively favor those that appear to be best. On a stochastic task, each action must be tried many times to gain a reliable estimate of its expected reward. The exploration–exploitation dilemma has been intensively studied by mathematicians for many decades, yet remains unresolved. For now, we simply note that the entire issue of balancing exploration and exploitation does not even arise in supervised and unsupervised learning, at least in the purest forms of these paradigms.

Another key feature of reinforcement learning is that it explicitly considers the *whole* problem of a goal-directed agent interacting with an uncertain environment. This is in contrast to many approaches that consider subproblems without addressing how they might fit into a larger picture. For example, we have mentioned that much of machine learning research is concerned with supervised learning without explicitly specifying how such an ability would finally be useful. Other researchers have developed theories of planning with general goals, but without considering planning's role in real-time decision making, or the question of where the predictive models necessary for planning would come from. Although these approaches have yielded many useful results, their focus on isolated subproblems is a significant limitation.

Reinforcement learning takes the opposite tack, starting with a complete, interactive, goal-seeking agent. All reinforcement learning agents have explicit goals, can sense aspects of their environments, and can choose actions to influence their environments. Moreover, it is usually assumed from the beginning that the agent has to operate despite significant uncertainty about the environment it faces. When reinforcement learning involves planning, it has to address the interplay between planning and real-time action selection, as well as the question of how environment models are acquired and improved. When reinforcement learning involves supervised learning, it does so for specific reasons that determine which capabilities are critical and which are not. For learning research to make progress, important subproblems have to be isolated and studied, but they should be subproblems that play clear roles in complete, interactive, goal-seeking agents, even if all the details of the complete agent cannot yet be filled in.

By a complete, interactive, goal-seeking agent we do not always mean something like a complete organism or robot. These are clearly examples, but a complete, interactive, goal-seeking agent can also be a component of a larger behaving system. In this case, the agent directly interacts with the rest of the larger system and indirectly interacts with the larger system's environment. A simple example is an agent that monitors the charge level of robot's battery and sends commands to the robot's control architecture. This agent's environment is the rest of the robot together with the robot's environment. One must look beyond the most obvious examples of agents and their environments to

appreciate the generality of the reinforcement learning framework.

One of the most exciting aspects of modern reinforcement learning is its substantive and fruitful interactions with other engineering and scientific disciplines. Reinforcement learning is part of a decades-long trend within artificial intelligence and machine learning toward greater integration with statistics, optimization, and other mathematical subjects. For example, the ability of some reinforcement learning methods to learn with parameterized approximators addresses the classical "curse of dimensionality" in operations research and control theory. More distinctively, reinforcement learning has also interacted strongly with psychology and neuroscience, with substantial benefits going both ways. Of all the forms of machine learning, reinforcement learning is the closest to the kind of learning that humans and other animals do, and many of the core algorithms of reinforcement learning were originally inspired by biological learning systems. Reinforcement learning has also given back, both through a psychological model of animal learning that better matches some of the empirical data, and through an influential model of parts of the brain's reward system. The body of this book develops the ideas of reinforcement learning that pertain to engineering and artificial intelligence, with connections to psychology and neuroscience summarized in Chapters 14 and 15.

Finally, reinforcement learning is also part of a larger trend in artificial intelligence back toward simple general principles. Since the late 1960's, many artificial intelligence researchers presumed that there are no general principles to be discovered, that intelligence is instead due to the possession of a vast number of special purpose tricks, procedures, and heuristics. It was sometimes said that if we could just get enough relevant facts into a machine, say one million, or one billion, then it would become intelligent. Methods based on general principles, such as search or learning, were characterized as "weak methods," whereas those based on specific knowledge were called "strong methods." This view is still common today, but not dominant. From our point of view, it was simply premature: too little effort had been put into the search for general principles to conclude that there were none. Modern artificial intelligence now includes much research looking for general principles of learning, search, and decision making. It is not clear how far back the pendulum will swing, but reinforcement learning research is certainly part of the swing back toward simpler and fewer general principles of artificial intelligence.

1.2 Examples

A good way to understand reinforcement learning is to consider some of the examples and possible applications that have guided its development.

- A master chess player makes a move. The choice is informed both by planning—anticipating possible replies and counterreplies—and by immediate, intuitive judgments of the desirability of particular positions and moves.
- An adaptive controller adjusts parameters of a petroleum refinery's operation in real time. The controller optimizes the yield/cost/quality trade-off on the basis of specified marginal costs without sticking strictly to the set points originally suggested by engineers.

- A gazelle calf struggles to its feet minutes after being born. Half an hour later it is running at 20 miles per hour.

- A mobile robot decides whether it should enter a new room in search of more trash to collect or start trying to find its way back to its battery recharging station. It makes its decision based on the current charge level of its battery and how quickly and easily it has been able to find the recharger in the past.

- Phil prepares his breakfast. Closely examined, even this apparently mundane activity reveals a complex web of conditional behavior and interlocking goal–subgoal relationships: walking to the cupboard, opening it, selecting a cereal box, then reaching for, grasping, and retrieving the box. Other complex, tuned, interactive sequences of behavior are required to obtain a bowl, spoon, and milk carton. Each step involves a series of eye movements to obtain information and to guide reaching and locomotion. Rapid judgments are continually made about how to carry the objects or whether it is better to ferry some of them to the dining table before obtaining others. Each step is guided by goals, such as grasping a spoon or getting to the refrigerator, and is in service of other goals, such as having the spoon to eat with once the cereal is prepared and ultimately obtaining nourishment. Whether he is aware of it or not, Phil is accessing information about the state of his body that determines his nutritional needs, level of hunger, and food preferences.

These examples share features that are so basic that they are easy to overlook. All involve *interaction* between an active decision-making agent and its environment, within which the agent seeks to achieve a *goal* despite *uncertainty* about its environment. The agent's actions are permitted to affect the future state of the environment (e.g., the next chess position, the level of reservoirs of the refinery, the robot's next location and the future charge level of its battery), thereby affecting the actions and opportunities available to the agent at later times. Correct choice requires taking into account indirect, delayed consequences of actions, and thus may require foresight or planning.

At the same time, in all of these examples the effects of actions cannot be fully predicted; thus the agent must monitor its environment frequently and react appropriately. For example, Phil must watch the milk he pours into his cereal bowl to keep it from overflowing. All these examples involve goals that are explicit in the sense that the agent can judge progress toward its goal based on what it can sense directly. The chess player knows whether or not he wins, the refinery controller knows how much petroleum is being produced, the gazelle calf knows when it falls, the mobile robot knows when its batteries run down, and Phil knows whether or not he is enjoying his breakfast.

In all of these examples the agent can use its experience to improve its performance over time. The chess player refines the intuition he uses to evaluate positions, thereby improving his play; the gazelle calf improves the efficiency with which it can run; Phil learns to streamline making his breakfast. The knowledge the agent brings to the task at the start—either from previous experience with related tasks or built into it by design or evolution—influences what is useful or easy to learn, but interaction with the environment is essential for adjusting behavior to exploit specific features of the task.

1.3 Elements of Reinforcement Learning

Beyond the agent and the environment, one can identify four main subelements of a reinforcement learning system: a *policy*, a *reward signal*, a *value function*, and, optionally, a *model* of the environment.

A *policy* defines the learning agent's way of behaving at a given time. Roughly speaking, a policy is a mapping from perceived states of the environment to actions to be taken when in those states. It corresponds to what in psychology would be called a set of stimulus–response rules or associations. In some cases the policy may be a simple function or lookup table, whereas in others it may involve extensive computation such as a search process. The policy is the core of a reinforcement learning agent in the sense that it alone is sufficient to determine behavior. In general, policies may be stochastic, specifying probabilities for each action.

A *reward signal* defines the goal of a reinforcement learning problem. On each time step, the environment sends to the reinforcement learning agent a single number called the *reward*. The agent's sole objective is to maximize the total reward it receives over the long run. The reward signal thus defines what are the good and bad events for the agent. In a biological system, we might think of rewards as analogous to the experiences of pleasure or pain. They are the immediate and defining features of the problem faced by the agent. The reward signal is the primary basis for altering the policy; if an action selected by the policy is followed by low reward, then the policy may be changed to select some other action in that situation in the future. In general, reward signals may be stochastic functions of the state of the environment and the actions taken.

Whereas the reward signal indicates what is good in an immediate sense, a *value function* specifies what is good in the long run. Roughly speaking, the *value* of a state is the total amount of reward an agent can expect to accumulate over the future, starting from that state. Whereas rewards determine the immediate, intrinsic desirability of environmental states, values indicate the *long-term* desirability of states after taking into account the states that are likely to follow and the rewards available in those states. For example, a state might always yield a low immediate reward but still have a high value because it is regularly followed by other states that yield high rewards. Or the reverse could be true. To make a human analogy, rewards are somewhat like pleasure (if high) and pain (if low), whereas values correspond to a more refined and farsighted judgment of how pleased or displeased we are that our environment is in a particular state.

Rewards are in a sense primary, whereas values, as predictions of rewards, are secondary. Without rewards there could be no values, and the only purpose of estimating values is to achieve more reward. Nevertheless, it is values with which we are most concerned when making and evaluating decisions. Action choices are made based on value judgments. We seek actions that bring about states of highest value, not highest reward, because these actions obtain the greatest amount of reward for us over the long run. Unfortunately, it is much harder to determine values than it is to determine rewards. Rewards are basically given directly by the environment, but values must be estimated and re-estimated from the sequences of observations an agent makes over its entire lifetime. In fact, the most important component of almost all reinforcement learning algorithms we consider is a

method for efficiently estimating values. The central role of value estimation is arguably the most important thing that has been learned about reinforcement learning over the last six decades.

The fourth and final element of some reinforcement learning systems is a *model* of the environment. This is something that mimics the behavior of the environment, or more generally, that allows inferences to be made about how the environment will behave. For example, given a state and action, the model might predict the resultant next state and next reward. Models are used for *planning*, by which we mean any way of deciding on a course of action by considering possible future situations before they are actually experienced. Methods for solving reinforcement learning problems that use models and planning are called *model-based* methods, as opposed to simpler *model-free* methods that are explicitly trial-and-error learners—viewed as almost the *opposite* of planning. In Chapter 8 we explore reinforcement learning systems that simultaneously learn by trial and error, learn a model of the environment, and use the model for planning. Modern reinforcement learning spans the spectrum from low-level, trial-and-error learning to high-level, deliberative planning.

1.4 Limitations and Scope

Reinforcement learning relies heavily on the concept of state—as input to the policy and value function, and as both input to and output from the model. Informally, we can think of the state as a signal conveying to the agent some sense of "how the environment is" at a particular time. The formal definition of state as we use it here is given by the framework of Markov decision processes presented in Chapter 3. More generally, however, we encourage the reader to follow the informal meaning and think of the state as whatever information is available to the agent about its environment. In effect, we assume that the state signal is produced by some preprocessing system that is nominally part of the agent's environment. We do not address the issues of constructing, changing, or learning the state signal in this book (other than briefly in Section 17.3). We take this approach not because we consider state representation to be unimportant, but in order to focus fully on the decision-making issues. In other words, our concern in this book is not with designing the state signal, but with deciding what action to take as a function of whatever state signal is available.

Most of the reinforcement learning methods we consider in this book are structured around estimating value functions, but it is not strictly necessary to do this to solve reinforcement learning problems. For example, solution methods such as genetic algorithms, genetic programming, simulated annealing, and other optimization methods never estimate value functions. These methods apply multiple static policies each interacting over an extended period of time with a separate instance of the environment. The policies that obtain the most reward, and random variations of them, are carried over to the next generation of policies, and the process repeats. We call these *evolutionary* methods because their operation is analogous to the way biological evolution produces organisms with skilled behavior even if they do not learn during their individual lifetimes. If the space of policies is sufficiently small, or can be structured so that good policies are

common or easy to find—or if a lot of time is available for the search—then evolutionary methods can be effective. In addition, evolutionary methods have advantages on problems in which the learning agent cannot sense the complete state of its environment.

Our focus is on reinforcement learning methods that learn while interacting with the environment, which evolutionary methods do not do. Methods able to take advantage of the details of individual behavioral interactions can be much more efficient than evolutionary methods in many cases. Evolutionary methods ignore much of the useful structure of the reinforcement learning problem: they do not use the fact that the policy they are searching for is a function from states to actions; they do not notice which states an individual passes through during its lifetime, or which actions it selects. In some cases this information can be misleading (e.g., when states are misperceived), but more often it should enable more efficient search. Although evolution and learning share many features and naturally work together, we do not consider evolutionary methods by themselves to be especially well suited to reinforcement learning problems and, accordingly, we do not cover them in this book.

1.5 An Extended Example: Tic-Tac-Toe

To illustrate the general idea of reinforcement learning and contrast it with other approaches, we next consider a single example in more detail.

Consider the familiar child's game of tic-tac-toe. Two players take turns playing on a three-by-three board. One player plays Xs and the other Os until one player wins by placing three marks in a row, horizontally, vertically, or diagonally, as the X player has in the game shown to the right. If the board fills up with neither player getting three in a row, then the game is a draw. Because a skilled player can play so as never to lose, let us assume that we are playing against an imperfect player, one whose play is sometimes incorrect and allows us to win. For the moment, in fact, let us consider draws and losses to be equally bad for us. How might we construct a player that will find the imperfections in its opponent's play and learn to maximize its chances of winning?

X	O	O
O	X	X
		X

Although this is a simple problem, it cannot readily be solved in a satisfactory way through classical techniques. For example, the classical "minimax" solution from game theory is not correct here because it assumes a particular way of playing by the opponent. For example, a minimax player would never reach a game state from which it could lose, even if in fact it always won from that state because of incorrect play by the opponent. Classical optimization methods for sequential decision problems, such as dynamic programming, can *compute* an optimal solution for any opponent, but require as input a complete specification of that opponent, including the probabilities with which the opponent makes each move in each board state. Let us assume that this information is not available a priori for this problem, as it is not for the vast majority of problems of practical interest. On the other hand, such information can be estimated from experience, in this case by playing many games against the opponent. About the best one can do

on this problem is first to learn a model of the opponent's behavior, up to some level of confidence, and then apply dynamic programming to compute an optimal solution given the approximate opponent model. In the end, this is not that different from some of the reinforcement learning methods we examine later in this book.

An evolutionary method applied to this problem would directly search the space of possible policies for one with a high probability of winning against the opponent. Here, a policy is a rule that tells the player what move to make for every state of the game—every possible configuration of Xs and Os on the three-by-three board. For each policy considered, an estimate of its winning probability would be obtained by playing some number of games against the opponent. This evaluation would then direct which policy or policies were considered next. A typical evolutionary method would hill-climb in policy space, successively generating and evaluating policies in an attempt to obtain incremental improvements. Or, perhaps, a genetic-style algorithm could be used that would maintain and evaluate a population of policies. Literally hundreds of different optimization methods could be applied.

Here is how the tic-tac-toe problem would be approached with a method making use of a value function. First we would set up a table of numbers, one for each possible state of the game. Each number will be the latest estimate of the probability of our winning from that state. We treat this estimate as the state's *value*, and the whole table is the learned value function. State A has higher value than state B, or is considered "better" than state B, if the current estimate of the probability of our winning from A is higher than it is from B. Assuming we always play Xs, then for all states with three Xs in a row the probability of winning is 1, because we have already won. Similarly, for all states with three Os in a row, or that are filled up, the correct probability is 0, as we cannot win from them. We set the initial values of all the other states to 0.5, representing a guess that we have a 50% chance of winning.

We then play many games against the opponent. To select our moves we examine the states that would result from each of our possible moves (one for each blank space on the board) and look up their current values in the table. Most of the time we move *greedily*, selecting the move that leads to the state with greatest value, that is, with the highest estimated probability of winning. Occasionally, however, we select randomly from among the other moves instead. These are called *exploratory* moves because they cause us to experience states that we might otherwise never see. A sequence of moves made and considered during a game can be diagrammed as in Figure 1.1.

While we are playing, we change the values of the states in which we find ourselves during the game. We attempt to make them more accurate estimates of the probabilities of winning. To do this, we "back up" the value of the state after each greedy move to the state before the move, as suggested by the arrows in Figure 1.1. More precisely, the current value of the earlier state is updated to be closer to the value of the later state. This can be done by moving the earlier state's value a fraction of the way toward the value of the later state. If we let S_t denote the state before the greedy move, and S_{t+1} the state after the move, then the update to the estimated value of S_t, denoted $V(S_t)$, can be written as

$$V(S_t) \leftarrow V(S_t) + \alpha\Big[V(S_{t+1}) - V(S_t)\Big],$$

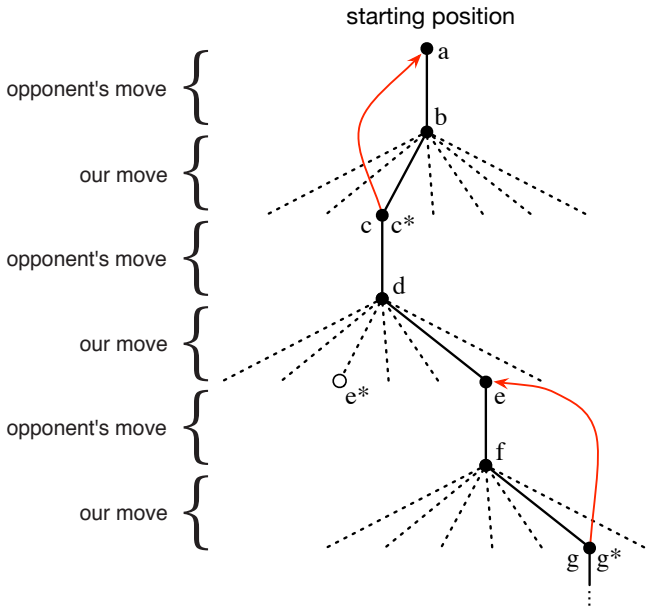


Figure 1.1: A sequence of tic-tac-toe moves. The solid black lines represent the moves taken during a game; the dashed lines represent moves that we (our reinforcement learning player) considered but did not make. Our second move was an exploratory move, meaning that it was taken even though another sibling move, the one leading to e^*, was ranked higher. Exploratory moves do not result in any learning, but each of our other moves does, causing updates as suggested by the red arrows in which estimated values are moved up the tree from later nodes to earlier nodes as detailed in the text.

where α is a small positive fraction called the *step-size parameter*, which influences the rate of learning. This update rule is an example of a *temporal-difference* learning method, so called because its changes are based on a difference, $V(S_{t+1}) - V(S_t)$, between estimates at two successive times.

The method described above performs quite well on this task. For example, if the step-size parameter is reduced properly over time, then this method converges, for any fixed opponent, to the true probabilities of winning from each state given optimal play by our player. Furthermore, the moves then taken (except on exploratory moves) are in fact the optimal moves against this (imperfect) opponent. In other words, the method converges to an optimal policy for playing the game against this opponent. If the step-size parameter is not reduced all the way to zero over time, then this player also plays well against opponents that slowly change their way of playing.

This example illustrates the differences between evolutionary methods and methods that learn value functions. To evaluate a policy an evolutionary method holds the policy fixed and plays many games against the opponent, or simulates many games using a model of the opponent. The frequency of wins gives an unbiased estimate of the probability

of winning with that policy, and can be used to direct the next policy selection. But each policy change is made only after many games, and only the final outcome of each game is used: what happens *during* the games is ignored. For example, if the player wins, then *all* of its behavior in the game is given credit, independently of how specific moves might have been critical to the win. Credit is even given to moves that never occurred! Value function methods, in contrast, allow individual states to be evaluated. In the end, evolutionary and value function methods both search the space of policies, but learning a value function takes advantage of information available during the course of play.

This simple example illustrates some of the key features of reinforcement learning methods. First, there is the emphasis on learning while interacting with an environment, in this case with an opponent player. Second, there is a clear goal, and correct behavior requires planning or foresight that takes into account delayed effects of one's choices. For example, the simple reinforcement learning player would learn to set up multi-move traps for a shortsighted opponent. It is a striking feature of the reinforcement learning solution that it can achieve the effects of planning and lookahead without using a model of the opponent and without conducting an explicit search over possible sequences of future states and actions.

While this example illustrates some of the key features of reinforcement learning, it is so simple that it might give the impression that reinforcement learning is more limited than it really is. Although tic-tac-toe is a two-person game, reinforcement learning also applies in the case in which there is no external adversary, that is, in the case of a "game against nature." Reinforcement learning also is not restricted to problems in which behavior breaks down into separate episodes, like the separate games of tic-tac-toe, with reward only at the end of each episode. It is just as applicable when behavior continues indefinitely and when rewards of various magnitudes can be received at any time. Reinforcement learning is also applicable to problems that do not even break down into discrete time steps like the plays of tic-tac-toe. The general principles apply to continuous-time problems as well, although the theory gets more complicated and we omit it from this introductory treatment.

Tic-tac-toe has a relatively small, finite state set, whereas reinforcement learning can be used when the state set is very large, or even infinite. For example, Gerry Tesauro (1992, 1995) combined the algorithm described above with an artificial neural network to learn to play backgammon, which has approximately 10^{20} states. With this many states it is impossible ever to experience more than a small fraction of them. Tesauro's program learned to play far better than any previous program and eventually better than the world's best human players (Section 16.1). The artificial neural network provides the program with the ability to generalize from its experience, so that in new states it selects moves based on information saved from similar states faced in the past, as determined by its network. How well a reinforcement learning system can work in problems with such large state sets is intimately tied to how appropriately it can generalize from past experience. It is in this role that we have the greatest need for supervised learning methods with reinforcement learning. Artificial neural networks and deep learning (Section 9.6) are not the only, or necessarily the best, way to do this.

In this tic-tac-toe example, learning started with no prior knowledge beyond the

rules of the game, but reinforcement learning by no means entails a tabula rasa view of learning and intelligence. On the contrary, prior information can be incorporated into reinforcement learning in a variety of ways that can be critical for efficient learning (e.g., see Sections 9.5, 17.4, and 13.1). We also have access to the true state in the tic-tac-toe example, whereas reinforcement learning can also be applied when part of the state is hidden, or when different states appear to the learner to be the same.

Finally, the tic-tac-toe player was able to look ahead and know the states that would result from each of its possible moves. To do this, it had to have a model of the game that allowed it to foresee how its environment would change in response to moves that it might never make. Many problems are like this, but in others even a short-term model of the effects of actions is lacking. Reinforcement learning can be applied in either case. A model is not required, but models can easily be used if they are available or can be learned (Chapter 8).

On the other hand, there are reinforcement learning methods that do not need any kind of environment model at all. Model-free systems cannot even think about how their environments will change in response to a single action. The tic-tac-toe player is model-free in this sense with respect to its opponent: it has no model of its opponent of any kind. Because models have to be reasonably accurate to be useful, model-free methods can have advantages over more complex methods when the real bottleneck in solving a problem is the difficulty of constructing a sufficiently accurate environment model. Model-free methods are also important building blocks for model-based methods. In this book we devote several chapters to model-free methods before we discuss how they can be used as components of more complex model-based methods.

Reinforcement learning can be used at both high and low levels in a system. Although the tic-tac-toe player learned only about the basic moves of the game, nothing prevents reinforcement learning from working at higher levels where each of the "actions" may itself be the application of a possibly elaborate problem-solving method. In hierarchical learning systems, reinforcement learning can work simultaneously on several levels.

Exercise 1.1: Self-Play Suppose, instead of playing against a random opponent, the reinforcement learning algorithm described above played against itself, with both sides learning. What do you think would happen in this case? Would it learn a different policy for selecting moves? □

Exercise 1.2: Symmetries Many tic-tac-toe positions appear different but are really the same because of symmetries. How might we amend the learning process described above to take advantage of this? In what ways would this change improve the learning process? Now think again. Suppose the opponent did not take advantage of symmetries. In that case, should we? Is it true, then, that symmetrically equivalent positions should necessarily have the same value? □

Exercise 1.3: Greedy Play Suppose the reinforcement learning player was *greedy*, that is, it always played the move that brought it to the position that it rated the best. Might it learn to play better, or worse, than a nongreedy player? What problems might occur? □

Exercise 1.4: Learning from Exploration Suppose learning updates occurred after *all* moves, including exploratory moves. If the step-size parameter is appropriately reduced

over time (but not the tendency to explore), then the state values would converge to a different set of probabilities. What (conceptually) are the two sets of probabilities computed when we do, and when we do not, learn from exploratory moves? Assuming that we do continue to make exploratory moves, which set of probabilities might be better to learn? Which would result in more wins? □

Exercise 1.5: Other Improvements Can you think of other ways to improve the reinforcement learning player? Can you think of any better way to solve the tic-tac-toe problem as posed? □

1.6 Summary

Reinforcement learning is a computational approach to understanding and automating goal-directed learning and decision making. It is distinguished from other computational approaches by its emphasis on learning by an agent from direct interaction with its environment, without requiring exemplary supervision or complete models of the environment. In our opinion, reinforcement learning is the first field to seriously address the computational issues that arise when learning from interaction with an environment in order to achieve long-term goals.

Reinforcement learning uses the formal framework of Markov decision processes to define the interaction between a learning agent and its environment in terms of states, actions, and rewards. This framework is intended to be a simple way of representing essential features of the artificial intelligence problem. These features include a sense of cause and effect, a sense of uncertainty and nondeterminism, and the existence of explicit goals.

The concepts of value and value function are key to most of the reinforcement learning methods that we consider in this book. We take the position that value functions are important for efficient search in the space of policies. The use of value functions distinguishes reinforcement learning methods from evolutionary methods that search directly in policy space guided by evaluations of entire policies.

1.7 Early History of Reinforcement Learning

The early history of reinforcement learning has two main threads, both long and rich, that were pursued independently before intertwining in modern reinforcement learning. One thread concerns learning by trial and error, and originated in the psychology of animal learning. This thread runs through some of the earliest work in artificial intelligence and led to the revival of reinforcement learning in the early 1980s. The second thread concerns the problem of optimal control and its solution using value functions and dynamic programming. For the most part, this thread did not involve learning. The two threads were mostly independent, but became interrelated to some extent around a third, less distinct thread concerning temporal-difference methods such as that used in the tic-tac-toe example in this chapter. All three threads came together in the late 1980s to produce the modern field of reinforcement learning as we present it in this book.

The thread focusing on trial-and-error learning is the one with which we are most familiar and about which we have the most to say in this brief history. Before doing that, however, we briefly discuss the optimal control thread.

The term "optimal control" came into use in the late 1950s to describe the problem of designing a controller to minimize or maximize a measure of a dynamical system's behavior over time. One of the approaches to this problem was developed in the mid-1950s by Richard Bellman and others through extending a nineteenth century theory of Hamilton and Jacobi. This approach uses the concepts of a dynamical system's state and of a value function, or "optimal return function," to define a functional equation, now often called the Bellman equation. The class of methods for solving optimal control problems by solving this equation came to be known as dynamic programming (Bellman, 1957a). Bellman (1957b) also introduced the discrete stochastic version of the optimal control problem known as Markov decision processes (MDPs). Ronald Howard (1960) devised the policy iteration method for MDPs. All of these are essential elements underlying the theory and algorithms of modern reinforcement learning.

Dynamic programming is widely considered the only feasible way of solving general stochastic optimal control problems. It suffers from what Bellman called "the curse of dimensionality," meaning that its computational requirements grow exponentially with the number of state variables, but it is still far more efficient and more widely applicable than any other general method. Dynamic programming has been extensively developed since the late 1950s, including extensions to partially observable MDPs (surveyed by Lovejoy, 1991), many applications (surveyed by White, 1985, 1988, 1993), approximation methods (surveyed by Rust, 1996), and asynchronous methods (Bertsekas, 1982, 1983). Many excellent modern treatments of dynamic programming are available (e.g., Bertsekas, 2005, 2012; Puterman, 1994; Ross, 1983; and Whittle, 1982, 1983). Bryson (1996) provides an authoritative history of optimal control.

Connections between optimal control and dynamic programming, on the one hand, and learning, on the other, were slow to be recognized. We cannot be sure about what accounted for this separation, but its main cause was likely the separation between the disciplines involved and their different goals. Also contributing may have been the prevalent view of dynamic programming as an offline computation depending essentially on accurate system models and analytic solutions to the Bellman equation. Further, the simplest form of dynamic programming is a computation that proceeds backwards in time, making it difficult to see how it could be involved in a learning process that must proceed in a forward direction. Some of the earliest work in dynamic programming, such as that by Bellman and Dreyfus (1959), might now be classified as following a learning approach. Witten's (1977) work (discussed below) certainly qualifies as a combination of learning and dynamic-programming ideas. Werbos (1987) argued explicitly for greater interrelation of dynamic programming and learning methods and for dynamic programming's relevance to understanding neural and cognitive mechanisms. For us the full integration of dynamic programming methods with online learning did not occur until the work of Chris Watkins in 1989, whose treatment of reinforcement learning using the MDP formalism has been widely adopted. Since then these relationships have been extensively developed by many researchers, most particularly by Dimitri Bertsekas

and John Tsitsiklis (1996), who coined the term "neurodynamic programming" to refer to the combination of dynamic programming and artificial neural networks. Another term currently in use is "approximate dynamic programming." These various approaches emphasize different aspects of the subject, but they all share with reinforcement learning an interest in circumventing the classical shortcomings of dynamic programming.

We consider all of the work in optimal control also to be, in a sense, work in reinforcement learning. We define a reinforcement learning method as any effective way of solving reinforcement learning problems, and it is now clear that these problems are closely related to optimal control problems, particularly stochastic optimal control problems such as those formulated as MDPs. Accordingly, we must consider the solution methods of optimal control, such as dynamic programming, also to be reinforcement learning methods. Because almost all of the conventional methods require complete knowledge of the system to be controlled, it feels a little unnatural to say that they are part of reinforcement *learning*. On the other hand, many dynamic programming algorithms are incremental and iterative. Like learning methods, they gradually reach the correct answer through successive approximations. As we show in the rest of this book, these similarities are far more than superficial. The theories and solution methods for the cases of complete and incomplete knowledge are so closely related that we feel they must be considered together as part of the same subject matter.

Let us return now to the other major thread leading to the modern field of reinforcement learning, the thread centered on the idea of trial-and-error learning. We only touch on the major points of contact here, taking up this topic in more detail in Section 14.3. According to American psychologist R. S. Woodworth (1938) the idea of trial-and-error learning goes as far back as the 1850s to Alexander Bain's discussion of learning by "groping and experiment" and more explicitly to the British ethologist and psychologist Conway Lloyd Morgan's 1894 use of the term to describe his observations of animal behavior. Perhaps the first to succinctly express the essence of trial-and-error learning as a principle of learning was Edward Thorndike:

> Of several responses made to the same situation, those which are accompanied or closely followed by satisfaction to the animal will, other things being equal, be more firmly connected with the situation, so that, when it recurs, they will be more likely to recur; those which are accompanied or closely followed by discomfort to the animal will, other things being equal, have their connections with that situation weakened, so that, when it recurs, they will be less likely to occur. The greater the satisfaction or discomfort, the greater the strengthening or weakening of the bond. (Thorndike, 1911, p. 244)

Thorndike called this the "Law of Effect" because it describes the effect of reinforcing events on the tendency to select actions. Thorndike later modified the law to better account for subsequent data on animal learning (such as differences between the effects of reward and punishment), and the law in its various forms has generated considerable controversy among learning theorists (e.g., see Gallistel, 2005; Herrnstein, 1970; Kimble, 1961, 1967; Mazur, 1994). Despite this, the Law of Effect—in one form or another—is widely regarded as a basic principle underlying much behavior (e.g., Hilgard and Bower, 1975; Dennett, 1978; Campbell, 1960; Cziko, 1995). It is the basis of the influential

learning theories of Clark Hull (1943, 1952) and the influential experimental methods of B. F. Skinner (1938).

The term "reinforcement" in the context of animal learning came into use well after Thorndike's expression of the Law of Effect, first appearing in this context (to the best of our knowledge) in the 1927 English translation of Pavlov's monograph on conditioned reflexes. Pavlov described reinforcement as the strengthening of a pattern of behavior due to an animal receiving a stimulus—a reinforcer—in an appropriate temporal relationship with another stimulus or with a response. Some psychologists extended the idea of reinforcement to include weakening as well as strengthening of behavior, and extended the idea of a reinforcer to include possibly the omission or termination of stimulus. To be considered reinforcer, the strengthening or weakening must persist after the reinforcer is withdrawn; a stimulus that merely attracts an animal's attention or that energizes its behavior without producing lasting changes would not be considered a reinforcer.

The idea of implementing trial-and-error learning in a computer appeared among the earliest thoughts about the possibility of artificial intelligence. In a 1948 report, Alan Turing described a design for a "pleasure-pain system" that worked along the lines of the Law of Effect:

> When a configuration is reached for which the action is undetermined, a random choice for the missing data is made and the appropriate entry is made in the description, tentatively, and is applied. When a pain stimulus occurs all tentative entries are cancelled, and when a pleasure stimulus occurs they are all made permanent. (Turing, 1948)

Many ingenious electro-mechanical machines were constructed that demonstrated trial-and-error learning. The earliest may have been a machine built by Thomas Ross (1933) that was able to find its way through a simple maze and remember the path through the settings of switches. In 1951 W. Grey Walter built a version of his "mechanical tortoise" (Walter, 1950) capable of a simple form of learning. In 1952 Claude Shannon demonstrated a maze-running mouse named Theseus that used trial and error to find its way through a maze, with the maze itself remembering the successful directions via magnets and relays under its floor (see also Shannon, 1951). J. A. Deutsch (1954) described a maze-solving machine based on his behavior theory (Deutsch, 1953) that has some properties in common with model-based reinforcement learning (Chapter 8). In his Ph.D. dissertation, Marvin Minsky (1954) discussed computational models of reinforcement learning and described his construction of an analog machine composed of components he called SNARCs (Stochastic Neural-Analog Reinforcement Calculators) meant to resemble modifiable synaptic connections in the brain (Chapter 15). The web site `cyberneticzoo.com` contains a wealth of information on these and many other electro-mechanical learning machines.

Building electro-mechanical learning machines gave way to programming digital computers to perform various types of learning, some of which implemented trial-and-error learning. Farley and Clark (1954) described a digital simulation of a neural-network learning machine that learned by trial and error. But their interests soon shifted from trial-and-error learning to generalization and pattern recognition, that is, from reinforcement learning to supervised learning (Clark and Farley, 1955). This began a pattern

of confusion about the relationship between these types of learning. Many researchers seemed to believe that they were studying reinforcement learning when they were actually studying supervised learning. For example, artificial neural network pioneers such as Rosenblatt (1962) and Widrow and Hoff (1960) were clearly motivated by reinforcement learning—they used the language of rewards and punishments—but the systems they studied were supervised learning systems suitable for pattern recognition and perceptual learning. Even today, some researchers and textbooks minimize or blur the distinction between these types of learning. For example, some artificial neural network textbooks have used the term "trial-and-error" to describe networks that learn from training examples. This is an understandable confusion because these networks use error information to update connection weights, but this misses the essential character of trial-and-error learning as selecting actions on the basis of evaluative feedback that does not rely on knowledge of what the correct action should be.

Partly as a result of these confusions, research into genuine trial-and-error learning became rare in the 1960s and 1970s, although there were notable exceptions. In the 1960s the terms "reinforcement" and "reinforcement learning" were used in the engineering literature for the first time to describe engineering uses of trial-and-error learning (e.g., Waltz and Fu, 1965; Mendel, 1966; Fu, 1970; Mendel and McClaren, 1970). Particularly influential was Minsky's paper "Steps Toward Artificial Intelligence" (Minsky, 1961), which discussed several issues relevant to trial-and-error learning, including prediction, expectation, and what he called the *basic credit-assignment problem for complex reinforcement learning systems*: How do you distribute credit for success among the many decisions that may have been involved in producing it? All of the methods we discuss in this book are, in a sense, directed toward solving this problem. Minsky's paper is well worth reading today.

In the next few paragraphs we discuss some of the other exceptions and partial exceptions to the relative neglect of computational and theoretical study of genuine trial-and-error learning in the 1960s and 1970s.

One exception was the work of the New Zealand researcher John Andreae, who developed a system called STeLLA that learned by trial and error in interaction with its environment. This system included an internal model of the world and, later, an "internal monologue" to deal with problems of hidden state (Andreae, 1963, 1969a,b). Andreae's later work (1977) placed more emphasis on learning from a teacher, but still included learning by trial and error, with the generation of novel events being one of the system's goals. A feature of this work was a "leakback process," elaborated more fully in Andreae (1998), that implemented a credit-assignment mechanism similar to the backing-up update operations that we describe. Unfortunately, his pioneering research was not well known and did not greatly impact subsequent reinforcement learning research. Recent summaries are available (Andreae, 2017a,b).

More influential was the work of Donald Michie. In 1961 and 1963 he described a simple trial-and-error learning system for learning how to play tic-tac-toe (or naughts and crosses) called MENACE (for Matchbox Educable Naughts and Crosses Engine). It consisted of a matchbox for each possible game position, each matchbox containing a number of colored beads, a different color for each possible move from that position. By

drawing a bead at random from the matchbox corresponding to the current game position, one could determine MENACE's move. When a game was over, beads were added to or removed from the boxes used during play to reward or punish MENACE's decisions. Michie and Chambers (1968) described another tic-tac-toe reinforcement learner called GLEE (Game Learning Expectimaxing Engine) and a reinforcement learning controller called BOXES. They applied BOXES to the task of learning to balance a pole hinged to a movable cart on the basis of a failure signal occurring only when the pole fell or the cart reached the end of a track. This task was adapted from the earlier work of Widrow and Smith (1964), who used supervised learning methods, assuming instruction from a teacher already able to balance the pole. Michie and Chambers's version of pole-balancing is one of the best early examples of a reinforcement learning task under conditions of incomplete knowledge. It influenced much later work in reinforcement learning, beginning with some of our own studies (Barto, Sutton, and Anderson, 1983; Sutton, 1984). Michie consistently emphasized the role of trial and error and learning as essential aspects of artificial intelligence (Michie, 1974).

Widrow, Gupta, and Maitra (1973) modified the Least-Mean-Square (LMS) algorithm of Widrow and Hoff (1960) to produce a reinforcement learning rule that could learn from success and failure signals instead of from training examples. They called this form of learning "selective bootstrap adaptation" and described it as "learning with a critic" instead of "learning with a teacher." They analyzed this rule and showed how it could learn to play blackjack. This was an isolated foray into reinforcement learning by Widrow, whose contributions to supervised learning were much more influential. Our use of the term "critic" is derived from Widrow, Gupta, and Maitra's paper. Buchanan, Mitchell, Smith, and Johnson (1978) independently used the term critic in the context of machine learning (see also Dietterich and Buchanan, 1984), but for them a critic is an expert system able to do more than evaluate performance.

Research on *learning automata* had a more direct influence on the trial-and-error thread leading to modern reinforcement learning research. These are methods for solving a nonassociative, purely selectional learning problem known as the *k-armed bandit* by analogy to a slot machine, or "one-armed bandit," except with k levers (see Chapter 2). Learning automata are simple, low-memory machines for improving the probability of reward in these problems. Learning automata originated with work in the 1960s of the Russian mathematician and physicist M. L. Tsetlin and colleagues (published posthumously in Tsetlin, 1973) and has been extensively developed since then within engineering (see Narendra and Thathachar, 1974, 1989). These developments included the study of *stochastic learning automata*, which are methods for updating action probabilities on the basis of reward signals. Although not developed in the tradition of stochastic learning automata, Harth and Tzanakou's (1974) Alopex algorithm (for *Al*gorithm *o*f *p*attern *ex*traction) is a stochastic method for detecting correlations between actions and reinforcement that influenced some of our early research (Barto, Sutton, and Brouwer, 1981). Stochastic learning automata were foreshadowed by earlier work in psychology, beginning with William Estes' (1950) effort toward a statistical theory of learning and further developed by others (e.g., Bush and Mosteller, 1955; Sternberg, 1963).

The statistical learning theories developed in psychology were adopted by researchers in

economics, leading to a thread of research in that field devoted to reinforcement learning. This work began in 1973 with the application of Bush and Mosteller's learning theory to a collection of classical economic models (Cross, 1973). One goal of this research was to study artificial agents that act more like real people than do traditional idealized economic agents (Arthur, 1991). This approach expanded to the study of reinforcement learning in the context of game theory. Reinforcement learning in economics developed largely independently of the early work in reinforcement learning in artificial intelligence, though game theory remains a topic of interest in both fields (beyond the scope of this book). Camerer (2011) discusses the reinforcement learning tradition in economics, and Nowé, Vrancx, and De Hauwere (2012) provide an overview of the subject from the point of view of multi-agent extensions to the approach that we introduce in this book. Reinforcement in the context of game theory is a much different subject than reinforcement learning used in programs to play tic-tac-toe, checkers, and other recreational games. See, for example, Szita (2012) for an overview of this aspect of reinforcement learning and games.

John Holland (1975) outlined a general theory of adaptive systems based on selectional principles. His early work concerned trial and error primarily in its nonassociative form, as in evolutionary methods and the k-armed bandit. In 1976 and more fully in 1986, he introduced *classifier systems*, true reinforcement learning systems including association and value functions. A key component of Holland's classifier systems was the "bucket-brigade algorithm" for credit assignment, which is closely related to the temporal difference algorithm used in our tic-tac-toe example and discussed in Chapter 6. Another key component was a *genetic algorithm*, an evolutionary method whose role was to evolve useful representations. Classifier systems have been extensively developed by many researchers to form a major branch of reinforcement learning research (reviewed by Urbanowicz and Moore, 2009), but genetic algorithms—which we do not consider to be reinforcement learning systems by themselves—have received much more attention, as have other approaches to evolutionary computation (e.g., Fogel, Owens and Walsh, 1966, and Koza, 1992).

The individual most responsible for reviving the trial-and-error thread to reinforcement learning within artificial intelligence was Harry Klopf (1972, 1975, 1982). Klopf recognized that essential aspects of adaptive behavior were being lost as learning researchers came to focus almost exclusively on supervised learning. What was missing, according to Klopf, were the hedonic aspects of behavior, the drive to achieve some result from the environment, to control the environment toward desired ends and away from undesired ends (see Section 15.9). This is the essential idea of trial-and-error learning. Klopf's ideas were especially influential on the authors because our assessment of them (Barto and Sutton, 1981a) led to our appreciation of the distinction between supervised and reinforcement learning, and to our eventual focus on reinforcement learning. Much of the early work that we and colleagues accomplished was directed toward showing that reinforcement learning and supervised learning were indeed different (Barto, Sutton, and Brouwer, 1981; Barto and Sutton, 1981b; Barto and Anandan, 1985). Other studies showed how reinforcement learning could address important problems in artificial neural network learning, in particular, how it could produce learning algorithms for multilayer networks (Barto, Anderson, and Sutton, 1982; Barto and Anderson, 1985; Barto, 1985, 1986; Barto and Jordan, 1987; see Section 15.10).

We turn now to the third thread to the history of reinforcement learning, that concerning temporal-difference learning. Temporal-difference learning methods are distinctive in being driven by the difference between temporally successive estimates of the same quantity—for example, of the probability of winning in the tic-tac-toe example. This thread is smaller and less distinct than the other two, but it has played a particularly important role in the field, in part because temporal-difference methods seem to be new and unique to reinforcement learning.

The origins of temporal-difference learning are in part in animal learning psychology, in particular, in the notion of *secondary reinforcers*. A secondary reinforcer is a stimulus that has been paired with a primary reinforcer such as food or pain and, as a result, has come to take on similar reinforcing properties. Minsky (1954) may have been the first to realize that this psychological principle could be important for artificial learning systems. Arthur Samuel (1959) was the first to propose and implement a learning method that included temporal-difference ideas, as part of his celebrated checkers-playing program (Section 16.2).

Samuel made no reference to Minsky's work or to possible connections to animal learning. His inspiration apparently came from Claude Shannon's (1950) suggestion that a computer could be programmed to use an evaluation function to play chess, and that it might be able to improve its play by modifying this function online. (It is possible that these ideas of Shannon's also influenced Bellman, but we know of no evidence for this.) Minsky (1961) extensively discussed Samuel's work in his "Steps" paper, suggesting the connection to secondary reinforcement theories, both natural and artificial.

As we have discussed, in the decade following the work of Minsky and Samuel, little computational work was done on trial-and-error learning, and apparently no computational work at all was done on temporal-difference learning. In 1972, Klopf brought trial-and-error learning together with an important component of temporal-difference learning. Klopf was interested in principles that would scale to learning in large systems, and thus was intrigued by notions of local reinforcement, whereby subcomponents of an overall learning system could reinforce one another. He developed the idea of "generalized reinforcement," whereby every component (nominally, every neuron) views all of its inputs in reinforcement terms: excitatory inputs as rewards and inhibitory inputs as punishments. This is not the same idea as what we now know as temporal-difference learning, and in retrospect it is farther from it than was Samuel's work. On the other hand, Klopf linked the idea with trial-and-error learning and related it to the massive empirical database of animal learning psychology.

Sutton (1978a,b,c) developed Klopf's ideas further, particularly the links to animal learning theories, describing learning rules driven by changes in temporally successive predictions. He and Barto refined these ideas and developed a psychological model of classical conditioning based on temporal-difference learning (Sutton and Barto, 1981a; Barto and Sutton, 1982). There followed several other influential psychological models of classical conditioning based on temporal-difference learning (e.g., Klopf, 1988; Moore et al., 1986; Sutton and Barto, 1987, 1990). Some neuroscience models developed at this time are well interpreted in terms of temporal-difference learning (Hawkins and Kandel, 1984; Byrne, Gingrich, and Baxter, 1990; Gelperin, Hopfield, and Tank, 1985; Tesauro,

1986; Friston et al., 1994), although in most cases there was no historical connection.

Our early work on temporal-difference learning was strongly influenced by animal learning theories and by Klopf's work. Relationships to Minsky's "Steps" paper and to Samuel's checkers players were recognized only afterward. By 1981, however, we were fully aware of all the prior work mentioned above as part of the temporal-difference and trial-and-error threads. At this time we developed a method for using temporal-difference learning combined with trial-and-error learning, known as the *actor–critic architecture*, and applied this method to Michie and Chambers's pole-balancing problem (Barto, Sutton, and Anderson, 1983). This method was extensively studied in Sutton's (1984) Ph.D. dissertation and extended to use backpropagation neural networks in Anderson's (1986) Ph.D. dissertation. Around this time, Holland (1986) incorporated temporal-difference ideas explicitly into his classifier systems in the form of his bucket-brigade algorithm. A key step was taken by Sutton (1988) by separating temporal-difference learning from control, treating it as a general prediction method. That paper also introduced the TD(λ) algorithm and proved some of its convergence properties.

As we were finalizing our work on the actor–critic architecture in 1981, we discovered a paper by Ian Witten (1977, 1976a) which appears to be the earliest publication of a temporal-difference learning rule. He proposed the method that we now call tabular TD(0) for use as part of an adaptive controller for solving MDPs. This work was first submitted for journal publication in 1974 and also appeared in Witten's 1976 PhD dissertation. Witten's work was a descendant of Andreae's early experiments with STeLLA and other trial-and-error learning systems. Thus, Witten's 1977 paper spanned both major threads of reinforcement learning research—trial-and-error learning and optimal control—while making a distinct early contribution to temporal-difference learning.

The temporal-difference and optimal control threads were fully brought together in 1989 with Chris Watkins's development of Q-learning. This work extended and integrated prior work in all three threads of reinforcement learning research. Paul Werbos (1987) contributed to this integration by arguing for the convergence of trial-and-error learning and dynamic programming since 1977. By the time of Watkins's work there had been tremendous growth in reinforcement learning research, primarily in the machine learning subfield of artificial intelligence, but also in artificial neural networks and artificial intelligence more broadly. In 1992, the remarkable success of Gerry Tesauro's backgammon playing program, TD-Gammon, brought additional attention to the field.

In the time since publication of the first edition of this book, a flourishing subfield of neuroscience developed that focuses on the relationship between reinforcement learning algorithms and reinforcement learning in the nervous system. Most responsible for this is an uncanny similarity between the behavior of temporal-difference algorithms and the activity of dopamine producing neurons in the brain, as pointed out by a number of researchers (Friston et al., 1994; Barto, 1995a; Houk, Adams, and Barto, 1995; Montague, Dayan, and Sejnowski, 1996; and Schultz, Dayan, and Montague, 1997). Chapter 15 provides an introduction to this exciting aspect of reinforcement learning. Other important contributions made in the recent history of reinforcement learning are too numerous to mention in this brief account; we cite many of these at the end of the individual chapters in which they arise.

Bibliographical Remarks

For additional general coverage of reinforcement learning, we refer the reader to the books by Szepesvári (2010), Bertsekas and Tsitsiklis (1996), Kaelbling (1993a), and Sugiyama, Hachiya, and Morimura (2013). Books that take a control or operations research perspective include those of Si, Barto, Powell, and Wunsch (2004), Powell (2011), Lewis and Liu (2012), and Bertsekas (2012). Cao's (2009) review places reinforcement learning in the context of other approaches to learning and optimization of stochastic dynamic systems. Three special issues of the journal *Machine Learning* focus on reinforcement learning: Sutton (1992a), Kaelbling (1996), and Singh (2002). Useful surveys are provided by Barto (1995b); Kaelbling, Littman, and Moore (1996); and Keerthi and Ravindran (1997). The volume edited by Weiring and van Otterlo (2012) provides an excellent overview of recent developments.

1.2 The example of Phil's breakfast in this chapter was inspired by Agre (1988).

1.5 The temporal-difference method used in the tic-tac-toe example is developed in Chapter 6.

Part II: Approximate Solution Methods

In the second part of the book we extend the tabular methods presented in the first part to apply to problems with arbitrarily large state spaces. In many of the tasks to which we would like to apply reinforcement learning the state space is combinatorial and enormous; the number of possible camera images, for example, is much larger than the number of atoms in the universe. In such cases we cannot expect to find an optimal policy or the optimal value function even in the limit of infinite time and data; our goal instead is to find a good approximate solution using limited computational resources. In this part of the book we explore such approximate solution methods.

The problem with large state spaces is not just the memory needed for large tables, but the time and data needed to fill them accurately. In many of our target tasks, almost every state encountered will never have been seen before. To make sensible decisions in such states it is necessary to generalize from previous encounters with different states that are in some sense similar to the current one. In other words, the key issue is that of *generalization*. How can experience with a limited subset of the state space be usefully generalized to produce a good approximation over a much larger subset?

Fortunately, generalization from examples has already been extensively studied, and we do not need to invent totally new methods for use in reinforcement learning. To some extent we need only combine reinforcement learning methods with existing generalization methods. The kind of generalization we require is often called *function approximation* because it takes examples from a desired function (e.g., a value function) and attempts to generalize from them to construct an approximation of the entire function. Function approximation is an instance of *supervised learning*, the primary topic studied in machine learning, artificial neural networks, pattern recognition, and statistical curve fitting. In theory, any of the methods studied in these fields can be used in the role of function approximator within reinforcement learning algorithms, although in practice some fit more easily into this role than others.

Reinforcement learning with function approximation involves a number of new issues that do not normally arise in conventional supervised learning, such as nonstationarity, bootstrapping, and delayed targets. We introduce these and other issues successively over the five chapters of this part. Initially we restrict attention to on-policy training, treating in Chapter 9 the prediction case, in which the policy is given and only its value function is approximated, and then in Chapter 10 the control case, in which an approximation to the optimal policy is found. The challenging problem of off-policy learning with function

approximation is treated in Chapter 11. In each of these three chapters we will have to return to first principles and re-examine the objectives of the learning to take into account function approximation. Chapter 12 introduces and analyzes the algorithmic mechanism of *eligibility traces*, which dramatically improves the computational properties of multi-step reinforcement learning methods in many cases. The final chapter of this part explores a different approach to control, *policy-gradient methods*, which approximate the optimal policy directly and need never form an approximate value function (although they may be much more efficient if they do approximate a value function as well the policy).

Chapter 9

On-policy Prediction with Approximation

In this chapter, we begin our study of function approximation in reinforcement learning by considering its use in estimating the state-value function from on-policy data, that is, in approximating v_π from experience generated using a known policy π. The novelty in this chapter is that the approximate value function is represented not as a table but as a parameterized functional form with weight vector $\mathbf{w} \in \mathbb{R}^d$. We will write $\hat{v}(s,\mathbf{w}) \approx v_\pi(s)$ for the approximate value of state s given weight vector $\mathbf{w}$. For example, $\hat{v}$ might be a linear function in features of the state, with $\mathbf{w}$ the vector of feature weights. More generally, $\hat{v}$ might be the function computed by a multi-layer artificial neural network, with $\mathbf{w}$ the vector of connection weights in all the layers. By adjusting the weights, any of a wide range of different functions can be implemented by the network. Or $\hat{v}$ might be the function computed by a decision tree, where $\mathbf{w}$ is all the numbers defining the split points and leaf values of the tree. Typically, the number of weights (the dimensionality of $\mathbf{w}$) is much less than the number of states ($d \ll |\mathcal{S}|$), and changing one weight changes the estimated value of many states. Consequently, when a single state is updated, the change generalizes from that state to affect the values of many other states. Such *generalization* makes the learning potentially more powerful but also potentially more difficult to manage and understand.

Perhaps surprisingly, extending reinforcement learning to function approximation also makes it applicable to partially observable problems, in which the full state is not available to the agent. If the parameterized function form for $\hat{v}$ does not allow the estimated value to depend on certain aspects of the state, then it is just as if those aspects are unobservable. In fact, all the theoretical results for methods using function approximation presented in this part of the book apply equally well to cases of partial observability. What function approximation can't do, however, is augment the state representation with memories of past observations. Some such possible further extensions are discussed briefly in Section 17.3.

9.1 Value-function Approximation

All of the prediction methods covered in this book have been described as updates to an estimated value function that shift its value at particular states toward a "backed-up value," or *update target*, for that state. Let us refer to an individual update by the notation $s \mapsto u$, where s is the state updated and u is the update target that s's estimated value is shifted toward. For example, the Monte Carlo update for value prediction is $S_t \mapsto G_t$, the TD(0) update is $S_t \mapsto R_{t+1} + \gamma\hat{v}(S_{t+1},\mathbf{w}_t)$, and the n-step TD update is $S_t \mapsto G_{t:t+n}$. In the DP (dynamic programming) policy-evaluation update, $s \mapsto \mathbb{E}_\pi[R_{t+1} + \gamma\hat{v}(S_{t+1},\mathbf{w}_t) \mid S_t = s]$, an arbitrary state s is updated, whereas in the other cases the state encountered in actual experience, S_t, is updated.

It is natural to interpret each update as specifying an example of the desired input–output behavior of the value function. In a sense, the update $s \mapsto u$ means that the estimated value for state s should be more like the update target u. Up to now, the actual update has been trivial: the table entry for s's estimated value has simply been shifted a fraction of the way toward u, and the estimated values of all other states were left unchanged. Now we permit arbitrarily complex and sophisticated methods to implement the update, and updating at s generalizes so that the estimated values of many other states are changed as well. Machine learning methods that learn to mimic input–output examples in this way are called *supervised learning* methods, and when the outputs are numbers, like u, the process is often called *function approximation*. Function approximation methods expect to receive examples of the desired input–output behavior of the function they are trying to approximate. We use these methods for value prediction simply by passing to them the $s \mapsto g$ of each update as a training example. We then interpret the approximate function they produce as an estimated value function.

Viewing each update as a conventional training example in this way enables us to use any of a wide range of existing function approximation methods for value prediction. In principle, we can use any method for supervised learning from examples, including artificial neural networks, decision trees, and various kinds of multivariate regression. However, not all function approximation methods are equally well suited for use in reinforcement learning. The most sophisticated artificial neural network and statistical methods all assume a static training set over which multiple passes are made. In reinforcement learning, however, it is important that learning be able to occur online, while the agent interacts with its environment or with a model of its environment. To do this requires methods that are able to learn efficiently from incrementally acquired data. In addition, reinforcement learning generally requires function approximation methods able to handle nonstationary target functions (target functions that change over time). For example, in control methods based on GPI (generalized policy iteration) we often seek to learn q_π while π changes. Even if the policy remains the same, the target values of training examples are nonstationary if they are generated by bootstrapping methods (DP and TD learning). Methods that cannot easily handle such nonstationarity are less suitable for reinforcement learning.

9.2 The Prediction Objective ($\overline{\text{VE}}$)

Up to now we have not specified an explicit objective for prediction. In the tabular case a continuous measure of prediction quality was not necessary because the learned value function could come to equal the true value function exactly. Moreover, the learned values at each state were decoupled—an update at one state affected no other. But with genuine approximation, an update at one state affects many others, and it is not possible to get the values of all states exactly correct. By assumption we have far more states than weights, so making one state's estimate more accurate invariably means making others' less accurate. We are obligated then to say which states we care most about. We must specify a state distribution $\mu(s) \geq 0$, $\sum_s \mu(s) = 1$, representing how much we care about the error in each state s. By the error in a state s we mean the square of the difference between the approximate value $\hat{v}(s,\mathbf{w})$ and the true value $v_\pi(s)$. Weighting this over the state space by μ, we obtain a natural objective function, the *Mean Squared Value Error*, denoted $\overline{\text{VE}}$:

$$\overline{\text{VE}}(\mathbf{w}) \doteq \sum_{s \in \mathcal{S}} \mu(s) \Big[v_\pi(s) - \hat{v}(s,\mathbf{w}) \Big]^2. \tag{9.1}$$

The square root of this measure, the root $\overline{\text{VE}}$, gives a rough measure of how much the approximate values differ from the true values and is often used in plots. Often $\mu(s)$ is chosen to be the fraction of time spent in s. Under on-policy training this is called the *on-policy distribution*; we focus entirely on this case in this chapter. In continuing tasks, the on-policy distribution is the stationary distribution under π.

The on-policy distribution in episodic tasks

In an episodic task, the on-policy distribution is a little different in that it depends on how the initial states of episodes are chosen. Let $h(s)$ denote the probability that an episode begins in each state s, and let $\eta(s)$ denote the number of time steps spent, on average, in state s in a single episode. Time is spent in a state s if episodes start in s, or if transitions are made into s from a preceding state $\bar{s}$ in which time is spent:

$$\eta(s) = h(s) + \sum_{\bar{s}} \eta(\bar{s}) \sum_a \pi(a|\bar{s}) p(s \,|\, \bar{s}, a), \quad \text{for all } s \in \mathcal{S}. \tag{9.2}$$

This system of equations can be solved for the expected number of visits $\eta(s)$. The on-policy distribution is then the fraction of time spent in each state normalized to sum to one:

$$\mu(s) = \frac{\eta(s)}{\sum_{s'} \eta(s')}, \quad \text{for all } s \in \mathcal{S}. \tag{9.3}$$

This is the natural choice without discounting. If there is discounting ($\gamma < 1$) it should be treated as a form of termination, which can be done simply by including a factor of γ in the second term of (9.2).

The two cases, continuing and episodic, behave similarly, but with approximation they must be treated separately in formal analyses, as we will see repeatedly in this part of the book. This completes the specification of the learning objective.

But it is not completely clear that the $\overline{\text{VE}}$ is the right performance objective for reinforcement learning. Remember that our ultimate purpose—the reason we are learning a value function—is to find a better policy. The best value function for this purpose is not necessarily the best for minimizing $\overline{\text{VE}}$. Nevertheless, it is not yet clear what a more useful alternative goal for value prediction might be. For now, we will focus on $\overline{\text{VE}}$.

An ideal goal in terms of $\overline{\text{VE}}$ would be to find a *global optimum*, a weight vector $\mathbf{w}^*$ for which $\overline{\text{VE}}(\mathbf{w}^*) \leq \overline{\text{VE}}(\mathbf{w})$ for all possible $\mathbf{w}$. Reaching this goal is sometimes possible for simple function approximators such as linear ones, but is rarely possible for complex function approximators such as artificial neural networks and decision trees. Short of this, complex function approximators may seek to converge instead to a *local optimum*, a weight vector $\mathbf{w}^*$ for which $\overline{\text{VE}}(\mathbf{w}^*) \leq \overline{\text{VE}}(\mathbf{w})$ for all $\mathbf{w}$ in some neighborhood of $\mathbf{w}^*$. Although this guarantee is only slightly reassuring, it is typically the best that can be said for nonlinear function approximators, and often it is enough. Still, for many cases of interest in reinforcement learning there is no guarantee of convergence to an optimum, or even to within a bounded distance of an optimum. Some methods may in fact diverge, with their $\overline{\text{VE}}$ approaching infinity in the limit.

In the last two sections we outlined a framework for combining a wide range of reinforcement learning methods for value prediction with a wide range of function approximation methods, using the updates of the former to generate training examples for the latter. We also described a $\overline{\text{VE}}$ performance measure which these methods may aspire to minimize. The range of possible function approximation methods is far too large to cover all, and anyway too little is known about most of them to make a reliable evaluation or recommendation. Of necessity, we consider only a few possibilities. In the rest of this chapter we focus on function approximation methods based on gradient principles, and on linear gradient-descent methods in particular. We focus on these methods in part because we consider them to be particularly promising and because they reveal key theoretical issues, but also because they are simple and our space is limited.

9.3 Stochastic-gradient and Semi-gradient Methods

We now develop in detail one class of learning methods for function approximation in value prediction, those based on stochastic gradient descent (SGD). SGD methods are among the most widely used of all function approximation methods and are particularly well suited to online reinforcement learning.

In gradient-descent methods, the weight vector is a column vector with a fixed number of real valued components, $\mathbf{w} \doteq (w_1, w_2, \ldots, w_d)^\top$,[1] and the approximate value function $\hat{v}(s,\mathbf{w})$ is a differentiable function of $\mathbf{w}$ for all $s \in \mathcal{S}$. We will be updating $\mathbf{w}$ at each of a series of discrete time steps, $t = 0, 1, 2, 3, \ldots$, so we will need a notation $\mathbf{w}_t$ for the

[1] The $^\top$ denotes transpose, needed here to turn the horizontal row vector in the text into a vertical column vector; in this book vectors are generally taken to be column vectors unless explicitly written out horizontally or transposed.

weight vector at each step. For now, let us assume that, on each step, we observe a new example $S_t \mapsto v_\pi(S_t)$ consisting of a (possibly randomly selected) state S_t and its true value under the policy. These states might be successive states from an interaction with the environment, but for now we do not assume so. Even though we are given the exact, correct values, $v_\pi(S_t)$ for each S_t, there is still a difficult problem because our function approximator has limited resources and thus limited resolution. In particular, there is generally no $\mathbf{w}$ that gets all the states, or even all the examples, exactly correct. In addition, we must generalize to all the other states that have not appeared in examples.

We assume that states appear in examples with the same distribution, μ, over which we are trying to minimize the $\overline{\text{VE}}$ as given by (9.1). A good strategy in this case is to try to minimize error on the observed examples. *Stochastic gradient-descent* (SGD) methods do this by adjusting the weight vector after each example by a small amount in the direction that would most reduce the error on that example:

$$\mathbf{w}_{t+1} \doteq \mathbf{w}_t - \frac{1}{2}\alpha\nabla\Big[v_\pi(S_t) - \hat{v}(S_t,\mathbf{w}_t)\Big]^2 \tag{9.4}$$

$$= \mathbf{w}_t + \alpha\Big[v_\pi(S_t) - \hat{v}(S_t,\mathbf{w}_t)\Big]\nabla\hat{v}(S_t,\mathbf{w}_t), \tag{9.5}$$

where α is a positive step-size parameter, and $\nabla f(\mathbf{w})$, for any scalar expression $f(\mathbf{w})$ that is a function of a vector (here $\mathbf{w}$), denotes the column vector of partial derivatives of the expression with respect to the components of the vector:

$$\nabla f(\mathbf{w}) \doteq \left(\frac{\partial f(\mathbf{w})}{\partial w_1}, \frac{\partial f(\mathbf{w})}{\partial w_2}, \ldots, \frac{\partial f(\mathbf{w})}{\partial w_d}\right)^\top. \tag{9.6}$$

This derivative vector is the *gradient* of f with respect to $\mathbf{w}$. SGD methods are "gradient descent" methods because the overall step in $\mathbf{w}_t$ is proportional to the negative gradient of the example's squared error (9.4). This is the direction in which the error falls most rapidly. Gradient descent methods are called "stochastic" when the update is done, as here, on only a single example, which might have been selected stochastically. Over many examples, making small steps, the overall effect is to minimize an average performance measure such as the $\overline{\text{VE}}$.

It may not be immediately apparent why SGD takes only a small step in the direction of the gradient. Could we not move all the way in this direction and completely eliminate the error on the example? In many cases this could be done, but usually it is not desirable. Remember that we do not seek or expect to find a value function that has zero error for all states, but only an approximation that balances the errors in different states. If we completely corrected each example in one step, then we would not find such a balance. In fact, the convergence results for SGD methods assume that α decreases over time. If it decreases in such a way as to satisfy the standard stochastic approximation conditions (2.7), then the SGD method (9.5) is guaranteed to converge to a local optimum.

We turn now to the case in which the target output, here denoted $U_t \in \mathbb{R}$, of the tth training example, $S_t \mapsto U_t$, is not the true value, $v_\pi(S_t)$, but some, possibly random, approximation to it. For example, U_t might be a noise-corrupted version of $v_\pi(S_t)$, or it might be one of the bootstrapping targets using $\hat{v}$ mentioned in the previous section. In

these cases we cannot perform the exact update (9.5) because $v_\pi(S_t)$ is unknown, but we can approximate it by substituting U_t in place of $v_\pi(S_t)$. This yields the following general SGD method for state-value prediction:

$$\mathbf{w}_{t+1} \doteq \mathbf{w}_t + \alpha\Big[U_t - \hat{v}(S_t,\mathbf{w}_t)\Big]\nabla\hat{v}(S_t,\mathbf{w}_t). \tag{9.7}$$

If U_t is an *unbiased* estimate, that is, if $\mathbb{E}[U_t|S_t=s] = v_\pi(S_t)$, for each t, then $\mathbf{w}_t$ is guaranteed to converge to a local optimum under the usual stochastic approximation conditions (2.7) for decreasing α.

For example, suppose the states in the examples are the states generated by interaction (or simulated interaction) with the environment using policy π. Because the true value of a state is the expected value of the return following it, the Monte Carlo target $U_t \doteq G_t$ is by definition an unbiased estimate of $v_\pi(S_t)$. With this choice, the general SGD method (9.7) converges to a locally optimal approximation to $v_\pi(S_t)$. Thus, the gradient-descent version of Monte Carlo state-value prediction is guaranteed to find a locally optimal solution. Pseudocode for a complete algorithm is shown in the box below.

Gradient Monte Carlo Algorithm for Estimating $\hat{v} \approx v_\pi$

Input: the policy π to be evaluated
Input: a differentiable function $\hat{v} : \mathcal{S} \times \mathbb{R}^d \to \mathbb{R}$
Algorithm parameter: step size $\alpha > 0$
Initialize value-function weights $\mathbf{w} \in \mathbb{R}^d$ arbitrarily (e.g., $\mathbf{w} = \mathbf{0}$)

Loop forever (for each episode):
  Generate an episode $S_0, A_0, R_1, S_1, A_1, \ldots, R_T, S_T$ using π
  Loop for each step of episode, $t = 0, 1, \ldots, T-1$:
    $\mathbf{w} \leftarrow \mathbf{w} + \alpha\big[G_t - \hat{v}(S_t,\mathbf{w})\big]\nabla\hat{v}(S_t,\mathbf{w})$

One does not obtain the same guarantees if a bootstrapping estimate of $v_\pi(S_t)$ is used as the target U_t in (9.7). Bootstrapping targets such as n-step returns $G_{t:t+n}$ or the DP target $\sum_{a,s',r}\pi(a|S_t)p(s',r|S_t,a)[r+\gamma\hat{v}(s',\mathbf{w}_t)]$ all depend on the current value of the weight vector $\mathbf{w}_t$, which implies that they will be biased and that they will not produce a true gradient-descent method. One way to look at this is that the key step from (9.4) to (9.5) relies on the target being independent of $\mathbf{w}_t$. This step would not be valid if a bootstrapping estimate were used in place of $v_\pi(S_t)$. Bootstrapping methods are not in fact instances of true gradient descent (Barnard, 1993). They take into account the effect of changing the weight vector $\mathbf{w}_t$ on the estimate, but ignore its effect on the target. They include only a part of the gradient and, accordingly, we call them *semi-gradient methods*.

Although semi-gradient (bootstrapping) methods do not converge as robustly as gradient methods, they do converge reliably in important cases such as the linear case discussed in the next section. Moreover, they offer important advantages that make them often clearly preferred. One reason for this is that they typically enable significantly faster learning, as we have seen in Chapters 6 and 7. Another is that they enable learning to

be continual and online, without waiting for the end of an episode. This enables them to be used on continuing problems and provides computational advantages. A prototypical semi-gradient method is semi-gradient TD(0), which uses $U_t \doteq R_{t+1} + \gamma\hat{v}(S_{t+1},\mathbf{w})$ as its target. Complete pseudocode for this method is given in the box below.

Semi-gradient TD(0) for estimating $\hat{v} \approx v_\pi$

Input: the policy π to be evaluated
Input: a differentiable function $\hat{v} : \mathcal{S}^+ \times \mathbb{R}^d \to \mathbb{R}$ such that $\hat{v}(\text{terminal},\cdot) = 0$
Algorithm parameter: step size $\alpha > 0$
Initialize value-function weights $\mathbf{w} \in \mathbb{R}^d$ arbitrarily (e.g., $\mathbf{w} = \mathbf{0}$)

Loop for each episode:
  Initialize S
  Loop for each step of episode:
    Choose $A \sim \pi(\cdot|S)$
    Take action A, observe R, S'
    $\mathbf{w} \leftarrow \mathbf{w} + \alpha\big[R + \gamma\hat{v}(S',\mathbf{w}) - \hat{v}(S,\mathbf{w})\big]\nabla\hat{v}(S,\mathbf{w})$
    $S \leftarrow S'$
  until S is terminal

State aggregation is a simple form of generalizing function approximation in which states are grouped together, with one estimated value (one component of the weight vector $\mathbf{w}$) for each group. The value of a state is estimated as its group's component, and when the state is updated, that component alone is updated. State aggregation is a special case of SGD (9.7) in which the gradient, $\nabla\hat{v}(S_t,\mathbf{w}_t)$, is 1 for S_t's group's component and 0 for the other components.

Example 9.1: State Aggregation on the 1000-state Random Walk Consider a 1000-state version of the random walk task (Examples 6.2 and 7.1 on pages 125 and 144). The states are numbered from 1 to 1000, left to right, and all episodes begin near the center, in state 500. State transitions are from the current state to one of the 100 neighboring states to its left, or to one of the 100 neighboring states to its right, all with equal probability. Of course, if the current state is near an edge, then there may be fewer than 100 neighbors on that side of it. In this case, all the probability that would have gone into those missing neighbors goes into the probability of terminating on that side (thus, state 1 has a 0.5 chance of terminating on the left, and state 950 has a 0.25 chance of terminating on the right). As usual, termination on the left produces a reward of -1, and termination on the right produces a reward of $+1$. All other transitions have a reward of zero. We use this task as a running example throughout this section.

Figure 9.1 shows the true value function v_π for this task. It is nearly a straight line, but curving slightly toward the horizontal for the last 100 states at each end. Also shown is the final approximate value function learned by the gradient Monte-Carlo algorithm with state aggregation after 100,000 episodes with a step size of $\alpha = 2 \times 10^{-5}$. For the state aggregation, the 1000 states were partitioned into 10 groups of 100 states each (i.e., states 1–100 were one group, states 101–200 were another, and so on). The staircase effect

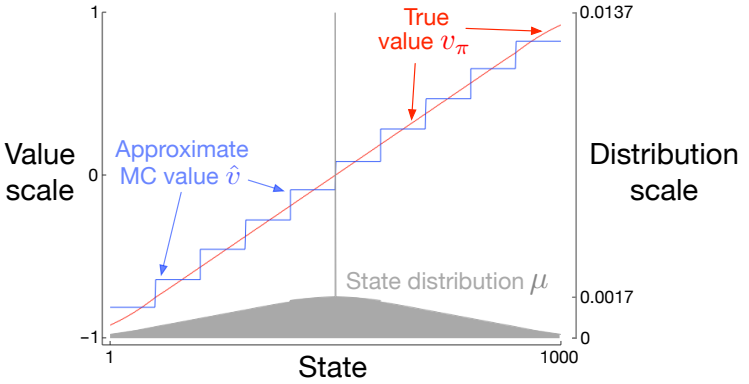


Figure 9.1: Function approximation by state aggregation on the 1000-state random walk task, using the gradient Monte Carlo algorithm (page 202).

shown in the figure is typical of state aggregation; within each group, the approximate value is constant, and it changes abruptly from one group to the next. These approximate values are close to the global minimum of the $\overline{\text{VE}}$ (9.1).

Some of the details of the approximate values are best appreciated by reference to the state distribution μ for this task, shown in the lower portion of the figure with a right-side scale. State 500, in the center, is the first state of every episode, but is rarely visited again. On average, about 1.37% of the time steps are spent in the start state. The states reachable in one step from the start state are the second most visited, with about 0.17% of the time steps being spent in each of them. From there μ falls off almost linearly, reaching about 0.0147% at the extreme states 1 and 1000. The most visible effect of the distribution is on the leftmost groups, whose values are clearly shifted higher than the unweighted average of the true values of states within the group, and on the rightmost groups, whose values are clearly shifted lower. This is due to the states in these areas having the greatest asymmetry in their weightings by μ. For example, in the leftmost group, state 100 is weighted more than 3 times more strongly than state 1. Thus the estimate for the group is biased toward the true value of state 100, which is higher than the true value of state 1. ■

9.4 Linear Methods

One of the most important special cases of function approximation is that in which the approximate function, $\hat{v}(\cdot,\mathbf{w})$, is a linear function of the weight vector, $\mathbf{w}$. Corresponding to every state s, there is a real-valued vector $\mathbf{x}(s) \doteq (x_1(s), x_2(s), \ldots, x_d(s))^\top$, with the same number of components as $\mathbf{w}$. Linear methods approximate state-value function by

the inner product between $\mathbf{w}$ and $\mathbf{x}(s)$:

$$\hat{v}(s,\mathbf{w}) \doteq \mathbf{w}^\top \mathbf{x}(s) \doteq \sum_{i=1}^{d} w_i x_i(s). \tag{9.8}$$

In this case the approximate value function is said to be *linear in the weights*, or simply *linear*.

The vector $\mathbf{x}(s)$ is called a *feature vector* representing state s. Each component $x_i(s)$ of $\mathbf{x}(s)$ is the value of a function $x_i : \mathcal{S} \to \mathbb{R}$. We think of a *feature* as the entirety of one of these functions, and we call its value for a state s a *feature of* s. For linear methods, features are *basis functions* because they form a linear basis for the set of approximate functions. Constructing d-dimensional feature vectors to represent states is the same as selecting a set of d basis functions. Features may be defined in many different ways; we cover a few possibilities in the next sections.

It is natural to use SGD updates with linear function approximation. The gradient of the approximate value function with respect to $\mathbf{w}$ in this case is

$$\nabla \hat{v}(s,\mathbf{w}) = \mathbf{x}(s).$$

Thus, in the linear case the general SGD update (9.7) reduces to a particularly simple form:

$$\mathbf{w}_{t+1} \doteq \mathbf{w}_t + \alpha\Big[U_t - \hat{v}(S_t,\mathbf{w}_t)\Big]\mathbf{x}(S_t).$$

Because it is so simple, the linear SGD case is one of the most favorable for mathematical analysis. Almost all useful convergence results for learning systems of all kinds are for linear (or simpler) function approximation methods.

In particular, in the linear case there is only one optimum (or, in degenerate cases, one set of equally good optima), and thus any method that is guaranteed to converge to or near a local optimum is automatically guaranteed to converge to or near the global optimum. For example, the gradient Monte Carlo algorithm presented in the previous section converges to the global optimum of the $\overline{\text{VE}}$ under linear function approximation if α is reduced over time according to the usual conditions.

The semi-gradient TD(0) algorithm presented in the previous section also converges under linear function approximation, but this does not follow from general results on SGD; a separate theorem is necessary. The weight vector converged to is also not the global optimum, but rather a point near the local optimum. It is useful to consider this important case in more detail, specifically for the continuing case. The update at each time t is

$$\begin{aligned}\mathbf{w}_{t+1} &\doteq \mathbf{w}_t + \alpha\Big(R_{t+1} + \gamma \mathbf{w}_t^\top \mathbf{x}_{t+1} - \mathbf{w}_t^\top \mathbf{x}_t\Big)\mathbf{x}_t \\ &= \mathbf{w}_t + \alpha\Big(R_{t+1}\mathbf{x}_t - \mathbf{x}_t\big(\mathbf{x}_t - \gamma \mathbf{x}_{t+1}\big)^\top \mathbf{w}_t\Big),\end{aligned} \tag{9.9}$$

where here we have used the notational shorthand $\mathbf{x}_t = \mathbf{x}(S_t)$. Once the system has reached steady state, for any given $\mathbf{w}_t$, the expected next weight vector can be written

$$\mathbb{E}[\mathbf{w}_{t+1}|\mathbf{w}_t] = \mathbf{w}_t + \alpha(\mathbf{b} - \mathbf{A}\mathbf{w}_t), \tag{9.10}$$

where

$$\mathbf{b} \doteq \mathbb{E}[R_{t+1}\mathbf{x}_t] \in \mathbb{R}^d \quad \text{and} \quad \mathbf{A} \doteq \mathbb{E}\Big[\mathbf{x}_t\big(\mathbf{x}_t - \gamma\mathbf{x}_{t+1}\big)^\top\Big] \in \mathbb{R}^d \times \mathbb{R}^d \tag{9.11}$$

From (9.10) it is clear that, if the system converges, it must converge to the weight vector $\mathbf{w}_{\text{TD}}$ at which

$$\begin{aligned} & \mathbf{b} - \mathbf{A}\mathbf{w}_{\text{TD}} = \mathbf{0} \\ \Rightarrow \quad & \mathbf{b} = \mathbf{A}\mathbf{w}_{\text{TD}} \\ \Rightarrow \quad & \mathbf{w}_{\text{TD}} \doteq \mathbf{A}^{-1}\mathbf{b}. \end{aligned} \tag{9.12}$$

This quantity is called the *TD fixed point*. In fact linear semi-gradient TD(0) converges to this point. Some of the theory proving its convergence, and the existence of the inverse above, is given in the box.

Proof of Convergence of Linear TD(0)

What properties assure convergence of the linear TD(0) algorithm (9.9)? Some insight can be gained by rewriting (9.10) as

$$\mathbb{E}[\mathbf{w}_{t+1}|\mathbf{w}_t] = (\mathbf{I} - \alpha\mathbf{A})\mathbf{w}_t + \alpha\mathbf{b}. \tag{9.13}$$

Note that the matrix $\mathbf{A}$ multiplies the weight vector $\mathbf{w}_t$ and not $\mathbf{b}$; only $\mathbf{A}$ is important to convergence. To develop intuition, consider the special case in which $\mathbf{A}$ is a diagonal matrix. If any of the diagonal elements are negative, then the corresponding diagonal element of $\mathbf{I} - \alpha\mathbf{A}$ will be greater than one, and the corresponding component of $\mathbf{w}_t$ will be amplified, which will lead to divergence if continued. On the other hand, if the diagonal elements of $\mathbf{A}$ are all positive, then α can be chosen smaller than one over the largest of them, such that $\mathbf{I} - \alpha\mathbf{A}$ is diagonal with all diagonal elements between 0 and 1. In this case the first term of the update tends to shrink $\mathbf{w}_t$, and stability is assured. In general, $\mathbf{w}_t$ will be reduced toward zero whenever $\mathbf{A}$ is *positive definite*, meaning $y^\top\mathbf{A}y > 0$ for any real vector $y \neq 0$. Positive definiteness also ensures that the inverse $\mathbf{A}^{-1}$ exists.

For linear TD(0), in the continuing case with $\gamma < 1$, the $\mathbf{A}$ matrix (9.11) can be written

$$\begin{aligned} \mathbf{A} &= \sum_s \mu(s) \sum_a \pi(a|s) \sum_{r,s'} p(r, s'|s, a)\mathbf{x}(s)\big(\mathbf{x}(s) - \gamma\mathbf{x}(s')\big)^\top \\ &= \sum_s \mu(s) \sum_{s'} p(s'|s)\mathbf{x}(s)\big(\mathbf{x}(s) - \gamma\mathbf{x}(s')\big)^\top \\ &= \sum_s \mu(s)\mathbf{x}(s)\Big(\mathbf{x}(s) - \gamma\sum_{s'} p(s'|s)\mathbf{x}(s')\Big)^\top \\ &= \mathbf{X}^\top\mathbf{D}(\mathbf{I} - \gamma\mathbf{P})\mathbf{X}, \end{aligned}$$

where $\mu(s)$ is the stationary distribution under π, $p(s'|s)$ is the probability of

transition from s to s' under policy π, $\mathbf{P}$ is the $|\mathcal{S}| \times |\mathcal{S}|$ matrix of these probabilities, $\mathbf{D}$ is the $|\mathcal{S}| \times |\mathcal{S}|$ diagonal matrix with the $\mu(s)$ on its diagonal, and $\mathbf{X}$ is the $|\mathcal{S}| \times d$ matrix with $\mathbf{x}(s)$ as its rows. From here it is clear that the inner matrix $\mathbf{D}(\mathbf{I} - \gamma\mathbf{P})$ is key to determining the positive definiteness of $\mathbf{A}$.

For a key matrix of this form, positive definiteness is assured if all of its columns sum to a nonnegative number. This was shown by Sutton (1988, p. 27) based on two previously established theorems. One theorem says that any matrix $\mathbf{M}$ is positive definite if and only if the symmetric matrix $\mathbf{S} = \mathbf{M} + \mathbf{M}^\top$ is positive definite (Sutton 1988, appendix). The second theorem says that any symmetric real matrix $\mathbf{S}$ is positive definite if all of its diagonal entries are positive and greater than the sum of the absolute values of the corresponding off-diagonal entries (Varga 1962, p. 23). For our key matrix, $\mathbf{D}(\mathbf{I} - \gamma\mathbf{P})$, the diagonal entries are positive and the off-diagonal entries are negative, so all we have to show is that each row sum plus the corresponding column sum is positive. The row sums are all positive because $\mathbf{P}$ is a stochastic matrix and $\gamma < 1$. Thus it only remains to show that the column sums are nonnegative. Note that the row vector of the column sums of any matrix $\mathbf{M}$ can be written as $\mathbf{1}^\top\mathbf{M}$, where $\mathbf{1}$ is the column vector with all components equal to 1. Let $\boldsymbol{\mu}$ denote the $|\mathcal{S}|$-vector of the $\mu(s)$, where $\boldsymbol{\mu} = \mathbf{P}^\top\boldsymbol{\mu}$ by virtue of μ being the stationary distribution. The column sums of our key matrix, then, are:

$$\begin{aligned}
\mathbf{1}^\top\mathbf{D}(\mathbf{I} - \gamma\mathbf{P}) &= \boldsymbol{\mu}^\top(\mathbf{I} - \gamma\mathbf{P}) \\
&= \boldsymbol{\mu}^\top - \gamma\boldsymbol{\mu}^\top\mathbf{P} \\
&= \boldsymbol{\mu}^\top - \gamma\boldsymbol{\mu}^\top \qquad \text{(because } \boldsymbol{\mu} \text{ is the stationary distribution)} \\
&= (1-\gamma)\boldsymbol{\mu}^\top,
\end{aligned}$$

all components of which are positive. Thus, the key matrix and its $\mathbf{A}$ matrix are positive definite, and on-policy TD(0) is stable. (Additional conditions and a schedule for reducing α over time are needed to prove convergence with probability one.)

At the TD fixed point, it has also been proven (in the continuing case) that the $\overline{\text{VE}}$ is within a bounded expansion of the lowest possible error:

$$\overline{\text{VE}}(\mathbf{w}_{\text{TD}}) \;\leq\; \frac{1}{1-\gamma} \min_{\mathbf{w}} \overline{\text{VE}}(\mathbf{w}). \tag{9.14}$$

That is, the asymptotic error of the TD method is no more than $\frac{1}{1-\gamma}$ times the smallest possible error, that attained in the limit by the Monte Carlo method. Because γ is often near one, this expansion factor can be quite large, so there is substantial potential loss in asymptotic performance with the TD method. On the other hand, recall that the TD methods are often of vastly reduced variance compared to Monte Carlo methods, and thus faster, as we saw in Chapters 6 and 7. Which method will be best depends on the nature of the approximation and problem, and on how long learning continues.

A bound analogous to (9.14) applies to other on-policy bootstrapping methods as well. For example, linear semi-gradient DP (Eq. 9.7 with $U_t \doteq \sum_a \pi(a|S_t) \sum_{s',r} p(s', r \,|\, S_t, a)[r + \gamma \hat{v}(s', \mathbf{w}_t)]$) with updates according to the on-policy distribution will also converge to the TD fixed point. One-step semi-gradient *action-value* methods, such as semi-gradient Sarsa(0) covered in the next chapter converge to an analogous fixed point and an analogous bound. For episodic tasks, there is a slightly different but related bound (see Bertsekas and Tsitsiklis, 1996). There are also a few technical conditions on the rewards, features, and decrease in the step-size parameter, which we have omitted here. The full details can be found in the original paper (Tsitsiklis and Van Roy, 1997).

Critical to the these convergence results is that states are updated according to the on-policy distribution. For other update distributions, bootstrapping methods using function approximation may actually diverge to infinity. Examples of this and a discussion of possible solution methods are given in Chapter 11.

Example 9.2: Bootstrapping on the 1000-state Random Walk State aggregation is a special case of linear function approximation, so let's return to the 1000-state random walk to illustrate some of the observations made in this chapter. The left panel of Figure 9.2 shows the final value function learned by the semi-gradient TD(0) algorithm (page 203) using the same state aggregation as in Example 9.1. We see that the near-asymptotic TD approximation is indeed farther from the true values than the Monte Carlo approximation shown in Figure 9.1.

Nevertheless, TD methods retain large potential advantages in learning rate, and generalize Monte Carlo methods, as we investigated fully with n-step TD methods in Chapter 7. The right panel of Figure 9.2 shows results with an n-step semi-gradient TD method using state aggregation on the 1000-state random walk that are strikingly similar to those we obtained earlier with tabular methods and the 19-state random walk (Figure 7.2). To obtain such quantitatively similar results we switched the state aggregation to 20 groups of 50 states each. The 20 groups were then quantitatively close

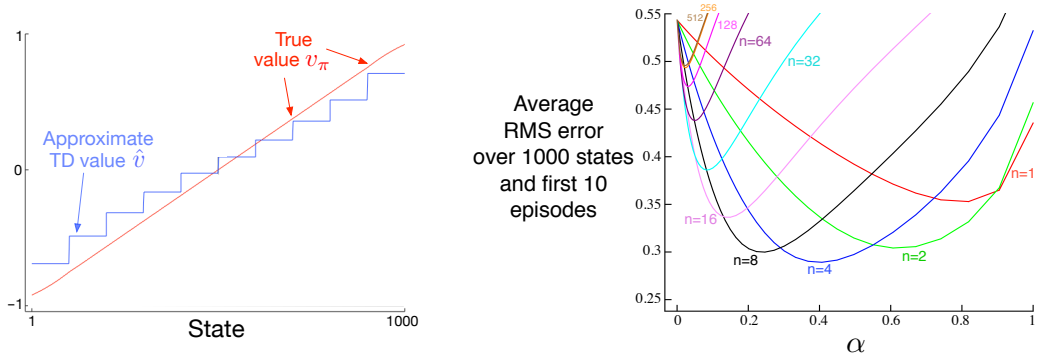


Figure 9.2: Bootstrapping with state aggregation on the 1000-state random walk task. *Left*: Asymptotic values of semi-gradient TD are worse than the asymptotic Monte Carlo values in Figure 9.1. *Right*: Performance of n-step methods with state-aggregation are strikingly similar to those with tabular representations (cf. Figure 7.2). These data are averages over 100 runs.

to the 19 states of the tabular problem. In particular, recall that state transitions were up to 100 states to the left or right. A typical transition would then be of 50 states to the right or left, which is quantitatively analogous to the single-state state transitions of the 19-state tabular system. To complete the match, we use here the same performance measure—an unweighted average of the RMS error over all states and over the first 10 episodes—rather than a $\overline{\text{VE}}$ objective as is otherwise more appropriate when using function approximation. ■

The semi-gradient n-step TD algorithm used in the example above is the natural extension of the tabular n-step TD algorithm presented in Chapter 7 to semi-gradient function approximation. Pseudocode is given in the box below.

n-step semi-gradient TD for estimating $\hat{v} \approx v_\pi$

Input: the policy π to be evaluated
Input: a differentiable function $\hat{v} : \mathcal{S}^+ \times \mathbb{R}^d \to \mathbb{R}$ such that $\hat{v}(\text{terminal},\cdot) = 0$
Algorithm parameters: step size $\alpha > 0$, a positive integer n
Initialize value-function weights $\mathbf{w}$ arbitrarily (e.g., $\mathbf{w} = \mathbf{0}$)
All store and access operations (S_t and R_t) can take their index mod $n+1$

Loop for each episode:
  Initialize and store $S_0 \neq$ terminal
  $T \leftarrow \infty$
  Loop for $t = 0, 1, 2, \ldots$:
  | If $t < T$, then:
  | Take an action according to $\pi(\cdot|S_t)$
  | Observe and store the next reward as R_{t+1} and the next state as S_{t+1}
  | If S_{t+1} is terminal, then $T \leftarrow t+1$
  | $\tau \leftarrow t - n + 1$ (τ is the time whose state's estimate is being updated)
  | If $\tau \geq 0$:
  | $G \leftarrow \sum_{i=\tau+1}^{\min(\tau+n,T)} \gamma^{i-\tau-1} R_i$
  | If $\tau + n < T$, then: $G \leftarrow G + \gamma^n \hat{v}(S_{\tau+n},\mathbf{w})$ $(G_{\tau:\tau+n})$
  | $\mathbf{w} \leftarrow \mathbf{w} + \alpha\,[G - \hat{v}(S_\tau,\mathbf{w})]\,\nabla\hat{v}(S_\tau,\mathbf{w})$
  Until $\tau = T - 1$

The key equation of this algorithm, analogous to (7.2), is

$$\mathbf{w}_{t+n} \doteq \mathbf{w}_{t+n-1} + \alpha\,[G_{t:t+n} - \hat{v}(S_t,\mathbf{w}_{t+n-1})]\,\nabla\hat{v}(S_t,\mathbf{w}_{t+n-1}), \qquad 0 \leq t < T, \tag{9.15}$$

where the n-step return is generalized from (7.1) to

$$G_{t:t+n} \doteq R_{t+1} + \gamma R_{t+2} + \cdots + \gamma^{n-1} R_{t+n} + \gamma^n \hat{v}(S_{t+n},\mathbf{w}_{t+n-1}), \quad 0 \leq t \leq T-n. \tag{9.16}$$

Exercise 9.1 Show that tabular methods such as presented in Part I of this book are a special case of linear function approximation. What would the feature vectors be? □

9.5 Feature Construction for Linear Methods

Linear methods are interesting because of their convergence guarantees, but also because in practice they can be very efficient in terms of both data and computation. Whether or not this is so depends critically on how the states are represented in terms of features, which we investigate in this large section. Choosing features appropriate to the task is an important way of adding prior domain knowledge to reinforcement learning systems. Intuitively, the features should correspond to the aspects of the state space along which generalization may be appropriate. If we are valuing geometric objects, for example, we might want to have features for each possible shape, color, size, or function. If we are valuing states of a mobile robot, then we might want to have features for locations, degrees of remaining battery power, recent sonar readings, and so on.

A limitation of the linear form is that it cannot take into account any interactions between features, such as the presence of feature i being good only in the absence of feature j. For example, in the pole-balancing task (Example 3.4) high angular velocity can be either good or bad depending on the angle. If the angle is high, then high angular velocity means an imminent danger of falling—a bad state—whereas if the angle is low, then high angular velocity means the pole is righting itself—a good state. A linear value function could not represent this if its features coded separately for the angle and the angular velocity. It needs instead, or in addition, features for combinations of these two underlying state dimensions. In the following subsections we consider a variety of general ways of doing this.

9.5.1 Polynomials

The states of many problems are initially expressed as numbers, such as positions and velocities in the pole-balancing task (Example 3.4), the number of cars in each lot in the Jack's car rental problem (Example 4.2), or the gambler's capital in the gambler problem (Example 4.3). In these types of problems, function approximation for reinforcement learning has much in common with the familiar tasks of interpolation and regression. Various families of features commonly used for interpolation and regression can also be used in reinforcement learning. Polynomials make up one of the simplest families of features used for interpolation and regression. While the basic polynomial features we discuss here do not work as well as other types of features in reinforcement learning, they serve as a good introduction because they are simple and familiar.

As an example, suppose a reinforcement learning problem has states with two numerical dimensions. For a single representative state s, let its two numbers be $s_1 \in \mathbb{R}$ and $s_2 \in \mathbb{R}$. You might choose to represent s simply by its two state dimensions, so that $\mathbf{x}(s) = (s_1, s_2)^\top$, but then you would not be able to take into account any interactions between these dimensions. In addition, if both s_1 and s_2 were zero, then the approximate value would have to also be zero. Both limitations can be overcome by instead representing s by the four-dimensional feature vector $\mathbf{x}(s) = (1, s_1, s_2, s_1 s_2)^\top$. The initial 1 feature allows the representation of affine functions in the original state numbers, and the final product feature, $s_1 s_2$, enables interactions to be taken into account. Or you might choose to use higher-dimensional feature vectors like $\mathbf{x}(s) = (1, s_1, s_2, s_1 s_2, s_1^2, s_2^2, s_1 s_2^2, s_1^2 s_2, s_1^2 s_2^2)^\top$ to

take more complex interactions into account. Such feature vectors enable approximations as arbitrary quadratic functions of the state numbers—even though the approximation is still linear in the weights that have to be learned. Generalizing this example from two to k numbers, we can represent highly-complex interactions among a problem's state dimensions:

> Suppose each state s corresponds to k numbers, s_1, s_2, ..., s_k, with each $s_i \in \mathbb{R}$. For this k-dimensional state space, each order-n polynomial-basis feature x_i can be written as
>
> $$x_i(s) = \Pi_{j=1}^{k} s_j^{c_{i,j}}, \tag{9.17}$$
>
> where each $c_{i,j}$ is an integer in the set $\{0, 1, \ldots, n\}$ for an integer $n \geq 0$. These features make up the order-n polynomial basis for dimension k, which contains $(n+1)^k$ different features.

Higher-order polynomial bases allow for more accurate approximations of more complicated functions. But because the number of features in an order-n polynomial basis grows exponentially with the dimension k of the natural state space (if $n > 0$), it is generally necessary to select a subset of them for function approximation. This can be done using prior beliefs about the nature of the function to be approximated, and some automated selection methods developed for polynomial regression can be adapted to deal with the incremental and nonstationary nature of reinforcement learning.

Exercise 9.2 Why does (9.17) define $(n+1)^k$ distinct features for dimension k? □

Exercise 9.3 What n and $c_{i,j}$ produce the feature vectors $\mathbf{x}(s) = (1, s_1, s_2, s_1 s_2, s_1^2, s_2^2, s_1 s_2^2, s_1^2 s_2, s_1^2 s_2^2)^\top$? □

9.5.2 Fourier Basis

Another linear function approximation method is based on the time-honored Fourier series, which expresses periodic functions as weighted sums of sine and cosine basis functions (features) of different frequencies. (A function f is periodic if $f(x) = f(x+\tau)$ for all x and some period τ.) The Fourier series and the more general Fourier transform are widely used in applied sciences in part because if a function to be approximated is known, then the basis function weights are given by simple formulae and, further, with enough basis functions essentially any function can be approximated as accurately as desired. In reinforcement learning, where the functions to be approximated are unknown, Fourier basis functions are of interest because they are easy to use and can perform well in a range of reinforcement learning problems.

First consider the one-dimensional case. The usual Fourier series representation of a function of one dimension having period τ represents the function as a linear combination of sine and cosine functions that are each periodic with periods that evenly divide τ (in other words, whose frequencies are integer multiples of a fundamental frequency $1/\tau$). But if you are interested in approximating an aperiodic function defined over a bounded interval, then you can use these Fourier basis features with τ set to the length the interval.

The function of interest is then just one period of the periodic linear combination of the sine and cosine features.

Furthermore, if you set τ to twice the length of the interval of interest and restrict attention to the approximation over the half interval $[0, \tau/2]$, then you can use just the cosine features. This is possible because you can represent any *even* function, that is, any function that is symmetric about the origin, with just the cosine basis. So any function over the half-period $[0, \tau/2]$ can be approximated as closely as desired with enough cosine features. (Saying "any function" is not exactly correct because the function has to be mathematically well-behaved, but we skip this technicality here.) Alternatively, it is possible to use just sine features, linear combinations of which are always *odd* functions, that is functions that are anti-symmetric about the origin. But it is generally better to keep just the cosine features because "half-even" functions tend to be easier to approximate than "half-odd" functions because the latter are often discontinuous at the origin. Of course, this does not rule out using both sine and cosine features to approximate over the interval $[0, \tau/2]$, which might be advantageous in some circumstances.

Following this logic and letting $\tau = 2$ so that the features are defined over the half-τ interval $[0, 1]$, the one-dimensional order-n Fourier cosine basis consists of the $n + 1$ features

$$x_i(s) = \cos(i\pi s), \quad s \in [0, 1],$$

for $i = 0, \ldots, n$. Figure 9.3 shows one-dimensional Fourier cosine features x_i, for $i = 1, 2, 3, 4$; x_0 is a constant function.

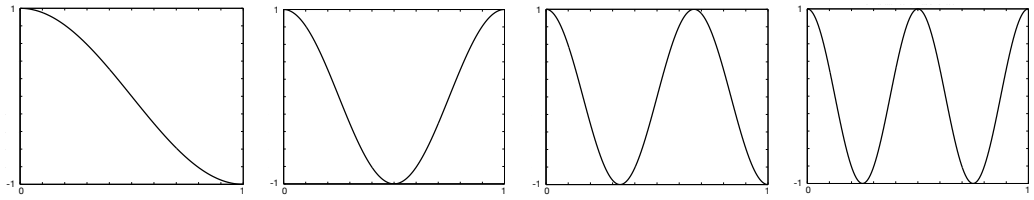


Figure 9.3: One-dimensional Fourier cosine-basis features x_i, $i = 1, 2, 3, 4$, for approximating functions over the interval $[0, 1]$. After Konidaris et al. (2011).

This same reasoning applies to the Fourier cosine series approximation in the multi-dimensional case as described in the box below.

Suppose each state s corresponds to a vector of k numbers, $\mathbf{s} = (s_1, s_2, ..., s_k)^\top$, with each $s_i \in [0, 1]$. The ith feature in the order-n Fourier cosine basis can then be written

$$x_i(s) = \cos\left(\pi \mathbf{s}^\top \mathbf{c}^i\right), \tag{9.18}$$

where $\mathbf{c}^i = (c_1^i, \ldots, c_k^i)^\top$, with $c_j^i \in \{0, \ldots, n\}$ for $j = 1, \ldots, k$ and $i = 0, \ldots, (n+1)^k$. This defines a feature for each of the $(n + 1)^k$ possible integer vectors $\mathbf{c}^i$. The inner

product $\mathbf{s}^\top \mathbf{c}^i$ has the effect of assigning an integer in $\{0, \ldots, n\}$ to each dimension of $\mathbf{s}$. As in the one-dimensional case, this integer determines the feature's frequency along that dimension. The features can of course be shifted and scaled to suit the bounded state space of a particular application.

As an example, consider the $k = 2$ case in which $\mathbf{s} = (s_1, s_2)^\top$, where each $\mathbf{c}^i = (c_1^i, c_2^i)^\top$. Figure 9.4 shows a selection of six Fourier cosine features, each labeled by the vector $\mathbf{c}^i$ that defines it (s_1 is the horizontal axis and $\mathbf{c}^i$ is shown as a row vector with the index i omitted). Any zero in $\mathbf{c}$ means the feature is constant along that state dimension. So if $\mathbf{c} = (0, 0)^\top$, the feature is constant over both dimensions; if $\mathbf{c} = (c_1, 0)^\top$ the feature is constant over the second dimension and varies over the first with frequency depending on c_1; and similarly, for $\mathbf{c} = (0, c_2)^\top$. When $\mathbf{c} = (c_1, c_2)^\top$ with neither $c_j = 0$, the feature varies along both dimensions and represents an interaction between the two state variables. The values of c_1 and c_2 determine the frequency along each dimension, and their ratio gives the direction of the interaction.

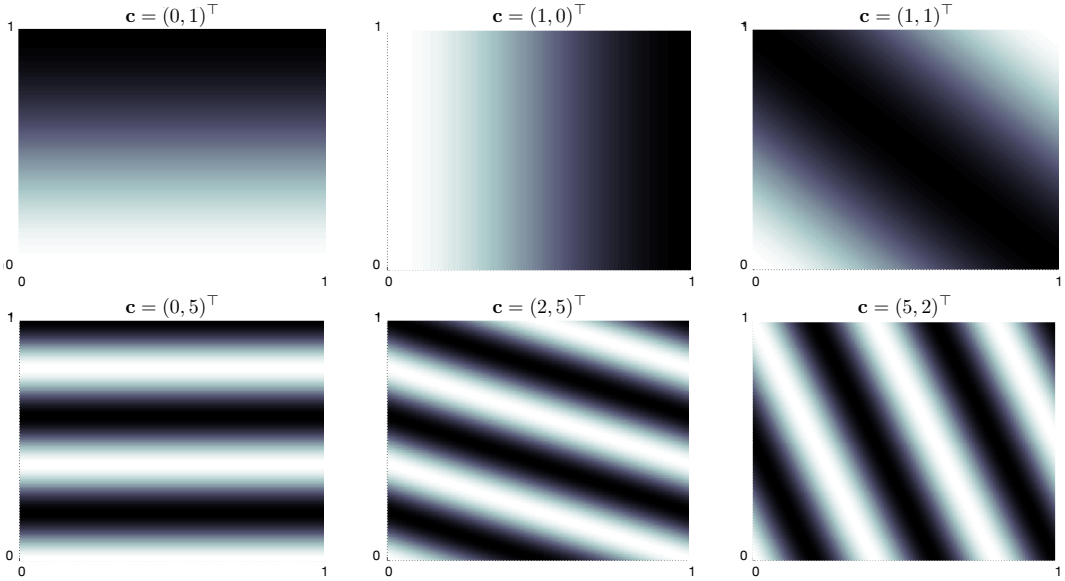


Figure 9.4: A selection of six two-dimensional Fourier cosine features, each labeled by the vector $\mathbf{c}^i$ that defines it (s_1 is the horizontal axis, and $\mathbf{c}^i$ is shown with the index i omitted). After Konidaris et al. (2011).

When using Fourier cosine features with a learning algorithm such as (9.7), semi-gradient TD(0), or semi-gradient Sarsa, it may be helpful to use a different step-size parameter for each feature. If α is the basic step-size parameter, then Konidaris, Osentoski, and Thomas (2011) suggest setting the step-size parameter for feature x_i to $\alpha_i = \alpha / \sqrt{(c_1^i)^2 + \cdots + (c_k^i)^2}$ (except when each $c_j^i = 0$, in which case $\alpha_i = \alpha$).

Fourier cosine features with Sarsa can produce good performance compared to several

other collections of basis functions, including polynomial and radial basis functions. Not surprisingly, however, Fourier features have trouble with discontinuities because it is difficult to avoid "ringing" around points of discontinuity unless very high frequency basis functions are included.

The number of features in the order-n Fourier basis grows exponentially with the dimension of the state space, but if that dimension is small enough (e.g., $k \leq 5$), then one can select n so that all of the order-n Fourier features can be used. This makes the selection of features more-or-less automatic. For high dimension state spaces, however, it is necessary to select a subset of these features. This can be done using prior beliefs about the nature of the function to be approximated, and some automated selection methods can be adapted to deal with the incremental and nonstationary nature of reinforcement learning. An advantage of Fourier basis features in this regard is that it is easy to select features by setting the $\mathbf{c}^i$ vectors to account for suspected interactions among the state variables and by limiting the values in the $\mathbf{c}^j$ vectors so that the approximation can filter out high frequency components considered to be noise. On the other hand, because Fourier features are non-zero over the entire state space (with the few zeros excepted), they represent global properties of states, which can make it difficult to find good ways to represent local properties.

Figure 9.5 shows learning curves comparing the Fourier and polynomial bases on the 1000-state random walk example. In general, we do not recommend using polynomials for online learning.[2]

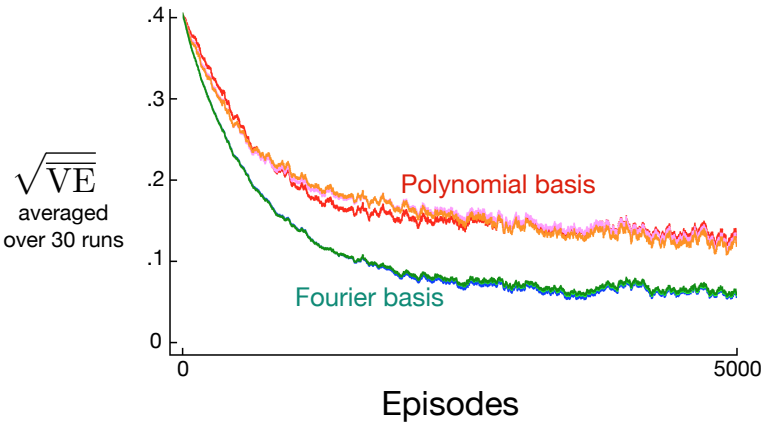


Figure 9.5: Fourier basis vs polynomials on the 1000-state random walk. Shown are learning curves for the gradient Monte Carlo method with Fourier and polynomial bases of order 5, 10, and 20. The step-size parameters were roughly optimized for each case: $\alpha = 0.0001$ for the polynomial basis and $\alpha = 0.00005$ for the Fourier basis. The performance measure (y-axis) is the root mean squared value error (9.1).

[2]There are families of polynomials more complicated than those we have discussed, for example, different families of orthogonal polynomials, and these might work better, but at present there is little experience with them in reinforcement learning.

9.5.3 Coarse Coding

Consider a task in which the natural representation of the state set is a continuous two-dimensional space. One kind of representation for this case is made up of features corresponding to *circles* in state space, as shown to the right. If the state is inside a circle, then the corresponding feature has the value 1 and is said to be *present*; otherwise the feature is 0 and is said to be *absent*. This kind of 1–0-valued feature is called a *binary feature*. Given a state, which binary features are present indicate within which circles the state lies, and thus coarsely code for its location. Representing a state with features that overlap in this way (although they need not be circles or binary) is known as *coarse coding*.

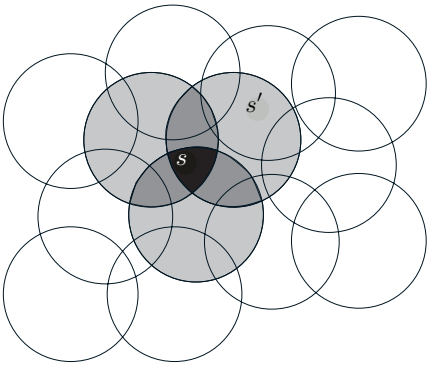


Figure 9.6: Coarse coding. Generalization from state s to state s' depends on the number of their features whose receptive fields (in this case, circles) overlap. These states have one feature in common, so there will be slight generalization between them.

Assuming linear gradient-descent function approximation, consider the effect of the size and density of the circles. Corresponding to each circle is a single weight (a component of $\mathbf{w}$) that is affected by learning. If we train at one state, a point in the space, then the weights of all circles intersecting that state will be affected. Thus, by (9.8), the approximate value function will be affected at all states within the union of the circles, with a greater effect the more circles a point has "in common" with the state, as shown in Figure 9.6. If the circles are small, then the generalization will be over a short distance, as in Figure 9.7 (left), whereas if they are large, it will be over a large distance, as in Figure 9.7 (middle). Moreover,

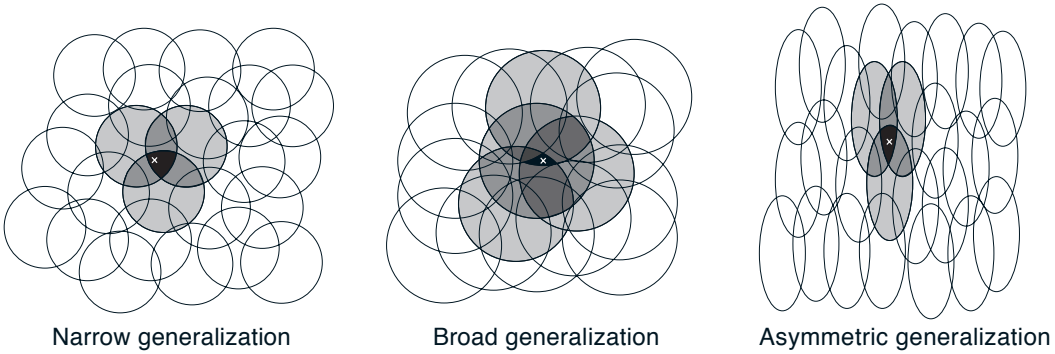


Figure 9.7: Generalization in linear function approximation methods is determined by the sizes and shapes of the features' receptive fields. All three of these cases have roughly the same number and density of features.

the shape of the features will determine the nature of the generalization. For example, if they are not strictly circular, but are elongated in one direction, then generalization will be similarly affected, as in Figure 9.7 (right).

Features with large receptive fields give broad generalization, but might also seem to limit the learned function to a coarse approximation, unable to make discriminations much finer than the width of the receptive fields. Happily, this is not the case. Initial generalization from one point to another is indeed controlled by the size and shape of the receptive fields, but acuity, the finest discrimination ultimately possible, is controlled more by the total number of features.

Example 9.3: Coarseness of Coarse Coding This example illustrates the effect on learning of the size of the receptive fields in coarse coding. Linear function approximation based on coarse coding and (9.7) was used to learn a one-dimensional square-wave function (shown at the top of Figure 9.8). The values of this function were used as the targets, U_t. With just one dimension, the receptive fields were intervals rather than circles. Learning was repeated with three different sizes of the intervals: narrow, medium, and broad, as shown at the bottom of the figure. All three cases had the same density of features, about 50 over the extent of the function being learned. Training examples were generated uniformly at random over this extent. The step-size parameter was $\alpha = \frac{0.2}{n}$, where n is the number of features that were present at one time. Figure 9.8 shows the functions learned in all three cases over the course of learning. Note that the width of the features had a strong effect early in learning. With broad features, the generalization tended to be broad; with narrow features, only the close neighbors of each trained point were changed, causing the function learned to be more bumpy. However, the final function learned was affected only slightly by the width of the features. Receptive field shape tends to have a strong effect on generalization but little effect on asymptotic solution quality.

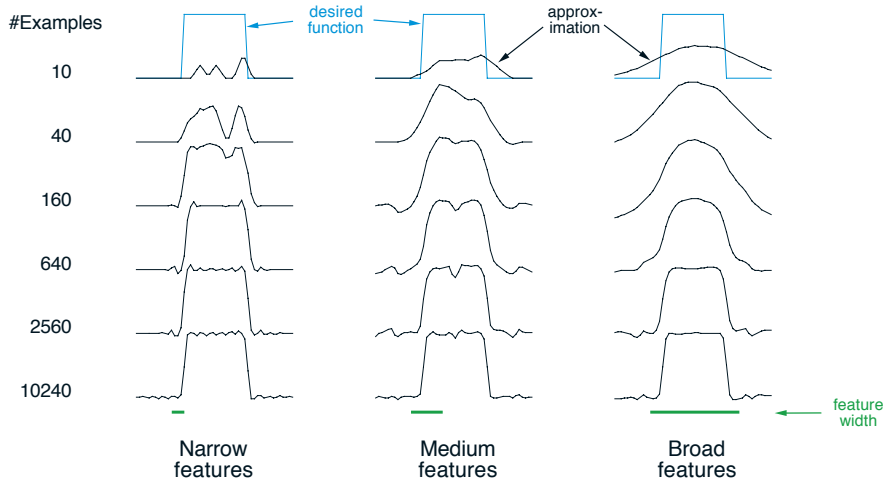


Figure 9.8: Example of feature width's strong effect on initial generalization (first row) and weak effect on asymptotic accuracy (last row). ■

9.5.4 Tile Coding

Tile coding is a form of coarse coding for multi-dimensional continuous spaces that is flexible and computationally efficient. It may be the most practical feature representation for modern sequential digital computers.

In tile coding the receptive fields of the features are grouped into partitions of the state space. Each such partition is called a *tiling*, and each element of the partition is called a *tile*. For example, the simplest tiling of a two-dimensional state space is a uniform grid such as that shown on the left side of Figure 9.9. The tiles or receptive field here are squares rather than the circles in Figure 9.6. If just this single tiling were used, then the state indicated by the white spot would be represented by the single feature whose tile it falls within; generalization would be complete to all states within the same tile and nonexistent to states outside it. With just one tiling, we would not have coarse coding but just a case of state aggregation.

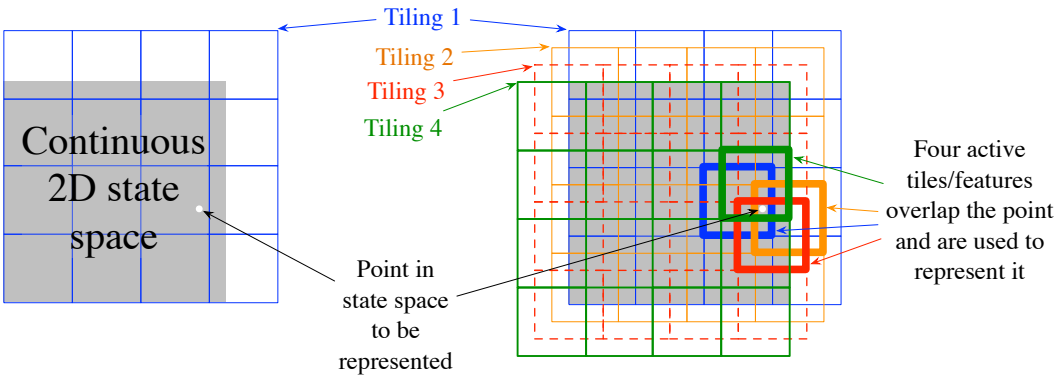


Figure 9.9: Multiple, overlapping grid-tilings on a limited two-dimensional space. These tilings are offset from one another by a uniform amount in each dimension.

To get the strengths of coarse coding requires overlapping receptive fields, and by definition the tiles of a partition do not overlap. To get true coarse coding with tile coding, multiple tilings are used, each offset by a fraction of a tile width. A simple case with four tilings is shown on the right side of Figure 9.9. Every state, such as that indicated by the white spot, falls in exactly one tile in each of the four tilings. These four tiles correspond to four features that become active when the state occurs. Specifically, the feature vector $\mathbf{x}(s)$ has one component for each tile in each tiling. In this example there are $4 \times 4 \times 4 = 64$ components, all of which will be 0 except for the four corresponding to the tiles that s falls within. Figure 9.10 shows the advantage of multiple offset tilings (coarse coding) over a single tiling on the 1000-state random walk example.

An immediate practical advantage of tile coding is that, because it works with partitions, the overall number of features that are active at one time is the same for any state. Exactly one feature is present in each tiling, so the total number of features present is always the same as the number of tilings. This allows the step-size parameter, α, to be set in an easy, intuitive way. For example, choosing $\alpha = \frac{1}{n}$, where n is the number

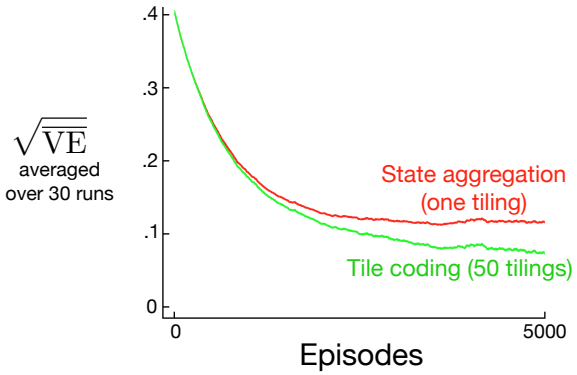


Figure 9.10: Why we use coarse coding. Shown are learning curves on the 1000-state random walk example for the gradient Monte Carlo algorithm with a single tiling and with multiple tilings. The space of 1000 states was treated as a single continuous dimension, covered with tiles each 200 states wide. The multiple tilings were offset from each other by 4 states. The step-size parameter was set so that the initial learning rate in the two cases was the same, $\alpha = 0.0001$ for the single tiling and $\alpha = 0.0001/50$ for the 50 tilings.

of tilings, results in exact one-trial learning. If the example $s \mapsto v$ is trained on, then whatever the prior estimate, $\hat{v}(s,\mathbf{w}_t)$, the new estimate will be $\hat{v}(s,\mathbf{w}_{t+1}) = v$. Usually one wishes to change more slowly than this, to allow for generalization and stochastic variation in target outputs. For example, one might choose $\alpha = \frac{1}{10n}$, in which case the estimate for the trained state would move one-tenth of the way to the target in one update, and neighboring states will be moved less, proportional to the number of tiles they have in common.

Tile coding also gains computational advantages from its use of binary feature vectors. Because each component is either 0 or 1, the weighted sum making up the approximate value function (9.8) is almost trivial to compute. Rather than performing d multiplications and additions, one simply computes the indices of the $n \ll d$ active features and then adds up the n corresponding components of the weight vector.

Generalization occurs to states other than the one trained if those states fall within any of the same tiles, proportional to the number of tiles in common. Even the choice of how to offset the tilings from each other affects generalization. If they are offset uniformly in each dimension, as they were in Figure 9.9, then different states can generalize in qualitatively different ways, as shown in the upper half of Figure 9.11. Each of the eight subfigures show the pattern of generalization from a trained state to nearby points. In this example there are eight tilings, thus 64 subregions within a tile that generalize distinctly, but all according to one of these eight patterns. Note how uniform offsets result in a strong effect along the diagonal in many patterns. These artifacts can be avoided if the tilings are offset asymmetrically, as shown in the lower half of the figure. These lower generalization patterns are better because they are all well centered on the trained state with no obvious asymmetries.

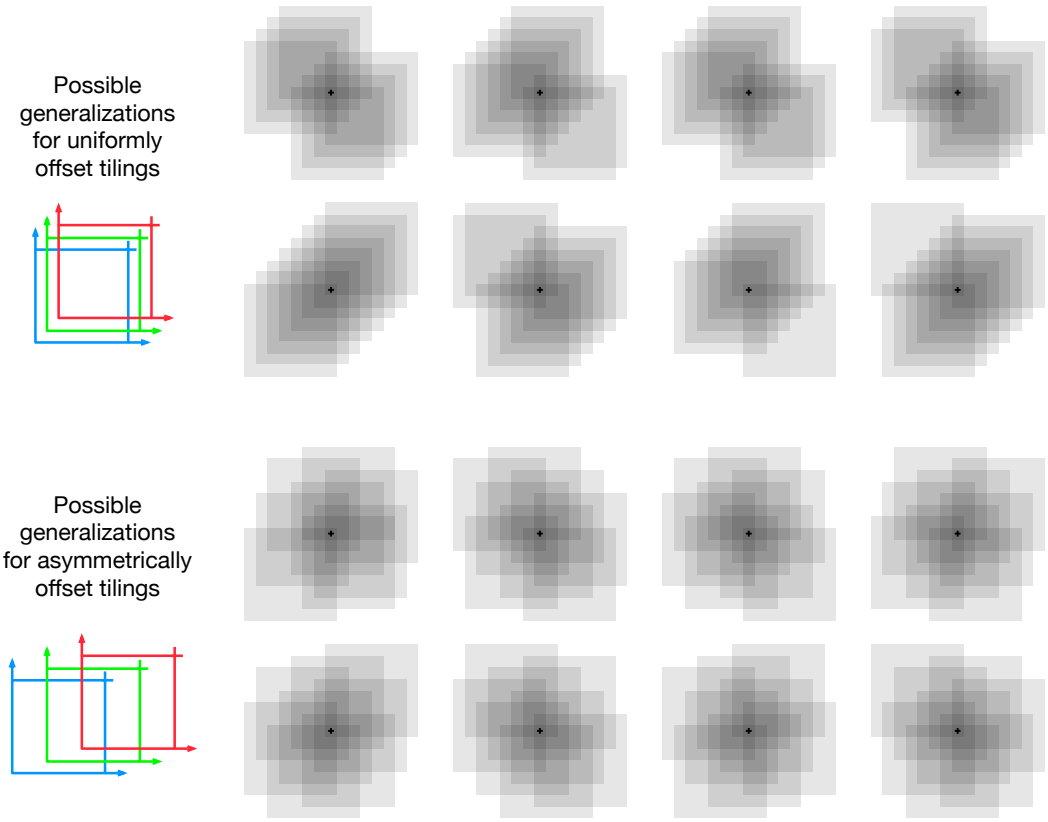


Figure 9.11: Why tile asymmetrical offsets are preferred in tile coding. Shown is the strength of generalization from a trained state, indicated by the small black plus, to nearby states, for the case of eight tilings. If the tilings are uniformly offset (above), then there are diagonal artifacts and substantial variations in the generalization, whereas with asymmetrically offset tilings the generalization is more spherical and homogeneous.

Tilings in all cases are offset from each other by a fraction of a tile width in each dimension. If w denotes the tile width and n the number of tilings, then $\frac{w}{n}$ is a fundamental unit. Within small squares $\frac{w}{n}$ on a side, all states activate the same tiles, have the same feature representation, and the same approximated value. If a state is moved by $\frac{w}{n}$ in any cartesian direction, the feature representation changes by one component/tile. Uniformly offset tilings are offset from each other by exactly this unit distance. For a two-dimensional space, we say that each tiling is offset by the displacement vector $(1, 1)$, meaning that it is offset from the previous tiling by $\frac{w}{n}$ times this vector. In these terms, the asymmetrically offset tilings shown in the lower part of Figure 9.11 are offset by a displacement vector of $(1, 3)$.

Extensive studies have been made of the effect of different displacement vectors on the generalization of tile coding (Parks and Militzer, 1991; An, 1991; An, Miller and Parks,

1991; Miller, An, Glanz and Carter, 1990), assessing their homegeneity and tendency toward diagonal artifacts like those seen for the $(1, 1)$ displacement vectors. Based on this work, Miller and Glanz (1996) recommend using displacement vectors consisting of the first odd integers. In particular, for a continuous space of dimension k, a good choice is to use the first odd integers $(1, 3, 5, 7, \ldots, 2k-1)$, with n (the number of tilings) set to an integer power of 2 greater than or equal to $4k$. This is what we have done to produce the tilings in the lower half of Figure 9.11, in which $k = 2$, $n = 2^3 \geq 4k$, and the displacement vector is $(1, 3)$. In a three-dimensional case, the first four tilings would be offset in total from a base position by $(0, 0, 0)$, $(1, 3, 5)$, $(2, 6, 10)$, and $(3, 9, 15)$. Open-source software that can efficiently make tilings like this for any k is readily available.

In choosing a tiling strategy, one has to pick the number of the tilings and the shape of the tiles. The number of tilings, along with the size of the tiles, determines the resolution or fineness of the asymptotic approximation, as in general coarse coding and illustrated in Figure 9.8. The shape of the tiles will determine the nature of generalization as in Figure 9.7. Square tiles will generalize roughly equally in each dimension as indicated in Figure 9.11 (lower). Tiles that are elongated along one dimension, such as the stripe tilings in Figure 9.12 (middle), will promote generalization along that dimension. The tilings in Figure 9.12 (middle) are also denser and thinner on the left, promoting discrimination along the horizontal dimension at lower values along that dimension. The diagonal stripe tiling in Figure 9.12 (right) will promote generalization along one diagonal. In higher dimensions, axis-aligned stripes correspond to ignoring some of the dimensions in some of the tilings, that is, to hyperplanar slices. Irregular tilings such as shown in Figure 9.12 (left) are also possible, though rare in practice and beyond the standard software.

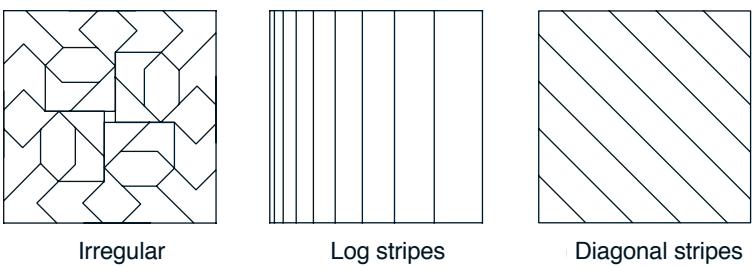
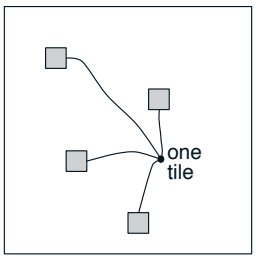


Figure 9.12: Tilings need not be grids. They can be arbitrarily shaped and non-uniform, while still in many cases being computationally efficient to compute.

In practice, it is often desirable to use different shaped tiles in different tilings. For example, one might use some vertical stripe tilings and some horizontal stripe tilings. This would encourage generalization along either dimension. However, with stripe tilings alone it is not possible to learn that a particular conjunction of horizontal and vertical coordinates has a distinctive value (whatever is learned for it will bleed into states with the same horizontal and vertical coordinates). For this one needs the conjunctive rectangular tiles such as originally shown in Figure 9.9. With multiple tilings—some horizontal, some vertical, and some conjunctive—one can get everything: a preference for generalizing along each dimension, yet the ability to learn specific values for conjunctions (see Sutton,

1996 for examples). The choice of tilings determines generalization, and until this choice can be effectively automated, it is important that tile coding enables the choice to be made flexibly and in a way that makes sense to people.

Another useful trick for reducing memory requirements is *hashing*—a consistent pseudo-random collapsing of a large tiling into a much smaller set of tiles. Hashing produces tiles consisting of noncontiguous, disjoint regions randomly spread throughout the state space, but that still form an exhaustive partition. For example, one tile might consist of the four subtiles shown to the right. Through hashing, memory requirements are often reduced by large factors with little loss of performance. This is possible because high resolution is needed in only a small fraction of the state space. Hashing frees us from the curse of dimensionality in the sense that memory requirements need not be exponential in the number of dimensions, but need merely match the real demands of the task. Open-source implementations of tile coding commonly include efficient hashing.



Exercise 9.4 Suppose we believe that one of two state dimensions is more likely to have an effect on the value function than is the other, that generalization should be primarily across this dimension rather than along it. What kind of tilings could be used to take advantage of this prior knowledge? □

9.5.5 Radial Basis Functions

Radial basis functions (RBFs) are the natural generalization of coarse coding to continuous-valued features. Rather than each feature being either 0 or 1, it can be anything in the interval $[0, 1]$, reflecting various *degrees* to which the feature is present. A typical RBF feature, x_i, has a Gaussian (bell-shaped) response $x_i(s)$ dependent only on the distance between the state, s, and the feature's prototypical or center state, c_i, and relative to the feature's width, σ_i:

$$x_i(s) \doteq \exp\left(-\frac{||s - c_i||^2}{2\sigma_i^2}\right).$$

The norm or distance metric of course can be chosen in whatever way seems most appropriate to the states and task at hand. The figure below shows a one-dimensional example with a Euclidean distance metric.

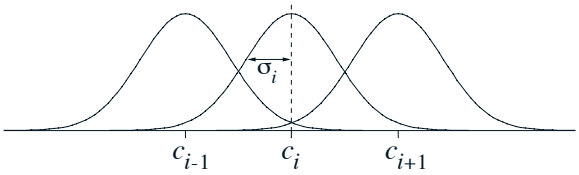


Figure 9.13: One-dimensional radial basis functions.

The primary advantage of RBFs over binary features is that they produce approximate functions that vary smoothly and are differentiable. Although this is appealing, in most cases it has no practical significance. Nevertheless, extensive studies have been made of graded response functions such as RBFs in the context of tile coding (An, 1991; Miller et al., 1991; An et al., 1991; Lane, Handelman and Gelfand, 1992). All of these methods require substantial additional computational complexity (over tile coding) and often reduce performance when there are more than two state dimensions. In high dimensions the edges of tiles are much more important, and it has proven difficult to obtain well controlled graded tile activations near the edges.

An *RBF network* is a linear function approximator using RBFs for its features. Learning is defined by equations (9.7) and (9.8), exactly as in other linear function approximators. In addition, some learning methods for RBF networks change the centers and widths of the features as well, bringing them into the realm of nonlinear function approximators. Nonlinear methods may be able to fit target functions much more precisely. The downside to RBF networks, and to nonlinear RBF networks especially, is greater computational complexity and, often, more manual tuning before learning is robust and efficient.

9.6 Selecting Step-Size Parameters Manually

Most SGD methods require the designer to select an appropriate step-size parameter α. Ideally this selection would be automated, and in some cases it has been, but for most cases it is still common practice to set it manually. To do this, and to better understand the algorithms, it is useful to develop some intuitive sense of the role of the step-size parameter. Can we say in general how it should be set?

Theoretical considerations are unfortunately of little help. The theory of stochastic approximation gives us conditions (2.7) on a slowly decreasing step-size sequence that are sufficient to guarantee convergence, but these tend to result in learning that is too slow. The classical choice $\alpha_t = 1/t$, which produces sample averages in tabular MC methods, is not appropriate for TD methods, for nonstationary problems, or for any method using function approximation. For linear methods, there are recursive least-squares methods that set an optimal *matrix* step size, and these methods can be extended to temporal-difference learning as in the LSTD method described in Section 9.8, but these require $O(d^2)$ step-size parameters, or d times more parameters than we are learning. For this reason we rule them out for use on large problems where function approximation is most needed.

To get some intuitive feel for how to set the step-size parameter manually, it is best to go back momentarily to the tabular case. There we can understand that a step size of $\alpha = 1$ will result in a complete elimination of the sample error after one target (see (2.4) with a step size of one). As discussed on page 201, we usually want to learn slower than this. In the tabular case, a step size of $\alpha = \frac{1}{10}$ would take about 10 experiences to converge approximately to their mean target, and if we wanted to learn in 100 experiences we would use $\alpha = \frac{1}{100}$. In general, if $\alpha = \frac{1}{\tau}$, then the tabular estimate for a state will approach the mean of its targets, with the most recent targets having the greatest effect, after about τ experiences with the state.

With general function approximation there is not such a clear notion of *number* of experiences with a state, as each state may be similar to and dissimilar from all the others to various degrees. However, there is a similar rule that gives similar behavior in the case of linear function approximation. Suppose you wanted to learn in about τ experiences with substantially the same feature vector. A good rule of thumb for setting the step-size parameter of linear SGD methods is then

$$\alpha \doteq \big(\tau \mathbb{E}\big[\mathbf{x}^\top \mathbf{x}\big]\big)^{-1}, \tag{9.19}$$

where $\mathbf{x}$ is a random feature vector chosen from the same distribution as input vectors will be in the SGD. This method works best if the feature vectors do not vary greatly in length; ideally $\mathbf{x}^\top \mathbf{x}$ is a constant.

Exercise 9.5 Suppose you are using tile coding to transform a seven-dimensional continuous state space into binary feature vectors to estimate a state value function $\hat{v}(s,\mathbf{w}) \approx v_\pi(s)$. You believe that the dimensions do not interact strongly, so you decide to use eight tilings of each dimension separately (stripe tilings), for $7 \times 8 = 56$ tilings. In addition, in case there are some pairwise interactions between the dimensions, you also take all $\binom{7}{2} = 21$ pairs of dimensions and tile each pair conjunctively with rectangular tiles. You make two tilings for each pair of dimensions, making a grand total of $21 \times 2 + 56 = 98$ tilings. Given these feature vectors, you suspect that you still have to average out some noise, so you decide that you want learning to be gradual, taking about 10 presentations with the same feature vector before learning nears its asymptote. What step-size parameter α should you use? Why? □

9.7 Nonlinear Function Approximation: Artificial Neural Networks

Artificial neural networks (ANNs) are widely used for nonlinear function approximation. An ANN is a network of interconnected units that have some of the properties of neurons, the main components of nervous systems. ANNs have a long history, with the latest advances in training deeply-layered ANNs (deep learning) being responsible for some of the most impressive abilities of machine learning systems, including reinforcement learning systems. In Chapter 16 we describe several impressive examples of reinforcement learning systems that use ANN function approximation.

Figure 9.14 shows a generic feedforward ANN, meaning that there are no loops in the network, that is, there are no paths within the network by which a unit's output can influence its input. The network in the figure has an output layer consisting of two output units, an input layer with four input units, and two "hidden layers": layers that are neither input nor output layers. A real-valued weight is associated with each link. A weight roughly corresponds to the efficacy of a synaptic connection in a real neural network (see Section 15.1). If an ANN has at least one loop in its connections, it is a recurrent rather than a feedforward ANN. Although both feedforward and recurrent ANNs have been used in reinforcement learning, here we look only at the simpler feedforward case.

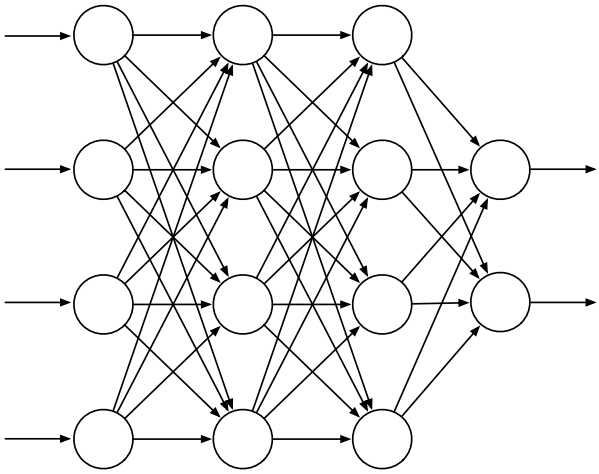


Figure 9.14: A generic feedforward ANN with four input units, two output units, and two hidden layers.

The units (the circles in Figure 9.14) are typically semi-linear units, meaning that they compute a weighted sum of their input signals and then apply to the result a nonlinear function, called the *activation function*, to produce the unit's output, or activation. Different activation functions are used, but they are typically S-shaped, or sigmoid, functions such as the logistic function $f(x) = 1/(1 + e^{-x})$, though sometimes the rectifier nonlinearity $f(x) = \max(0, x)$ is used. A step function like $f(x) = 1$ if $x \geq \theta$, and 0 otherwise, results in a binary unit with threshold θ. The units in a network's input layer are somewhat different in having their activations set to externally-supplied values that are the inputs to the function the network is approximating.

The activation of each output unit of a feedforward ANN is a nonlinear function of the activation patterns over the network's input units. The functions are parameterized by the network's connection weights. An ANN with no hidden layers can represent only a very small fraction of the possible input-output functions. However an ANN with a single hidden layer containing a large enough finite number of sigmoid units can approximate any continuous function on a compact region of the network's input space to any degree of accuracy (Cybenko, 1989). This is also true for other nonlinear activation functions that satisfy mild conditions, but nonlinearity is essential: if all the units in a multi-layer feedforward ANN have linear activation functions, the entire network is equivalent to a network with no hidden layers (because linear functions of linear functions are themselves linear).

Despite this "universal approximation" property of one-hidden-layer ANNs, both experience and theory show that approximating the complex functions needed for many artificial intelligence tasks is made easier—indeed may require—abstractions that are hierarchical compositions of many layers of lower-level abstractions, that is, abstractions produced by deep architectures such as ANNs with many hidden layers. (See Bengio, 2009, for a thorough review.) The successive layers of a deep ANN compute increasingly

abstract representations of the network's "raw" input, with each unit providing a feature contributing to a hierarchical representation of the overall input-output function of the network.

Training the hidden layers of an ANN is therefore a way to automatically create features appropriate for a given problem so that hierarchical representations can be produced without relying exclusively on hand-crafted features. This has been an enduring challenge for artificial intelligence and explains why learning algorithms for ANNs with hidden layers have received so much attention over the years. ANNs typically learn by a stochastic gradient method (Section 9.3). Each weight is adjusted in a direction aimed at improving the network's overall performance as measured by an objective function to be either minimized or maximized. In the most common supervised learning case, the objective function is the expected error, or loss, over a set of labeled training examples. In reinforcement learning, ANNs can use TD errors to learn value functions, or they can aim to maximize expected reward as in a gradient bandit (Section 2.8) or a policy-gradient algorithm (Chapter 13). In all of these cases it is necessary to estimate how a change in each connection weight would influence the network's overall performance, in other words, to estimate the partial derivative of an objective function with respect to each weight, given the current values of all the network's weights. The gradient is the vector of these partial derivatives.

The most successful way to do this for ANNs with hidden layers (provided the units have differentiable activation functions) is the backpropagation algorithm, which consists of alternating forward and backward passes through the network. Each forward pass computes the activation of each unit given the current activations of the network's input units. After each forward pass, a backward pass efficiently computes a partial derivative for each weight. (As in other stochastic gradient learning algorithms, the vector of these partial derivatives is an estimate of the true gradient.) In Section 15.10 we discuss methods for training ANNs with hidden layers that use reinforcement learning principles instead of backpropagation. These methods are less efficient than the backpropagation algorithm, but they may be closer to how real neural networks learn.

The backpropagation algorithm can produce good results for shallow networks having 1 or 2 hidden layers, but it may not work well for deeper ANNs. In fact, training a network with $k + 1$ hidden layers can actually result in poorer performance than training a network with k hidden layers, even though the deeper network can represent all the functions that the shallower network can (Bengio, 2009). Explaining results like these is not easy, but several factors are important. First, the large number of weights in a typical deep ANN makes it difficult to avoid the problem of overfitting, that is, the problem of failing to generalize correctly to cases on which the network has not been trained. Second, backpropagation does not work well for deep ANNs because the partial derivatives computed by its backward passes either decay rapidly toward the input side of the network, making learning by deep layers extremely slow, or the partial derivatives grow rapidly toward the input side of the network, making learning unstable. Methods for dealing with these problems are largely responsible for many impressive recent results achieved by systems that use deep ANNs.

Overfitting is a problem for any function approximation method that adjusts functions

with many degrees of freedom on the basis of limited training data. It is less of a problem for online reinforcement learning that does not rely on limited training sets, but generalizing effectively is still an important issue. Overfitting is a problem for ANNs in general, but especially so for deep ANNs because they tend to have very large numbers of weights. Many methods have been developed for reducing overfitting. These include stopping training when performance begins to decrease on validation data different from the training data (cross validation), modifying the objective function to discourage complexity of the approximation (regularization), and introducing dependencies among the weights to reduce the number of degrees of freedom (e.g., weight sharing).

A particularly effective method for reducing overfitting by deep ANNs is the dropout method introduced by Srivastava, Hinton, Krizhevsky, Sutskever, and Salakhutdinov (2014). During training, units are randomly removed from the network (dropped out) along with their connections. This can be thought of as training a large number of "thinned" networks. Combining the results of these thinned networks at test time is a way to improve generalization performance. The dropout method efficiently approximates this combination by multiplying each outgoing weight of a unit by the probability that that unit was retained during training. Srivastava et al. found that this method significantly improves generalization performance. It encourages individual hidden units to learn features that work well with random collections of other features. This increases the versatility of the features formed by the hidden units so that the network does not overly specialize to rarely-occurring cases.

Hinton, Osindero, and Teh (2006) took a major step toward solving the problem of training the deep layers of a deep ANN in their work with deep belief networks, layered networks closely related to the deep ANNs discussed here. In their method, the deepest layers are trained one at a time using an unsupervised learning algorithm. Without relying on the overall objective function, unsupervised learning can extract features that capture statistical regularities of the input stream. The deepest layer is trained first, then with input provided by this trained layer, the next deepest layer is trained, and so on, until the weights in all, or many, of the network's layers are set to values that now act as initial values for supervised learning. The network is then fine-tuned by backpropagation with respect to the overall objective function. Studies show that this approach generally works much better than backpropagation with weights initialized with random values. The better performance of networks trained with weights initialized this way could be due to many factors, but one idea is that this method places the network in a region of weight space from which a gradient-based algorithm can make good progress.

Batch normalization (Ioffe and Szegedy, 2015) is another technique that makes it easier to train deep ANNs. It has long been known that ANN learning is easier if the network input is normalized, for example, by adjusting each input variable to have zero mean and unit variance. Batch normalization for training deep ANNs normalizes the output of deep layers before they feed into the following layer. Ioffe and Szegedy (2015) used statistics from subsets, or "mini-batches," of training examples to normalize these between-layer signals to improve the learning rate of deep ANNs.

Another technique useful for training deep ANNs is *deep residual learning* (He, Zhang, Ren, and Sun, 2016). Sometimes it is easier to learn how a function differs from the

identity function than to learn the function itself. Then adding this difference, or residual function, to the input produces the desired function. In deep ANNs, a block of layers can be made to learn a residual function simply by adding shortcut, or skip, connections around the block. These connections add the input to the block to its output, and no additional weights are needed. He et al. (2016) evaluated this method using deep convolutional networks with skip connections around every pair of adjacent layers, finding substantial improvement over networks without the skip connections on benchmark image classification tasks. Both batch normalization and deep residual learning were used in the reinforcement learning application to the game of Go that we describe in Chapter 16.

A type of deep ANN that has proven to be very successful in applications, including impressive reinforcement learning applications (Chapter 16), is the *deep convolutional network*. This type of network is specialized for processing high-dimensional data arranged in spatial arrays, such as images. It was inspired by how early visual processing works in the brain (LeCun, Bottou, Bengio and Haffner, 1998). Because of its special architecture, a deep convolutional network can be trained by backpropagation without resorting to methods like those described above to train the deep layers.

Figure 9.15 illustrates the architecture of a deep convolutional network. This instance, from LeCun et al. (1998), was designed to recognize hand-written characters. It consists of alternating convolutional and subsampling layers, followed by several fully connected final layers. Each convolutional layer produces a number of feature maps. A feature map is a pattern of activity over an array of units, where each unit performs the same operation on data in its receptive field, which is the part of the data it "sees" from the preceding layer (or from the external input in the case of the first convolutional layer). The units of a feature map are identical to one another except that their receptive fields, which are all the same size and shape, are shifted to different locations on the arrays of incoming data. Units in the same feature map share the same weights. This means that a feature map detects the same feature no matter where it is located in the input array. In the network in Figure 9.15, for example, the first convolutional layer produces 6 feature maps, each consisting of 28×28 units. Each unit in each feature map has a 5×5 receptive field, and these receptive fields overlap (in this case by four columns and four

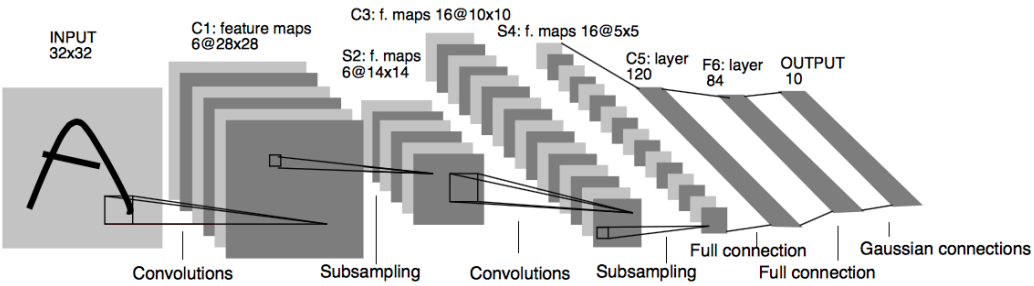


Figure 9.15: Deep Convolutional Network. Republished with permission of Proceedings of the IEEE, from Gradient-based learning applied to document recognition, LeCun, Bottou, Bengio, and Haffner, volume 86, 1998; permission conveyed through Copyright Clearance Center, Inc.

rows). Consequently, each of the 6 feature maps is specified by just 25 adjustable weights.

The subsampling layers of a deep convolutional network reduce the spatial resolution of the feature maps. Each feature map in a subsampling layer consists of units that average over a receptive field of units in the feature maps of the preceding convolutional layer. For example, each unit in each of the 6 feature maps in the first subsampling layer of the network of Figure 9.15 averages over a 2×2 non-overlapping receptive field over one of the feature maps produced by the first convolutional layer, resulting in six 14×14 feature maps. Subsampling layers reduce the network's sensitivity to the spatial locations of the features detected, that is, they help make the network's responses spatially invariant. This is useful because a feature detected at one place in an image is likely to be useful at other places as well.

Advances in the design and training of ANNs—of which we have only mentioned a few—all contribute to reinforcement learning. Although current reinforcement learning theory is mostly limited to methods using tabular or linear function approximation methods, the impressive performances of notable reinforcement learning applications owe much of their success to nonlinear function approximation by multi-layer ANNs. We discuss several of these applications in Chapter 16.

9.8 Least-Squares TD

All the methods we have discussed so far in this chapter have required computation per time step proportional to the number of parameters. With more computation, however, one can do better. In this section we present a method for linear function approximation that is arguably the best that can be done for this case.

As we established in Section 9.4 TD(0) with linear function approximation converges asymptotically (for appropriately decreasing step sizes) to the TD fixed point:

$$\mathbf{w}_{\text{TD}} = \mathbf{A}^{-1}\mathbf{b},$$

where

$$\mathbf{A} \doteq \mathbb{E}\big[\mathbf{x}_t(\mathbf{x}_t - \gamma\mathbf{x}_{t+1})^\top\big] \quad \text{and} \quad \mathbf{b} \doteq \mathbb{E}[R_{t+1}\mathbf{x}_t]\,.$$

Why, one might ask, must we compute this solution iteratively? This is wasteful of data! Could one not do better by computing estimates of $\mathbf{A}$ and $\mathbf{b}$, and then directly computing the TD fixed point? The *Least-Squares TD* algorithm, commonly known as *LSTD*, does exactly this. It forms the natural estimates

$$\widehat{\mathbf{A}}_t \doteq \sum_{k=0}^{t-1} \mathbf{x}_k(\mathbf{x}_k - \gamma\mathbf{x}_{k+1})^\top + \varepsilon\mathbf{I} \quad \text{and} \quad \widehat{\mathbf{b}}_t \doteq \sum_{k=0}^{t-1} R_{k+1}\mathbf{x}_k, \tag{9.20}$$

where $\mathbf{I}$ is the identity matrix, and $\varepsilon\mathbf{I}$, for some small $\varepsilon > 0$, ensures that $\widehat{\mathbf{A}}_t$ is always invertible. It might seem that these estimates should both be divided by t, and indeed they should; as defined here, these are really estimates of t *times* $\mathbf{A}$ and t *times* $\mathbf{b}$.

However, the extra t factors cancel out when LSTD uses these estimates to estimate the TD fixed point as

$$\mathbf{w}_t \doteq \widehat{\mathbf{A}}_t^{-1}\widehat{\mathbf{b}}_t. \tag{9.21}$$

This algorithm is the most data efficient form of linear TD(0), but it is also more expensive computationally. Recall that semi-gradient TD(0) requires memory and per-step computation that is only $O(d)$.

How complex is LSTD? As it is written above the complexity seems to increase with t, but the two approximations in (9.20) could be implemented incrementally using the techniques we have covered earlier (e.g., in Chapter 2) so that they can be done in constant time per step. Even so, the update for $\widehat{\mathbf{A}}_t$ would involve an outer product (a column vector times a row vector) and thus would be a matrix update; its computational complexity would be $O(d^2)$, and of course the memory required to hold the $\widehat{\mathbf{A}}_t$ matrix would be $O(d^2)$.

A potentially greater problem is that our final computation (9.21) uses the inverse of $\widehat{\mathbf{A}}_t$, and the computational complexity of a general inverse computation is $O(d^3)$. Fortunately, an inverse of a matrix of our special form—a sum of outer products—can also be updated incrementally with only $O(d^2)$ computations, as

$$\begin{aligned}
\widehat{\mathbf{A}}_t^{-1} &= \left(\widehat{\mathbf{A}}_{t-1} + \mathbf{x}_t(\mathbf{x}_t - \gamma\mathbf{x}_{t+1})^\top\right)^{-1} && \text{(from (9.20))} \\
&= \widehat{\mathbf{A}}_{t-1}^{-1} - \frac{\widehat{\mathbf{A}}_{t-1}^{-1}\mathbf{x}_t(\mathbf{x}_t - \gamma\mathbf{x}_{t+1})^\top\widehat{\mathbf{A}}_{t-1}^{-1}}{1 + (\mathbf{x}_t - \gamma\mathbf{x}_{t+1})^\top\widehat{\mathbf{A}}_{t-1}^{-1}\mathbf{x}_t}, && (9.22)
\end{aligned}$$

for $t > 0$, with $\widehat{\mathbf{A}}_0 \doteq \varepsilon\mathbf{I}$. Although the identity (9.22), known as *the Sherman-Morrison formula*, is superficially complicated, it involves only vector-matrix and vector-vector multiplications and thus is only $O(d^2)$. Thus we can store the inverse matrix $\widehat{\mathbf{A}}_t^{-1}$, maintain it with (9.22), and then use it in (9.21), all with only $O(d^2)$ memory and per-step computation. The complete algorithm is given in the box on the next page.

Of course, $O(d^2)$ is still significantly more expensive than the $O(d)$ of semi-gradient TD. Whether the greater data efficiency of LSTD is worth this computational expense depends on how large d is, how important it is to learn quickly, and the expense of other parts of the system. The fact that LSTD requires no step-size parameter is sometimes also touted, but the advantage of this is probably overstated. LSTD does not require a step size, but it does requires ε; if ε is chosen too small the sequence of inverses can vary wildly, and if ε is chosen too large then learning is slowed. In addition, LSTD's lack of a step-size parameter means that it never forgets. This is sometimes desirable, but it is problematic if the target policy π changes as it does in reinforcement learning and GPI. In control applications, LSTD typically has to be combined with some other mechanism to induce forgetting, mooting any initial advantage of not requiring a step-size parameter.

LSTD for estimating $\hat{v} = \mathbf{w}^\top \mathbf{x}(\cdot) \approx v_\pi$ ($O(d^2)$ version)

Input: feature representation $\mathbf{x} : \mathcal{S}^+ \to \mathbb{R}^d$ such that $\mathbf{x}(terminal) = \mathbf{0}$
Algorithm parameter: small $\varepsilon > 0$

$\widehat{\mathbf{A}^{-1}} \leftarrow \varepsilon^{-1}\mathbf{I}$ — A $d \times d$ matrix
$\widehat{\mathbf{b}} \leftarrow \mathbf{0}$ — A d-dimensional vector
Loop for each episode:
  Initialize S; $\mathbf{x} \leftarrow \mathbf{x}(S)$
  Loop for each step of episode:
    Choose and take action $A \sim \pi(\cdot|S)$, observe R, S'; $\mathbf{x}' \leftarrow \mathbf{x}(S')$
    $\mathbf{v} \leftarrow \widehat{\mathbf{A}^{-1}}^\top (\mathbf{x} - \gamma\mathbf{x}')$
    $\widehat{\mathbf{A}^{-1}} \leftarrow \widehat{\mathbf{A}^{-1}} - \big(\widehat{\mathbf{A}^{-1}}\mathbf{x}\big)\mathbf{v}^\top / \big(1 + \mathbf{v}^\top \mathbf{x}\big)$
    $\widehat{\mathbf{b}} \leftarrow \widehat{\mathbf{b}} + R\mathbf{x}$
    $\mathbf{w} \leftarrow \widehat{\mathbf{A}^{-1}}\widehat{\mathbf{b}}$
    $S \leftarrow S'$; $\mathbf{x} \leftarrow \mathbf{x}'$
  until S' is terminal

9.9 Memory-based Function Approximation

So far we have discussed the *parametric* approach to approximating value functions. In this approach, a learning algorithm adjusts the parameters of a functional form intended to approximate the value function over a problem's entire state space. Each update, $s \mapsto g$, is a training example used by the learning algorithm to change the parameters with the aim of reducing the approximation error. After the update, the training example can be discarded (although it might be saved to be used again). When an approximate value of a state (which we will call the *query state*) is needed, the function is simply evaluated at that state using the latest parameters produced by the learning algorithm.

Memory-based function approximation methods are very different. They simply save training examples in memory as they arrive (or at least save a subset of the examples) without updating any parameters. Then, whenever a query state's value estimate is needed, a set of examples is retrieved from memory and used to compute a value estimate for the query state. This approach is sometimes called *lazy learning* because processing training examples is postponed until the system is queried to provide an output.

Memory-based function approximation methods are prime examples of *nonparametric* methods. Unlike parametric methods, the approximating function's form is not limited to a fixed parameterized class of functions, such as linear functions or polynomials, but is instead determined by the training examples themselves, together with some means for combining them to output estimated values for query states. As more training examples accumulate in memory, one expects nonparametric methods to produce increasingly accurate approximations of any target function.

There are many different memory-based methods depending on how the stored training examples are selected and how they are used to respond to a query. Here, we focus on *local-learning* methods that approximate a value function only locally in the neighborhood of the current query state. These methods retrieve a set of training examples from memory whose states are judged to be the most relevant to the query state, where relevance usually depends on the distance between states: the closer a training example's state is to the query state, the more relevant it is considered to be, where distance can be defined in many different ways. After the query state is given a value, the local approximation is discarded.

The simplest example of the memory-based approach is the *nearest neighbor* method, which simply finds the example in memory whose state is closest to the query state and returns that example's value as the approximate value of the query state. In other words, if the query state is s, and $s' \mapsto g$ is the example in memory in which s' is the closest state to s, then g is returned as the approximate value of s. Slightly more complicated are *weighted average* methods that retrieve a set of nearest neighbor examples and return a weighted average of their target values, where the weights generally decrease with increasing distance between their states and the query state. *Locally weighted regression* is similar, but it fits a surface to the values of a set of nearest states by means of a parametric approximation method that minimizes a weighted error measure like (9.1), where the weights depend on distances from the query state. The value returned is the evaluation of the locally-fitted surface at the query state, after which the local approximation surface is discarded.

Being nonparametric, memory-based methods have the advantage over parametric methods of not limiting approximations to pre-specified functional forms. This allows accuracy to improve as more data accumulates. Memory-based *local* approximation methods have other properties that make them well suited for reinforcement learning. Because trajectory sampling is of such importance in reinforcement learning, as discussed in Section 8.6, memory-based local methods can focus function approximation on local neighborhoods of states (or state–action pairs) visited in real or simulated trajectories. There may be no need for global approximation because many areas of the state space will never (or almost never) be reached. In addition, memory-based methods allow an agent's experience to have a relatively immediate affect on value estimates in the neighborhood of the current state, in contrast with a parametric method's need to incrementally adjust parameters of a global approximation.

Avoiding global approximation is also a way to address the curse of dimensionality. For example, for a state space with k dimensions, a tabular method storing a global approximation requires memory exponential in k. On the other hand, in storing examples for a memory-based method, each example requires memory proportional to k, and the memory required to store, say, n examples is linear in n. Nothing is exponential in k or n. Of course, the critical remaining issue is whether a memory-based method can answer queries quickly enough to be useful to an agent. A related concern is how speed degrades as the size of the memory grows. Finding nearest neighbors in a large database can take too long to be practical in many applications.

Proponents of memory-based methods have developed ways to accelerate the nearest neighbor search. Using parallel computers or special purpose hardware is one approach; another is the use of special multi-dimensional data structures to store the training data. One data structure studied for this application is the k-d tree (short for k-dimensional tree), which recursively splits a k-dimensional space into regions arranged as nodes of a binary tree. Depending on the amount of data and how it is distributed over the state space, nearest-neighbor search using k-d trees can quickly eliminate large regions of the space in the search for neighbors, making the searches feasible in some problems where naive searches would take too long.

Locally weighted regression additionally requires fast ways to do the local regression computations which have to be repeated to answer each query. Researchers have developed many ways to address these problems, including methods for forgetting entries in order to keep the size of the database within bounds. The Bibliographic and Historical Comments section at the end of this chapter points to some of the relevant literature, including a selection of papers describing applications of memory-based learning to reinforcement learning.

9.10 Kernel-based Function Approximation

Memory-based methods such as the weighted average and locally weighted regression methods described above depend on assigning weights to examples $s' \mapsto g$ in the database depending on the distance between s' and a query states s. The function that assigns these weights is called a *kernel function*, or simply a *kernel*. In the weighted average and locally weighted regressions methods, for example, a kernel function $k : \mathbb{R} \to \mathbb{R}$ assigns weights to distances between states. More generally, weights do not have to depend on distances; they can depend on some other measure of similarity between states. In this case, $k : \mathcal{S} \times \mathcal{S} \to \mathbb{R}$, so that $k(s, s')$ is the weight given to data about s' in its influence on answering queries about s.

Viewed slightly differently, $k(s, s')$ is a measure of the strength of generalization from s' to s. Kernel functions numerically express how *relevant* knowledge about any state is to any other state. As an example, the strengths of generalization for tile coding shown in Figure 9.11 correspond to different kernel functions resulting from uniform and asymmetrical tile offsets. Although tile coding does not explicitly use a kernel function in its operation, it generalizes according to one. In fact, as we discuss more below, the strength of generalization resulting from linear parametric function approximation can always be described by a kernel function.

Kernel regression is the memory-based method that computes a kernel weighted average of the targets of *all* examples stored in memory, assigning the result to the query state. If $\mathcal{D}$ is the set of stored examples, and $g(s')$ denotes the target for state s' in a stored example, then kernel regression approximates the target function, in this case a value function depending on $\mathcal{D}$, as

$$\hat{v}(s,\mathcal{D}) = \sum_{s' \in \mathcal{D}} k(s, s')g(s'). \tag{9.23}$$

The weighted average method described above is a special case in which $k(s, s')$ is non-zero only when s and s' are close to one another so that the sum need not be computed over all of $\mathcal{D}$.

A common kernel is the Gaussian radial basis function (RBF) used in RBF function approximation as described in Section 9.5.5. In the method described there, RBFs are features whose centers and widths are either fixed from the start, with centers presumably concentrated in areas where many examples are expected to fall, or are adjusted in some way during learning. Barring methods that adjust centers and widths, this is a linear parametric method whose parameters are the weights of each RBF, which are typically learned by stochastic gradient, or semi-gradient, descent. The form of the approximation is a linear combination of the pre-determined RBFs. Kernel regression with an RBF kernel differs from this in two ways. First, it is memory-based: the RBFs are centered on the states of the stored examples. Second, it is nonparametric: there are no parameters to learn; the response to a query is given by (9.23).

Of course, many issues have to be addressed for practical implementation of kernel regression, issues that are beyond the scope of our brief discussion. However, it turns out that any linear parametric regression method like those we described in Section 9.4, with states represented by feature vectors $\mathbf{x}(s) = (x_1(s), x_2(s), \ldots, x_d(s))^\top$, can be recast as kernel regression where $k(s, s')$ is the inner product of the feature vector representations of s and s'; that is

$$k(s, s') = \mathbf{x}(s)^\top \mathbf{x}(s'). \tag{9.24}$$

Kernel regression with this kernel function produces the same approximation that a linear parametric method would if it used these feature vectors and learned with the same training data.

We skip the mathematical justification for this, which can be found in any modern machine learning text, such as Bishop (2006), and simply point out an important implication. Instead of constructing features for linear parametric function approximators, one can instead construct kernel functions directly without referring at all to feature vectors. Not all kernel functions can be expressed as inner products of feature vectors as in (9.24), but a kernel function that can be expressed like this can offer significant advantages over the equivalent parametric method. For many sets of feature vectors, (9.24) has a compact functional form that can be evaluated without any computation taking place in the d-dimensional feature space. In these cases, kernel regression is much less complex than directly using a linear parametric method with states represented by these feature vectors. This is the so-called "kernel trick" that allows effectively working in the high-dimension of an expansive feature space while actually working only with the set of stored training examples. The kernel trick is the basis of many machine learning methods, and researchers have shown how it can sometimes benefit reinforcement learning.

9.11 Looking Deeper at On-policy Learning: Interest and Emphasis

The algorithms we have considered so far in this chapter have treated all the states encountered equally, as if they were all equally important. In some cases, however, we are more interested in some states than others. In discounted episodic problems, for example, we may be more interested in accurately valuing early states in the episode than in later states where discounting may have made the rewards much less important to the value of the start state. Or, if an action-value function is being learned, it may be less important to accurately value poor actions whose value is much less than the greedy action. Function approximation resources are always limited, and if they were used in a more targeted way, then performance could be improved.

One reason we have treated all states encountered equally is that then we are updating according to the on-policy distribution, for which stronger theoretical results are available for semi-gradient methods. Recall that the on-policy distribution was defined as the distribution of states encountered in an MDP while following the target policy. Now we will generalize this concept significantly. Rather than having one on-policy distribution for the MDP, we will have many. All of them will have in common that they are a distribution of states encountered in trajectories while following the target policy, but they will vary in how the trajectories are, in a sense, initiated.

We now introduce some new concepts. First we introduce a non-negative scalar measure, a random variable I_t called *interest*, indicating the degree to which we are interested in accurately valuing the state (or state–action pair) at time t. If we don't care at all about the state, then the interest should be zero; if we fully care, it might be one, though it is formally allowed to take any non-negative value. The interest can be set in any causal way; for example, it may depend on the trajectory up to time t or the learned parameters at time t. The distribution μ in the $\overline{\text{VE}}$ (9.1) is then defined as the distribution of states encountered while following the target policy, weighted by the interest. Second, we introduce another non-negative scalar random variable, the *emphasis* M_t. This scalar multiplies the learning update and thus emphasizes or de-emphasizes the learning done at time t. The general n-step learning rule, replacing (9.15), is

$$\mathbf{w}_{t+n} \doteq \mathbf{w}_{t+n-1} + \alpha M_t \left[G_{t:t+n} - \hat{v}(S_t, \mathbf{w}_{t+n-1})\right] \nabla \hat{v}(S_t, \mathbf{w}_{t+n-1}), \qquad 0 \leq t < T, \tag{9.25}$$

with the n-step return given by (9.16) and the emphasis determined recursively from the interest by:

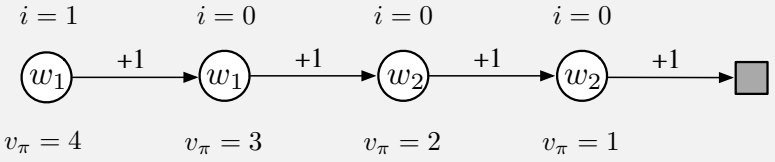
$$M_t = I_t + \gamma^n M_{t-n}, \qquad 0 \leq t < T, \tag{9.26}$$

with $M_t \doteq 0$, for all $t < 0$. These equations are taken to include the Monte Carlo case, for which $G_{t:t+n} = G_t$, all the updates are made at end of the episode, $n = T - t$, and $M_t = I_t$.

Example 9.4 illustrates how interest and emphasis can result in more accurate value estimates.

Example 9.4: Interest and Emphasis

To see the potential benefits of using interest and emphasis, consider the four-state Markov reward process shown below:



Episodes start in the leftmost state, then transition one state to the right, with a reward of +1, on each step until the terminal state is reached. The true value of the first state is thus 4, of the second state 3, and so on as shown below each state. These are the true values; the estimated values can only approximate these because they are constrained by the parameterization. There are two components to the parameter vector $\mathbf{w} = (w_1, w_2)^\top$, and the parameterization is as written inside each state. The estimated values of the first two states are given by w_1 alone and thus must be the same even though their true values are different. Similarly, the estimated values of the third and fourth states are given by w_2 alone and must be the same even though their true values are different. Suppose that we are interested in accurately valuing only the leftmost state; we assign it an interest of 1 while all the other states are assigned an interest of 0, as indicated above the states.

First consider applying gradient Monte Carlo algorithms to this problem. The algorithms presented earlier in this chapter that do not take into account interest and emphasis (in (9.7) and the box on page 202) will converge (for decreasing step sizes) to the parameter vector $\mathbf{w}_\infty = (3.5, 1.5)$, which gives the first state—the only one we are interested in—a value of 3.5 (i.e., intermediate between the true values of the first and second states). The methods presented in this section that do use interest and emphasis, on the other hand, will learn the value of the first state exactly correctly; w_1 will converge to 4 while w_2 will never be updated because the emphasis is zero in all states save the leftmost.

Now consider applying two-step semi-gradient TD methods. The methods from earlier in this chapter without interest and emphasis (in (9.15) and (9.16) and the box on page 209) will again converge to $\mathbf{w}_\infty = (3.5, 1.5)$, while the methods with interest and emphasis converge to $\mathbf{w}_\infty = (4, 2)$. The latter produces the exactly correct values for the first state and for the third state (which the first state bootstraps from) while never making any updates corresponding to the second or fourth states.

9.12 Summary

Reinforcement learning systems must be capable of *generalization* if they are to be applicable to artificial intelligence or to large engineering applications. To achieve this, any of a broad range of existing methods for *supervised-learning function approximation* can be used simply by treating each update as a training example.

Perhaps the most suitable supervised learning methods are those using *parameterized function approximation*, in which the policy is parameterized by a weight vector $\mathbf{w}$. Although the weight vector has many components, the state space is much larger still, and we must settle for an approximate solution. We defined the *mean squared value error*, $\overline{\text{VE}}(\mathbf{w})$, as a measure of the error in the values $v_{\pi_\mathbf{w}}(s)$ for a weight vector $\mathbf{w}$ under the *on-policy distribution*, μ. The $\overline{\text{VE}}$ gives us a clear way to rank different value-function approximations in the on-policy case.

To find a good weight vector, the most popular methods are variations of *stochastic gradient descent* (SGD). In this chapter we have focused on the *on-policy* case with a *fixed policy*, also known as policy evaluation or prediction; a natural learning algorithm for this case is *n-step semi-gradient TD*, which includes gradient Monte Carlo and semi-gradient TD(0) algorithms as the special cases when $n=\infty$ and $n=1$ respectively. Semi-gradient TD methods are not true gradient methods. In such bootstrapping methods (including DP), the weight vector appears in the update target, yet this is not taken into account in computing the gradient—thus they are *semi*-gradient methods. As such, they cannot rely on classical SGD results.

Nevertheless, good results can be obtained for semi-gradient methods in the special case of *linear* function approximation, in which the value estimates are sums of features times corresponding weights. The linear case is the most well understood theoretically and works well in practice when provided with appropriate features. Choosing the features is one of the most important ways of adding prior domain knowledge to reinforcement learning systems. They can be chosen as polynomials, but this case generalizes poorly in the online learning setting typically considered in reinforcement learning. Better is to choose features according the Fourier basis, or according to some form of coarse coding with sparse overlapping receptive fields. Tile coding is a form of coarse coding that is particularly computationally efficient and flexible. Radial basis functions are useful for one- or two-dimensional tasks in which a smoothly varying response is important. LSTD is the most data-efficient linear TD prediction method, but requires computation proportional to the square of the number of weights, whereas all the other methods are of complexity linear in the number of weights. Nonlinear methods include artificial neural networks trained by backpropagation and variations of SGD; these methods have become very popular in recent years under the name *deep reinforcement learning*.

Linear semi-gradient n-step TD is guaranteed to converge under standard conditions, for all n, to a $\overline{\text{VE}}$ that is within a bound of the optimal error (achieved asymptotically by Monte Carlo methods). This bound is always tighter for higher n and approaches zero as $n \to \infty$. However, in practice very high n results in very slow learning, and some degree of bootstrapping ($n < \infty$) is usually preferrable, just as we saw in comparisons of tabular n-step methods in Chapter 7 and in comparisons of tabular TD and Monte Carlo methods in Chapter 6.

Bibliographical and Historical Remarks

Generalization and function approximation have always been an integral part of reinforcement learning. Bertsekas and Tsitsiklis (1996), Bertsekas (2012), and Sugiyama et al. (2013) present the state of the art in function approximation in reinforcement learning. Some of the early work with function approximation in reinforcement learning is discussed at the end of this section.

9.3 Gradient-descent methods for minimizing mean-squared error in supervised learning are well known. Widrow and Hoff (1960) introduced the least-mean-square (LMS) algorithm, which is the prototypical incremental gradient-descent algorithm. Details of this and related algorithms are provided in many texts (e.g., Widrow and Stearns, 1985; Bishop, 1995; Duda and Hart, 1973).

Semi-gradient TD(0) was first explored by Sutton (1984, 1988), as part of the linear TD(λ) algorithm that we will treat in Chapter 12. The term "semi-gradient" to describe these bootstrapping methods is new to the second edition of this book.

The earliest use of state aggregation in reinforcement learning may have been Michie and Chambers's BOXES system (1968). The theory of state aggregation in reinforcement learning has been developed by Singh, Jaakkola, and Jordan (1995) and Tsitsiklis and Van Roy (1996). State aggregation has been used in dynamic programming from its earliest days (e.g., Bellman, 1957a).

9.4 Sutton (1988) proved convergence of linear TD(0) in the mean to the minimal $\overline{\text{VE}}$ solution for the case in which the feature vectors, $\{\mathbf{x}(s) : s \in \mathcal{S}\}$, are linearly independent. Convergence with probability 1 was proved by several researchers at about the same time (Peng, 1993; Dayan and Sejnowski, 1994; Tsitsiklis, 1994; Gurvits, Lin, and Hanson, 1994). In addition, Jaakkola, Jordan, and Singh (1994) proved convergence under online updating. All of these results assumed linearly independent feature vectors, which implies at least as many components to $\mathbf{w}_t$ as there are states. Convergence for the more important case of general (dependent) feature vectors was first shown by Dayan (1992). A significant generalization and strengthening of Dayan's result was proved by Tsitsiklis and Van Roy (1997). They proved the main result presented in this section, the bound on the asymptotic error of linear bootstrapping methods.

9.5 Our presentation of the range of possibilities for linear function approximation is based on that by Barto (1990).

9.5.2 Konidaris, Osentoski, and Thomas (2011) introduced the Fourier basis in a simple form suitable for reinforcement learning problems with multi-dimensional continuous state spaces and functions that do not have to be periodic.

9.5.3 The term *coarse coding* is due to Hinton (1984), and our Figure 9.6 is based on one of his figures. Waltz and Fu (1965) provide an early example of this type of function approximation in a reinforcement learning system.

9.5.4 Tile coding, including hashing, was introduced by Albus (1971, 1981). He described it in terms of his "cerebellar model articulator controller," or CMAC, as tile coding is sometimes known in the literature. The term "tile coding" was new to the first edition of this book, though the idea of describing CMAC in these terms is taken from Watkins (1989). Tile coding has been used in many reinforcement learning systems (e.g., Shewchuk and Dean, 1990; Lin and Kim, 1991; Miller, Scalera, and Kim, 1994; Sofge and White, 1992; Tham, 1994; Sutton, 1996; Watkins, 1989) as well as in other types of learning control systems (e.g., Kraft and Campagna, 1990; Kraft, Miller, and Dietz, 1992). This section draws heavily on the work of Miller and Glanz (1996). General software for tile coding is available in several languages (e.g., see `http://incompleteideas.net/tiles/tiles3.html`).

9.5.5 Function approximation using radial basis functions has received wide attention ever since being related to ANNs by Broomhead and Lowe (1988). Powell (1987) reviewed earlier uses of RBFs, and Poggio and Girosi (1989, 1990) extensively developed and applied this approach.

9.6 Automatic methods for adapting the step-size parameter include RMSprop (Tieleman and Hinton, 2012), Adam (Kingma and Ba, 2015), stochastic meta-descent methods such as Delta-Bar-Delta (Jacobs, 1988), its incremental generalization (Sutton, 1992b, c; Mahmood et al., 2012), and nonlinear generalizations (Schraudolph, 1999, 2002). Methods explicitly designed for reinforcement learning include AlphaBound (Dabney and Barto, 2012), SID and NOSID (Dabney, 2014), TIDBD (Kearney et al., in preparation) and the application of stochastic meta-descent to policy gradient learning (Schraudolph, Yu, and Aberdeen, 2006).

9.6 The introduction of the threshold logic unit as an abstract model neuron by McCulloch and Pitts (1943) was the beginning of ANNs. The history of ANNs as learning methods for classification or regression has passed through several stages: roughly, the Perceptron (Rosenblatt, 1962) and ADALINE (ADAptive LINear Element) (Widrow and Hoff, 1960) stage of learning by single-layer ANNs, the error-backpropagation stage (LeCun, 1985; Rumelhart, Hinton, and Williams, 1986) of learning by multi-layer ANNs, and the current deep-learning stage with its emphasis on representation learning (e.g., Bengio, Courville, and Vincent, 2012; Goodfellow, Bengio, and Courville, 2016). Examples of the many books on ANNs are Haykin (1994), Bishop (1995), and Ripley (2007).

ANNs as function approximation for reinforcement learning goes back to the early work of Farley and Clark (1954), who used reinforcement-like learning to modify the weights of linear threshold functions representing policies. Widrow, Gupta, and Maitra (1973) presented a neuron-like linear threshold unit implementing a learning process they called *learning with a critic* or *selective bootstrap adaptation*, a reinforcement-learning variant of the ADALINE algorithm. Werbos (1987, 1994) developed an approach to prediction and control that uses ANNs trained by error backpropation to learn policies and value functions using TD-like algorithms. Barto, Sutton, and Brouwer (1981) and Barto and Sutton (1981b) extended the

idea of an associative memory network (e.g., Kohonen, 1977; Anderson, Silverstein, Ritz, and Jones, 1977) to reinforcement learning. Barto, Anderson, and Sutton (1982) used a two-layer ANN to learn a nonlinear control policy, and emphasized the first layer's role of learning a suitable representation. Hampson (1983, 1989) was an early proponent of multilayer ANNs for learning value functions. Barto, Sutton, and Anderson (1983) presented an actor–critic algorithm in the form of an ANN learning to balance a simulated pole (see Sections 15.7 and 15.8). Barto and Anandan (1985) introduced a stochastic version of Widrow et al.'s (1973) selective bootstrap algorithm called the *associative reward-penalty* (A_{R-P}) *algorithm*. Barto (1985, 1986) and Barto and Jordan (1987) described multi-layer ANNs consisting of $\mathrm{A_{R-P}}$ units trained with a globally-broadcast reinforcement signal to learn classification rules that are not linearly separable. Barto (1985) discussed this approach to ANNs and how this type of learning rule is related to others in the literature at that time. (See Section 15.10 for additional discussion of this approach to training multi-layer ANNs.) Anderson (1986, 1987, 1989) evaluated numerous methods for training multilayer ANNs and showed that an actor–critic algorithm in which both the actor and critic were implemented by two-layer ANNs trained by error backpropagation outperformed single-layer ANNs in the pole-balancing and tower of Hanoi tasks. Williams (1988) described several ways that backpropagation and reinforcement learning can be combined for training ANNs. Gullapalli (1990) and Williams (1992) devised reinforcement learning algorithms for neuron-like units having continuous, rather than binary, outputs. Barto, Sutton, and Watkins (1990) argued that ANNs can play significant roles for approximating functions required for solving sequential decision problems. Williams (1992) related REINFORCE learning rules (Section 13.3) to the error backpropagation method for training multi-layer ANNs. Tesauro's TD-Gammon (Tesauro 1992, 1994; Section 16.1) influentially demonstrated the learning abilities of TD(λ) algorithm with function approximation by multi-layer ANNs in learning to play backgammon. The *AlphaGo*, *AlphaGo Zero*, and *AlphaZero* programs of Silver et al. (2016, 2017a, b; Section 16.6) used reinforcement learning with deep convolutional ANNs in achieving impressive results with the game of Go. Schmidhuber (2015) reviews applications of ANNs in reinforcement learning, including applications of recurrent ANNs.

9.8 LSTD is due to Bradtke and Barto (see Bradtke, 1993, 1994; Bradtke and Barto, 1996; Bradtke, Ydstie, and Barto, 1994), and was further developed by Boyan (1999, 2002), Nedić and Bertsekas (2003), and Yu (2010). The incremental update of the inverse matrix has been known at least since 1949 (Sherman and Morrison, 1949). An extension of least-squares methods to control was introduced by Lagoudakis and Parr (2003; Buşoniu, Lazaric, Ghavamzadeh, Munos, Babuška, and De Schutter, 2012).

9.9 Our discussion of memory-based function approximation is largely based on the review of locally weighted learning by Atkeson, Moore, and Schaal (1997). Atkeson (1992) discussed the use of locally weighted regression in memory-based robot learning and supplied an extensive bibliography covering the history of the idea. Stanfill and Waltz (1986) influentially argued for the importance of memory based methods in artificial intelligence, especially in light of parallel architectures then becoming available, such as the Connection Machine. Baird and Klopf (1993) introduced a novel memory-based approach and used it as the function approximation method for Q-learning applied to the pole-balancing task. Schaal and Atkeson (1994) applied locally weighted regression to a robot juggling control problem, where it was used to learn a system model. Peng (1995) used the pole-balancing task to experiment with several nearest-neighbor methods for approximating value functions, policies, and environment models. Tadepalli and Ok (1996) obtained promising results with locally-weighted linear regression to learn a value function for a simulated automatic guided vehicle task. Bottou and Vapnik (1992) demonstrated surprising efficiency of several local learning algorithms compared to non-local algorithms in some pattern recognition tasks, discussing the impact of local learning on generalization.

Bentley (1975) introduced k-d trees and reported observing average running time of $O(\log n)$ for nearest neighbor search over n records. Friedman, Bentley, and Finkel (1977) clarified the algorithm for nearest neighbor search with k-d trees. Omohundro (1987) discussed efficiency gains possible with hierarchical data structures such as k-d-trees. Moore, Schneider, and Deng (1997) introduced the use of k-d trees for efficient locally weighted regression.

9.10 The origin of kernel regression is the *method of potential functions* of Aizerman, Braverman, and Rozonoer (1964). They likened the data to point electric charges of various signs and magnitudes distributed over space. The resulting electric potential over space produced by summing the potentials of the point charges corresponded to the interpolated surface. In this analogy, the kernel function is the potential of a point charge, which falls off as the reciprocal of the distance from the charge. Connell and Utgoff (1987) applied an actor–critic method to the pole-balancing task in which the critic approximated the value function using kernel regression with an inverse-distance weighting. Predating widespread interest in kernel regression in machine learning, these authors did not use the term kernel, but referred to "Shepard's method" (Shepard, 1968). Other kernel-based approaches to reinforcement learning include those of Ormoneit and Sen (2002), Dietterich and Wang (2002), Xu, Xie, Hu, and Lu (2005), Taylor and Parr (2009), Barreto, Precup, and Pineau (2011), and Bhat, Farias, and Moallemi (2012).

9.11 For Emphatic-TD methods, see the bibliographical notes to Section 11.8.

The earliest example we know of in which function approximation methods were used for learning value functions was Samuel's checkers player (1959, 1967). Samuel followed Shannon's (1950) suggestion that a value function did not have to be exact to be a useful guide to selecting moves in a game and that it might be approximated by linear combination of features. In addition to linear function approximation, Samuel experimented with lookup tables and hierarchical lookup tables called signature tables (Griffith, 1966, 1974; Page, 1977; Biermann, Fairfield, and Beres, 1982).

At about the same time as Samuel's work, Bellman and Dreyfus (1959) proposed using function approximation methods with DP. (It is tempting to think that Bellman and Samuel had some influence on one another, but we know of no reference to the other in the work of either.) There is now a fairly extensive literature on function approximation methods and DP, such as multigrid methods and methods using splines and orthogonal polynomials (e.g., Bellman and Dreyfus, 1959; Bellman, Kalaba, and Kotkin, 1973; Daniel, 1976; Whitt, 1978; Reetz, 1977; Schweitzer and Seidmann, 1985; Chow and Tsitsiklis, 1991; Kushner and Dupuis, 1992; Rust, 1996).

Holland's (1986) classifier system used a selective feature-match technique to generalize evaluation information across state–action pairs. Each classifier matched a subset of states having specified values for a subset of features, with the remaining features having arbitrary values ("wild cards"). These subsets were then used in a conventional state-aggregation approach to function approximation. Holland's idea was to use a genetic algorithm to evolve a set of classifiers that collectively would implement a useful action-value function. Holland's ideas influenced the early research of the authors on reinforcement learning, but we focused on different approaches to function approximation. As function approximators, classifiers are limited in several ways. First, they are state-aggregation methods, with concomitant limitations in scaling and in representing smooth functions efficiently. In addition, the matching rules of classifiers can implement only aggregation boundaries that are parallel to the feature axes. Perhaps the most important limitation of conventional classifier systems is that the classifiers are learned via the genetic algorithm, an evolutionary method. As we discussed in Chapter 1, there is available during learning much more detailed information about how to learn than can be used by evolutionary methods. This perspective led us to instead adapt supervised learning methods for use in reinforcement learning, specifically gradient-descent and ANN methods. These differences between Holland's approach and ours are not surprising because Holland's ideas were developed during a period when ANNs were generally regarded as being too weak in computational power to be useful, whereas our work was at the beginning of the period that saw widespread questioning of that conventional wisdom. There remain many opportunities for combining aspects of these different approaches.

Christensen and Korf (1986) experimented with regression methods for modifying coefficients of linear value function approximations in the game of chess. Chapman and Kaelbling (1991) and Tan (1991) adapted decision-tree methods for learning value functions. Explanation-based learning methods have also been adapted for learning value functions, yielding compact representations (Yee, Saxena, Utgoff, and Barto, 1990; Dietterich and Flann, 1995).

Chapter 10

On-policy Control with Approximation

In this chapter we return to the control problem, now with parametric approximation of the action-value function $\hat{q}(s, a, \mathbf{w}) \approx q_*(s, a)$, where $\mathbf{w} \in \mathbb{R}^d$ is a finite-dimensional weight vector. We continue to restrict attention to the on-policy case, leaving off-policy methods to Chapter 11. The present chapter features the semi-gradient Sarsa algorithm, the natural extension of semi-gradient TD(0) (last chapter) to action values and to on-policy control. In the episodic case, the extension is straightforward, but in the continuing case we have to take a few steps backward and re-examine how we have used discounting to define an optimal policy. Surprisingly, once we have genuine function approximation we have to give up discounting and switch to a new "average-reward" formulation of the control problem, with new "differential" value functions.

Starting first in the episodic case, we extend the function approximation ideas presented in the last chapter from state values to action values. Then we extend them to control following the general pattern of on-policy GPI, using ε-greedy for action selection. We show results for n-step linear Sarsa on the Mountain Car problem. Then we turn to the continuing case and repeat the development of these ideas for the average-reward case with differential values.

10.1 Episodic Semi-gradient Control

The extension of the semi-gradient prediction methods of Chapter 9 to action values is straightforward. In this case it is the approximate action-value function, $\hat{q} \approx q_\pi$, that is represented as a parameterized functional form with weight vector $\mathbf{w}$. Whereas before we considered random training examples of the form $S_t \mapsto U_t$, now we consider examples of the form $S_t, A_t \mapsto U_t$. The update target U_t can be any approximation of $q_\pi(S_t, A_t)$, including the usual backed-up values such as the full Monte Carlo return (G_t) or any of the n-step Sarsa returns (7.4). The general gradient-descent update for action-value

prediction is

$$\mathbf{w}_{t+1} \doteq \mathbf{w}_t + \alpha\Big[U_t - \hat{q}(S_t, A_t, \mathbf{w}_t)\Big]\nabla\hat{q}(S_t, A_t, \mathbf{w}_t). \tag{10.1}$$

For example, the update for the one-step Sarsa method is

$$\mathbf{w}_{t+1} \doteq \mathbf{w}_t + \alpha\Big[R_{t+1} + \gamma\hat{q}(S_{t+1}, A_{t+1}, \mathbf{w}_t) - \hat{q}(S_t, A_t, \mathbf{w}_t)\Big]\nabla\hat{q}(S_t, A_t, \mathbf{w}_t). \tag{10.2}$$

We call this method *episodic semi-gradient one-step Sarsa*. For a constant policy, this method converges in the same way that TD(0) does, with the same kind of error bound (9.14).

To form control methods, we need to couple such action-value prediction methods with techniques for policy improvement and action selection. Suitable techniques applicable to continuous actions, or to actions from large discrete sets, are a topic of ongoing research with as yet no clear resolution. On the other hand, if the action set is discrete and not too large, then we can use the techniques already developed in previous chapters. That is, for each possible action a available in the current state S_t, we can compute $\hat{q}(S_t, a, \mathbf{w}_t)$ and then find the greedy action $A_t^* = \arg\max_a \hat{q}(S_t, a, \mathbf{w}_t)$. Policy improvement is then done (in the on-policy case treated in this chapter) by changing the estimation policy to a soft approximation of the greedy policy such as the ε-greedy policy. Actions are selected according to this same policy. Pseudocode for the complete algorithm is given in the box.

Episodic Semi-gradient Sarsa for Estimating $\hat{q} \approx q_*$

Input: a differentiable action-value function parameterization $\hat{q} : \mathcal{S} \times \mathcal{A} \times \mathbb{R}^d \to \mathbb{R}$
Algorithm parameters: step size $\alpha > 0$, small $\varepsilon > 0$
Initialize value-function weights $\mathbf{w} \in \mathbb{R}^d$ arbitrarily (e.g., $\mathbf{w} = \mathbf{0}$)

Loop for each episode:
  $S, A \leftarrow$ initial state and action of episode (e.g., ε-greedy)
  Loop for each step of episode:
    Take action A, observe R, S'
    If S' is terminal:
      $\mathbf{w} \leftarrow \mathbf{w} + \alpha\big[R - \hat{q}(S, A, \mathbf{w})\big]\nabla\hat{q}(S, A, \mathbf{w})$
      Go to next episode
    Choose A' as a function of $\hat{q}(S', \cdot, \mathbf{w})$ (e.g., ε-greedy)
    $\mathbf{w} \leftarrow \mathbf{w} + \alpha\big[R + \gamma\hat{q}(S', A', \mathbf{w}) - \hat{q}(S, A, \mathbf{w})\big]\nabla\hat{q}(S, A, \mathbf{w})$
    $S \leftarrow S'$
    $A \leftarrow A'$

Example 10.1: Mountain Car Task Consider the task of driving an underpowered car up a steep mountain road, as suggested by the diagram in the upper left of Figure 10.1. The difficulty is that gravity is stronger than the car's engine, and even at full throttle the car cannot accelerate up the steep slope. The only solution is to first move away from the goal and up the opposite slope on the left. Then, by applying full throttle the car

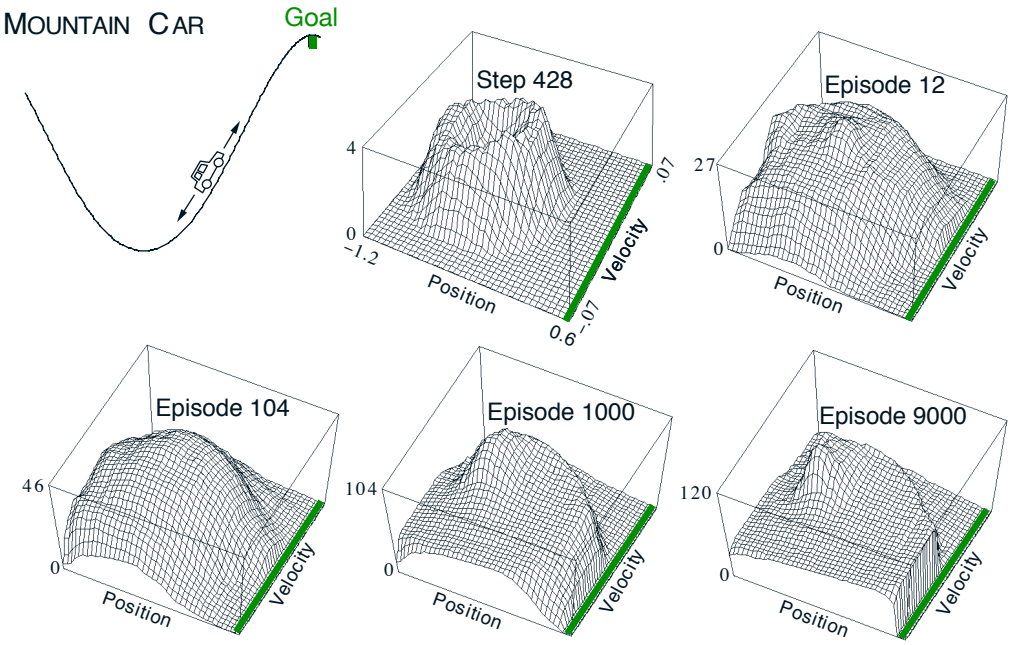


Figure 10.1: The Mountain Car task (upper left panel) and the cost-to-go function ($-\max_a \hat{q}(s, a, \mathbf{w})$) learned during one run.

can build up enough inertia to carry it up the steep slope even though it is slowing down the whole way. This is a simple example of a continuous control task where things have to get worse in a sense (farther from the goal) before they can get better. Many control methodologies have great difficulties with tasks of this kind unless explicitly aided by a human designer.

The reward in this problem is -1 on all time steps until the car moves past its goal position at the top of the mountain, which ends the episode. There are three possible actions: full throttle forward ($+1$), full throttle reverse (-1), and zero throttle (0). The car moves according to a simplified physics. Its position, x_t, and velocity, $\dot{x}_t$, are updated by

$$x_{t+1} \doteq \mathit{bound}\big[x_t + \dot{x}_{t+1}\big]$$

$$\dot{x}_{t+1} \doteq \mathit{bound}\big[\dot{x}_t + 0.001 A_t - 0.0025\cos(3x_t)\big],$$

where the *bound* operation enforces $-1.2 \leq x_{t+1} \leq 0.5$ and $-0.07 \leq \dot{x}_{t+1} \leq 0.07$. In addition, when x_{t+1} reached the left bound, $\dot{x}_{t+1}$ was reset to zero. When it reached the right bound, the goal was reached and the episode was terminated. Each episode started from a random position $x_t \in [-0.6, -0.4)$ and zero velocity. To convert the two continuous state variables to binary features, we used grid-tilings as in Figure 9.9. We used 8 tilings, with each tile covering 1/8th of the bounded distance in each dimension,

and asymmetrical offsets as described in Section 9.5.4.[1] The feature vectors $\mathbf{x}(s,a)$ created by tile coding were then combined linearly with the parameter vector to approximate the action-value function:

$$\hat{q}(s,a,\mathbf{w}) \doteq \mathbf{w}^\top \mathbf{x}(s,a) = \sum_{i=1}^{d} w_i \cdot x_i(s,a), \tag{10.3}$$

for each pair of state, s, and action, a.

Figure 10.1 shows what typically happens while learning to solve this task with this form of function approximation.[2] Shown is the negative of the value function (the *cost-to-go* function) learned on a single run. The initial action values were all zero, which was optimistic (all true values are negative in this task), causing extensive exploration to occur even though the exploration parameter, ε, was 0. This can be seen in the middle-top panel of the figure, labeled "Step 428". At this time not even one episode had been completed, but the car has oscillated back and forth in the valley, following circular trajectories in state space. All the states visited frequently are valued worse than unexplored states, because the actual rewards have been worse than what was (unrealistically) expected. This continually drives the agent away from wherever it has been, to explore new states, until a solution is found.

Figure 10.2 shows several learning curves for semi-gradient Sarsa on this problem, with various step sizes.

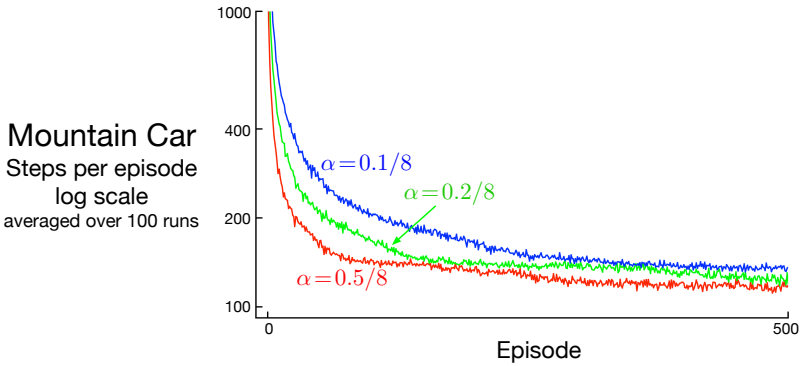


Figure 10.2: Mountain Car learning curves for the semi-gradient Sarsa method with tile-coding function approximation and ε-greedy action selection. ■

[1] In particular, we used the tile-coding software, available at `http://incompleteideas.net/tiles/tiles3.html`, with `iht=IHT(4096)` and `tiles(iht,8,[8*x/(0.5+1.2),8*xdot/(0.07+0.07)],A)` to get the indices of the ones in the feature vector for state (`x`, `xdot`) and action `A`.

[2] This data is actually from the "semi-gradient Sarsa(λ)" algorithm that we will not meet until Chapter 12, but semi-gradient Sarsa would behave similarly.

10.2 Semi-gradient n-step Sarsa

We can obtain an n-step version of episodic semi-gradient Sarsa by using an n-step return as the update target in the semi-gradient Sarsa update equation (10.1). The n-step return immediately generalizes from its tabular form (7.4) to a function approximation form:

$$G_{t:t+n} \doteq R_{t+1}+\gamma R_{t+2}+\cdots+\gamma^{n-1}R_{t+n}+\gamma^n \hat{q}(S_{t+n}, A_{t+n}, \mathbf{w}_{t+n-1}), \quad t+n<T, \tag{10.4}$$

with $G_{t:t+n} \doteq G_t$ if $t+n \geq T$, as usual. The n-step update equation is

$$\mathbf{w}_{t+n} \doteq \mathbf{w}_{t+n-1} + \alpha \left[G_{t:t+n} - \hat{q}(S_t, A_t, \mathbf{w}_{t+n-1})\right] \nabla \hat{q}(S_t, A_t, \mathbf{w}_{t+n-1}), \qquad 0 \leq t < T. \tag{10.5}$$

Complete pseudocode is given in the box below.

Episodic semi-gradient n-step Sarsa for estimating $\hat{q} \approx q_*$ or q_π

Input: a differentiable action-value function parameterization $\hat{q} : \mathcal{S} \times \mathcal{A} \times \mathbb{R}^d \to \mathbb{R}$
Input: a policy π (if estimating q_π)
Algorithm parameters: step size $\alpha > 0$, small $\varepsilon > 0$, a positive integer n
Initialize value-function weights $\mathbf{w} \in \mathbb{R}^d$ arbitrarily (e.g., $\mathbf{w} = \mathbf{0}$)
All store and access operations (S_t, A_t, and R_t) can take their index mod $n+1$

Loop for each episode:
  Initialize and store $S_0 \neq$ terminal
  Select and store an action $A_0 \sim \pi(\cdot|S_0)$ or ε-greedy wrt $\hat{q}(S_0, \cdot, \mathbf{w})$
  $T \leftarrow \infty$
  Loop for $t = 0, 1, 2, \ldots$:
    If $t < T$, then:
      Take action A_t
      Observe and store the next reward as R_{t+1} and the next state as S_{t+1}
      If S_{t+1} is terminal, then:
        $T \leftarrow t+1$
      else:
        Select and store $A_{t+1} \sim \pi(\cdot|S_{t+1})$ or ε-greedy wrt $\hat{q}(S_{t+1}, \cdot, \mathbf{w})$
    $\tau \leftarrow t-n+1$ (τ is the time whose estimate is being updated)
    If $\tau \geq 0$:
      $G \leftarrow \sum_{i=\tau+1}^{\min(\tau+n,T)} \gamma^{i-\tau-1} R_i$
      If $\tau+n < T$, then $G \leftarrow G + \gamma^n \hat{q}(S_{\tau+n}, A_{\tau+n}, \mathbf{w})$ $(G_{\tau:\tau+n})$
      $\mathbf{w} \leftarrow \mathbf{w} + \alpha \left[G - \hat{q}(S_\tau, A_\tau, \mathbf{w})\right] \nabla \hat{q}(S_\tau, A_\tau, \mathbf{w})$
  Until $\tau = T-1$

As we have seen before, performance is best if an intermediate level of bootstrapping is used, corresponding to an n larger than 1. Figure 10.3 shows how this algorithm tends to learn faster and obtain a better asymptotic performance at $n=8$ than at $n=1$ on the Mountain Car task. Figure 10.4 shows the results of a more detailed study of the effect of the parameters α and n on the rate of learning on this task.

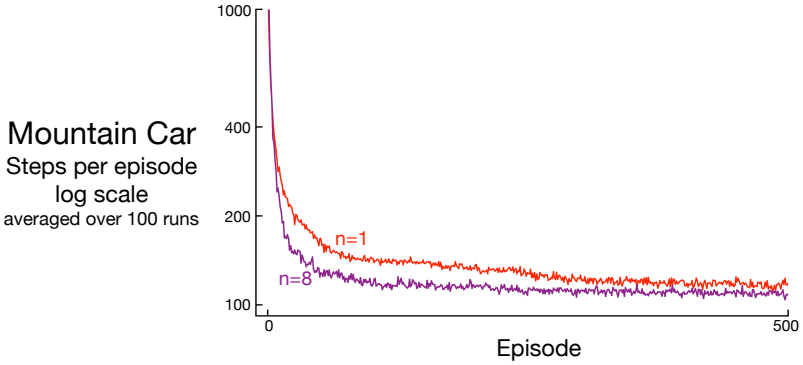


Figure 10.3: Performance of one-step vs 8-step semi-gradient Sarsa on the Mountain Car task. Good step sizes were used: $\alpha = 0.5/8$ for $n = 1$ and $\alpha = 0.3/8$ for $n = 8$.

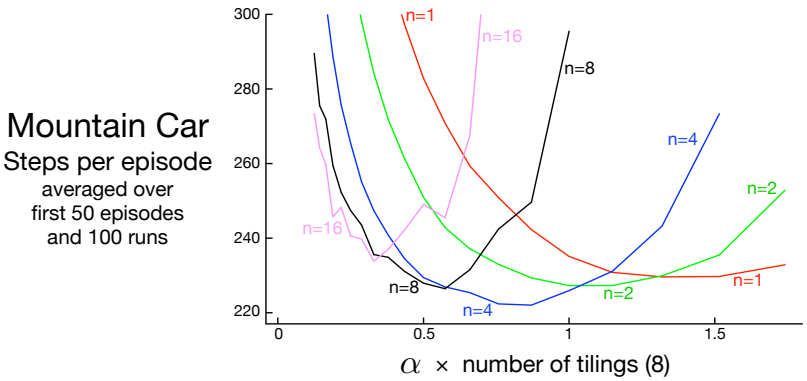


Figure 10.4: Effect of the α and n on early performance of n-step semi-gradient Sarsa and tile-coding function approximation on the Mountain Car task. As usual, an intermediate level of bootstrapping ($n = 4$) performed best. These results are for selected α values, on a log scale, and then connected by straight lines. The standard errors ranged from 0.5 (less than the line width) for $n = 1$ to about 4 for $n = 16$, so the main effects are all statistically significant.

Exercise 10.1 We have not explicitly considered or given pseudocode for any Monte Carlo methods or in this chapter. What would they be like? Why is it reasonable not to give pseudocode for them? How would they perform on the Mountain Car task? □

Exercise 10.2 Give pseudocode for semi-gradient one-step *Expected* Sarsa for control. □

Exercise 10.3 Why do the results shown in Figure 10.4 have higher standard errors at large n than at small n? □

10.3 Average Reward: A New Problem Setting for Continuing Tasks

We now introduce a third classical setting—alongside the episodic and discounted settings—for formulating the goal in Markov decision problems (MDPs). Like the discounted setting, the *average reward* setting applies to continuing problems, problems for which the interaction between agent and environment goes on and on forever without termination or start states. Unlike that setting, however, there is no discounting—the agent cares just as much about delayed rewards as it does about immediate reward. The average-reward setting is one of the major settings commonly considered in the classical theory of dynamic programming and less-commonly in reinforcement learning. As we discuss in the next section, the discounted setting is problematic with function approximation, and thus the average-reward setting is needed to replace it.

In the average-reward setting, the quality of a policy π is defined as the average rate of reward, or simply *average reward*, while following that policy, which we denote as $r(\pi)$:

$$\begin{aligned} r(\pi) &\doteq \lim_{h\to\infty} \frac{1}{h} \sum_{t=1}^{h} \mathbb{E}[R_t \mid S_0, A_{0:t-1} \sim \pi] && (10.6) \\ &= \lim_{t\to\infty} \mathbb{E}[R_t \mid S_0, A_{0:t-1} \sim \pi], && (10.7) \\ &= \sum_s \mu_\pi(s) \sum_a \pi(a|s) \sum_{s',r} p(s', r \,|\, s, a) r, \end{aligned}$$

where the expectations are conditioned on the initial state, S_0, and on the subsequent actions, $A_0, A_1, \ldots, A_{t-1}$, being taken according to π. μ_π is the steady-state distribution, $\mu_\pi(s) \doteq \lim_{t\to\infty} \Pr\{S_t = s \mid A_{0:t-1} \sim \pi\}$, which is assumed to exist for any π and to be independent of S_0. This assumption about the MDP is known as *ergodicity*. It means that where the MDP starts or any early decision made by the agent can have only a temporary effect; in the long run the expectation of being in a state depends only on the policy and the MDP transition probabilities. Ergodicity is sufficient to guarantee the existence of the limits in the equations above.

There are subtle distinctions that can be drawn between different kinds of optimality in the undiscounted continuing case. Nevertheless, for most practical purposes it may be adequate simply to order policies according to their average reward per time step, in other words, according to their $r(\pi)$. This quantity is essentially the average reward under π, as suggested by (10.7). In particular, we consider all policies that attain the maximal value of $r(\pi)$ to be optimal.

Note that the steady state distribution is the special distribution under which, if you select actions according to π, you remain in the same distribution. That is, for which

$$\sum_s \mu_\pi(s) \sum_a \pi(a|s) p(s' \,|\, s, a) = \mu_\pi(s'). \tag{10.8}$$

In the average-reward setting, returns are defined in terms of differences between

rewards and the average reward:

$$G_t \doteq R_{t+1} - r(\pi) + R_{t+2} - r(\pi) + R_{t+3} - r(\pi) + \cdots. \tag{10.9}$$

This is known as the *differential* return, and the corresponding value functions are known as *differential* value functions. They are defined in the same way and we will use the same notation for them as we have all along: $v_\pi(s) \doteq \mathbb{E}_\pi[G_t|S_t = s]$ and $q_\pi(s,a) \doteq \mathbb{E}_\pi[G_t|S_t = s, A_t = a]$ (similarly for v_* and q_*). Differential value functions also have Bellman equations, just slightly different from those we have seen earlier. We simply remove all γs and replace all rewards by the difference between the reward and the true average reward:

$$v_\pi(s) = \sum_a \pi(a|s) \sum_{r,s'} p(s', r \,|\, s, a)\Big[r - r(\pi) + v_\pi(s')\Big],$$

$$q_\pi(s,a) = \sum_{r,s'} p(s', r \,|\, s, a)\Big[r - r(\pi) + \sum_{a'} \pi(a'|s') q_\pi(s', a')\Big],$$

$$v_*(s) = \max_a \sum_{r,s'} p(s', r \,|\, s, a)\Big[r - \max_\pi r(\pi) + v_*(s')\Big], \text{ and}$$

$$q_*(s,a) = \sum_{r,s'} p(s', r \,|\, s, a)\Big[r - \max_\pi r(\pi) + \max_{a'} q_*(s', a')\Big]$$

(cf. (3.14), Exercise 3.17, (3.19), and (3.20)).

There is also a differential form of the two TD errors:

$$\delta_t \doteq R_{t+1} - \bar{R}_t + \hat{v}(S_{t+1}, \mathbf{w}_t) - \hat{v}(S_t, \mathbf{w}_t), \tag{10.10}$$

and

$$\delta_t \doteq R_{t+1} - \bar{R}_t + \hat{q}(S_{t+1}, A_{t+1}, \mathbf{w}_t) - \hat{q}(S_t, A_t, \mathbf{w}_t), \tag{10.11}$$

where $\bar{R}_t$ is an estimate at time t of the average reward $r(\pi)$. With these alternate definitions, most of our algorithms and many theoretical results carry through to the average-reward setting without change.

For example, the average reward version of semi-gradient Sarsa is defined just as in (10.2) except with the differential version of the TD error. That is, by

$$\mathbf{w}_{t+1} \doteq \mathbf{w}_t + \alpha \delta_t \nabla \hat{q}(S_t, A_t, \mathbf{w}_t), \tag{10.12}$$

with δ_t given by (10.11). The pseudocode for the complete algorithm is given in the box on the next page.

Exercise 10.4 Give pseudocode for a differential version of semi-gradient Q-learning. □

Exercise 10.5 What equations are needed (beyond 10.10) to specify the differential version of TD(0)? □

Differential semi-gradient Sarsa for estimating $\hat{q} \approx q_*$

Input: a differentiable action-value function parameterization $\hat{q} : \mathcal{S} \times \mathcal{A} \times \mathbb{R}^d \to \mathbb{R}$
Algorithm parameters: step sizes $\alpha, \beta > 0$
Initialize value-function weights $\mathbf{w} \in \mathbb{R}^d$ arbitrarily (e.g., $\mathbf{w} = \mathbf{0}$)
Initialize average reward estimate $\bar{R} \in \mathbb{R}$ arbitrarily (e.g., $\bar{R} = 0$)

Initialize state S, and action A
Loop for each step:
  Take action A, observe R, S'
  Choose A' as a function of $\hat{q}(S', \cdot, \mathbf{w})$ (e.g., ε-greedy)
  $\delta \leftarrow R - \bar{R} + \hat{q}(S', A', \mathbf{w}) - \hat{q}(S, A, \mathbf{w})$
  $\bar{R} \leftarrow \bar{R} + \beta\delta$
  $\mathbf{w} \leftarrow \mathbf{w} + \alpha\delta\nabla\hat{q}(S, A, \mathbf{w})$
  $S \leftarrow S'$
  $A \leftarrow A'$

Exercise 10.6 Consider a Markov reward process consisting of a ring of three states A, B, and C, with state transitions going deterministically around the ring. A reward of +1 is received upon arrival in A and otherwise the reward is 0. What are the differential values of the three states? □

Example 10.2: An Access-Control Queuing Task This is a decision task involving access control to a set of 10 servers. Customers of four different priorities arrive at a single queue. If given access to a server, the customers pay a reward of 1, 2, 4, or 8 to the server, depending on their priority, with higher priority customers paying more. In each time step, the customer at the head of the queue is either accepted (assigned to one of the servers) or rejected (removed from the queue, with a reward of zero). In either case, on the next time step the next customer in the queue is considered. The queue never empties, and the priorities of the customers in the queue are equally randomly distributed. Of course a customer cannot be served if there is no free server; the customer is always rejected in this case. Each busy server becomes free with probability $p = 0.06$ on each time step. Although we have just described them for definiteness, let us assume the statistics of arrivals and departures are unknown. The task is to decide on each step whether to accept or reject the next customer, on the basis of his priority and the number of free servers, so as to maximize long-term reward without discounting.

In this example we consider a tabular solution to this problem. Although there is no generalization between states, we can still consider it in the general function approximation setting as this setting generalizes the tabular setting. Thus we have a differential action-value estimate for each pair of state (number of free servers and priority of the customer at the head of the queue) and action (accept or reject). Figure 10.5 shows the solution found by differential semi-gradient Sarsa with parameters $\alpha = 0.01$, $\beta = 0.01$, and $\varepsilon = 0.1$. The initial action values and $\bar{R}$ were zero.

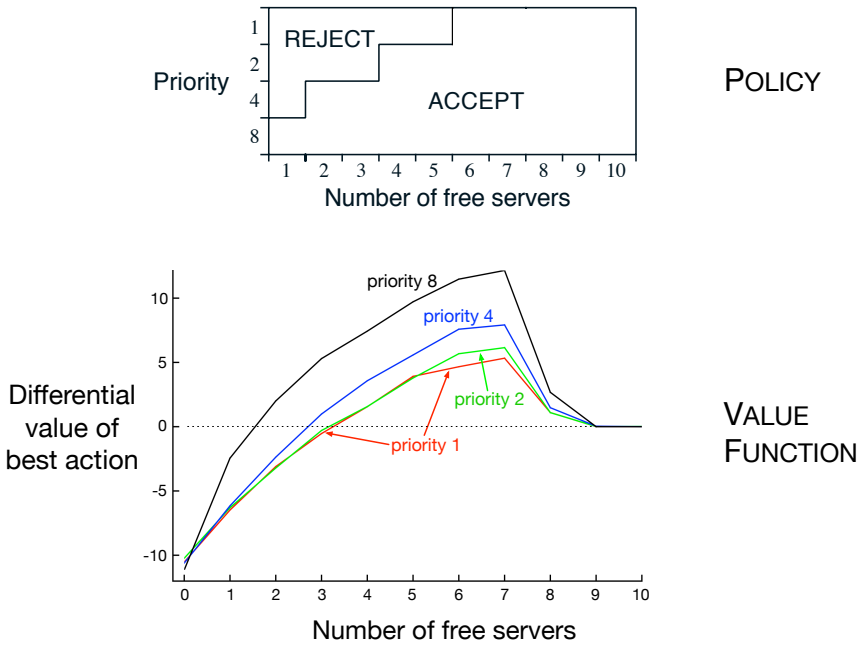


Figure 10.5: The policy and value function found by differential semi-gradient one-step Sarsa on the access-control queuing task after 2 million steps. The drop on the right of the graph is probably due to insufficient data; many of these states were never experienced. The value learned for $\bar{R}$ was about 2.31. ■

Exercise 10.7 Suppose there is an MDP that under any policy produces the deterministic sequence of rewards $+1, 0, +1, 0, +1, 0, \ldots$ going on forever. Technically, this is not allowed because it violates ergodicity; there is no stationary limiting distribution μ_π and the limit (10.7) does not exist. Nevertheless, the average reward (10.6) is well defined; What is it? Now consider two states in this MDP. From A, the reward sequence is exactly as described above, starting with a $+1$, whereas, from B, the reward sequence starts with a 0 and then continues with $+1, 0, +1, 0, \ldots$. The differential return (10.9) is not well defined for this case as the limit does not exist. To repair this, one could alternately define the value of a state as

$$v_\pi(s) \doteq \lim_{\gamma \to 1} \lim_{h \to \infty} \sum_{t=0}^{h} \gamma^t \Big(\mathbb{E}_\pi[R_{t+1} | S_0 = s] - r(\pi) \Big). \tag{10.13}$$

Under this definition, what are the values of states A and B? □

Exercise 10.8 The pseudocode in the box on page 251 updates $\bar{R}_t$ using δ_t as an error rather than simply $R_{t+1} - \bar{R}_t$. Both errors work, but using δ_t is better. To see why, consider the ring MRP of three states from Exercise 10.6. The estimate of the average reward should tend towards its true value of $\frac{1}{3}$. Suppose it was already there and was

held stuck there. What would the sequence of $R_{t+1} - \bar{R}_t$ errors be? What would the sequence of δ_t errors be (using (10.10))? Which error sequence would produce a more stable estimate of the average reward if the estimate were allowed to change in response to the errors? Why? □

10.4 Deprecating the Discounted Setting

The continuing, discounted problem formulation has been very useful in the tabular case, in which the returns from each state can be separately identified and averaged. But in the approximate case it is questionable whether one should ever use this problem formulation.

To see why, consider an infinite sequence of returns with no beginning or end, and no clearly identified states. The states might be represented only by feature vectors, which may do little to distinguish the states from each other. As a special case, all of the feature vectors may be the same. Thus one really has only the reward sequence (and the actions), and performance has to be assessed purely from these. How could it be done? One way is by averaging the rewards over a long interval—this is the idea of the average-reward setting. How could discounting be used? Well, for each time step we could measure the discounted return. Some returns would be small and some big, so again we would have to average them over a sufficiently large time interval. In the continuing setting there are no starts and ends, and no special time steps, so there is nothing else that could be done. However, if you do this, it turns out that the average of the discounted returns is proportional to the average reward. In fact, for policy π, the average of the discounted returns is always $r(\pi)/(1-\gamma)$, that is, it is essentially the average reward, $r(\pi)$. In particular, the *ordering* of all policies in the average discounted return setting would be exactly the same as in the average-reward setting. The discount rate γ thus has no effect on the problem formulation. It could in fact be *zero* and the ranking would be unchanged.

This surprising fact is proven in the box on the next page, but the basic idea can be seen via a symmetry argument. Each time step is exactly the same as every other. With discounting, every reward will appear exactly once in each position in some return. The tth reward will appear undiscounted in the $t-1$st return, discounted once in the $t-2$nd return, and discounted 999 times in the $t-1000$th return. The weight on the tth reward is thus $1+\gamma+\gamma^2+\gamma^3+\cdots = 1/(1-\gamma)$. Because all states are the same, they are all weighted by this, and thus the average of the returns will be this times the average reward, or $r(\pi)/(1-\gamma)$.

This example and the more general argument in the box show that if we optimized discounted value over the on-policy distribution, then the effect would be identical to optimizing *undiscounted* average reward; the actual value of γ would have no effect. This strongly suggests that discounting has no role to play in the definition of the control problem with function approximation. One can nevertheless go ahead and use discounting in solution methods. The discounting parameter γ changes from a problem parameter to a solution method parameter! However, in this case we unfortunately would not be guaranteed to optimize average reward (or the equivalent discounted value over the on-policy distribution).

The Futility of Discounting in Continuing Problems

Perhaps discounting can be saved by choosing an objective that sums discounted values over the distribution with which states occur under the policy:

$$\begin{aligned}
J(\pi) &= \sum_s \mu_\pi(s) v_\pi^\gamma(s) && \text{(where } v_\pi^\gamma \text{ is the discounted value function)} \\
&= \sum_s \mu_\pi(s) \sum_a \pi(a|s) \sum_{s'} \sum_r p(s', r \,|\, s, a) \left[r + \gamma v_\pi^\gamma(s') \right] && \text{(Bellman Eq.)} \\
&= r(\pi) + \sum_s \mu_\pi(s) \sum_a \pi(a|s) \sum_{s'} \sum_r p(s', r \,|\, s, a) \gamma v_\pi^\gamma(s') && \text{(from (10.7))} \\
&= r(\pi) + \gamma \sum_{s'} v_\pi^\gamma(s') \sum_s \mu_\pi(s) \sum_a \pi(a|s) p(s' \,|\, s, a) && \text{(from (3.4))} \\
&= r(\pi) + \gamma \sum_{s'} v_\pi^\gamma(s') \mu_\pi(s') && \text{(from (10.8))} \\
&= r(\pi) + \gamma J(\pi) \\
&= r(\pi) + \gamma r(\pi) + \gamma^2 J(\pi) \\
&= r(\pi) + \gamma r(\pi) + \gamma^2 r(\pi) + \gamma^3 r(\pi) + \cdots \\
&= \frac{1}{1-\gamma} r(\pi).
\end{aligned}$$

The proposed discounted objective orders policies identically to the undiscounted (average reward) objective. The discount rate γ does not influence the ordering!

The root cause of the difficulties with the discounted control setting is that with function approximation we have lost the policy improvement theorem (Section 4.2). It is no longer true that if we change the policy to improve the discounted value of one state then we are guaranteed to have improved the overall policy in any useful sense. That guarantee was key to the theory of our reinforcement learning control methods. With function approximation we have lost it!

In fact, the lack of a policy improvement theorem is also a theoretical lacuna for the total-episodic and average-reward settings. Once we introduce function approximation we can no longer guarantee improvement for any setting. In Chapter 13 we introduce an alternative class of reinforcement learning algorithms based on parameterized policies, and there we have a theoretical guarantee called the "policy-gradient theorem" which plays a similar role as the policy improvement theorem. But for methods that learn action values we seem to be currently without a local improvement guarantee (possibly the approach taken by Perkins and Precup (2003) may provide a part of the answer). We do know that ε-greedification may sometimes result in an inferior policy, as policies may chatter among good policies rather than converge (Gordon, 1996a). This is an area with multiple open theoretical questions.

10.5 Differential Semi-gradient n-step Sarsa

In order to generalize to *n-step* bootstrapping, we need an n-step version of the TD error. We begin by generalizing the n-step return (7.4) to its differential form, with function approximation:

$$G_{t:t+n} \doteq R_{t+1} - \bar{R}_{t+n-1} + \cdots + R_{t+n} - \bar{R}_{t+n-1} + \hat{q}(S_{t+n}, A_{t+n}, \mathbf{w}_{t+n-1}), \quad (10.14)$$

where $\bar{R}$ is an estimate of $r(\pi)$, $n \geq 1$, and $t + n < T$. If $t + n \geq T$, then we define $G_{t:t+n} \doteq G_t$ as usual. The n-step TD error is then

$$\delta_t \doteq G_{t:t+n} - \hat{q}(S_t, A_t, \mathbf{w}), \quad (10.15)$$

after which we can apply our usual semi-gradient Sarsa update (10.12). Pseudocode for the complete algorithm is given in the box.

Differential semi-gradient n-step Sarsa for estimating $\hat{q} \approx q_\pi$ or q_*

Input: a differentiable function $\hat{q} : \mathcal{S} \times \mathcal{A} \times \mathbb{R}^d \to \mathbb{R}$, a policy π
Initialize value-function weights $\mathbf{w} \in \mathbb{R}^d$ arbitrarily (e.g., $\mathbf{w} = \mathbf{0}$)
Initialize average-reward estimate $\bar{R} \in \mathbb{R}$ arbitrarily (e.g., $\bar{R} = 0$)
Algorithm parameters: step size $\alpha, \beta > 0$, a positive integer n
All store and access operations (S_t, A_t, and R_t) can take their index mod $n + 1$

Initialize and store S_0 and A_0
Loop for each step, $t = 0, 1, 2, \ldots$:
  Take action A_t
  Observe and store the next reward as R_{t+1} and the next state as S_{t+1}
  Select and store an action $A_{t+1} \sim \pi(\cdot|S_{t+1})$, or ε-greedy wrt $\hat{q}(S_{t+1}, \cdot, \mathbf{w})$
  $\tau \leftarrow t - n + 1$ (τ is the time whose estimate is being updated)
  If $\tau \geq 0$:
    $\delta \leftarrow \sum_{i=\tau+1}^{\tau+n} (R_i - \bar{R}) + \hat{q}(S_{\tau+n}, A_{\tau+n}, \mathbf{w}) - \hat{q}(S_\tau, A_\tau, \mathbf{w})$
    $\bar{R} \leftarrow \bar{R} + \beta\delta$
    $\mathbf{w} \leftarrow \mathbf{w} + \alpha\delta\nabla\hat{q}(S_\tau, A_\tau, \mathbf{w})$

Exercise 10.9 In the differential semi-gradient n-step Sarsa algorithm, the step-size parameter on the average reward, β, needs to be quite small so that $\bar{R}$ becomes a good long-term estimate of the average reward. Unfortunately, $\bar{R}$ will then be biased by its initial value for many steps, which may make learning inefficient. Alternatively, one could use a sample average of the observed rewards for $\bar{R}$. That would initially adapt rapidly but in the long run would also adapt slowly. As the policy slowly changed, $\bar{R}$ would also change; the potential for such long-term nonstationarity makes sample-average methods ill-suited. In fact, the step-size parameter on the average reward is a perfect place to use the unbiased constant-step-size trick from Exercise 2.7. Describe the specific changes needed to the boxed algorithm for differential semi-gradient n-step Sarsa to use this trick. □

10.6 Summary

In this chapter we have extended the ideas of parameterized function approximation and semi-gradient descent, introduced in the previous chapter, to control. The extension is immediate for the episodic case, but for the continuing case we have to introduce a whole new problem formulation based on maximizing the *average reward setting* per time step. Surprisingly, the discounted formulation cannot be carried over to control in the presence of approximations. In the approximate case most policies cannot be represented by a value function. The arbitrary policies that remain need to be ranked, and the scalar average reward $r(\pi)$ provides an effective way to do this.

The average reward formulation involves new *differential* versions of value functions, Bellman equations, and TD errors, but all of these parallel the old ones, and the conceptual changes are small. There is also a new parallel set of differential algorithms for the average-reward case.

Bibliographical and Historical Remarks

10.1 Semi-gradient Sarsa with function approximation was first explored by Rummery and Niranjan (1994). Linear semi-gradient Sarsa with ε-greedy action selection does not converge in the usual sense, but does enter a bounded region near the best solution (Gordon, 1996a, 2001). Precup and Perkins (2003) showed convergence in a differentiable action selection setting. See also Perkins and Pendrith (2002) and Melo, Meyn, and Ribeiro (2008). The mountain–car example is based on a similar task studied by Moore (1990), but the exact form used here is from Sutton (1996).

10.2 Episodic n-step semi-gradient Sarsa is based on the forward Sarsa(λ) algorithm of van Seijen (2016). The empirical results shown here are new to the second edition of this text.

10.3 The average-reward formulation has been described for dynamic programming (e.g., Puterman, 1994) and from the point of view of reinforcement learning (Mahadevan, 1996; Tadepalli and Ok, 1994; Bertsekas and Tsitiklis, 1996; Tsitsiklis and Van Roy, 1999). The algorithm described here is the on-policy analog of the "R-learning" algorithm introduced by Schwartz (1993). The name R-learning was probably meant to be the alphabetic successor to Q-learning, but we prefer to think of it as a reference to the learning of differential or *relative* values. The access-control queuing example was suggested by the work of Carlström and Nordström (1997).

10.4 The recognition of the limitations of discounting as a formulation of the reinforcement learning problem with function approximation became apparent to the authors shortly after the publication of the first edition of this text. Singh, Jaakkola, and Jordan (1994) may have been the first to observe it in print.

Chapter 11

*Off-policy Methods with Approximation

This book has treated on-policy and off-policy learning methods since Chapter 5 primarily as two alternative ways of handling the conflict between exploitation and exploration inherent in learning forms of generalized policy iteration. The two chapters preceding this have treated the *on*-policy case with function approximation, and in this chapter we treat the *off*-policy case with function approximation. The extension to function approximation turns out to be significantly different and harder for off-policy learning than it is for on-policy learning. The tabular off-policy methods developed in Chapters 6 and 7 readily extend to semi-gradient algorithms, but these algorithms do not converge as robustly as they do under on-policy training. In this chapter we explore the convergence problems, take a closer look at the theory of linear function approximation, introduce a notion of learnability, and then discuss new algorithms with stronger convergence guarantees for the off-policy case. In the end we will have improved methods, but the theoretical results will not be as strong, nor the empirical results as satisfying, as they are for on-policy learning. Along the way, we will gain a deeper understanding of approximation in reinforcement learning for on-policy learning as well as off-policy learning.

Recall that in off-policy learning we seek to learn a value function for a *target policy* π, given data due to a different *behavior policy* b. In the prediction case, both policies are static and given, and we seek to learn either state values $\hat{v} \approx v_\pi$ or action values $\hat{q} \approx q_\pi$. In the control case, action values are learned, and both policies typically change during learning—π being the greedy policy with respect to $\hat{q}$, and b being something more exploratory such as the ε-greedy policy with respect to $\hat{q}$.

The challenge of off-policy learning can be divided into two parts, one that arises in the tabular case and one that arises only with function approximation. The first part of the challenge has to do with the target of the update (not to be confused with the target policy), and the second part has to do with the distribution of the updates. The techniques related to importance sampling developed in Chapters 5 and 7 deal with the first part; these may increase variance but are needed in all successful algorithms,

tabular and approximate. The extension of these techniques to function approximation are quickly dealt with in the first section of this chapter.

Something more is needed for the second part of the challenge of off-policy learning with function approximation because the distribution of updates in the off-policy case is not according to the on-policy distribution. The on-policy distribution is important to the stability of semi-gradient methods. Two general approaches have been explored to deal with this. One is to use importance sampling methods again, this time to warp the update distribution back to the on-policy distribution, so that semi-gradient methods are guaranteed to converge (in the linear case). The other is to develop true gradient methods that do not rely on any special distribution for stability. We present methods based on both approaches. This is a cutting-edge research area, and it is not clear which of these approaches is most effective in practice.

11.1 Semi-gradient Methods

We begin by describing how the methods developed in earlier chapters for the off-policy case extend readily to function approximation as semi-gradient methods. These methods address the first part of the challenge of off-policy learning (changing the update targets) but not the second part (changing the update distribution). Accordingly, these methods may diverge in some cases, and in that sense are not sound, but still they are often successfully used. Remember that these methods *are* guaranteed stable and asymptotically unbiased for the tabular case, which corresponds to a special case of function approximation. So it may still be possible to combine them with feature selection methods in such a way that the combined system could be assured stable. In any event, these methods are simple and thus a good place to start.

In Chapter 7 we described a variety of tabular off-policy algorithms. To convert them to semi-gradient form, we simply replace the update to an array (V or Q) to an update to a weight vector ($\mathbf{w}$), using the approximate value function ($\hat{v}$ or $\hat{q}$) and its gradient. Many of these algorithms use the per-step importance sampling ratio:

$$\rho_t \doteq \rho_{t:t} = \frac{\pi(A_t|S_t)}{b(A_t|S_t)}. \tag{11.1}$$

For example, the one-step, state-value algorithm is semi-gradient off-policy TD(0), which is just like the corresponding on-policy algorithm (page 203) except for the addition of ρ_t:

$$\mathbf{w}_{t+1} \doteq \mathbf{w}_t + \alpha\rho_t\delta_t\nabla\hat{v}(S_t,\mathbf{w}_t), \tag{11.2}$$

where δ_t is defined appropriately depending on whether the problem is episodic and discounted, or continuing and undiscounted using average reward:

$$\delta_t \doteq R_{t+1} + \gamma\hat{v}(S_{t+1},\mathbf{w}_t) - \hat{v}(S_t,\mathbf{w}_t), \text{ or} \tag{11.3}$$

$$\delta_t \doteq R_{t+1} - \bar{R}_t + \hat{v}(S_{t+1},\mathbf{w}_t) - \hat{v}(S_t,\mathbf{w}_t). \tag{11.4}$$

For action values, the one-step algorithm is semi-gradient Expected Sarsa:

$$\mathbf{w}_{t+1} \doteq \mathbf{w}_t + \alpha\delta_t \nabla \hat{q}(S_t, A_t, \mathbf{w}_t), \text{ with} \tag{11.5}$$

$$\delta_t \doteq R_{t+1} + \gamma \sum_a \pi(a|S_{t+1})\hat{q}(S_{t+1}, a, \mathbf{w}_t) - \hat{q}(S_t, A_t, \mathbf{w}_t), \text{ or} \qquad \text{(episodic)}$$

$$\delta_t \doteq R_{t+1} - \bar{R}_t + \sum_a \pi(a|S_{t+1})\hat{q}(S_{t+1}, a, \mathbf{w}_t) - \hat{q}(S_t, A_t, \mathbf{w}_t). \qquad \text{(continuing)}$$

Note that this algorithm does not use importance sampling. In the tabular case it is clear that this is appropriate because the only sample action is A_t, and in learning its value we do not have to consider any other actions. With function approximation it is less clear because we might want to weight different state–action pairs differently once they all contribute to the same overall approximation. Proper resolution of this issue awaits a more thorough understanding of the theory of function approximation in reinforcement learning.

In the multi-step generalizations of these algorithms, both the state-value and action-value algorithms involve importance sampling. For example, the n-step version of semi-gradient Expected Sarsa is

$$\mathbf{w}_{t+n} \doteq \mathbf{w}_{t+n-1} + \alpha\rho_{t+1}\cdots\rho_{t+n-1}\left[G_{t:t+n} - \hat{q}(S_t, A_t, \mathbf{w}_{t+n-1})\right]\nabla\hat{q}(S_t, A_t, \mathbf{w}_{t+n-1}) \tag{11.6}$$

with

$$G_{t:t+n} \doteq R_{t+1} + \cdots + \gamma^{n-1}R_{t+n} + \gamma^n \hat{q}(S_{t+n}, A_{t+n}, \mathbf{w}_{t+n-1}), \text{ or} \qquad \text{(episodic)}$$

$$G_{t:t+n} \doteq R_{t+1} - \bar{R}_t + \cdots + R_{t+n} - \bar{R}_{t+n-1} + \hat{q}(S_{t+n}, A_{t+n}, \mathbf{w}_{t+n-1}), \text{ (continuing)}$$

where here we are being slightly informal in our treatment of the ends of episodes. In the first equation, the ρ_ks for $k \geq T$ (where T is the last time step of the episode) should be taken to be 1, and $G_{t:n}$ should be taken to be G_t if $t + n \geq T$.

Recall that we also presented in Chapter 7 an off-policy algorithm that does not involve importance sampling at all: the n-step tree-backup algorithm. Here is its semi-gradient version:

$$\mathbf{w}_{t+n} \doteq \mathbf{w}_{t+n-1} + \alpha\left[G_{t:t+n} - \hat{q}(S_t, A_t, \mathbf{w}_{t+n-1})\right]\nabla\hat{q}(S_t, A_t, \mathbf{w}_{t+n-1}), \tag{11.7}$$

$$G_{t:t+n} \doteq \hat{q}(S_t, A_t, \mathbf{w}_{t-1}) + \sum_{k=t}^{t+n-1} \delta_k \prod_{i=t+1}^{k} \gamma\pi(A_i|S_i), \tag{11.8}$$

with δ_t as defined at the top of this page for Expected Sarsa. We also defined in Chapter 7 an algorithm that unifies all action-value algorithms: n-step $Q(\sigma)$. We leave the semi-gradient form of that algorithm, and also of the n-step state-value algorithm, as exercises for the reader.

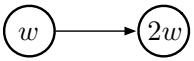
Exercise 11.1 Convert the equation of n-step off-policy TD (7.9) to semi-gradient form. Give accompanying definitions of the return for both the episodic and continuing cases. □

**Exercise 11.2* Convert the equations of n-step $Q(\sigma)$ (7.11 and 7.17) to semi-gradient form. Give definitions that cover both the episodic and continuing cases. □

11.2 Examples of Off-policy Divergence

In this section we begin to discuss the second part of the challenge of off-policy learning with function approximation—that the distribution of updates does not match the on-policy distribution. We describe some instructive counterexamples to off-policy learning—cases where semi-gradient and other simple algorithms are unstable and diverge.

To establish intuitions, it is best to consider first a very simple example. Suppose, perhaps as part of a larger MDP, there are two states whose estimated values are of the functional form w and $2w$, where the parameter vector $\mathbf{w}$ consists of only a single component w. This occurs under linear function approximation if the feature vectors for the two states are each simple numbers (single-component vectors), in this case 1 and 2. In the first state, only one action is available, and it results deterministically in a transition to the second state with a reward of 0:



where the expressions inside the two circles indicate the two state's values.

Suppose initially $w = 10$. The transition will then be from a state of estimated value 10 to a state of estimated value 20. It will look like a good transition, and w will be increased to raise the first state's estimated value. If γ is nearly 1, then the TD error will be nearly 10, and, if $\alpha = 0.1$, then w will be increased to nearly 11 in trying to reduce the TD error. However, the second state's estimated value will also be increased, to nearly 22. If the transition occurs again, then it will be from a state of estimated value $\approx$11 to a state of estimated value $\approx$22, for a TD error of $\approx$11—larger, not smaller, than before. It will look even more like the first state is undervalued, and its value will be increased again, this time to $\approx$12.1. This looks bad, and in fact with further updates w will diverge to infinity.

To see this definitively we have to look more carefully at the sequence of updates. The TD error on a transition between the two states is

$$\delta_t = R_{t+1} + \gamma\hat{v}(S_{t+1},\mathbf{w}_t) - \hat{v}(S_t,\mathbf{w}_t) = 0 + \gamma 2w_t - w_t = (2\gamma - 1)w_t,$$

and the off-policy semi-gradient TD(0) update (from (11.2)) is

$$w_{t+1} = w_t + \alpha\rho_t\delta_t\nabla\hat{v}(S_t,w_t) = w_t + \alpha \cdot 1 \cdot (2\gamma - 1)w_t \cdot 1 = \big(1 + \alpha(2\gamma - 1)\big)w_t.$$

Note that the importance sampling ratio, ρ_t, is 1 on this transition because there is only one action available from the first state, so its probabilities of being taken under the target and behavior policies must both be 1. In the final update above, the new parameter is the old parameter times a scalar constant, $1 + \alpha(2\gamma - 1)$. If this constant is greater than 1, then the system is unstable and w will go to positive or negative infinity depending on its initial value. Here this constant is greater than 1 whenever $\gamma > 0.5$. Note that stability does not depend on the specific step size, as long as $\alpha > 0$. Smaller or larger step sizes would affect the rate at which w goes to infinity, but not whether it goes there or not.

Key to this example is that the one transition occurs repeatedly without w being updated on other transitions. This is possible under off-policy training because the

behavior policy might select actions on those other transitions which the target policy never would. For these transitions, ρ_t would be zero and no update would be made. Under on-policy training, however, ρ_t is always one. Each time there is a transition from the w state to the $2w$ state, increasing w, there would also have to be a transition out of the $2w$ state. That transition would reduce w, unless it were to a state whose value was higher (because $\gamma < 1$) than $2w$, and then that state would have to be followed by a state of even higher value, or else again w would be reduced. Each state can support the one before only by creating a higher expectation. Eventually the piper must be paid. In the on-policy case the promise of future reward must be kept and the system is kept in check. But in the off-policy case, a promise can be made and then, after taking an action that the target policy never would, forgotten and forgiven.

This simple example communicates much of the reason why off-policy training can lead to divergence, but it is not completely convincing because it is not complete—it is just a fragment of a complete MDP. Can there really be a complete system with instability? A simple complete example of divergence is *Baird's counterexample*. Consider the episodic seven-state, two-action MDP shown in Figure 11.1. The dashed action takes the system to one of the six upper states with equal probability, whereas the solid action takes the system to the seventh state. The behavior policy b selects the dashed and solid actions with probabilities $\frac{6}{7}$ and $\frac{1}{7}$, so that the next-state distribution under it is uniform (the same for all nonterminal states), which is also the starting distribution for each episode. The target policy π always takes the solid action, and so the on-policy distribution (for π) is concentrated in the seventh state. The reward is zero on all transitions. The discount rate is $\gamma = 0.99$.

Consider estimating the state-value under the linear parameterization indicated by the expression shown in each state circle. For example, the estimated value of the leftmost state is $2w_1 + w_8$, where the subscript corresponds to the component of the

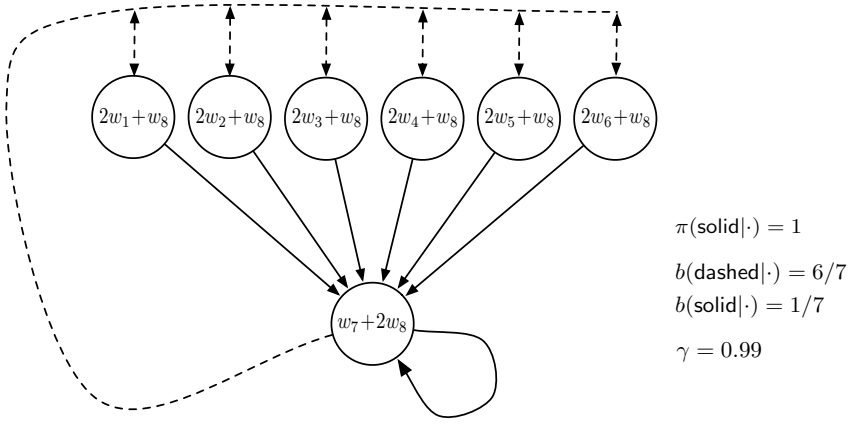


Figure 11.1: Baird's counterexample. The approximate state-value function for this Markov process is of the form shown by the linear expressions inside each state. The solid action usually results in the seventh state, and the dashed action usually results in one of the other six states, each with equal probability. The reward is always zero.

overall weight vector $\mathbf{w} \in \mathbb{R}^8$; this corresponds to a feature vector for the first state being $\mathbf{x}(1) = (2,0,0,0,0,0,0,1)^\top$. The reward is zero on all transitions, so the true value function is $v_\pi(s) = 0$, for all s, which can be exactly approximated if $\mathbf{w} = \mathbf{0}$. In fact, there are many solutions, as there are more components to the weight vector (8) than there are nonterminal states (7). Moreover, the set of feature vectors, $\{\mathbf{x}(s) : s \in \mathcal{S}\}$, is a linearly independent set. In all these ways this task seems a favorable case for linear function approximation.

If we apply semi-gradient TD(0) to this problem (11.2), then the weights diverge to infinity, as shown in Figure 11.2 (left). The instability occurs for any positive step size, no matter how small. In fact, it even occurs if an expected update is done as in dynamic programming (DP), as shown in Figure 11.2 (right). That is, if the weight vector, $\mathbf{w}_k$, is updated for all states at the same time in a semi-gradient way, using the DP (expectation-based) target:

$$\mathbf{w}_{k+1} \doteq \mathbf{w}_k + \frac{\alpha}{|\mathcal{S}|} \sum_s \Big(\mathbb{E}_\pi[R_{t+1} + \gamma \hat{v}(S_{t+1}, \mathbf{w}_k) \mid S_t = s] - \hat{v}(s, \mathbf{w}_k) \Big) \nabla \hat{v}(s, \mathbf{w}_k). \quad (11.9)$$

In this case, there is no randomness and no asynchrony, just as in a classical DP update. The method is conventional except in its use of semi-gradient function approximation. Yet still the system is unstable.

If we alter just the distribution of DP updates in Baird's counterexample, from the uniform distribution to the on-policy distribution (which generally requires asynchronous updating), then convergence is guaranteed to a solution with error bounded by (9.14). This example is striking because the TD and DP methods used are arguably the simplest

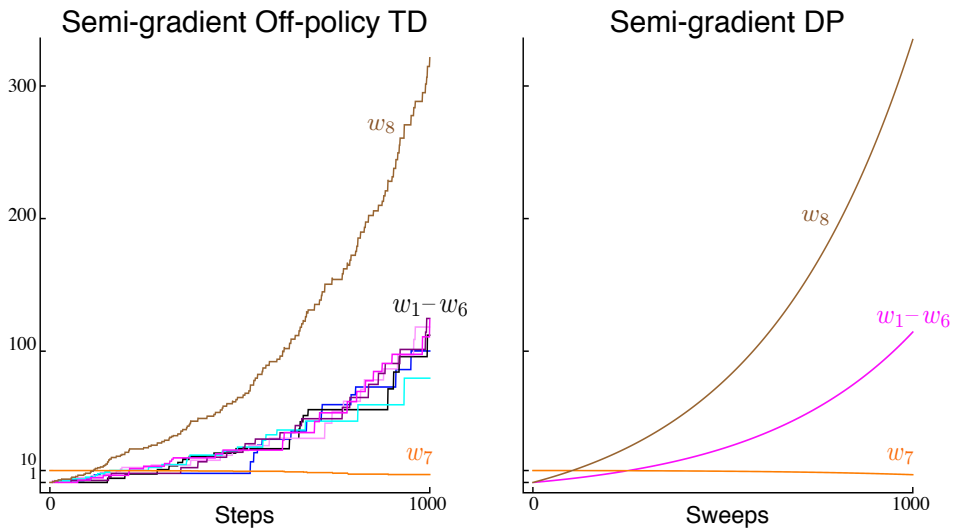


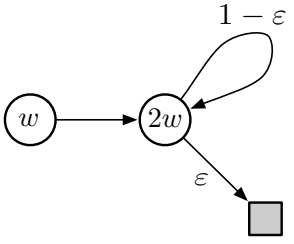
Figure 11.2: Demonstration of instability on Baird's counterexample. Shown are the evolution of the components of the parameter vector $\mathbf{w}$ of the two semi-gradient algorithms. The step size was $\alpha = 0.01$, and the initial weights were $\mathbf{w} = (1,1,1,1,1,1,10,1)^\top$.

and best-understood bootstrapping methods, and the linear, semi-descent method used is arguably the simplest and best-understood kind of function approximation. The example shows that even the simplest combination of bootstrapping and function approximation can be unstable if the updates are not done according to the on-policy distribution.

There are also counterexamples similar to Baird's showing divergence for Q-learning. This is cause for concern because otherwise Q-learning has the best convergence guarantees of all control methods. Considerable effort has gone into trying to find a remedy to this problem or to obtain some weaker, but still workable, guarantee. For example, it may be possible to guarantee convergence of Q-learning as long as the behavior policy is sufficiently close to the target policy, for example, when it is the ε-greedy policy. To the best of our knowledge, Q-learning has never been found to diverge in this case, but there has been no theoretical analysis. In the rest of this section we present several other ideas that have been explored.

Suppose that instead of taking just a step toward the expected one-step return on each iteration, as in Baird's counterexample, we actually change the value function all the way to the best, least-squares approximation. Would this solve the instability problem? Of course it would if the feature vectors, $\{\mathbf{x}(s) : s \in \mathcal{S}\}$, formed a linearly independent set, as they do in Baird's counterexample, because then exact approximation is possible on each iteration and the method reduces to standard tabular DP. But of course the point here is to consider the case when an exact solution is *not* possible. In this case stability is not guaranteed even when forming the best approximation at each iteration, as shown in the example.

Example 11.1: Tsitsiklis and Van Roy's Counterexample This example shows that linear function approximation would not work with DP even if the least-squares solution was found at each step. The counterexample is formed by extending the w-to-$2w$ example (from earlier in this section) with a terminal state, as shown to the right. As before, the estimated value of the first state is w, and the estimated value of the second state is $2w$. The reward is zero on all transitions, so the true values are zero at both states, which is exactly representable with $w = 0$. If we set w_{k+1} at each step so as to minimize the $\overline{\text{VE}}$ between the estimated value and the expected one-step return, then we have



$$\begin{aligned} w_{k+1} &= \operatorname*{argmin}_{w\in\mathbb{R}} \sum_{s\in\mathcal{S}} \Big(\hat{v}(s,w) - \mathbb{E}_\pi\big[R_{t+1} + \gamma\hat{v}(S_{t+1},w_k) \;\big|\; S_t = s\big]\Big)^2 \\ &= \operatorname*{argmin}_{w\in\mathbb{R}} \big(w - \gamma 2w_k\big)^2 + \big(2w - (1-\varepsilon)\gamma 2w_k\big)^2 \\ &= \frac{6-4\varepsilon}{5}\gamma w_k. \end{aligned} \tag{11.10}$$

The sequence $\{w_k\}$ diverges when $\gamma > \frac{5}{6-4\varepsilon}$ and $w_0 \neq 0$. ■

Another way to try to prevent instability is to use special methods for function approximation. In particular, stability is guaranteed for function approximation methods that do not extrapolate from the observed targets. These methods, called *averagers*, include nearest neighbor methods and locally weighted regression, but not popular methods such as tile coding and artificial neural networks (ANNs).

Exercise 11.3 (programming) Apply one-step semi-gradient Q-learning to Baird's counterexample and show empirically that its weights diverge. □

11.3 The Deadly Triad

Our discussion so far can be summarized by saying that the danger of instability and divergence arises whenever we combine all of the following three elements, making up what we call *the deadly triad*:

Function approximation A powerful, scalable way of generalizing from a state space much larger than the memory and computational resources (e.g., linear function approximation or ANNs).

Bootstrapping Update targets that include existing estimates (as in dynamic programming or TD methods) rather than relying exclusively on actual rewards and complete returns (as in MC methods).

Off-policy training Training on a distribution of transitions other than that produced by the target policy. Sweeping through the state space and updating all states uniformly, as in dynamic programming, does not respect the target policy and is an example of off-policy training.

In particular, note that the danger is *not* due to control or to generalized policy iteration. Those cases are more complex to analyze, but the instability arises in the simpler prediction case whenever it includes all three elements of the deadly triad. The danger is also *not* due to learning or to uncertainties about the environment, because it occurs just as strongly in planning methods, such as dynamic programming, in which the environment is completely known.

If any two elements of the deadly triad are present, but not all three, then instability can be avoided. It is natural, then, to go through the three and see if there is any one that can be given up.

Of the three, *function approximation* most clearly cannot be given up. We need methods that scale to large problems and to great expressive power. We need at least linear function approximation with many features and parameters. State aggregation or nonparametric methods whose complexity grows with data are too weak or too expensive. Least-squares methods such as LSTD are of quadratic complexity and are therefore too expensive for large problems.

Doing without *bootstrapping* is possible, at the cost of computational and data efficiency. Perhaps most important are the losses in computational efficiency. Monte Carlo (non-bootstrapping) methods require memory to save everything that happens between making

each prediction and obtaining the final return, and all their computation is done once the final return is obtained. The cost of these computational issues is not apparent on serial von Neumann computers, but would be on specialized hardware. With bootstrapping and eligibility traces (Chapter 12), data can be dealt with when and where it is generated, then need never be used again. The savings in communication and memory made possible by bootstrapping are great.

The losses in data efficiency by giving up *bootstrapping* are also significant. We have seen this repeatedly, such as in Chapters 7 (Figure 7.2) and 9 (Figure 9.2), where some degree of bootstrapping performed much better than Monte Carlo methods on the random-walk prediction task, and in Chapter 10 where the same was seen on the Mountain-Car control task (Figure 10.4). Many other problems show much faster learning with bootstrapping (e.g., see Figure 12.14). Bootstrapping often results in faster learning because it allows learning to take advantage of the state property, the ability to recognize a state upon returning to it. On the other hand, bootstrapping can impair learning on problems where the state representation is poor and causes poor generalization (e.g., this seems to be the case on Tetris, see Şimşek, Algórta, and Kothiyal, 2016). A poor state representation can also result in bias; this is the reason for the poorer bound on the asymptotic approximation quality of bootstrapping methods (Equation 9.14). On balance, the ability to bootstrap has to be considered extremely valuable. One may sometimes choose not to use it by selecting long n-step updates (or a large bootstrapping parameter, $\lambda \approx 1$; see Chapter 12) but often bootstrapping greatly increases efficiency. It is an ability that we would very much like to keep in our toolkit.

Finally, there is *off-policy learning*; can we give that up? On-policy methods are often adequate. For model-free reinforcement learning, one can simply use Sarsa rather than Q-learning. Off-policy methods free behavior from the target policy. This could be considered an appealing convenience but not a necessity. However, off-policy learning *is essential* to other anticipated use cases, cases that we have not yet mentioned in this book but may be important to the larger goal of creating a powerful intelligent agent.

In these use cases, the agent learns not just a single value function and single policy, but large numbers of them in parallel. There is extensive psychological evidence that people and animals learn to predict many different sensory events, not just rewards. We can be surprised by unusual events, and correct our predictions about them, even if they are of neutral valence (neither good nor bad). This kind of prediction presumably underlies predictive models of the world such as are used in planning. We predict what we will see after eye movements, how long it will take to walk home, the probability of making a jump shot in basketball, and the satisfaction we will get from taking on a new project. In all these cases, the events we would like to predict depend on our acting in a certain way. To learn them all, in parallel, requires learning from the one stream of experience. There are many target policies, and thus the one behavior policy cannot equal all of them. Yet parallel learning is conceptually possible because the behavior policy may overlap in part with many of the target policies. To take full advantage of this requires off-policy learning.

11.4 Linear Value-function Geometry

To better understand the stability challenge of off-policy learning, it is helpful to think about value function approximation more abstractly and independently of how learning is done. We can imagine the space of all possible state-value functions—all functions from states to real numbers $v : \mathcal{S} \to \mathbb{R}$. Most of these value functions do not correspond to any policy. More important for our purposes is that most are not representable by the function approximator, which by design has far fewer parameters than there are states.

Given an enumeration of the state space $\mathcal{S} = \{s_1, s_2, \ldots, s_{|\mathcal{S}|}\}$, any value function v corresponds to a vector listing the value of each state in order $[v(s_1), v(s_2), \ldots, v(s_{|\mathcal{S}|})]^\top$. This vector representation of a value function has as many components as there are states. In most cases where we want to use function approximation, this would be far too many components to represent the vector explicitly. Nevertheless, the idea of this vector is conceptually useful. In the following, we treat a value function and its vector representation interchangeably.

To develop intuitions, consider the case with three states $\mathcal{S} = \{s_1, s_2, s_3\}$ and two parameters $\mathbf{w} = (w_1, w_2)^\top$. We can then view all value functions/vectors as points in a three-dimensional space. The parameters provide an alternative coordinate system over a two-dimensional subspace. Any weight vector $\mathbf{w} = (w_1, w_2)^\top$ is a point in the two-dimensional subspace and thus also a complete value function $v_\mathbf{w}$ that assigns values to all three states. With general function approximation the relationship between the full space and the subspace of representable functions could be complex, but in the case of *linear* value-function approximation the subspace is a simple plane, as suggested by Figure 11.3.

Now consider a single fixed policy π. We assume that its true value function, v_π, is too complex to be represented exactly as an approximation. Thus v_π is not in the subspace; in the figure it is depicted as being above the planar subspace of representable functions.

If v_π cannot be represented exactly, what representable value function is closest to it? This turns out to be a subtle question with multiple answers. To begin, we need a measure of the distance between two value functions. Given two value functions v_1 and v_2, we can talk about the vector difference between them, $v = v_1 - v_2$. If v is small, then the two value functions are close to each other. But how are we to measure the size of this difference vector? The conventional Euclidean norm is not appropriate because, as discussed in Section 9.2, some states are more important than others because they occur more frequently or because we are more interested in them (Section 9.11). As in Section 9.2, let us use the distribution $\mu : \mathcal{S} \to [0, 1]$ to specify the degree to which we care about different states being accurately valued (often taken to be the on-policy distribution). We can then define the distance between value functions using the norm

$$\|v\|_\mu^2 \doteq \sum_{s \in \mathcal{S}} \mu(s) v(s)^2. \tag{11.11}$$

Note that the $\overline{\text{VE}}$ from Section 9.2 can be written simply using this norm as $\overline{\text{VE}}(\mathbf{w}) = \|v_\mathbf{w} - v_\pi\|_\mu^2$. For any value function v, the operation of finding its closest value function in the subspace of representable value functions is a projection operation. We define a

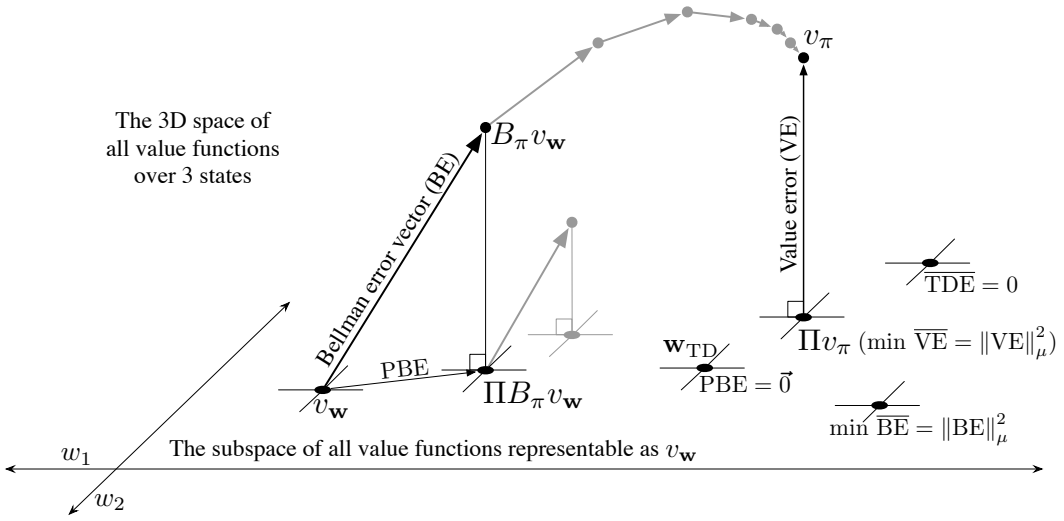


Figure 11.3: The geometry of linear value-function approximation. Shown is the three-dimensional space of all value functions over three states, while shown as a plane is the subspace of all value functions representable by a linear function approximator with parameter $\mathbf{w} = (w_1, w_2)^\top$. The true value function v_π is in the larger space and can be projected down (into the subspace, using a projection operator Π) to its best approximation in the value error (VE) sense. The best approximators in the Bellman error (BE), projected Bellman error (PBE), and temporal difference error (TDE) senses are all potentially different and are shown in the lower right. (VE, BE, and PBE are all treated as the corresponding vectors in this figure.) The Bellman operator takes a value function in the plane to one outside, which can then be projected back. If you iteratively applied the Bellman operator outside the space (shown in gray above) you would reach the true value function, as in conventional dynamic programming. If instead you kept projecting back into the subspace at each step, as in the lower step shown in gray, then the fixed point would be the point of vector-zero PBE.

projection operator Π that takes an arbitrary value function to the representable function that is closest in our norm:

$$\Pi v \doteq v_{\mathbf{w}} \quad \text{where} \quad \mathbf{w} = \operatorname*{arg\,min}_{\mathbf{w} \in \mathbb{R}^d} \|v - v_{\mathbf{w}}\|^2_\mu \,. \tag{11.12}$$

The representable value function that is closest to the true value function v_π is thus its projection, Πv_π, as suggested in Figure 11.3. This is the solution asymptotically found by Monte Carlo methods, albeit often very slowly. The projection operation is discussed more fully in the box on the next page.

TD methods find different solutions. To understand their rationale, recall that the Bellman equation for value function v_π is

$$v_\pi(s) = \sum_a \pi(a|s) \sum_{s',r} p(s', r \,|\, s, a) \left[r + \gamma v_\pi(s') \right], \qquad \text{for all } s \in \mathcal{S}. \tag{11.13}$$

The projection matrix

For a linear function approximator, the projection operation is linear, which implies that it can be represented as an $|\mathcal{S}| \times |\mathcal{S}|$ matrix:

$$\Pi \doteq \mathbf{X}\left(\mathbf{X}^\top \mathbf{D}\mathbf{X}\right)^{-1}\mathbf{X}^\top \mathbf{D}, \tag{11.14}$$

where, as in Section 9.4, $\mathbf{D}$ denotes the $|\mathcal{S}| \times |\mathcal{S}|$ diagonal matrix with the $\mu(s)$ on the diagonal, and $\mathbf{X}$ denotes the $|\mathcal{S}| \times d$ matrix whose rows are the feature vectors $\mathbf{x}(s)^\top$, one for each state s. If the inverse in (11.14) does not exist, then the pseudoinverse is substituted. Using these matrices, the squared norm of a vector can be written

$$\|v\|_\mu^2 = v^\top \mathbf{D} v, \tag{11.15}$$

and the approximate linear value function can be written

$$v_{\mathbf{w}} = \mathbf{X}\mathbf{w}. \tag{11.16}$$

The true value function v_π is the only value function that solves (11.13) exactly. If an approximate value function $v_{\mathbf{w}}$ were substituted for v_π, the difference between the right and left sides of the modified equation could be used as a measure of how far off $v_{\mathbf{w}}$ is from v_π. We call this the *Bellman error* at state s:

$$\bar{\delta}_{\mathbf{w}}(s) \doteq \left(\sum_a \pi(a|s) \sum_{s',r} p(s', r \,|\, s, a)\left[r + \gamma v_{\mathbf{w}}(s')\right]\right) \; - \; v_{\mathbf{w}}(s) \tag{11.17}$$

$$= \mathbb{E}_\pi\big[R_{t+1} + \gamma v_{\mathbf{w}}(S_{t+1}) - v_{\mathbf{w}}(S_t) \;\big|\; S_t = s, A_t \sim \pi\big], \tag{11.18}$$

which shows clearly the relationship of the Bellman error to the TD error (11.3). The Bellman error is the expectation of the TD error.

The vector of all the Bellman errors, at all states, $\bar{\delta}_{\mathbf{w}} \in \mathbb{R}^{|\mathcal{S}|}$, is called the *Bellman error vector* (shown as BE in Figure 11.3). The overall size of this vector, in the norm, is an overall measure of the error in the value function, called the *Mean Squared Bellman Error*:

$$\overline{\text{BE}}(\mathbf{w}) = \left\|\bar{\delta}_{\mathbf{w}}\right\|_\mu^2. \tag{11.19}$$

It is not possible in general to reduce the $\overline{\text{BE}}$ to zero (at which point $v_{\mathbf{w}} = v_\pi$), but for linear function approximation there is a unique value of $\mathbf{w}$ for which the $\overline{\text{BE}}$ is minimized. This point in the representable-function subspace (labeled $\min \overline{\text{BE}}$ in Figure 11.3) is different in general from that which minimizes the $\overline{\text{VE}}$ (shown as Πv_π). Methods that seek to minimize the $\overline{\text{BE}}$ are discussed in the next two sections.

The Bellman error vector is shown in Figure 11.3 as the result of applying the *Bellman operator* $B_\pi : \mathbb{R}^{|\mathcal{S}|} \to \mathbb{R}^{|\mathcal{S}|}$ to the approximate value function. The Bellman operator is

defined by

$$(B_\pi v)(s) \doteq \sum_a \pi(a|s) \sum_{s',r} p(s', r \,|\, s, a) \left[r + \gamma v(s') \right], \tag{11.20}$$

for all $s \in \mathcal{S}$ and $v : \mathcal{S} \to \mathbb{R}$. The Bellman error vector for v can be written $\bar{\delta}_{\mathbf{w}} = B_\pi v_{\mathbf{w}} - v_{\mathbf{w}}$.

If the Bellman operator is applied to a value function in the representable subspace, then, in general, it will produce a new value function that is outside the subspace, as suggested in the figure. In dynamic programming (without function approximation), this operator is applied repeatedly to the points outside the representable space, as suggested by the gray arrows in the top of Figure 11.3. Eventually that process converges to the true value function v_π, the only fixed point for the Bellman operator, the only value function for which

$$v_\pi = B_\pi v_\pi, \tag{11.21}$$

which is just another way of writing the Bellman equation for π (11.13).

With function approximation, however, the intermediate value functions lying outside the subspace cannot be represented. The gray arrows in the upper part of Figure 11.3 cannot be followed because after the first update (dark line) the value function must be projected back into something representable. The next iteration then begins within the subspace; the value function is again taken outside of the subspace by the Bellman operator and then mapped back by the projection operator, as suggested by the lower gray arrow and line. Following these arrows is a DP-like process with approximation.

In this case we are interested in the projection of the Bellman error vector back into the representable space. This is the projected Bellman error vector $\Pi \bar{\delta}_{v_{\mathbf{w}}}$, shown in Figure 11.3 as PBE. The size of this vector, in the norm, is another measure of error in the approximate value function. For any approximate value function v, we define the *Mean Square Projected Bellman Error*, denoted $\overline{\text{PBE}}$, as

$$\overline{\text{PBE}}(\mathbf{w}) = \left\| \Pi \bar{\delta}_{\mathbf{w}} \right\|_\mu^2 . \tag{11.22}$$

With linear function approximation there always exists an approximate value function (within the subspace) with zero $\overline{\text{PBE}}$; this is the TD fixed point, $\mathbf{w}_{\text{TD}}$, introduced in Section 9.4. As we have seen, this point is not always stable under semi-gradient TD methods and off-policy training. As shown in the figure, this value function is generally different from those minimizing $\overline{\text{VE}}$ or $\overline{\text{BE}}$. Methods that are guaranteed to converge to it are discussed in Sections 11.7 and 11.8.

11.5 Gradient Descent in the Bellman Error

Armed with a better understanding of value function approximation and its various objectives, we return now to the challenge of stability in off-policy learning. We would like to apply the approach of stochastic gradient descent (SGD, Section 9.3), in which updates are made that in expectation are equal to the negative gradient of an objective

function. These methods always go downhill (in expectation) in the objective and because of this are typically stable with excellent convergence properties. Among the algorithms investigated so far in this book, only the Monte Carlo methods are true SGD methods. These methods converge robustly under both on-policy and off-policy training as well as for general nonlinear (differentiable) function approximators, though they are often slower than semi-gradient methods with bootstrapping, which are not SGD methods. Semi-gradient methods may diverge under off-policy training, as we have seen earlier in this chapter, and under contrived cases of nonlinear function approximation (Tsitsiklis and Van Roy, 1997). With a true SGD method such divergence would not be possible.

The appeal of SGD is so strong that great effort has gone into finding a practical way of harnessing it for reinforcement learning. The starting place of all such efforts is the choice of an error or objective function to optimize. In this and the next section we explore the origins and limits of the most popular proposed objective function, that based on the *Bellman error* introduced in the previous section. Although this has been a popular and influential approach, the conclusion that we reach here is that it is a misstep and yields no good learning algorithms. On the other hand, this approach fails in an interesting way that offers insight into what might constitute a good approach.

To begin, let us consider not the Bellman error, but something more immediate and naive. Temporal difference learning is driven by the TD error. Why not take the minimization of the expected square of the TD error as the objective? In the general function-approximation case, the one-step TD error with discounting is

$$\delta_t = R_{t+1} + \gamma \hat{v}(S_{t+1},\mathbf{w}_t) - \hat{v}(S_t,\mathbf{w}_t).$$

A possible objective function then is what one might call the *Mean Squared TD Error*:

$$\begin{aligned}
\overline{\text{TDE}}(\mathbf{w}) &= \sum_{s\in\mathcal{S}} \mu(s)\mathbb{E}\big[\delta_t^2 \;\big|\; S_t{=}s, A_t{\sim}\pi\big] \\
&= \sum_{s\in\mathcal{S}} \mu(s)\mathbb{E}\big[\rho_t\delta_t^2 \;\big|\; S_t{=}s, A_t{\sim}b\big] \\
&= \mathbb{E}_b\big[\rho_t\delta_t^2\big]. \qquad \text{(if } \mu \text{ is the distribution encountered under } b\text{)}
\end{aligned}$$

The last equation is of the form needed for SGD; it gives the objective as an expectation that can be sampled from experience (remember the experience is due to the behavior policy b). Thus, following the standard SGD approach, one can derive the per-step update based on a sample of this expected value:

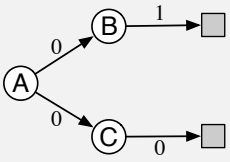
$$\begin{aligned}
\mathbf{w}_{t+1} &= \mathbf{w}_t - \frac{1}{2}\alpha\nabla(\rho_t\delta_t^2) \\
&= \mathbf{w}_t - \alpha\rho_t\delta_t\nabla\delta_t \\
&= \mathbf{w}_t + \alpha\rho_t\delta_t\big(\nabla\hat{v}(S_t,\mathbf{w}_t) - \gamma\nabla\hat{v}(S_{t+1},\mathbf{w}_t)\big),
\end{aligned} \tag{11.23}$$

which you will recognize as the same as the semi-gradient TD algorithm (11.2) except for the additional final term. This term completes the gradient and makes this a true SGD algorithm with excellent convergence guarantees. Let us call this algorithm the *naive*

residual-gradient algorithm (after Baird, 1995). Although the naive residual-gradient algorithm converges robustly, it does not necessarily converge to a desirable place.

Example 11.2: A-split example, showing the naiveté of the naive residual-gradient algorithm

Consider the three-state episodic MRP shown to the right. Episodes begin in state A and then 'split' stochastically, half the time going to B (and then invariably going on to terminate with a reward of 1) and half the time going to state C (and then invariably terminating with a reward of zero). Reward for the first transition, out of A, is always zero whichever way the episode goes. As this is an episodic problem, we can take γ to be 1. We also assume on-policy training, so that ρ_t is always 1, and tabular function approximation, so that the learning algorithms are free to give arbitrary, independent values to all three states. Thus, this should be an easy problem.



What should the values be? From A, half the time the return is 1, and half the time the return is 0; A should have value $\frac{1}{2}$. From B the return is always 1, so its value should be 1, and similarly from C the return is always 0, so its value should be 0. These are the true values and, as this is a tabular problem, all the methods presented previously converge to them exactly.

However, the naive residual-gradient algorithm finds different values for B and C. It converges with B having a value of $\frac{3}{4}$ and C having a value of $\frac{1}{4}$ (A converges correctly to $\frac{1}{2}$). These are in fact the values that minimize the $\overline{\text{TDE}}$.

Let us compute the $\overline{\text{TDE}}$ for these values. The first transition of each episode is either up from A's $\frac{1}{2}$ to B's $\frac{3}{4}$, a change of $\frac{1}{4}$, or down from A's $\frac{1}{2}$ to C's $\frac{1}{4}$, a change of $-\frac{1}{4}$. Because the reward is zero on these transitions, and $\gamma = 1$, these changes *are* the TD errors, and thus the squared TD error is always $\frac{1}{16}$ on the first transition. The second transition is similar; it is either up from B's $\frac{3}{4}$ to a reward of 1 (and a terminal state of value 0), or down from C's $\frac{1}{4}$ to a reward of 0 (again with a terminal state of value 0). Thus, the TD error is always $\pm\frac{1}{4}$, for a squared error of $\frac{1}{16}$ on the second step. Thus, for this set of values, the $\overline{\text{TDE}}$ on both steps is $\frac{1}{16}$.

Now let's compute the $\overline{\text{TDE}}$ for the true values (B at 1, C at 0, and A at $\frac{1}{2}$). In this case the first transition is either from $\frac{1}{2}$ up to 1, at B, or from $\frac{1}{2}$ down to 0, at C; in either case the absolute error is $\frac{1}{2}$ and the squared error is $\frac{1}{4}$. The second transition has zero error because the starting value, either 1 or 0 depending on whether the transition is from B or C, always exactly matches the immediate reward and return. Thus the squared TD error is $\frac{1}{4}$ on the first transition and 0 on the second, for a mean reward over the two transitions of $\frac{1}{8}$. As $\frac{1}{8}$ is bigger that $\frac{1}{16}$, this solution is worse according to the $\overline{\text{TDE}}$. On this simple problem, the true values do not have the smallest $\overline{\text{TDE}}$.

A tabular representation is used in the A-split example, so the true state values can be exactly represented, yet the naive residual-gradient algorithm finds different values, and these values have lower $\overline{\text{TDE}}$ than do the true values. Minimizing the $\overline{\text{TDE}}$ is naive; by penalizing all TD errors it achieves something more like temporal smoothing than accurate prediction.

A better idea would seem to be minimizing the Bellman error. If the exact values are learned, the Bellman error is zero everywhere. Thus, a Bellman-error-minimizing algorithm should have no trouble with the A-split example. We cannot expect to achieve zero Bellman error in general, as it would involve finding the true value function, which we presume is outside the space of representable value functions. But getting close to this ideal is a natural-seeming goal. As we have seen, the Bellman error is also closely related to the TD error. The Bellman error for a state is the expected TD error in that state. So let's repeat the derivation above with the expected TD error (all expectations here are implicitly conditional on S_t):

$$\begin{aligned}
\mathbf{w}_{t+1} &= \mathbf{w}_t - \frac{1}{2}\alpha\nabla\big(\mathbb{E}_\pi[\delta_t]^2\big) \\
&= \mathbf{w}_t - \frac{1}{2}\alpha\nabla\big(\mathbb{E}_b[\rho_t\delta_t]^2\big) \\
&= \mathbf{w}_t - \alpha\mathbb{E}_b[\rho_t\delta_t]\,\nabla\mathbb{E}_b[\rho_t\delta_t] \\
&= \mathbf{w}_t - \alpha\mathbb{E}_b\big[\rho_t(R_{t+1} + \gamma\hat{v}(S_{t+1},\mathbf{w}) - \hat{v}(S_t,\mathbf{w}))\big]\,\mathbb{E}_b[\rho_t\nabla\delta_t] \\
&= \mathbf{w}_t + \alpha\Big[\mathbb{E}_b\big[\rho_t(R_{t+1} + \gamma\hat{v}(S_{t+1},\mathbf{w}))\big] - \hat{v}(S_t,\mathbf{w})\Big]\Big[\nabla\hat{v}(S_t,\mathbf{w}) - \gamma\mathbb{E}_b\big[\rho_t\nabla\hat{v}(S_{t+1},\mathbf{w})\big]\Big].
\end{aligned}$$

This update and various ways of sampling it are referred to as the *residual-gradient algorithm*. If you simply used the sample values in all the expectations, then the equation above reduces almost exactly to (11.23), the naive residual-gradient algorithm.[1] But this is naive, because the equation above involves the next state, S_{t+1}, appearing in two expectations that are multiplied together. To get an unbiased sample of the product, two independent samples of the next state are required, but during normal interaction with an external environment only one is obtained. One expectation or the other can be sampled, but not both.

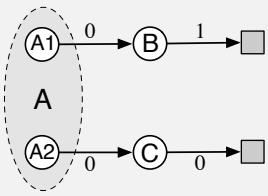
There are two ways to make the residual-gradient algorithm work. One is in the case of deterministic environments. If the transition to the next state is deterministic, then the two samples will necessarily be the same, and the naive algorithm is valid. The other way is to obtain *two* independent samples of the next state, S_{t+1}, from S_t, one for the first expectation and another for the second expectation. In real interaction with an environment, this would not seem possible, but when interacting with a simulated environment, it is. One simply rolls back to the previous state and obtains an alternate next state before proceeding forward from the first next state. In either of these cases the residual-gradient algorithm is guaranteed to converge to a minimum of the $\overline{\text{BE}}$ under the usual conditions on the step-size parameter. As a true SGD method, this convergence is

[1] For state values there remains a small difference in the treatment of the importance sampling ratio ρ_t. In the analagous action-value case (which is the most important case for control algorithms), the residual-gradient algorithm would reduce exactly to the naive version.

robust, applying to both linear and nonlinear function approximators. In the linear case, convergence is always to the *unique* $\mathbf{w}$ that minimizes the $\overline{\text{BE}}$.

However, there remain at least three ways in which the convergence of the residual-gradient method is unsatisfactory. The first of these is that empirically it is slow, much slower that semi-gradient methods. Indeed, proponents of this method have proposed increasing its speed by combining it with faster semi-gradient methods initially, then gradually switching over to residual gradient for the convergence guarantee (Baird and Moore, 1999). The second way in which the residual-gradient algorithm is unsatisfactory is that it still seems to converge to the wrong values. It does get the right values in all tabular cases, such as the A-split example, as for those an exact solution to the Bellman

Example 11.3: A-presplit example, a counterexample for the $\overline{\text{BE}}$



Consider the three-state episodic MRP shown to the right: Episodes start in either A1 or A2, with equal probability. These two states look exactly the same to the function approximator, like a single state A whose feature representation is distinct from and unrelated to the feature representation of the other two states, B and C, which are also distinct from each other. Specifically, the parameter of the function approximator has three components, one giving the value of state B, one giving the value of state C, and one giving the value of both states A1 and A2. Other than the selection of the initial state, the system is deterministic. If it starts in A1, then it transitions to B with a reward of 0 and then on to termination with a reward of 1. If it starts in A2, then it transitions to C, and then to termination, with both rewards zero.

To a learning algorithm, seeing only the features, the system looks identical to the A-split example. The system seems to always start in A, followed by either B or C with equal probability, and then terminating with a 1 or a 0 depending deterministically on the previous state. As in the A-split example, the true values of B and C are 1 and 0, and the best shared value of A1 and A2 is $\frac{1}{2}$, by symmetry.

Because this problem appears externally identical to the A-split example, we already know what values will be found by the algorithms. Semi-gradient TD converges to the ideal values just mentioned, while the naive residual-gradient algorithm converges to values of $\frac{3}{4}$ and $\frac{1}{4}$ for B and C respectively. All state transitions are deterministic, so the non-naive residual-gradient algorithm will also converge to these values (it is the same algorithm in this case). It follows then that this 'naive' solution must also be the one that minimizes the $\overline{\text{BE}}$, and so it is. On a deterministic problem, the Bellman errors and TD errors are all the same, so the $\overline{\text{BE}}$ is always the same as the $\overline{\text{TDE}}$. Optimizing the $\overline{\text{BE}}$ on this example gives rise to the same failure mode as with the naive residual-gradient algorithm on the A-split example.

equation is possible. But if we examine examples with genuine function approximation, then the residual-gradient algorithm, and indeed the $\overline{\text{BE}}$ objective, seem to find the wrong value functions. One of the most telling such examples is the variation on the A-split example known as the A-*pre*split example, shown on the preceding page, in which the residual-gradient algorithm finds the same poor solution as its naive version. This example shows intuitively that minimizing the $\overline{\text{BE}}$ (which the residual-gradient algorithm surely does) may not be a desirable goal.

The third way in which the convergence of the residual-gradient algorithm is not satisfactory is explained in the next section. Like the second way, the third way is also a problem with the $\overline{\text{BE}}$ objective itself rather than with any particular algorithm for achieving it.

11.6 The Bellman Error is Not Learnable

The concept of learnability that we introduce in this section is different from that commonly used in machine learning. There, a hypothesis is said to be "learnable" if it is *efficiently* learnable, meaning that it can be learned within a polynomial rather than an exponential number of examples. Here we use the term in a more basic way, to mean learnable at all, with any amount of experience. It turns out many quantities of apparent interest in reinforcement learning cannot be learned even from an infinite amount of experiential data. These quantities are well defined and can be computed given knowledge of the internal structure of the environment, but cannot be computed or estimated from the observed sequence of feature vectors, actions, and rewards.[2] We say that they are not *learnable*. It will turn out that the Bellman error objective ($\overline{\text{BE}}$) introduced in the last two sections is not learnable in this sense. That the Bellman error objective cannot be learned from the observable data is probably the strongest reason not to seek it.

To make the concept of learnability clear, let's start with some simple examples. Consider the two Markov reward processes[3] (MRPs) diagrammed below:

Where two edges leave a state, both transitions are assumed to occur with equal probability, and the numbers indicate the reward received. All the states appear the same; they all produce the same single-component feature vector $x = 1$ and have approximated value w. Thus, the only varying part of the data trajectory is the reward sequence. The left MRP stays in the same state and emits an endless stream of 0s and 2s at random, each with 0.5 probability. The right MRP, on every step, either stays in its current state or

[2] They would of course be estimated if the *state* sequence were observed rather than only the corresponding feature vectors.

[3] All MRPs can be considered MDPs with a single action in all states; what we conclude about MRPs here applies as well to MDPs.

switches to the other, with equal probability. The reward is deterministic in this MRP, always a 0 from one state and always a 2 from the other, but because the each state is equally likely on each step, the observable data is again an endless stream of 0s and 2s at random, identical to that produced by the left MRP. (We can assume the right MRP starts in one of two states at random with equal probability.) Thus, even given even an infinite amount of data, it would not be possible to tell which of these two MRPs was generating it. In particular, we could not tell if the MRP has one state or two, is stochastic or deterministic. These things are not learnable.

This pair of MRPs also illustrates that the $\overline{\text{VE}}$ objective (9.1) is not learnable. If $\gamma = 0$, then the true values of the three states (in both MRPs), left to right, are 1, 0, and 2. Suppose $w = 1$. Then the $\overline{\text{VE}}$ is 0 for the left MRP and 1 for the right MRP. Because the $\overline{\text{VE}}$ is different in the two problems, yet the data generated has the same distribution, the $\overline{\text{VE}}$ cannot be learned. The $\overline{\text{VE}}$ is not a unique function of the data distribution. And if it cannot be learned, then how could the $\overline{\text{VE}}$ possibly be useful as an objective for learning?

If an objective cannot be learned, it does indeed draw its utility into question. In the case of the $\overline{\text{VE}}$, however, there is a way out. Note that the same solution, $w = 1$, is optimal for both MRPs above (assuming μ is the same for the two indistinguishable states in the right MRP). Is this a coincidence, or could it be generally true that all MDPs with the same data distribution also have the same optimal parameter vector? If this is true—and we will show next that it is—then the $\overline{\text{VE}}$ remains a usable objective. The $\overline{\text{VE}}$ is not learnable, but the parameter that optimizes it is!

To understand this, it is useful to bring in another natural objective function, this time one that is clearly learnable. One error that is always observable is that between the value estimate at each time and the return from that time. The *Mean Square Return Error*, denoted $\overline{\text{RE}}$, is the expectation, under μ, of the square of this error. In the on-policy case the $\overline{\text{RE}}$ can be written

$$\begin{aligned}\overline{\text{RE}}(\mathbf{w}) &= \mathbb{E}\Big[\big(G_t - \hat{v}(S_t,\mathbf{w})\big)^2\Big] \\ &= \overline{\text{VE}}(\mathbf{w}) + \mathbb{E}\Big[\big(G_t - v_\pi(S_t)\big)^2\Big]. \end{aligned} \tag{11.24}$$

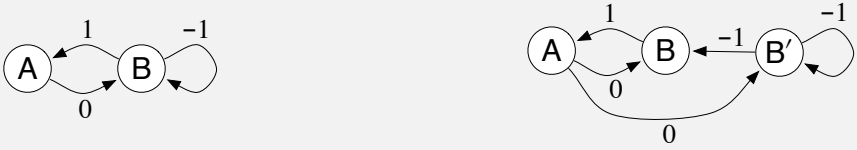
Thus, the two objectives are the same except for a variance term that does not depend on the parameter vector. The two objectives must therefore have the same optimal parameter value $\mathbf{w}^*$. The overall relationships are summarized in the left side of Figure 11.4.

**Exercise 11.4* Prove (11.24). Hint: Write the $\overline{\text{RE}}$ as an expectation over possible states s of the expectation of the squared error given that $S_t = s$. Then add and subtract the true value of state s from the error (before squaring), grouping the subtracted true value with the return and the added true value with the estimated value. Then, if you expand the square, the most complex term will end up being zero, leaving you with (11.24). □

Now let us return to the $\overline{\text{BE}}$. The $\overline{\text{BE}}$ is like the $\overline{\text{VE}}$ in that it can be computed from knowledge of the MDP but is not learnable from data. But it is not like the $\overline{\text{VE}}$ in that its minimum solution is not learnable. The box on the next page presents a counterexample—two MRPs that generate the same data distribution but whose minimizing parameter vector is different, proving that the optimal parameter vector is not a function of the

Example 11.4: Counterexample to the learnability of the Bellman error

To show the full range of possibilities we need a slightly more complex pair of Markov reward processes (MRPs) than those considered earlier. Consider the following two MRPs:



Where two edges leave a state, both transitions are assumed to occur with equal probability, and the numbers indicate the reward received. The MRP on the left has two states that are represented distinctly. The MRP on the right has three states, two of which, B and B′, appear the same and must be given the same approximate value. Specifically, $\mathbf{w}$ has two components and the value of state A is given by the first component and the value of B and B′ is given by the second. The second MRP has been designed so that equal time is spent in all three states, so we can take $\mu(s) = \frac{1}{3}$, for all s.

Note that the observable data distribution is identical for the two MRPs. In both cases the agent will see single occurrences of A followed by a 0, then some number of apparent Bs, each followed by a -1 except the last, which is followed by a 1, then we start all over again with a single A and a 0, etc. All the statistical details are the same as well; in both MRPs, the probability of a string of k Bs is 2^{-k}.

Now suppose $\mathbf{w} = \mathbf{0}$. In the first MRP, this is an exact solution, and the $\overline{\text{BE}}$ is zero. In the second MRP, this solution produces a squared error in both B and B′ of 1, such that $\overline{\text{BE}} = \mu(\text{B})1 + \mu(\text{B}')1 = \frac{2}{3}$. These two MRPs, which generate the same data distribution, have different $\overline{\text{BE}}$s; the $\overline{\text{BE}}$ is not learnable.

Moreover (and unlike the earlier example for the $\overline{\text{VE}}$) the minimizing value of $\mathbf{w}$ is different for the two MRPs. For the first MRP, $\mathbf{w} = \mathbf{0}$ minimizes the $\overline{\text{BE}}$ for any γ. For the second MRP, the minimizing $\mathbf{w}$ is a complicated function of γ, but in the limit, as $\gamma \to 1$, it is $(-\frac{1}{2}, 0)^\top$. Thus the solution that minimizes $\overline{\text{BE}}$ cannot be estimated from data alone; knowledge of the MRP beyond what is revealed in the data is required. In this sense, it is impossible in principle to pursue the $\overline{\text{BE}}$ as an objective for learning.

It may be surprising that in the second MRP the $\overline{\text{BE}}$-minimizing value of A is so far from zero. Recall that A has a dedicated weight and thus its value is unconstrained by function approximation. A is followed by a reward of 0 and transition to a state with a value of nearly 0, which suggests $v_{\mathbf{w}}(\text{A})$ should be 0; why is its optimal value substantially negative rather than 0? The answer is that making $v_{\mathbf{w}}(\text{A})$ negative reduces the error upon arriving in A from B. The reward on this deterministic transition is 1, which implies that B should have a value 1 more than A. Because B's value is approximately zero, A's value is driven toward -1. The $\overline{\text{BE}}$-minimizing value of $\approx -\frac{1}{2}$ for A is a compromise between reducing the errors on leaving and on entering A.

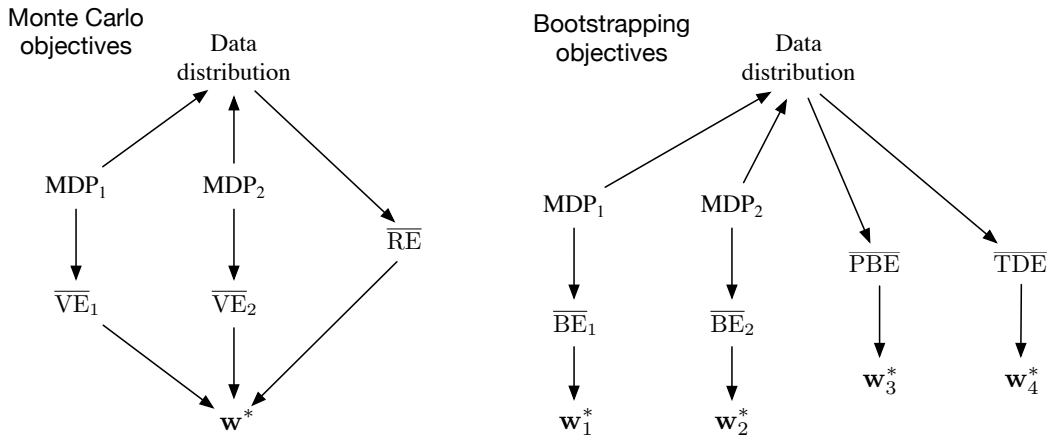


Figure 11.4: Causal relationships among the data distribution, MDPs, and various objectives. **Left, Monte Carlo objectives:** Two different MDPs can produce the same data distribution yet also produce different $\overline{\text{VE}}$s, proving that the $\overline{\text{VE}}$ objective cannot be determined from data and is not learnable. However, all such $\overline{\text{VE}}$s must have the same optimal parameter vector, $\mathbf{w}^*$! Moreover, this same $\mathbf{w}^*$ can be determined from another objective, the $\overline{\text{RE}}$, which *is* uniquely determined from the data distribution. Thus $\mathbf{w}^*$ and the $\overline{\text{RE}}$ are learnable even though the $\overline{\text{VE}}$s are not. **Right, Bootstrapping objectives:** Two different MDPs can produce the same data distribution yet also produce different $\overline{\text{BE}}$s *and* have different minimizing parameter vectors; these are not learnable from the data distribution. The $\overline{\text{PBE}}$ and $\overline{\text{TDE}}$ objectives and their (different) minima can be directly determined from data and thus are learnable.

data and thus cannot be learned from it. The other bootstrapping objectives that we have considered, the $\overline{\text{PBE}}$ and $\overline{\text{TDE}}$, can be determined from data (are learnable) and determine optimal solutions that are in general different from each other and the $\overline{\text{BE}}$ minimums. The general case is summarized in the right side of Figure 11.4.

Thus, the $\overline{\text{BE}}$ is not learnable; it cannot be estimated from feature vectors and other observable data. This limits the $\overline{\text{BE}}$ to model-based settings. There can be no algorithm that minimizes the $\overline{\text{BE}}$ without access to the underlying MDP states beyond the feature vectors. The residual-gradient algorithm is only able to minimize $\overline{\text{BE}}$ because it is allowed to double sample from the same state—not a state that has the same feature vector, but one that is guaranteed to be the same underlying state. We can see now that there is no way around this. Minimizing the $\overline{\text{BE}}$ requires some such access to the nominal, underlying MDP. This is an important limitation of the $\overline{\text{BE}}$ beyond that identified in the A-presplit example on page 273. All this directs more attention toward the $\overline{\text{PBE}}$.

11.7 Gradient-TD Methods

We now consider SGD methods for minimizing the $\overline{\text{PBE}}$. As true SGD methods, these *Gradient-TD methods* have robust convergence properties even under off-policy training and nonlinear function approximation. Remember that in the linear case there is always an exact solution, the TD fixed point $\mathbf{w}_{\text{TD}}$, at which the $\overline{\text{PBE}}$ is zero. This solution could be found by least-squares methods (Section 9.8), but only by methods of quadratic $O(d^2)$ complexity in the number of parameters. We seek instead an SGD method, which should be $O(d)$ and have robust convergence properties. Gradient-TD methods come close to achieving these goals, at the cost of a rough doubling of computational complexity.

To derive an SGD method for the $\overline{\text{PBE}}$ (assuming linear function approximation) we begin by expanding and rewriting the objective (11.22) in matrix terms:

$$\begin{aligned}
\overline{\text{PBE}}(\mathbf{w}) &= \left\|\Pi\bar{\delta}_{\mathbf{w}}\right\|_{\mu}^2 \\
&= (\Pi\bar{\delta}_{\mathbf{w}})^\top \mathbf{D}\Pi\bar{\delta}_{\mathbf{w}} && \text{(from (11.15))} \\
&= \bar{\delta}_{\mathbf{w}}^\top \Pi^\top \mathbf{D}\Pi\bar{\delta}_{\mathbf{w}} \\
&= \bar{\delta}_{\mathbf{w}}^\top \mathbf{D}\mathbf{X}\big(\mathbf{X}^\top\mathbf{D}\mathbf{X}\big)^{-1}\mathbf{X}^\top\mathbf{D}\bar{\delta}_{\mathbf{w}} && (11.25) \\
&\text{(using (11.14) and the identity } \Pi^\top\mathbf{D}\Pi = \mathbf{D}\mathbf{X}\left(\mathbf{X}^\top\mathbf{D}\mathbf{X}\right)^{-1}\mathbf{X}^\top\mathbf{D}\text{)} \\
&= \big(\mathbf{X}^\top\mathbf{D}\bar{\delta}_{\mathbf{w}}\big)^\top\big(\mathbf{X}^\top\mathbf{D}\mathbf{X}\big)^{-1}\big(\mathbf{X}^\top\mathbf{D}\bar{\delta}_{\mathbf{w}}\big). && (11.26)
\end{aligned}$$

The gradient with respect to $\mathbf{w}$ is

$$\nabla\overline{\text{PBE}}(\mathbf{w}) = 2\nabla\big[\mathbf{X}^\top\mathbf{D}\bar{\delta}_{\mathbf{w}}\big]^\top\big(\mathbf{X}^\top\mathbf{D}\mathbf{X}\big)^{-1}\big(\mathbf{X}^\top\mathbf{D}\bar{\delta}_{\mathbf{w}}\big).$$

To turn this into an SGD method, we have to sample something on every time step that has this quantity as its expected value. Let us take μ to be the distribution of states visited under the behavior policy. All three of the factors above can then be written in terms of expectations under this distribution. For example, the last factor can be written

$$\mathbf{X}^\top\mathbf{D}\bar{\delta}_{\mathbf{w}} = \sum_s \mu(s)\mathbf{x}(s)\bar{\delta}_{\mathbf{w}}(s) = \mathbb{E}[\rho_t\delta_t\mathbf{x}_t]\,,$$

which is just the expectation of the semi-gradient TD(0) update (11.2). The first factor is the transpose of the gradient of this update:

$$\begin{aligned}
\nabla\mathbb{E}[\rho_t\delta_t\mathbf{x}_t]^\top &= \mathbb{E}\big[\rho_t\nabla\delta_t^\top\mathbf{x}_t^\top\big] \\
&= \mathbb{E}\big[\rho_t\nabla(R_{t+1} + \gamma\mathbf{w}^\top\mathbf{x}_{t+1} - \mathbf{w}^\top\mathbf{x}_t)^\top\mathbf{x}_t^\top\big] && \text{(using episodic } \delta_t\text{)} \\
&= \mathbb{E}\big[\rho_t(\gamma\mathbf{x}_{t+1} - \mathbf{x}_t)\mathbf{x}_t^\top\big]\,.
\end{aligned}$$

Finally, the middle factor is the inverse of the expected outer-product matrix of the feature vectors:

$$\mathbf{X}^\top\mathbf{D}\mathbf{X} = \sum_s \mu(s)\mathbf{x}_s\mathbf{x}_s^\top = \mathbb{E}\big[\mathbf{x}_t\mathbf{x}_t^\top\big]\,.$$

Substituting these expectations for the three factors in our expression for the gradient of the $\overline{\text{PBE}}$, we get

$$\nabla\overline{\text{PBE}}(\mathbf{w}) = 2\mathbb{E}\big[\rho_t(\gamma\mathbf{x}_{t+1} - \mathbf{x}_t)\mathbf{x}_t^\top\big]\,\mathbb{E}\big[\mathbf{x}_t\mathbf{x}_t^\top\big]^{-1}\,\mathbb{E}[\rho_t\delta_t\mathbf{x}_t]\,. \tag{11.27}$$

It might not be obvious that we have made any progress by writing the gradient in this form. It is a product of three expressions and the first and last are not independent. They both depend on the next feature vector $\mathbf{x}_{t+1}$; we cannot simply sample both of these expectations and then multiply the samples. This would give us a biased estimate of the gradient just as in the naive residual-gradient algorithm.

Another idea would be to estimate the three expectations separately and then combine them to produce an unbiased estimate of the gradient. This would work, but would require a lot of computational resources, particularly to store the first two expectations, which are $d \times d$ matrices, and to compute the inverse of the second. This idea can be improved. If two of the three expectations are estimated and stored, then the third could be sampled and used in conjunction with the two stored quantities. For example, you could store estimates of the second two quantities (using the increment inverse-updating techniques in Section 9.8) and then sample the first expression. Unfortunately, the overall algorithm would still be of quadratic complexity (of order $O(d^2)$).

The idea of storing some estimates separately and then combining them with samples is a good one and is also used in Gradient-TD methods. Gradient-TD methods estimate and store *the product* of the second two factors in (11.27). These factors are a $d \times d$ matrix and a d-vector, so their product is just a d-vector, like $\mathbf{w}$ itself. We denote this second learned vector as $\mathbf{v}$:

$$\mathbf{v} \approx \mathbb{E}\big[\mathbf{x}_t\mathbf{x}_t^\top\big]^{-1}\,\mathbb{E}[\rho_t\delta_t\mathbf{x}_t]\,. \tag{11.28}$$

This form is familiar to students of linear supervised learning. It is the solution to a linear least-squares problem that tries to approximate $\rho_t\delta_t$ from the features. The standard SGD method for incrementally finding the vector $\mathbf{v}$ that minimizes the expected squared error $\big(\mathbf{v}^\top\mathbf{x}_t - \rho_t\delta_t\big)^2$ is known as the Least Mean Square (LMS) rule (here augmented with an importance sampling ratio):

$$\mathbf{v}_{t+1} \doteq \mathbf{v}_t + \beta\rho_t\,\big(\delta_t - \mathbf{v}_t^\top\mathbf{x}_t\big)\,\mathbf{x}_t,$$

where $\beta > 0$ is another step-size parameter. We can use this method to effectively achieve (11.28) with $O(d)$ storage and per-step computation.

Given a stored estimate $\mathbf{v}_t$ approximating (11.28), we can update our main parameter vector $\mathbf{w}_t$ using SGD methods based on (11.27). The simplest such rule is

$$\begin{aligned}
\mathbf{w}_{t+1} &= \mathbf{w}_t - \frac{1}{2}\alpha\nabla\overline{\text{PBE}}(\mathbf{w}_t) && \text{(the general SGD rule)}\\
&= \mathbf{w}_t - \frac{1}{2}\alpha 2\mathbb{E}\big[\rho_t(\gamma\mathbf{x}_{t+1} - \mathbf{x}_t)\mathbf{x}_t^\top\big]\,\mathbb{E}\big[\mathbf{x}_t\mathbf{x}_t^\top\big]^{-1}\,\mathbb{E}[\rho_t\delta_t\mathbf{x}_t] && \text{(from (11.27))}\\
&= \mathbf{w}_t + \alpha\mathbb{E}\big[\rho_t(\mathbf{x}_t - \gamma\mathbf{x}_{t+1})\mathbf{x}_t^\top\big]\,\mathbb{E}\big[\mathbf{x}_t\mathbf{x}_t^\top\big]^{-1}\,\mathbb{E}[\rho_t\delta_t\mathbf{x}_t] && (11.29)\\
&\approx \mathbf{w}_t + \alpha\mathbb{E}\big[\rho_t(\mathbf{x}_t - \gamma\mathbf{x}_{t+1})\mathbf{x}_t^\top\big]\,\mathbf{v}_t && \text{(based on (11.28))}\\
&\approx \mathbf{w}_t + \alpha\rho_t\,(\mathbf{x}_t - \gamma\mathbf{x}_{t+1})\,\mathbf{x}_t^\top\mathbf{v}_t. && \text{(sampling)}
\end{aligned}$$

This algorithm is called *GTD2*. Note that if the final inner product $(\mathbf{x}_t^\top \mathbf{v}_t)$ is done first, then the entire algorithm is of $O(d)$ complexity.

A slightly better algorithm can be derived by doing a few more analytic steps before substituting in $\mathbf{v}_t$. Continuing from (11.29):

$$\begin{aligned}
\mathbf{w}_{t+1} &= \mathbf{w}_t + \alpha \mathbb{E}\big[\rho_t(\mathbf{x}_t - \gamma \mathbf{x}_{t+1})\mathbf{x}_t^\top\big]\, \mathbb{E}\big[\mathbf{x}_t \mathbf{x}_t^\top\big]^{-1}\, \mathbb{E}[\rho_t \delta_t \mathbf{x}_t] \\
&= \mathbf{w}_t + \alpha \left(\mathbb{E}\big[\rho_t \mathbf{x}_t \mathbf{x}_t^\top\big] - \gamma \mathbb{E}\big[\rho_t \mathbf{x}_{t+1} \mathbf{x}_t^\top\big]\right) \mathbb{E}\big[\mathbf{x}_t \mathbf{x}_t^\top\big]^{-1}\, \mathbb{E}[\rho_t \delta_t \mathbf{x}_t] \\
&= \mathbf{w}_t + \alpha \left(\mathbb{E}\big[\mathbf{x}_t \mathbf{x}_t^\top\big] - \gamma \mathbb{E}\big[\rho_t \mathbf{x}_{t+1} \mathbf{x}_t^\top\big]\right) \mathbb{E}\big[\mathbf{x}_t \mathbf{x}_t^\top\big]^{-1}\, \mathbb{E}[\rho_t \delta_t \mathbf{x}_t] \\
&= \mathbf{w}_t + \alpha \left(\mathbb{E}[\mathbf{x}_t \rho_t \delta_t] - \gamma \mathbb{E}\big[\rho_t \mathbf{x}_{t+1} \mathbf{x}_t^\top\big]\, \mathbb{E}\big[\mathbf{x}_t \mathbf{x}_t^\top\big]^{-1}\, \mathbb{E}[\rho_t \delta_t \mathbf{x}_t]\right) \\
&\approx \mathbf{w}_t + \alpha \left(\mathbb{E}[\mathbf{x}_t \rho_t \delta_t] - \gamma \mathbb{E}\big[\rho_t \mathbf{x}_{t+1} \mathbf{x}_t^\top\big]\, \mathbf{v}_t\right) && \text{(based on (11.28))} \\
&\approx \mathbf{w}_t + \alpha \rho_t \left(\delta_t \mathbf{x}_t - \gamma \mathbf{x}_{t+1} \mathbf{x}_t^\top \mathbf{v}_t\right), && \text{(sampling)}
\end{aligned}$$

which again is $O(d)$ if the final product $(\mathbf{x}_t^\top \mathbf{v}_t)$ is done first. This algorithm is known as either *TD(0) with gradient correction (TDC)* or, alternatively, as *GTD(0)*.

Figure 11.5 shows a sample and the expected behavior of TDC on Baird's counterexample. As intended, the $\overline{\text{PBE}}$ falls to zero, but note that the individual components of the parameter vector do not approach zero. In fact, these values are still far from

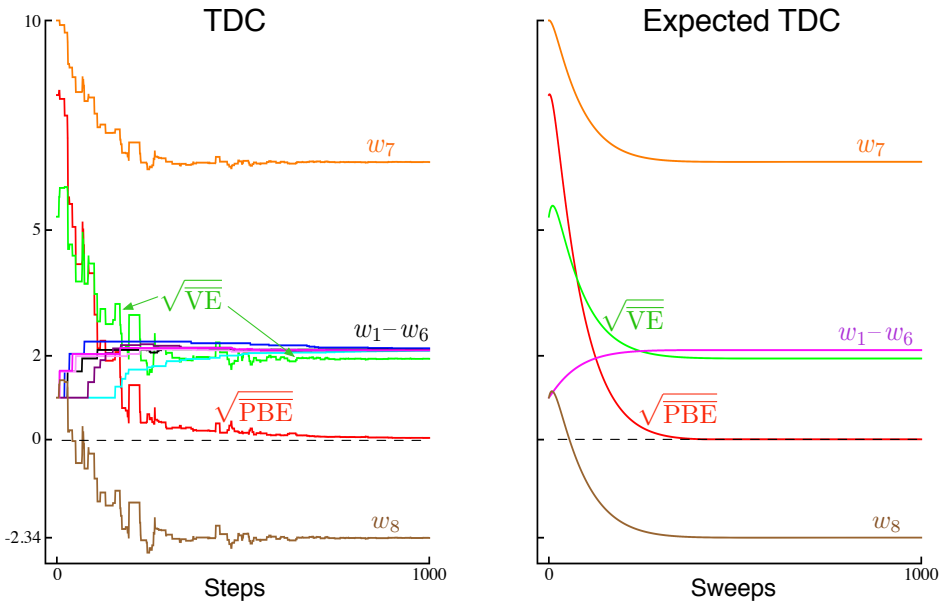


Figure 11.5: The behavior of the TDC algorithm on Baird's counterexample. On the left is shown a typical single run, and on the right is shown the expected behavior of this algorithm if the updates are done synchronously (analogous to (11.9), except for the two TDC parameter vectors). The step sizes were $\alpha = 0.005$ and $\beta = 0.05$.

an optimal solution, $\hat{v}(s) = 0$, for all s, for which $\mathbf{w}$ would have to be proportional to $(1, 1, 1, 1, 1, 1, 4, -2)^\top$. After 1000 iterations we are still far from an optimal solution, as we can see from the $\overline{\text{VE}}$, which remains almost 2. The system is actually converging to an optimal solution, but progress is extremely slow because the $\overline{\text{PBE}}$ is already so close to zero.

GTD2 and TDC both involve two learning processes, a primary one for $\mathbf{w}$ and a secondary one for $\mathbf{v}$. The logic of the primary learning process relies on the secondary learning process having finished, at least approximately, whereas the secondary learning process proceeds without being influenced by the first. We call this sort of asymmetrical dependence a *cascade*. In cascades we often assume that the secondary learning process is proceeding faster and thus is always at its asymptotic value, ready and accurate to assist the primary learning process. The convergence proofs for these methods often make this assumption explicitly. These are called *two-time-scale* proofs. The fast time scale is that of the secondary learning process, and the slower time scale is that of the primary learning process. If α is the step size of the primary learning process, and β is the step size of the secondary learning process, then these convergence proofs will typically require that in the limit $\beta \to 0$ and $\frac{\alpha}{\beta} \to 0$.

Gradient-TD methods are currently the most well understood and widely used stable off-policy methods. There are extensions to action values and control (GQ, Maei et al., 2010), to eligibility traces (GTD(λ) and GQ(λ), Maei, 2011; Maei and Sutton, 2010), and to nonlinear function approximation (Maei et al., 2009). There have also been proposed hybrid algorithms midway between semi-gradient TD and gradient TD (Hackman, 2012; White and White, 2016). Hybrid-TD algorithms behave like Gradient-TD algorithms in states where the target and behavior policies are very different, and behave like semi-gradient algorithms in states where the target and behavior policies are the same. Finally, the Gradient-TD idea has been combined with the ideas of proximal methods and control variates to produce more efficient methods (Mahadevan et al., 2014; Du et al., 2017).

11.8 Emphatic-TD Methods

We turn now to the second major strategy that has been extensively explored for obtaining a cheap and efficient off-policy learning method with function approximation. Recall that linear semi-gradient TD methods are efficient and stable when trained under the on-policy distribution, and that we showed in Section 9.4 that this has to do with the positive definiteness of the matrix $\mathbf{A}$ (9.11)[4] and the match between the on-policy state distribution μ_π and the state-transition probabilities $p(s \,|\, s, a)$ under the target policy. In off-policy learning, we reweight the state transitions using importance sampling so that they become appropriate for learning about the target policy, but the state distribution is still that of the behavior policy. There is a mismatch. A natural idea is to somehow reweight the states, emphasizing some and de-emphasizing others, so as to return the distribution of updates to the on-policy distribution. There would then be a match, and stability and convergence would follow from existing results. This is the idea of

[4] In the off-policy case, the matrix $\mathbf{A}$ is generally defined as $\mathbb{E}_{s\sim b}\big[\mathbf{x}(s)\mathbb{E}\big[\mathbf{x}(S_{t+1})^\top \,\big|\, S_t = s, A_t \sim \pi\big]\big]$.

Emphatic-TD methods, first introduced for on-policy training in Section 9.11.

Actually, the notion of "the on-policy distribution" is not quite right, as there are many on-policy distributions, and any one of these is sufficient to guarantee stability. Consider an undiscounted episodic problem. The way episodes terminate is fully determined by the transition probabilities, but there may be several different ways the episodes might begin. However the episodes start, if all state transitions are due to the target policy, then the state distribution that results is an on-policy distribution. You might start close to the terminal state and visit only a few states with high probability before ending the episode. Or you might start far away and pass through many states before terminating. Both are on-policy distributions, and training on both with a linear semi-gradient method would be guaranteed to be stable. However the process starts, an on-policy distribution results as long as all states encountered are updated up until termination.

If there is discounting, it can be treated as partial or probabilistic termination for these purposes. If $\gamma = 0.9$, then we can consider that with probability 0.1 the process terminates on every time step and then immediately restarts in the state that is transitioned to. A discounted problem is one that is continually terminating and restarting with probability $1 - \gamma$ on every step. This way of thinking about discounting is an example of a more general notion of *pseudo termination*—termination that does not affect the sequence of state transitions, but does affect the learning process and the quantities being learned. This kind of pseudo termination is important to off-policy learning because the restarting is optional—remember we can start any way we want to—and the termination relieves the need to keep including encountered states within the on-policy distribution. That is, if we don't consider the new states as restarts, then discounting quickly gives us a limited on-policy distribution.

The one-step Emphatic-TD algorithm for learning episodic state values is defined by:

$$\delta_t = R_{t+1} + \gamma \hat{v}(S_{t+1}, \mathbf{w}_t) - \hat{v}(S_t, \mathbf{w}_t),$$

$$\mathbf{w}_{t+1} = \mathbf{w}_t + \alpha M_t \rho_t \delta_t \nabla \hat{v}(S_t, \mathbf{w}_t),$$

$$M_t = \gamma \rho_{t-1} M_{t-1} + I_t,$$

with I_t, the *interest*, being arbitrary and M_t, the *emphasis*, being initialized to $M_{t-1} = 0$. How does this algorithm perform on Baird's counterexample? Figure 11.6 shows the trajectory in expectation of the components of the parameter vector (for the case in which $I_t = 1$, for all t). There are some oscillations but eventually everything converges and the $\overline{\text{VE}}$ goes to zero. These trajectories are obtained by iteratively computing the expectation of the parameter vector trajectory without any of the variance due to sampling of transitions and rewards. We do not show the results of applying the Emphatic-TD algorithm directly because its variance on Baird's counterexample is so high that it is nigh impossible to get consistent results in computational experiments. The algorithm converges to the optimal solution in theory on this problem, but in practice it does not. We turn to the topic of reducing the variance of all these algorithms in the next section.

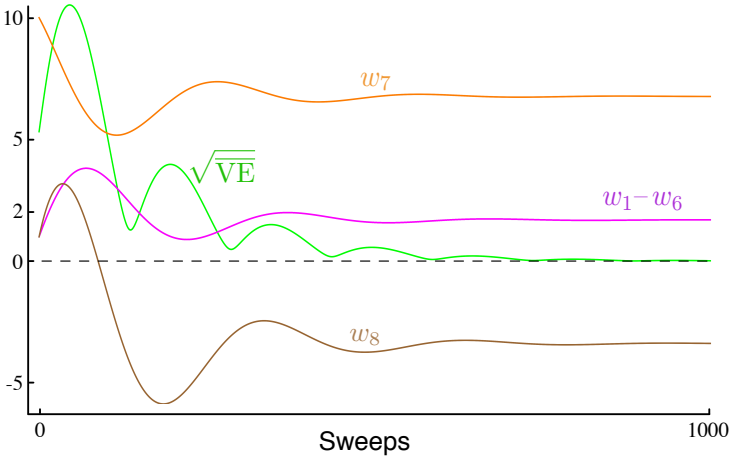


Figure 11.6: The behavior of the one-step Emphatic-TD algorithm in expectation on Baird's counterexample. The step size was $\alpha = 0.03$.

11.9 Reducing Variance

Off-policy learning is inherently of greater variance than on-policy learning. This is not surprising; if you receive data less closely related to a policy, you should expect to learn less about the policy's values. In the extreme, one may be able to learn nothing. You can't expect to learn how to drive by cooking dinner, for example. Only if the target and behavior policies are related, if they visit similar states and take similar actions, should one be able to make significant progress in off-policy training.

On the other hand, any policy has many neighbors, many similar policies with considerable overlap in states visited and actions chosen, and yet which are not identical. The raison d'être of off-policy learning is to enable generalization to this vast number of related-but-not-identical policies. The problem remains of how to make the best use of the experience. Now that we have some methods that are stable in expected value (if the step sizes are set right), attention naturally turns to reducing the variance of the estimates. There are many possible ideas, and we can just touch on a few of them in this introductory text.

Why is controlling variance especially critical in off-policy methods based on importance sampling? As we have seen, importance sampling often involves products of policy ratios. The ratios are always one in expectation (5.13), but their actual values may be very high or as low as zero. Successive ratios are uncorrelated, so their products are also always one in expected value, but they can be of very high variance. Recall that these ratios multiply the step size in SGD methods, so high variance means taking steps that vary greatly in their sizes. This is problematic for SGD because of the occasional very large steps. They must not be so large as to take the parameter to a part of the space with a very different gradient. SGD methods rely on averaging over multiple steps to get a good sense of the gradient, and if they make large moves from single samples they become unreliable.

If the step-size parameter is set small enough to prevent this, then the expected step can end up being very small, resulting in very slow learning. The notions of momentum (Derthick, 1984), of Polyak-Ruppert averaging (Polyak, 1990; Ruppert, 1988; Polyak and Juditsky, 1992), or further extensions of these ideas may significantly help. Methods for adaptively setting separate step sizes for different components of the parameter vector are also pertinent (e.g., Jacobs, 1988; Sutton, 1992b, c), as are the "importance weight aware" updates of Karampatziakis and Langford (2010).

In Chapter 5 we saw how weighted importance sampling is significantly better behaved, with lower variance updates, than ordinary importance sampling. However, adapting weighted importance sampling to function approximation is challenging and can probably only be done approximately with $O(d)$ complexity (Mahmood and Sutton, 2015).

The Tree Backup algorithm (Section 7.5) shows that it is possible to perform some off-policy learning without using importance sampling. This idea has been extended to the off-policy case to produce stable and more efficient methods by Munos, Stepleton, Harutyunyan, and Bellemare (2016) and by Mahmood, Yu and Sutton (2017).

Another, complementary strategy is to allow the target policy to be determined in part by the behavior policy, in such a way that it never can be so different from it to create large importance sampling ratios. For example, the target policy can be defined by reference to the behavior policy, as in the "recognizers" proposed by Precup et al. (2006).

11.10 Summary

Off-policy learning is a tempting challenge, testing our ingenuity in designing stable and efficient learning algorithms. Tabular Q-learning makes off-policy learning seem easy, and it has natural generalizations to Expected Sarsa and to the Tree Backup algorithm. But as we have seen in this chapter, the extension of these ideas to significant function approximation, even linear function approximation, involves new challenges and forces us to deepen our understanding of reinforcement learning algorithms.

Why go to such lengths? One reason to seek off-policy algorithms is to give flexibility in dealing with the tradeoff between exploration and exploitation. Another is to free behavior from learning, and avoid the tyranny of the target policy. TD learning appears to hold out the possibility of learning about multiple things in parallel, of using one stream of experience to solve many tasks simultaneously. We can certainly do this in special cases, just not in every case that we would like to or as efficiently as we would like to.

In this chapter we divided the challenge of off-policy learning into two parts. The first part, correcting the targets of learning for the behavior policy, is straightforwardly dealt with using the techniques devised earlier for the tabular case, albeit at the cost of increasing the variance of the updates and thereby slowing learning. High variance will probably always remains a challenge for off-policy learning.

The second part of the challenge of off-policy learning emerges as the instability of semi-gradient TD methods that involve bootstrapping. We seek powerful function

approximation, off-policy learning, and the efficiency and flexibility of bootstrapping TD methods, but it is challenging to combine all three aspects of this *deadly triad* in one algorithm without introducing the potential for instability. There have been several attempts. The most popular has been to seek to perform true stochastic gradient descent (SGD) in the Bellman error (a.k.a. the Bellman residual). However, our analysis concludes that this is not an appealing goal in many cases, and that anyway it is impossible to achieve with a learning algorithm—the gradient of the $\overline{\text{BE}}$ is not learnable from experience that reveals only feature vectors and not underlying states. Another approach, Gradient-TD methods, performs SGD in the *projected* Bellman error. The gradient of the $\overline{\text{PBE}}$ is learnable with $O(d)$ complexity, but at the cost of a second parameter vector with a second step size. The newest family of methods, Emphatic-TD methods, refine an old idea for reweighting updates, emphasizing some and de-emphasizing others. In this way they restore the special properties that make on-policy learning stable with computationally simple semi-gradient methods.

The whole area of off-policy learning is relatively new and unsettled. Which methods are best or even adequate is not yet clear. Are the complexities of the new methods introduced at the end of this chapter really necessary? Which of them can be combined effectively with variance reduction methods? The potential for off-policy learning remains tantalizing, the best way to achieve it still a mystery.

Bibliographical and Historical Remarks

11.1 The first semi-gradient method was linear TD(λ) (Sutton, 1988). The name "semi-gradient" is more recent (Sutton, 2015a). Semi-gradient off-policy TD(0) with general importance-sampling ratio may not have been explicitly stated until Sutton, Mahmood, and White (2016), but the action-value forms were introduced by Precup, Sutton, and Singh (2000), who also did eligibility trace forms of these algorithms (see Chapter 12). Their continuing, undiscounted forms have not been significantly explored. The n-step forms given here are new.

11.2 The earliest w-to-$2w$ example was given by Tsitsiklis and Van Roy (1996), who also introduced the specific counterexample in the box on page 263. Baird's counterexample is due to Baird (1995), though the version we present here is slightly modified. Averaging methods for function approximation were developed by Gordon (1995, 1996b). Other examples of instability with off-policy DP methods and more complex methods of function approximation are given by Boyan and Moore (1995). Bradtke (1993) gives an example in which Q-learning using linear function approximation in a linear quadratic regulation problem converges to a destabilizing policy.

11.3 The deadly triad was first identified by Sutton (1995b) and thoroughly analyzed by Tsitsiklis and Van Roy (1997). The name "deadly triad" is due to Sutton (2015a).

11.4 This kind of linear analysis was pioneered by Tsitsiklis and Van Roy (1996; 1997),

including the dynamic programming operator. Diagrams like Figure 11.3 were introduced by Lagoudakis and Parr (2003).

What we have called the Bellman operator, and denoted B_π, is more commonly denoted T^π and called a "dynamic programming operator," while a generalized form, denoted $T^{(\lambda)}$, is called the "TD(λ) operator" (Tsitsiklis and Van Roy, 1996, 1997).

11.5 The $\overline{\text{BE}}$ was first proposed as an objective function for dynamic programming by Schweitzer and Seidmann (1985). Baird (1995, 1999) extended it to TD learning based on stochastic gradient descent. In the literature, $\overline{\text{BE}}$ minimization is often referred to as Bellman residual minimization.

The earliest A-split example is due to Dayan (1992). The two forms given here were introduced by Sutton et al. (2009a).

11.6 The contents of this section are new to this text.

11.7 Gradient-TD methods were introduced by Sutton, Szepesvári, and Maei (2009b). The methods highlighted in this section were introduced by Sutton et al. (2009a) and Mahmood et al. (2014). A major extension to proximal TD methods was developed by Mahadeval et al. (2014). The most sensitive empirical investigations to date of Gradient-TD and related methods are given by Geist and Scherrer (2014), Dann, Neumann, and Peters (2014), White (2015), and Ghiassian, White, White, and Sutton (in preparation). Recent developments in the theory of Gradient-TD methods are presented by Yu (2017).

11.8 Emphatic-TD methods were introduced by Sutton, Mahmood, and White (2016). Full convergence proofs and other theory were later established by Yu (2015; 2016; Yu, Mahmood, and Sutton, 2017), Hallak, Tamar, and Mannor (2015), and Hallak, Tamar, Munos, and Mannor (2016).

Chapter 12

Eligibility Traces

Eligibility traces are one of the basic mechanisms of reinforcement learning. For example, in the popular TD(λ) algorithm, the λ refers to the use of an eligibility trace. Almost any temporal-difference (TD) method, such as Q-learning or Sarsa, can be combined with eligibility traces to obtain a more general method that may learn more efficiently.

Eligibility traces unify and generalize TD and Monte Carlo methods. When TD methods are augmented with eligibility traces, they produce a family of methods spanning a spectrum that has Monte Carlo methods at one end ($\lambda = 1$) and one-step TD methods at the other ($\lambda = 0$). In between are intermediate methods that are often better than either extreme method. Eligibility traces also provide a way of implementing Monte Carlo methods online and on continuing problems without episodes.

Of course, we have already seen one way of unifying TD and Monte Carlo methods: the n-step TD methods of Chapter 7. What eligibility traces offer beyond these is an elegant algorithmic mechanism with significant computational advantages. The mechanism is a short-term memory vector, the *eligibility trace* $\mathbf{z}_t \in \mathbb{R}^d$, that parallels the long-term weight vector $\mathbf{w}_t \in \mathbb{R}^d$. The rough idea is that when a component of $\mathbf{w}_t$ participates in producing an estimated value, then the corresponding component of $\mathbf{z}_t$ is bumped up and then begins to fade away. Learning will then occur in that component of $\mathbf{w}_t$ if a nonzero TD error occurs before the trace falls back to zero. The trace-decay parameter $\lambda \in [0, 1]$ determines the rate at which the trace falls.

The primary computational advantage of eligibility traces over n-step methods is that only a single trace vector is required rather than a store of the last n feature vectors. Learning also occurs continually and uniformly in time rather than being delayed and then catching up at the end of the episode. In addition learning can occur and affect behavior immediately after a state is encountered rather than being delayed n steps.

Eligibility traces illustrate that a learning algorithm can sometimes be implemented in a different way to obtain computational advantages. Many algorithms are most naturally formulated and understood as an update of a state's value based on events that follow that state over multiple future time steps. For example, Monte Carlo methods (Chapter 5) update a state based on all the future rewards, and n-step TD methods (Chapter 7)

update based on the next n rewards and state n steps in the future. Such formulations, based on looking forward from the updated state, are called *forward views*. Forward views are always somewhat complex to implement because the update depends on later things that are not available at the time. However, as we show in this chapter it is often possible to achieve nearly the same updates—and sometimes *exactly* the same updates—with an algorithm that uses the current TD error, looking backward to recently visited states using an eligibility trace. These alternate ways of looking at and implementing learning algorithms are called *backward views*. Backward views, transformations between forward views and backward views, and equivalences between them, date back to the introduction of temporal difference learning but have become much more powerful and sophisticated since 2014. Here we present the basics of the modern view.

As usual, first we fully develop the ideas for state values and prediction, then extend them to action values and control. We develop them first for the on-policy case then extend them to off-policy learning. Our treatment pays special attention to the case of linear function approximation, for which the results with eligibility traces are stronger. All these results apply also to the tabular and state aggregation cases because these are special cases of linear function approximation.

12.1 The λ-return

In Chapter 7 we defined an n-step return as the sum of the first n rewards plus the estimated value of the state reached in n steps, each appropriately discounted (7.1). The general form of that equation, for any parameterized function approximator, is

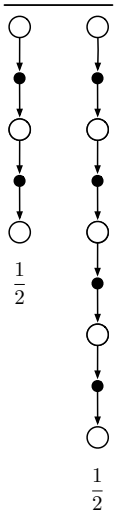
$$G_{t:t+n} \doteq R_{t+1} + \gamma R_{t+2} + \cdots + \gamma^{n-1} R_{t+n} + \gamma^n \hat{v}(S_{t+n}, \mathbf{w}_{t+n-1}), \quad 0 \leq t \leq T-n, \tag{12.1}$$

where $\hat{v}(s,\mathbf{w})$ is the approximate value of state s given weight vector $\mathbf{w}$ (Chapter 9), and T is the time of episode termination, if any. We noted in Chapter 7 that each n-step return, for $n \geq 1$, is a valid update target for a tabular learning update, just as it is for an approximate SGD learning update such as (9.7).

Now we note that a valid update can be done not just toward any n-step return, but toward any *average* of n-step returns for different ns. For example, an update can be done toward a target that is half of a two-step return and half of a four-step return: $\frac{1}{2}G_{t:t+2} + \frac{1}{2}G_{t:t+4}$. Any set of n-step returns can be averaged in this way, even an infinite set, as long as the weights on the component returns are positive and sum to 1. The composite return possesses an error reduction property similar to that of individual n-step returns (7.3) and thus can be used to construct updates with guaranteed convergence properties. Averaging produces a substantial new range of algorithms. For example, one could average one-step and infinite-step returns to obtain another way of interrelating TD and Monte Carlo methods. In principle, one could even average experience-based updates with DP updates to get a simple combination of experience-based and model-based methods (cf. Chapter 8).

An update that averages simpler component updates is called a *compound update*. The backup diagram for a compound update consists of the backup diagrams for each of the component updates with a horizontal line above them and the weighting fractions below.

For example, the compound update for the case mentioned at the start of this section, mixing half of a two-step return and half of a four-step return, has the diagram shown to the right. A compound update can only be done when the longest of its component updates is complete. The update at the right, for example, could only be done at time $t+4$ for the estimate formed at time t. In general one would like to limit the length of the longest component update because of the corresponding delay in the updates.



The TD(λ) algorithm can be understood as one particular way of averaging n-step updates. This average contains all the n-step updates, each weighted proportionally to λ^{n-1} (where $\lambda \in [0,1]$), and is normalized by a factor of $1-\lambda$ to ensure that the weights sum to 1 (Figure 12.1). The resulting update is toward a return, called the λ-*return*, defined in its state-based form by

$$G_t^\lambda \doteq (1-\lambda)\sum_{n=1}^{\infty} \lambda^{n-1} G_{t:t+n}. \tag{12.2}$$

Figure 12.2 further illustrates the weighting on the sequence of n-step returns in the λ-return. The one-step return is given the largest weight, $1-\lambda$; the two-step return is given the next largest weight, $(1-\lambda)\lambda$; the three-step return is given the weight $(1-\lambda)\lambda^2$; and so on. The weight fades by λ with each additional step. After a terminal state has been reached, all subsequent n-step returns are equal to the conventional return, G_t. If

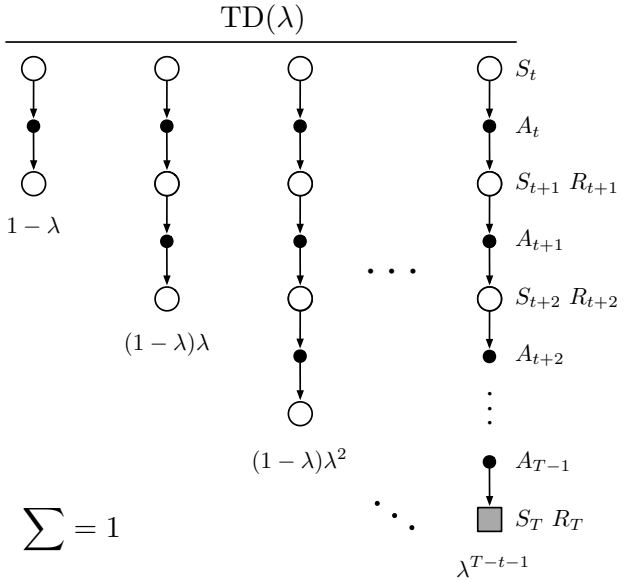


Figure 12.1: The backup digram for TD(λ). If $\lambda = 0$, then the overall update reduces to its first component, the one-step TD update, whereas if $\lambda = 1$, then the overall update reduces to its last component, the Monte Carlo update.

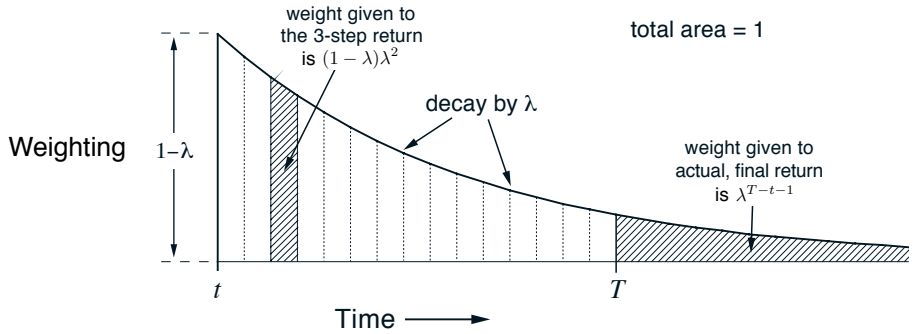


Figure 12.2: Weighting given in the λ-return to each of the n-step returns.

we want, we can separate these post-termination terms from the main sum, yielding

$$G_t^\lambda \;=\; (1-\lambda)\sum_{n=1}^{T-t-1}\lambda^{n-1}G_{t:t+n} \;+\; \lambda^{T-t-1}G_t, \tag{12.3}$$

as indicated in the figures. This equation makes it clearer what happens when $\lambda = 1$. In this case the main sum goes to zero, and the remaining term reduces to the conventional return. Thus, for $\lambda = 1$, updating according to the λ-return is a Monte Carlo algorithm. On the other hand, if $\lambda = 0$, then the λ-return reduces to $G_{t:t+1}$, the one-step return. Thus, for $\lambda = 0$, updating according to the λ-return is a one-step TD method.

Exercise 12.1 Just as the return can be written recursively in terms of the first reward and itself one-step later (3.9), so can the λ-return. Derive the analogous recursive relationship from (12.2) and (12.1). □

Exercise 12.2 The parameter λ characterizes how fast the exponential weighting in Figure 12.2 falls off, and thus how far into the future the λ-return algorithm looks in determining its update. But a rate factor such as λ is sometimes an awkward way of characterizing the speed of the decay. For some purposes it is better to specify a time constant, or half-life. What is the equation relating λ and the half-life, τ_λ, the time by which the weighting sequence will have fallen to half of its initial value? □

We are now ready to define our first learning algorithm based on the λ-return: the *offline λ-return algorithm*. As an offline algorithm, it makes no changes to the weight vector during the episode. Then, at the end of the episode, a whole sequence of offline updates are made according to our usual semi-gradient rule, using the λ-return as the target:

$$\mathbf{w}_{t+1} \doteq \mathbf{w}_t + \alpha\Big[G_t^\lambda - \hat{v}(S_t,\mathbf{w}_t)\Big]\nabla\hat{v}(S_t,\mathbf{w}_t), \quad t = 0,\ldots,T-1. \tag{12.4}$$

The λ-return gives us an alternative way of moving smoothly between Monte Carlo and one-step TD methods that can be compared with the n-step bootstrapping way developed in Chapter 7. There we assessed effectiveness on a 19-state random walk task (Example 7.1, page 144). Figure 12.3 shows the performance of the offline λ-return algorithm on this task alongside that of the n-step methods (repeated from Figure 7.2). The experiment was just as described earlier except that for the λ-return algorithm we varied λ instead of n. The performance measure used is the estimated root-mean-squared error between the correct and estimated values of each state measured at the end of the episode, averaged over the first 10 episodes and the 19 states. Note that overall performance of the offline λ-return algorithms is comparable to that of the n-step algorithms. In both cases we get best performance with an intermediate value of the bootstrapping parameter, n for n-step methods and λ for the offline λ-return algorithm.

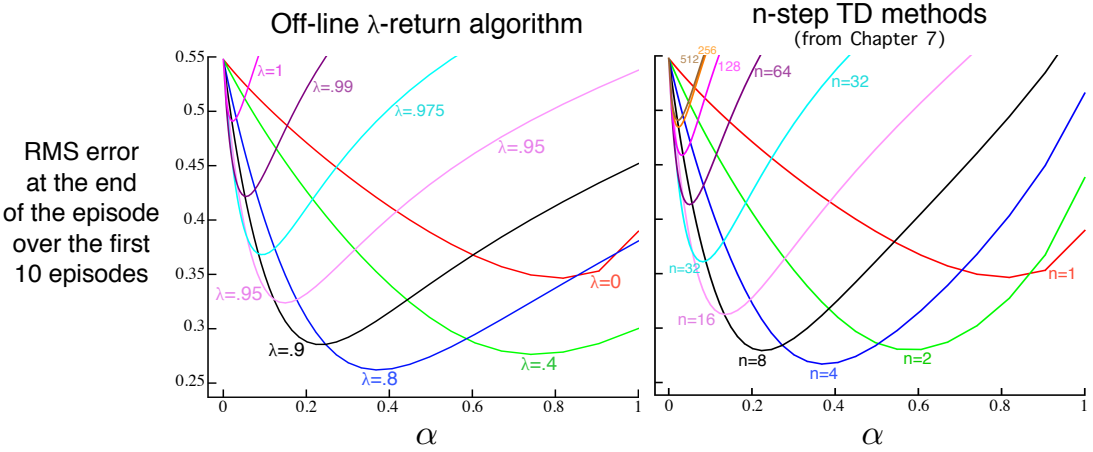


Figure 12.3: 19-state Random walk results (Example 7.1): Performance of the offline λ-return algorithm alongside that of the n-step TD methods. In both case, intermediate values of the bootstrapping parameter (λ or n) performed best. The results with the offline λ-return algorithm are slightly better at the best values of α and λ, and at high α.

The approach that we have been taking so far is what we call the theoretical, or *forward*, view of a learning algorithm. For each state visited, we look forward in time to all the future rewards and decide how best to combine them. We might imagine ourselves riding the stream of states, looking forward from each state to determine its update, as suggested by Figure 12.4. After looking forward from and updating one state, we move on to the next and never have to work with the preceding state again. Future states, on the other hand, are viewed and processed repeatedly, once from each vantage point preceding them.

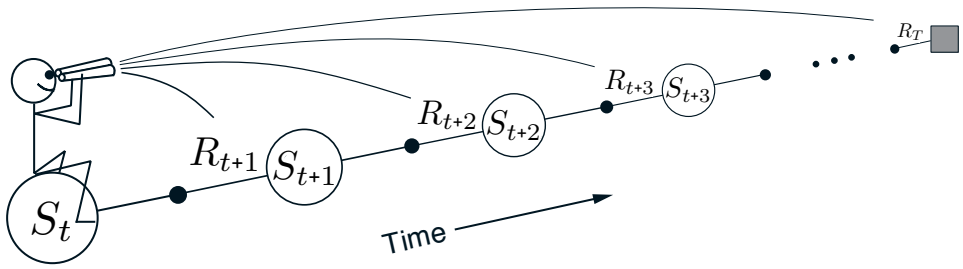


Figure 12.4: The forward view. We decide how to update each state by looking forward to future rewards and states.

12.2 TD(λ)

TD(λ) is one of the oldest and most widely used algorithms in reinforcement learning. It was the first algorithm for which a formal relationship was shown between a more theoretical forward view and a more computationally-congenial backward view using eligibility traces. Here we will show empirically that it approximates the offline λ-return algorithm presented in the previous section.

TD(λ) improves over the offline λ-return algorithm in three ways. First it updates the weight vector on every step of an episode rather than only at the end, and thus its estimates may be better sooner. Second, its computations are equally distributed in time rather than all at the end of the episode. And third, it can be applied to continuing problems rather than just to episodic problems. In this section we present the semi-gradient version of TD(λ) with function approximation.

With function approximation, the eligibility trace is a vector $\mathbf{z}_t \in \mathbb{R}^d$ with the same number of components as the weight vector $\mathbf{w}_t$. Whereas the weight vector is a long-term memory, accumulating over the lifetime of the system, the eligibility trace is a short-term memory, typically lasting less time than the length of an episode. Eligibility traces assist in the learning process; their only consequence is that they affect the weight vector, and then the weight vector determines the estimated value.

In TD(λ), the eligibility trace vector is initialized to zero at the beginning of the episode, is incremented on each time step by the value gradient, and then fades away by $\gamma\lambda$:

$$\begin{aligned} &\mathbf{z}_{-1} \doteq \mathbf{0}, \\ &\mathbf{z}_t \doteq \gamma\lambda\mathbf{z}_{t-1} + \nabla\hat{v}(S_t,\mathbf{w}_t), \quad 0 \leq t \leq T, \end{aligned} \tag{12.5}$$

where γ is the discount rate and λ is the parameter introduced in the previous section, which we henceforth call the trace-decay parameter. The eligibility trace keeps track of which components of the weight vector have contributed, positively or negatively, to recent state valuations, where "recent" is defined in terms of $\gamma\lambda$. (Recall that in linear function approximation, $\nabla\hat{v}(S_t,\mathbf{w}_t)$ is just the feature vector, $\mathbf{x}_t$, in which case the eligibility trace vector is just a sum of past, fading, input vectors.) The trace is said to indicate the eligibility of each component of the weight vector for undergoing learning

changes should a reinforcing event occur. The reinforcing events we are concerned with are the moment-by-moment one-step TD errors. The TD error for state-value prediction is

$$\delta_t \doteq R_{t+1} + \gamma \hat{v}(S_{t+1},\mathbf{w}_t) - \hat{v}(S_t,\mathbf{w}_t). \tag{12.6}$$

In TD(λ), the weight vector is updated on each step proportional to the scalar TD error and the vector eligibility trace:

$$\mathbf{w}_{t+1} \doteq \mathbf{w}_t + \alpha \delta_t \mathbf{z}_t. \tag{12.7}$$

Semi-gradient TD(λ) for estimating $\hat{v} \approx v_\pi$

Input: the policy π to be evaluated
Input: a differentiable function $\hat{v} : \mathcal{S}^+ \times \mathbb{R}^d \to \mathbb{R}$ such that $\hat{v}(\text{terminal},\cdot) = 0$
Algorithm parameters: step size $\alpha > 0$, trace decay rate $\lambda \in [0,1]$
Initialize value-function weights $\mathbf{w}$ arbitrarily (e.g., $\mathbf{w} = \mathbf{0}$)

Loop for each episode:
  Initialize S
  $\mathbf{z} \leftarrow \mathbf{0}$ (a d-dimensional vector)
  Loop for each step of episode:
  | Choose $A \sim \pi(\cdot|S)$
  | Take action A, observe R, S'
  | $\mathbf{z} \leftarrow \gamma\lambda\mathbf{z} + \nabla\hat{v}(S,\mathbf{w})$
  | $\delta \leftarrow R + \gamma\hat{v}(S',\mathbf{w}) - \hat{v}(S,\mathbf{w})$
  | $\mathbf{w} \leftarrow \mathbf{w} + \alpha\delta\mathbf{z}$
  | $S \leftarrow S'$
  until S' is terminal

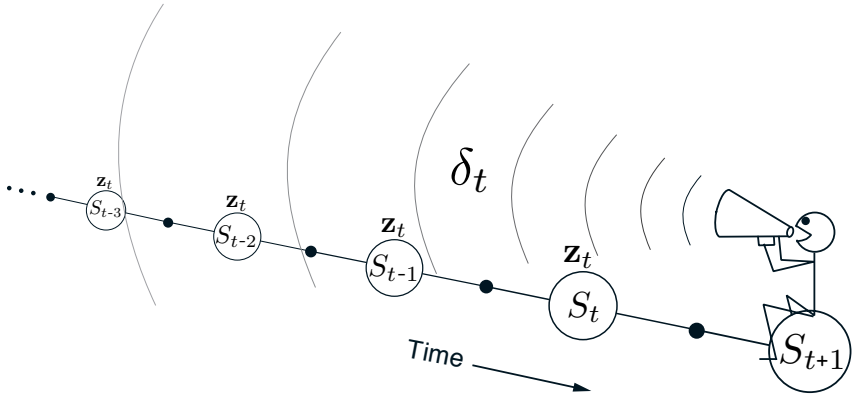


Figure 12.5: The backward or mechanistic view of TD(λ). Each update depends on the current TD error combined with the current eligibility traces of past events.

TD(λ) is oriented backward in time. At each moment we look at the current TD error and assign it backward to each prior state according to how much that state contributed to the current eligibility trace at that time. We might imagine ourselves riding along the stream of states, computing TD errors, and shouting them back to the previously visited states, as suggested by Figure 12.5. Where the TD error and traces come together, we get the update given by (12.7), changing the values of those past states for when they occur again in the future.

To better understand the backward view of TD(λ), consider what happens at various values of λ. If $\lambda = 0$, then by (12.5) the trace at t is exactly the value gradient corresponding to S_t. Thus the TD(λ) update (12.7) reduces to the one-step semi-gradient TD update treated in Chapter 9 (and, in the tabular case, to the simple TD rule (6.2)). This is why that algorithm was called TD(0). In terms of Figure 12.5, TD(0) is the case in which only the one state preceding the current one is changed by the TD error. For larger values of λ, but still $\lambda < 1$, more of the preceding states are changed, but each more temporally distant state is changed less because the corresponding eligibility trace is smaller, as suggested by the figure. We say that the earlier states are given less *credit* for the TD error.

If $\lambda = 1$, then the credit given to earlier states falls only by γ per step. This turns out to be just the right thing to do to achieve Monte Carlo behavior. For example, remember that the TD error, δ_t, includes an undiscounted term of R_{t+1}. In passing this back k steps it needs to be discounted, like any reward in a return, by γ^k, which is just what the falling eligibility trace achieves. If $\lambda = 1$ and $\gamma = 1$, then the eligibility traces do not decay at all with time. In this case the method behaves like a Monte Carlo method for an undiscounted, episodic task. If $\lambda = 1$, the algorithm is also known as TD(1).

TD(1) is a way of implementing Monte Carlo algorithms that is more general than those presented earlier and that significantly increases their range of applicability. Whereas the earlier Monte Carlo methods were limited to episodic tasks, TD(1) can be applied to discounted continuing tasks as well. Moreover, TD(1) can be performed incrementally and online. One disadvantage of Monte Carlo methods is that they learn nothing from an episode until it is over. For example, if a Monte Carlo control method takes an action that produces a very poor reward but does not end the episode, then the agent's tendency to repeat the action will be undiminished during the episode. Online TD(1), on the other hand, learns in an n-step TD way from the incomplete ongoing episode, where the n steps are all the way up to the current step. If something unusually good or bad happens during an episode, control methods based on TD(1) can learn immediately and alter their behavior on that same episode.

It is revealing to revisit the 19-state random walk example (Example 7.1) to see how well TD(λ) does in approximating the offline λ-return algorithm. The results for both algorithms are shown in Figure 12.6. For each λ value, if α is selected optimally for it (or smaller), then the two algorithms perform virtually identically. If α is chosen larger than is optimal, however, then the λ-return algorithm is only a little worse whereas TD(λ) is much worse and may even be unstable. This is not catastrophic for TD(λ) on this problem, as these higher parameter values are not what one would want to use anyway, but for other problems it can be a significant weakness.

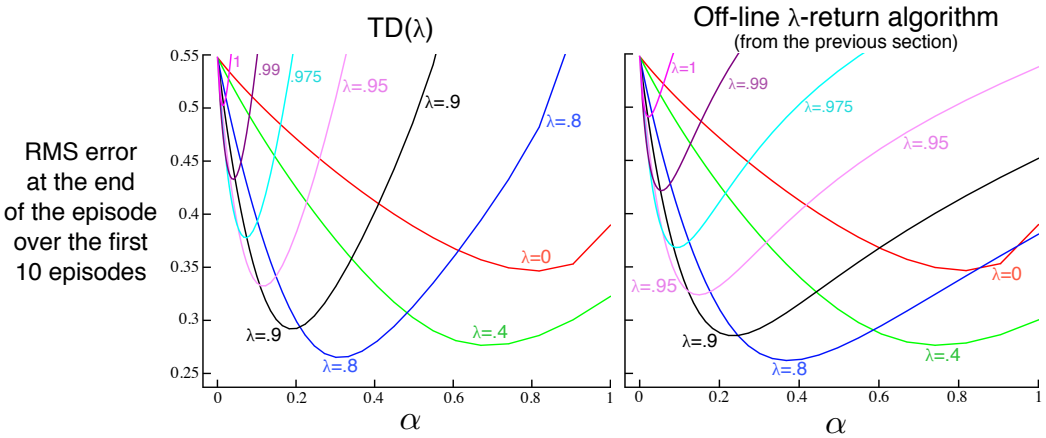


Figure 12.6: 19-state Random walk results (Example 7.1): Performance of TD(λ) alongside that of the offline λ-return algorithm. The two algorithms performed virtually identically at low (less than optimal) α values, but TD(λ) was worse at high α values.

Linear TD(λ) has been proved to converge in the on-policy case if the step-size parameter is reduced over time according to the usual conditions (2.7). Just as discussed in Section 9.4, convergence is not to the minimum-error weight vector, but to a nearby weight vector that depends on λ. The bound on solution quality presented in that section (9.14) can now be generalized to apply for any λ. For the continuing discounted case,

$$\overline{\text{VE}}(\mathbf{w}_\infty) \quad \leq \quad \frac{1-\gamma\lambda}{1-\gamma} \min_{\mathbf{w}} \overline{\text{VE}}(\mathbf{w}). \tag{12.8}$$

That is, the asymptotic error is no more than $\frac{1-\gamma\lambda}{1-\gamma}$ times the smallest possible error. As λ approaches 1, the bound approaches the minimum error (and it is loosest at $\lambda = 0$). In practice, however, $\lambda = 1$ is often the poorest choice, as will be illustrated later in Figure 12.14.

Exercise 12.3 Some insight into how TD(λ) can closely approximate the offline λ-return algorithm can be gained by seeing that the latter's error term (in brackets in (12.4)) can be written as the sum of TD errors (12.6) for a single fixed $\mathbf{w}$. Show this, following the pattern of (6.6), and using the recursive relationship for the λ-return you obtained in Exercise 12.1. □

Exercise 12.4 Use your result from the preceding exercise to show that, if the weight updates over an episode were computed on each step but not actually used to change the weights ($\mathbf{w}$ remained fixed), then the sum of TD(λ)'s weight updates would be the same as the sum of the offline λ-return algorithm's updates. □

12.3 *n*-step Truncated λ-return Methods

The offline λ-return algorithm is an important ideal, but it is of limited utility because it uses the λ-return (12.2), which is not known until the end of the episode. In the

continuing case, the λ-return is technically never known, as it depends on n-step returns for arbitrarily large n, and thus on rewards arbitrarily far in the future. However, the dependence becomes weaker for longer-delayed rewards, falling by $\gamma\lambda$ for each step of delay. A natural approximation, then, would be to truncate the sequence after some number of steps. Our existing notion of n-step returns provides a natural way to do this in which the missing rewards are replaced with estimated values.

In general, we define the *truncated λ-return* for time t, given data only up to some later horizon, h, as

$$G^{\lambda}_{t:h} \doteq (1-\lambda)\sum_{n=1}^{h-t-1}\lambda^{n-1}G_{t:t+n} + \lambda^{h-t-1}G_{t:h}, \qquad 0 \le t < h \le T. \tag{12.9}$$

If you compare this equation with the λ-return (12.3), it is clear that the horizon h is playing the same role as was previously played by T, the time of termination. Whereas in the λ-return there is a residual weight given to the conventional return G_t, here it is given to the longest available n-step return, $G_{t:h}$ (Figure 12.2).

The truncated λ-return immediately gives rise to a family of n-step λ-return algorithms similar to the n-step methods of Chapter 7. In all of these algorithms, updates are delayed by n steps and only take into account the first n rewards, but now all the k-step returns are included for $1 \le k \le n$ (whereas the earlier n-step algorithms used only the n-step return), weighted geometrically as in Figure 12.2. In the state-value case, this family of algorithms is known as truncated TD(λ), or TTD(λ). The compound backup diagram, shown in Figure 12.7, is similar to that for TD(λ) (Figure 12.1) except that the longest component update is at most n steps rather than always going all the way to the

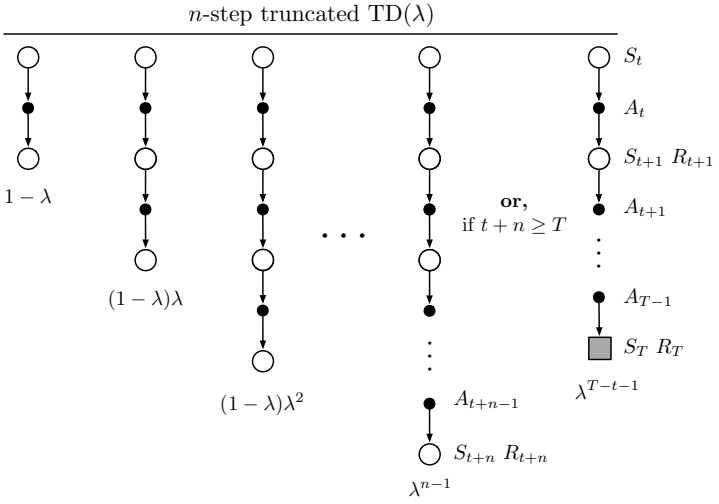


Figure 12.7: The backup diagram for truncated TD(λ).

end of the episode. TTD(λ) is defined by (cf. (9.15)):

$$\mathbf{w}_{t+n} \doteq \mathbf{w}_{t+n-1} + \alpha \left[G^{\lambda}_{t:t+n} - \hat{v}(S_t,\mathbf{w}_{t+n-1})\right] \nabla \hat{v}(S_t,\mathbf{w}_{t+n-1}), \qquad 0 \leq t < T.$$

This algorithm can be implemented efficiently so that per-step computation does not scale with n (though of course memory must). Much as in n-step TD methods, no updates are made on the first $n-1$ time steps of each episode, and $n-1$ additional updates are made upon termination. Efficient implementation relies on the fact that the k-step λ-return can be written exactly as

$$G^{\lambda}_{t:t+k} = \hat{v}(S_t,\mathbf{w}_{t-1}) + \sum_{i=t}^{t+k-1} (\gamma\lambda)^{i-t}\delta'_i, \tag{12.10}$$

where

$$\delta'_t \doteq R_{t+1} + \gamma\hat{v}(S_{t+1},\mathbf{w}_t) - \hat{v}(S_t,\mathbf{w}_{t-1}).$$

Exercise 12.5 Several times in this book (often in exercises) we have established that returns can be written as sums of TD errors if the value function is held constant. Why is (12.10) another instance of this? Prove (12.10). □

12.4 Redoing Updates: Online λ-return Algorithm

Choosing the truncation parameter n in truncated TD(λ) involves a tradeoff. n should be large so that the method closely approximates the offline λ-return algorithm, but it should also be small so that the updates can be made sooner and can influence behavior sooner. Can we get the best of both? Well, yes, in principle we can, albeit at the cost of computational complexity.

The idea is that, on each time step as you gather a new increment of data, you go back and redo all the updates since the beginning of the current episode. The new updates will be better than the ones you previously made because now they can take into account the time step's new data. That is, the updates are always towards an n-step truncated λ-return target, but they always use the latest horizon. In each pass over that episode you can use a slightly longer horizon and obtain slightly better results. Recall that the truncated λ-return is defined in (12.9) as

$$G^{\lambda}_{t:h} \doteq (1-\lambda) \sum_{n=1}^{h-t-1} \lambda^{n-1} G_{t:t+n} + \lambda^{h-t-1} G_{t:h}.$$

Let us step through how this target could ideally be used if computational complexity was not an issue. The episode begins with an estimate at time 0 using the weights $\mathbf{w}_0$ from the end of the previous episode. Learning begins when the data horizon is extended to time step 1. The target for the estimate at step 0, given the data up to horizon 1, could only be the one-step return $G_{0:1}$, which includes R_1 and bootstraps from the estimate

$\hat{v}(S_1,\mathbf{w}_0)$. Note that this is exactly what $G^{\lambda}_{0:1}$ is, with the sum in the first term of the equation degenerating to zero. Using this update target, we construct $\mathbf{w}_1$. Then, after advancing the data horizon to step 2, what do we do? We have new data in the form of R_2 and S_2, as well as the new $\mathbf{w}_1$, so now we can construct a better update target $G^{\lambda}_{0:2}$ for the first update from S_0 as well as a better update target $G^{\lambda}_{1:2}$ for the second update from S_1. Using these improved targets, we redo the updates at S_1 and S_2, starting again from $\mathbf{w}_0$, to produce $\mathbf{w}_2$. Now we advance the horizon to step 3 and repeat, going all the way back to produce three new targets, redoing all updates starting from the original $\mathbf{w}_0$ to produce $\mathbf{w}_3$, and so on. Each time the horizon is advanced, all the updates are redone starting from $\mathbf{w}_0$ using the weight vector from the preceding horizon.

This conceptual algorithm involves multiple passes over the episode, one at each horizon, each generating a different sequence of weight vectors. To describe it clearly we have to distinguish between the weight vectors computed at the different horizons. Let us use $\mathbf{w}^h_t$ to denote the weights used to generate the value at time t in the sequence up to horizon h. The first weight vector $\mathbf{w}^h_0$ in each sequence is that inherited from the previous episode (so they are the same for all h), and the last weight vector $\mathbf{w}^h_h$ in each sequence defines the ultimate weight-vector sequence of the algorithm. At the final horizon $h = T$ we obtain the final weights $\mathbf{w}^T_T$ which will be passed on to form the initial weights of the next episode. With these conventions, the three first sequences described in the previous paragraph can be given explicitly:

$$
\begin{aligned}
h = 1: \quad & \mathbf{w}^1_1 \doteq \mathbf{w}^1_0 + \alpha \left[G^{\lambda}_{0:1} - \hat{v}(S_0,\mathbf{w}^1_0)\right] \nabla \hat{v}(S_0,\mathbf{w}^1_0), \\
\\
h = 2: \quad & \mathbf{w}^2_1 \doteq \mathbf{w}^2_0 + \alpha \left[G^{\lambda}_{0:2} - \hat{v}(S_0,\mathbf{w}^2_0)\right] \nabla \hat{v}(S_0,\mathbf{w}^2_0), \\
& \mathbf{w}^2_2 \doteq \mathbf{w}^2_1 + \alpha \left[G^{\lambda}_{1:2} - \hat{v}(S_1,\mathbf{w}^2_1)\right] \nabla \hat{v}(S_1,\mathbf{w}^2_1), \\
\\
h = 3: \quad & \mathbf{w}^3_1 \doteq \mathbf{w}^3_0 + \alpha \left[G^{\lambda}_{0:3} - \hat{v}(S_0,\mathbf{w}^3_0)\right] \nabla \hat{v}(S_0,\mathbf{w}^3_0), \\
& \mathbf{w}^3_2 \doteq \mathbf{w}^3_1 + \alpha \left[G^{\lambda}_{1:3} - \hat{v}(S_1,\mathbf{w}^3_1)\right] \nabla \hat{v}(S_1,\mathbf{w}^3_1), \\
& \mathbf{w}^3_3 \doteq \mathbf{w}^3_2 + \alpha \left[G^{\lambda}_{2:3} - \hat{v}(S_2,\mathbf{w}^3_2)\right] \nabla \hat{v}(S_2,\mathbf{w}^3_2).
\end{aligned}
$$

The general form for the update is

$$\mathbf{w}^h_{t+1} \doteq \mathbf{w}^h_t + \alpha \left[G^{\lambda}_{t:h} - \hat{v}(S_t,\mathbf{w}^h_t)\right] \nabla \hat{v}(S_t,\mathbf{w}^h_t), \quad 0 \le t < h \le T.$$

This update, together with $\mathbf{w}_t \doteq \mathbf{w}^t_t$ defines the *online λ-return algorithm.*

The online λ-return algorithm is fully online, determining a new weight vector $\mathbf{w}_t$ at each step t during an episode, using only information available at time t. Its main drawback is that it is computationally complex, passing over the portion of the episode experienced so far on every step. Note that it is strictly more complex than the offline λ-return algorithm, which passes through all the steps at the time of termination but does not make any updates during the episode. In return, the online algorithm can be expected to perform better than the offline one, not only during the episode when it makes an update while the offline algorithm makes none, but also at the end of the episode because

the weight vector used in bootstrapping (in $G^{\lambda}_{t:h}$) has had a larger number of informative updates. This effect can be seen if one looks carefully at Figure 12.8, which compares the two algorithms on the 19-state random walk task.

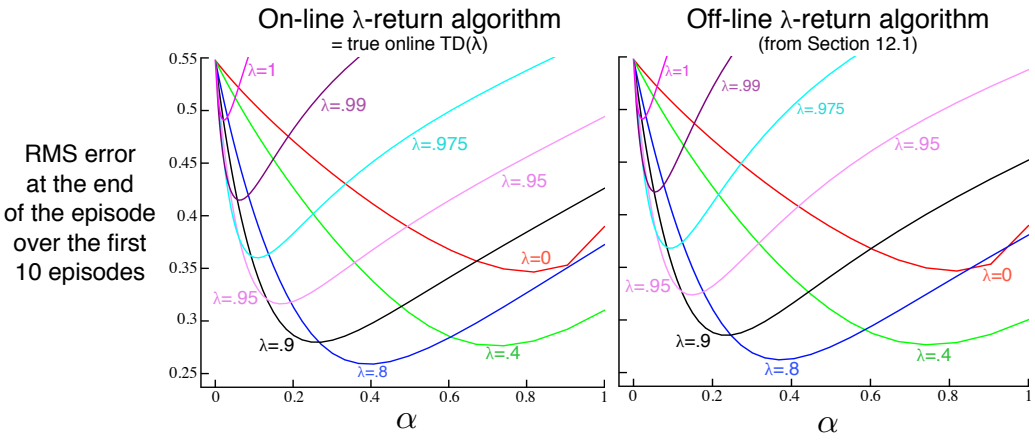


Figure 12.8: 19-state Random walk results (Example 7.1): Performance of online and offline λ-return algorithms. The performance measure here is the $\overline{\text{VE}}$ at the end of the episode, which should be the best case for the offline algorithm. Nevertheless, the online algorithm performs subtly better. For comparison, the $\lambda = 0$ line is the same for both methods.

12.5 True Online TD(λ)

The online λ-return algorithm just presented is currently the best performing temporal-difference algorithm. It is an ideal which online TD(λ) only approximates. As presented, however, the online λ-return algorithm is very complex. Is there a way to invert this forward-view algorithm to produce an efficient backward-view algorithm using eligibility traces? It turns out that there is indeed an exact computationally congenial implementation of the online λ-return algorithm for the case of linear function approximation. This implementation is known as the true online TD(λ) algorithm because it is "truer" to the ideal of the online λ-return algorithm than the TD(λ) algorithm is.

The derivation of true online TD(λ) is a little too complex to present here (see the next section and the appendix to the paper by van Seijen et al., 2016) but its strategy is simple. The sequence of weight vectors produced by the online λ-return algorithm can be arranged in a triangle:

$$
\begin{array}{ccccccc}
\mathbf{w}_0^0 & & & & & & \\
\mathbf{w}_0^1 & \mathbf{w}_1^1 & & & & & \\
\mathbf{w}_0^2 & \mathbf{w}_1^2 & \mathbf{w}_2^2 & & & & \\
\mathbf{w}_0^3 & \mathbf{w}_1^3 & \mathbf{w}_2^3 & \mathbf{w}_3^3 & & & \\
\vdots & \vdots & \vdots & \vdots & \ddots & & \\
\mathbf{w}_0^T & \mathbf{w}_1^T & \mathbf{w}_2^T & \mathbf{w}_3^T & \cdots & \mathbf{w}_T^T
\end{array}
$$

One row of this triangle is produced on each time step. It turns out that the weight vectors on the diagonal, the $\mathbf{w}_t^t$, are the only ones really needed. The first, $\mathbf{w}_0^0$, is the initial weight vector of the episode, the last, $\mathbf{w}_T^T$, is the final weight vector, and each weight vector along the way, $\mathbf{w}_t^t$, plays a role in bootstrapping in the n-step returns of the updates. In the final algorithm the diagonal weight vectors are renamed without a superscript, $\mathbf{w}_t \doteq \mathbf{w}_t^t$. The strategy then is to find a compact, efficient way of computing each $\mathbf{w}_t^t$ from the one before. If this is done, for the linear case in which $\hat{v}(s,\mathbf{w}) = \mathbf{w}^\top \mathbf{x}(s)$, then we arrive at the true online TD(λ) algorithm:

$$\mathbf{w}_{t+1} \doteq \mathbf{w}_t + \alpha\delta_t \mathbf{z}_t + \alpha\left(\mathbf{w}_t^\top \mathbf{x}_t - \mathbf{w}_{t-1}^\top \mathbf{x}_t\right)(\mathbf{z}_t - \mathbf{x}_t),$$

where we have used the shorthand $\mathbf{x}_t \doteq \mathbf{x}(S_t)$, δ_t is defined as in TD(λ) (12.6), and $\mathbf{z}_t$ is defined by

$$\mathbf{z}_t \doteq \gamma\lambda \mathbf{z}_{t-1} + \left(1 - \alpha\gamma\lambda \mathbf{z}_{t-1}^\top \mathbf{x}_t\right)\mathbf{x}_t. \tag{12.11}$$

This algorithm has been proven to produce exactly the same sequence of weight vectors, $\mathbf{w}_t, 0 \le t \le T$, as the online λ-return algorithm (van Seijen et al. 2016). Thus the results on the random walk task on the left of Figure 12.8 are also its results on that task. Now, however, the algorithm is much less expensive. The memory requirements of true online TD(λ) are identical to those of conventional TD(λ), while the per-step computation is increased by about 50% (there is one more inner product in the eligibility-trace update). Overall, the per-step computational complexity remains of $O(d)$, the same as TD(λ). Pseudocode for the complete algorithm is given in the box.

True online TD(λ) for estimating $\mathbf{w}^\top \mathbf{x} \approx v_\pi$

Input: the policy π to be evaluated
Input: a feature function $\mathbf{x} : \mathcal{S}^+ \to \mathbb{R}^d$ such that $\mathbf{x}(terminal, \cdot) = \mathbf{0}$
Algorithm parameters: step size $\alpha > 0$, trace decay rate $\lambda \in [0, 1]$
Initialize value-function weights $\mathbf{w} \in \mathbb{R}^d$ (e.g., $\mathbf{w} = \mathbf{0}$)

Loop for each episode:
  Initialize state and obtain initial feature vector $\mathbf{x}$
  $\mathbf{z} \leftarrow \mathbf{0}$ (a d-dimensional vector)
  $V_{old} \leftarrow 0$ (a temporary scalar variable)
  Loop for each step of episode:
  | Choose $A \sim \pi$
  | Take action A, observe R, $\mathbf{x}'$ (feature vector of the next state)
  | $V \leftarrow \mathbf{w}^\top \mathbf{x}$
  | $V' \leftarrow \mathbf{w}^\top \mathbf{x}'$
  | $\delta \leftarrow R + \gamma V' - V$
  | $\mathbf{z} \leftarrow \gamma\lambda \mathbf{z} + \left(1 - \alpha\gamma\lambda \mathbf{z}^\top \mathbf{x}\right)\mathbf{x}$
  | $\mathbf{w} \leftarrow \mathbf{w} + \alpha(\delta + V - V_{old})\mathbf{z} - \alpha(V - V_{old})\mathbf{x}$
  | $V_{old} \leftarrow V'$
  | $\mathbf{x} \leftarrow \mathbf{x}'$
  until $\mathbf{x}' = \mathbf{0}$ (signaling arrival at a terminal state)

The eligibility trace (12.11) used in true online TD(λ) is called a *dutch trace* to distinguish it from the trace (12.5) used in TD(λ), which is called an *accumulating trace*. Earlier work often used a third kind of trace called the *replacing trace*, defined only for the tabular case or for binary feature vectors such as those produced by tile coding. The replacing trace is defined on a component-by-component basis depending on whether the component of the feature vector was 1 or 0:

$$z_{i,t} \doteq \begin{cases} 1 & \text{if } x_{i,t} = 1 \\ \gamma\lambda z_{i,t-1} & \text{otherwise.} \end{cases} \tag{12.12}$$

Nowadays, we see replacing traces as crude approximations to dutch traces, which largely supercede them. Dutch traces usually perform better than replacing traces and have a clearer theoretical basis. Accumulating traces remain of interest for nonlinear function approximations where dutch traces are not available.

12.6 *Dutch Traces in Monte Carlo Learning

Although eligibility traces are closely associated historically with TD learning, in fact they have nothing to do with it. In fact, eligibility traces arise even in Monte Carlo learning, as we show in this section. We show that the linear MC algorithm (Chapter 9), taken as a forward view, can be used to derive an equivalent yet computationally cheaper backward-view algorithm using dutch traces. This is the only equivalence of forward- and backward-views that we explicitly demonstrate in this book. It gives some of the flavor of the proof of equivalence of true online TD(λ) and the online λ-return algorithm, but is much simpler.

The linear version of the gradient Monte Carlo prediction algorithm (page 202) makes the following sequence of updates, one for each time step of the episode:

$$\mathbf{w}_{t+1} \doteq \mathbf{w}_t + \alpha \left[G - \mathbf{w}_t^\top \mathbf{x}_t\right] \mathbf{x}_t, \quad 0 \leq t < T. \tag{12.13}$$

To simplify the example, we assume here that the return G is a single reward received at the end of the episode (this is why G is not subscripted by time) and that there is no discounting. In this case the update is also known as the Least Mean Square (LMS) rule. As a Monte Carlo algorithm, all the updates depend on the final reward/return, so none can be made until the end of the episode. The MC algorithm is an offline algorithm and we do not seek to improve this aspect of it. Rather we seek merely an implementation of this algorithm with computational advantages. We will still update the weight vector only at the end of the episode, but we will do some computation during each step of the episode and less at its end. This will give a more equal distribution of computation—$O(d)$ per step—and also remove the need to store the feature vectors at each step for use later at the end of each episode. Instead, we will introduce an additional vector memory, the eligibility trace, keeping in it a summary of all the feature vectors seen so far. This will be sufficient to efficiently recreate exactly the same overall update as the sequence of MC

updates (12.13), by the end of the episode:

$$\begin{aligned}
\mathbf{w}_T &= \mathbf{w}_{T-1} + \alpha\left(G - \mathbf{w}_{T-1}^\top \mathbf{x}_{T-1}\right)\mathbf{x}_{T-1} \\
&= \mathbf{w}_{T-1} + \alpha\mathbf{x}_{T-1}\left(-\mathbf{x}_{T-1}^\top \mathbf{w}_{T-1}\right) + \alpha G\mathbf{x}_{T-1} \\
&= \left(\mathbf{I} - \alpha\mathbf{x}_{T-1}\mathbf{x}_{T-1}^\top\right)\mathbf{w}_{T-1} + \alpha G\mathbf{x}_{T-1} \\
&= \mathbf{F}_{T-1}\mathbf{w}_{T-1} + \alpha G\mathbf{x}_{T-1}
\end{aligned}$$

where $\mathbf{F}_t \doteq \mathbf{I} - \alpha\mathbf{x}_t\mathbf{x}_t^\top$ is a *forgetting*, or *fading*, matrix. Now, recursing,

$$\begin{aligned}
&= \mathbf{F}_{T-1}\left(\mathbf{F}_{T-2}\mathbf{w}_{T-2} + \alpha G\mathbf{x}_{T-2}\right) + \alpha G\mathbf{x}_{T-1} \\
&= \mathbf{F}_{T-1}\mathbf{F}_{T-2}\mathbf{w}_{T-2} + \alpha G\left(\mathbf{F}_{T-1}\mathbf{x}_{T-2} + \mathbf{x}_{T-1}\right) \\
&= \mathbf{F}_{T-1}\mathbf{F}_{T-2}\left(\mathbf{F}_{T-3}\mathbf{w}_{T-3} + \alpha G\mathbf{x}_{T-3}\right) + \alpha G\left(\mathbf{F}_{T-1}\mathbf{x}_{T-2} + \mathbf{x}_{T-1}\right) \\
&= \mathbf{F}_{T-1}\mathbf{F}_{T-2}\mathbf{F}_{T-3}\mathbf{w}_{T-3} + \alpha G\left(\mathbf{F}_{T-1}\mathbf{F}_{T-2}\mathbf{x}_{T-3} + \mathbf{F}_{T-1}\mathbf{x}_{T-2} + \mathbf{x}_{T-1}\right) \\
&\;\;\vdots \\
&= \underbrace{\mathbf{F}_{T-1}\mathbf{F}_{T-2}\cdots\mathbf{F}_0\mathbf{w}_0}_{\mathbf{a}_{T-1}} \;+\; \underbrace{\alpha G\sum_{k=0}^{T-1}\mathbf{F}_{T-1}\mathbf{F}_{T-2}\cdots\mathbf{F}_{k+1}\mathbf{x}_k}_{\mathbf{z}_{T-1}} \\
&= \mathbf{a}_{T-1} + \alpha G\mathbf{z}_{T-1}, \qquad (12.14)
\end{aligned}$$

where $\mathbf{a}_{T-1}$ and $\mathbf{z}_{T-1}$ are the values at time $T-1$ of two auxiliary memory vectors that can be updated incrementally without knowledge of G and with $O(d)$ complexity per time step. The $\mathbf{z}_t$ vector is in fact a dutch-style eligibility trace. It is initialized to $\mathbf{z}_0 = \mathbf{x}_0$ and then updated according to

$$\begin{aligned}
\mathbf{z}_t &\doteq \sum_{k=0}^{t}\mathbf{F}_t\mathbf{F}_{t-1}\cdots\mathbf{F}_{k+1}\mathbf{x}_k, \qquad 1 \le t < T \\
&= \sum_{k=0}^{t-1}\mathbf{F}_t\mathbf{F}_{t-1}\cdots\mathbf{F}_{k+1}\mathbf{x}_k + \mathbf{x}_t \\
&= \mathbf{F}_t\sum_{k=0}^{t-1}\mathbf{F}_{t-1}\mathbf{F}_{t-2}\cdots\mathbf{F}_{k+1}\mathbf{x}_k + \mathbf{x}_t \\
&= \mathbf{F}_t\mathbf{z}_{t-1} + \mathbf{x}_t \\
&= \left(\mathbf{I} - \alpha\mathbf{x}_t\mathbf{x}_t^\top\right)\mathbf{z}_{t-1} + \mathbf{x}_t \\
&= \mathbf{z}_{t-1} - \alpha\mathbf{x}_t\mathbf{x}_t^\top\mathbf{z}_{t-1} + \mathbf{x}_t \\
&= \mathbf{z}_{t-1} - \alpha\left(\mathbf{z}_{t-1}^\top\mathbf{x}_t\right)\mathbf{x}_t + \mathbf{x}_t \\
&= \mathbf{z}_{t-1} + \left(1 - \alpha\mathbf{z}_{t-1}^\top\mathbf{x}_t\right)\mathbf{x}_t,
\end{aligned}$$

which is the dutch trace for the case of $\gamma\lambda{=}1$ (cf. Eq. 12.11). The $\mathbf{a}_t$ auxilary vector is initialized to $\mathbf{a}_0 = \mathbf{w}_0$ and then updated according to

$$\mathbf{a}_t \;\doteq\; \mathbf{F}_t\mathbf{F}_{t-1}\cdots\mathbf{F}_0\mathbf{w}_0 \;=\; \mathbf{F}_t\mathbf{a}_{t-1} \;=\; \mathbf{a}_{t-1} - \alpha\mathbf{x}_t\mathbf{x}_t^\top\mathbf{a}_{t-1}, \quad 1 \le t < T.$$

The auxiliary vectors, $\mathbf{a}_t$ and $\mathbf{z}_t$, are updated on each time step $t < T$ and then, at time T when G is observed, they are used in (12.14) to compute $\mathbf{w}_T$. In this way we achieve exactly the same final result as the MC/LMS algorithm that has poor computational properties (12.13), but now with an incremental algorithm whose time and memory complexity per step is $O(d)$. This is surprising and intriguing because the notion of an eligibility trace (and the dutch trace in particular) has arisen in a setting without temporal-difference (TD) learning (in contrast to van Seijen and Sutton, 2014). It seems eligibility traces are not specific to TD learning at all; they are more fundamental than that. The need for eligibility traces seems to arise whenever one tries to learn long-term predictions in an efficient manner.

12.7 Sarsa(λ)

Very few changes in the ideas already presented in this chapter are required in order to extend eligibility-traces to action-value methods. To learn approximate action values, $\hat{q}(s, a, \mathbf{w})$, rather than approximate state values, $\hat{v}(s,\mathbf{w})$, we need to use the action-value form of the n-step return, from Chapter 10:

$$G_{t:t+n} \doteq R_{t+1} + \cdots + \gamma^{n-1} R_{t+n} + \gamma^n \hat{q}(S_{t+n}, A_{t+n}, \mathbf{w}_{t+n-1}), \qquad t + n < T,$$

with $G_{t:t+n} \doteq G_t$ if $t + n \geq T$. Using this, we can form the action-value form of the truncated λ-return, which is otherwise identical to the state-value form (12.9). The action-value form of the offline λ-return algorithm (12.4) simply uses $\hat{q}$ rather than $\hat{v}$:

$$\mathbf{w}_{t+1} \doteq \mathbf{w}_t + \alpha \Big[G_t^\lambda - \hat{q}(S_t, A_t, \mathbf{w}_t) \Big] \nabla \hat{q}(S_t, A_t, \mathbf{w}_t), \quad t = 0, \ldots, T-1, \tag{12.15}$$

where $G_t^\lambda \doteq G_{t:\infty}^\lambda$. The compound backup diagram for this forward view is shown in Figure 12.9. Notice the similarity to the diagram of the TD(λ) algorithm (Figure 12.1). The first update looks ahead one full step, to the next state–action pair, the second looks ahead two steps, to the second state–action pair, and so on. A final update is based on the complete return. The weighting of each n-step update in the λ-return is just as in TD(λ) and the λ-return algorithm (12.3).

The temporal-difference method for action values, known as *Sarsa(λ)*, approximates this forward view. It has the same update rule as given earlier for TD(λ):

$$\mathbf{w}_{t+1} \doteq \mathbf{w}_t + \alpha \, \delta_t \, \mathbf{z}_t,$$

except, naturally, using the action-value form of the TD error:

$$\delta_t \doteq R_{t+1} + \gamma \hat{q}(S_{t+1}, A_{t+1}, \mathbf{w}_t) - \hat{q}(S_t, A_t, \mathbf{w}_t), \tag{12.16}$$

and the action-value form of the eligibility trace:

$$\begin{aligned} &\mathbf{z}_{-1} \doteq \mathbf{0}, \\ &\mathbf{z}_t \doteq \gamma \lambda \mathbf{z}_{t-1} + \nabla \hat{q}(S_t, A_t, \mathbf{w}_t), \quad 0 \leq t \leq T. \end{aligned}$$

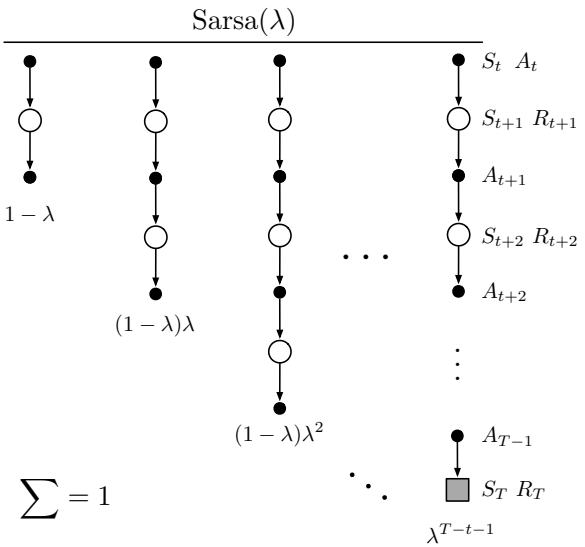
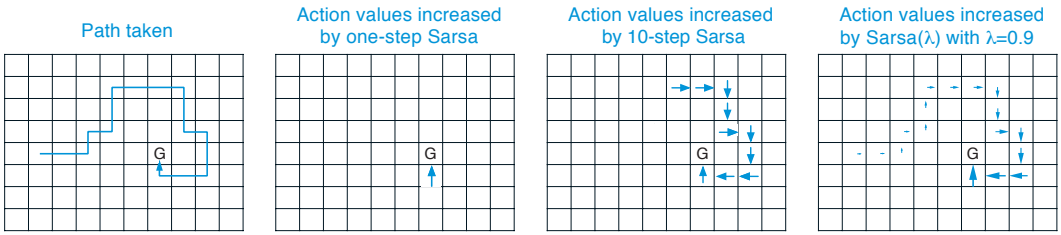


Figure 12.9: Sarsa(λ)'s backup diagram. Compare with Figure 12.1.

Complete pseudocode for Sarsa(λ) with linear function approximation, binary features, and either accumulating or replacing traces is given in the box on the next page. This pseudocode highlights a few optimizations possible in the special case of binary features (features are either active (=1) or inactive (=0).

Example 12.1: Traces in Gridworld The use of eligibility traces can substantially increase the efficiency of control algorithms over one-step methods and even over n-step methods. The reason for this is illustrated by the gridworld example below.



The first panel shows the path taken by an agent in a single episode. The initial estimated values were zero, and all rewards were zero except for a positive reward at the goal location marked by G. The arrows in the other panels show, for various algorithms, which action-values would be increased, and by how much, upon reaching the goal. A one-step method would increment only the last action value, whereas an n-step method would equally increment the last n actions' values, and an eligibility trace method would update all the action values up to the beginning of the episode, to different degrees, fading with recency. The fading strategy is often the best. ■

Sarsa(λ) with binary features and linear function approximation for estimating $\mathbf{w}^\top\mathbf{x} \approx q_\pi$ or q_*

Input: a function $\mathcal{F}(s,a)$ returning the set of (indices of) active features for s,a
Input: a policy π (if estimating q_π)
Algorithm parameters: step size $\alpha > 0$, trace decay rate $\lambda \in [0,1]$
Initialize: $\mathbf{w} = (w_1,\ldots,w_d)^\top \in \mathbb{R}^d$ (e.g., $\mathbf{w} = \mathbf{0}$), $\mathbf{z} = (z_1,\ldots,z_d)^\top \in \mathbb{R}^d$

Loop for each episode:
  Initialize S
  Choose $A \sim \pi(\cdot|S)$ or ε-greedy according to $\hat{q}(S,\cdot,\mathbf{w})$
  $\mathbf{z} \leftarrow \mathbf{0}$
  Loop for each step of episode:
    Take action A, observe R, S'
    $\delta \leftarrow R$
    Loop for i in $\mathcal{F}(S,A)$:
      $\delta \leftarrow \delta - w_i$
      $z_i \leftarrow z_i + 1$ (accumulating traces)
      or $z_i \leftarrow 1$ (replacing traces)
    If S' is terminal then:
      $\mathbf{w} \leftarrow \mathbf{w} + \alpha\delta\mathbf{z}$
      Go to next episode
    Choose $A' \sim \pi(\cdot|S')$ or near greedily $\sim \hat{q}(S',\cdot,\mathbf{w})$
    Loop for i in $\mathcal{F}(S',A')$: $\delta \leftarrow \delta + \gamma w_i$
    $\mathbf{w} \leftarrow \mathbf{w} + \alpha\delta\mathbf{z}$
    $\mathbf{z} \leftarrow \gamma\lambda\mathbf{z}$
    $S \leftarrow S'; A \leftarrow A'$

Exercise 12.6 Modify the pseudocode for Sarsa(λ) to use dutch traces (12.11) without the other distinctive features of a true online algorithm. Assume linear function approximation and binary features. □

Example 12.2: Sarsa(λ) on Mountain Car Figure 12.10 (left) on the next page shows results with Sarsa(λ) on the Mountain Car task introduced in Example 10.1. The function approximation, action selection, and environmental details were exactly as in Chapter 10, and thus it is appropriate to numerically compare these results with the Chapter 10 results for n-step Sarsa (right side of the figure). The earlier results varied the update length n whereas here for Sarsa(λ) we vary the trace parameter λ, which plays a similar role. The fading-trace bootstrapping strategy of Sarsa(λ) appears to result in more efficient learning on this problem. ■

There is also an action-value version of our ideal TD method, the online λ-return algorithm (Section 12.4) and its efficient implementation as true online TD(λ) (Section 12.5). Everything in Section 12.4 goes through without change other than to use the action-value form of the n-step return given at the beginning of the current section. The analyses in Sections 12.5 and 12.6 also carry through for action values, the only change being the use

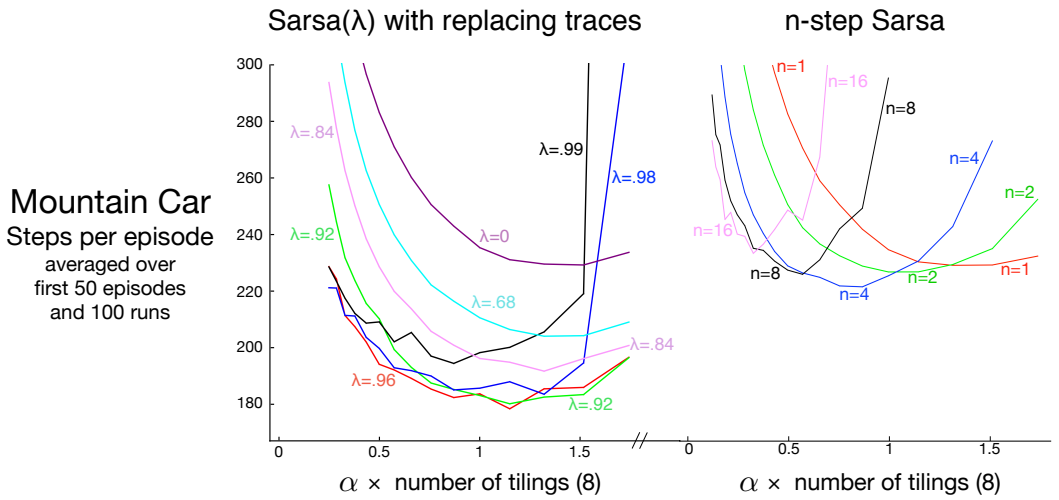


Figure 12.10: Early performance on the Mountain Car task of Sarsa(λ) with replacing traces and n-step Sarsa (copied from Figure 10.4) as a function of the step size, α.

of state–action feature vectors $\mathbf{x}_t = \mathbf{x}(S_t, A_t)$ instead of state feature vectors $\mathbf{x}_t = \mathbf{x}(S_t)$. Pseudocode for the resulting efficient algorithm, called *true online Sarsa(λ)* is given in the box on the next page. The figure below compares the performance of various versions of Sarsa(λ) on the Mountain Car example.

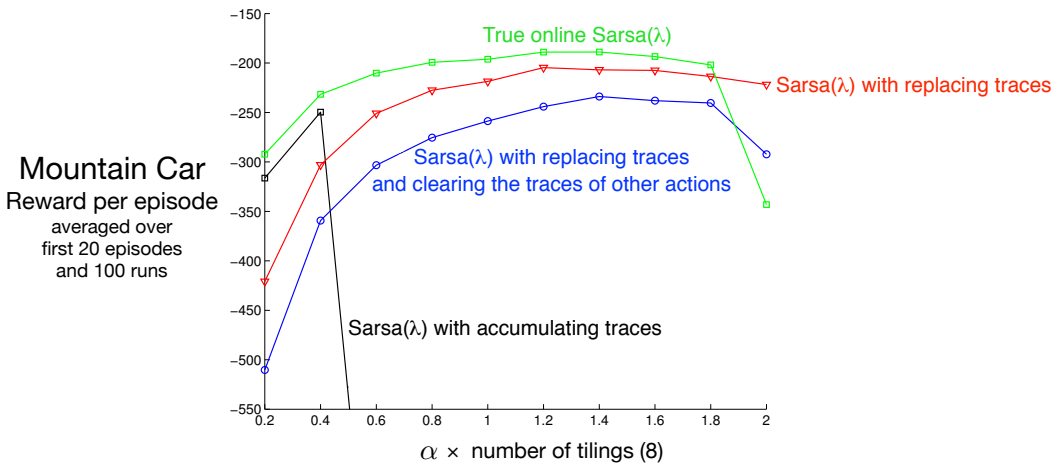


Figure 12.11: Summary comparison of Sarsa(λ) algorithms on the Mountain Car task. True online Sarsa(λ) performed better than regular Sarsa(λ) with both accumulating and replacing traces. Also included is a version of Sarsa(λ) with replacing traces in which, on each time step, the traces for the state and the actions not selected were set to zero.

True online Sarsa(λ) for estimating $\mathbf{w}^\top\mathbf{x} \approx q_\pi$ or q_*

Input: a feature function $\mathbf{x} : \mathcal{S}^+ \times \mathcal{A} \to \mathbb{R}^d$ such that $\mathbf{x}(terminal, \cdot) = \mathbf{0}$
Input: a policy π (if estimating q_π)
Algorithm parameters: step size $\alpha > 0$, trace decay rate $\lambda \in [0, 1]$
Initialize: $\mathbf{w} \in \mathbb{R}^d$ (e.g., $\mathbf{w} = \mathbf{0}$)

Loop for each episode:
  Initialize S
  Choose $A \sim \pi(\cdot|S)$ or near greedily from S using $\mathbf{w}$
  $\mathbf{x} \leftarrow \mathbf{x}(S, A)$
  $\mathbf{z} \leftarrow \mathbf{0}$
  $Q_{old} \leftarrow 0$
  Loop for each step of episode:
    Take action A, observe R, S'
    Choose $A' \sim \pi(\cdot|S')$ or near greedily from S' using $\mathbf{w}$
    $\mathbf{x}' \leftarrow \mathbf{x}(S', A')$
    $Q \leftarrow \mathbf{w}^\top\mathbf{x}$
    $Q' \leftarrow \mathbf{w}^\top\mathbf{x}'$
    $\delta \leftarrow R + \gamma Q' - Q$
    $\mathbf{z} \leftarrow \gamma\lambda\mathbf{z} + \big(1 - \alpha\gamma\lambda\mathbf{z}^\top\mathbf{x}\big)\,\mathbf{x}$
    $\mathbf{w} \leftarrow \mathbf{w} + \alpha(\delta + Q - Q_{old})\mathbf{z} - \alpha(Q - Q_{old})\mathbf{x}$
    $Q_{old} \leftarrow Q'$
    $\mathbf{x} \leftarrow \mathbf{x}'$
    $A \leftarrow A'$
  until S' is terminal

Finally, there is also a truncated version of Sarsa(λ), called *forward Sarsa(λ)* (van Seijen, 2016), which appears to be a particularly effective model-free control method for use in conjunction with multi-layer artificial neural networks.

12.8 Variable λ and γ

We are starting now to reach the end of our development of fundamental TD learning algorithms. To present the final algorithms in their most general forms, it is useful to generalize the degree of bootstrapping and discounting beyond constant parameters to functions potentially dependent on the state and action. That is, each time step will have a different λ and γ, denoted λ_t and γ_t. We change notation now so that $\lambda : \mathcal{S} \times \mathcal{A} \to [0, 1]$ is now a function from states and actions to the unit interval such that $\lambda_t \doteq \lambda(S_t, A_t)$, and similarly, $\gamma : \mathcal{S} \to [0, 1]$ is a function from states to the unit interval such that $\gamma_t \doteq \gamma(S_t)$.

Introducing the function γ, the *termination function*, is particularly significant because it changes the return, the fundamental random variable whose expectation we seek to

estimate. Now the return is defined more generally as

$$\begin{aligned} G_t &\doteq R_{t+1} + \gamma_{t+1} G_{t+1} \\ &= R_{t+1} + \gamma_{t+1} R_{t+2} + \gamma_{t+1}\gamma_{t+2} R_{t+3} + \gamma_{t+1}\gamma_{t+2}\gamma_{t+3} R_{t+4} + \cdots \\ &= \sum_{k=t}^{\infty} \left(\prod_{i=t+1}^{k} \gamma_i \right) R_{k+1}, \end{aligned} \tag{12.17}$$

where, to assure the sums are finite, we require that $\prod_{k=t}^{\infty} \gamma_k = 0$ with probability one for all t. One convenient aspect of this definition is that it enables the episodic setting and its algorithms to be presented in terms of a single stream of experience, without special terminal states, start distributions, or termination times. An erstwhile terminal state becomes a state at which $\gamma(s)=0$ and which transitions to the start distribution. In that way (and by choosing $\gamma(\cdot)$ as a constant in all other states) we can recover the classical episodic setting as a special case. State dependent termination includes other prediction cases such as *pseudo termination*, in which we seek to predict a quantity without altering the flow of the Markov process. Discounted returns can be thought of as such a quantity, in which case state dependent termination unifies the episodic and discounted-continuing cases. (The undiscounted-continuing case still needs some special treatment.)

The generalization to variable bootstrapping is not a change in the problem, like discounting, but a change in the solution strategy. The generalization affects the λ-returns for states and actions. The new state-based λ-return can be written recursively as

$$G_t^{\lambda s} \doteq R_{t+1} + \gamma_{t+1} \left((1-\lambda_{t+1}) \hat{v}(S_{t+1}, \mathbf{w}_t) + \lambda_{t+1} G_{t+1}^{\lambda s} \right), \tag{12.18}$$

where now we have added the "s" to the superscript λ to remind us that this is a return that bootstraps from state values, distinguishing it from returns that bootstrap from action values, which we present below with "a" in the superscript. This equation says that the λ-return is the first reward, undiscounted and unaffected by bootstrapping, plus possibly a second term to the extent that we are not discounting at the next state (that is, according to γ_{t+1}; recall that this is zero if the next state is terminal). To the extent that we aren't terminating at the next state, we have a second term which is itself divided into two cases depending on the degree of bootstrapping in the state. To the extent we are bootstrapping, this term is the estimated value at the state, whereas, to the extent that we not bootstrapping, the term is the λ-return for the next time step. The action-based λ-return is either the Sarsa form

$$G_t^{\lambda a} \doteq R_{t+1} + \gamma_{t+1} \Big((1-\lambda_{t+1}) \hat{q}(S_{t+1}, A_{t+1}, \mathbf{w}_t) + \lambda_{t+1} G_{t+1}^{\lambda a} \Big), \tag{12.19}$$

or the Expected Sarsa form,

$$G_t^{\lambda a} \doteq R_{t+1} + \gamma_{t+1} \Big((1-\lambda_{t+1}) \bar{V}_t(S_{t+1}) + \lambda_{t+1} G_{t+1}^{\lambda a} \Big), \tag{12.20}$$

where (7.8) is generalized to function approximation as

$$\bar{V}_t(s) \doteq \sum_a \pi(a|s) \hat{q}(s, a, \mathbf{w}_t). \tag{12.21}$$

Exercise 12.7 Generalize the three recursive equations above to their truncated versions, defining $G^{\lambda s}_{t:h}$ and $G^{\lambda a}_{t:h}$. □

12.9 *Off-policy Traces with Control Variates

The final step is to incorporate importance sampling. Unlike in the case of n-step methods, for full non-truncated λ-returns one does not have a practical option in which the importance sampling is done outside the target return. Instead, we move directly to the bootstrapping generalization of per-decision importance sampling with control variates (Section 7.4). In the state case, our final definition of the λ-return generalizes (12.18), after the model of (7.13), to

$$G^{\lambda s}_t \doteq \rho_t\Big(R_{t+1}+\gamma_{t+1}\big((1-\lambda_{t+1})\hat{v}(S_{t+1},\mathbf{w}_t)+\lambda_{t+1}G^{\lambda s}_{t+1}\big)\Big)+(1-\rho_t)\hat{v}(S_t,\mathbf{w}_t) \quad (12.22)$$

where $\rho_t = \frac{\pi(A_t|S_t)}{b(A_t|S_t)}$ is the usual single-step importance sampling ratio. Much like the other returns we have seen in this book, the truncated version of this return can be approximated simply in terms of sums of the state-based TD error,

$$\delta^s_t \doteq R_{t+1} + \gamma_{t+1}\hat{v}(S_{t+1},\mathbf{w}_t) - \hat{v}(S_t,\mathbf{w}_t), \quad (12.23)$$

as

$$G^{\lambda s}_t \approx \hat{v}(S_t,\mathbf{w}_t) + \rho_t \sum_{k=t}^{\infty} \delta^s_k \prod_{i=t+1}^{k} \gamma_i\lambda_i\rho_i \quad (12.24)$$

with the approximation becoming exact if the approximate value function does not change.

Exercise 12.8 Prove that (12.24) becomes exact if the value function does not change. To save writing, consider the case of $t = 0$, and use the notation $V_k \doteq \hat{v}(S_k,\mathbf{w})$. □

Exercise 12.9 The truncated version of the general off-policy return is denoted $G^{\lambda s}_{t:h}$. Guess the correct equation, based on (12.24). □

The above form of the λ-return (12.24) is convenient to use in a forward-view update,

$$\begin{aligned}\mathbf{w}_{t+1} &= \mathbf{w}_t + \alpha\left(G^{\lambda s}_t - \hat{v}(S_t,\mathbf{w}_t)\right)\nabla\hat{v}(S_t,\mathbf{w}_t)\\ &\approx \mathbf{w}_t + \alpha\rho_t\left(\sum_{k=t}^{\infty}\delta^s_k\prod_{i=t+1}^{k}\gamma_i\lambda_i\rho_i\right)\nabla\hat{v}(S_t,\mathbf{w}_t),\end{aligned}$$

which to the experienced eye looks like an eligibility-based TD update—the product is like an eligibility trace and it is multiplied by TD errors. But this is just one time step of a forward view. The relationship that we are looking for is that the forward-view update, summed over time, is approximately equal to a backward-view update, summed over time (this relationship is only approximate because again we ignore changes in the value

function). The sum of the forward-view update over time is

$$\begin{aligned}
\sum_{t=1}^{\infty}(\mathbf{w}_{t+1}-\mathbf{w}_t) &\approx \sum_{t=1}^{\infty}\sum_{k=t}^{\infty}\alpha\rho_t\delta_k^s\nabla\hat{v}(S_t,\mathbf{w}_t)\prod_{i=t+1}^{k}\gamma_i\lambda_i\rho_i \\
&= \sum_{k=1}^{\infty}\sum_{t=1}^{k}\alpha\rho_t\nabla\hat{v}(S_t,\mathbf{w}_t)\delta_k^s\prod_{i=t+1}^{k}\gamma_i\lambda_i\rho_i \\
&\qquad\qquad \text{(using the summation rule: } \textstyle\sum_{t=x}^{y}\sum_{k=t}^{y}=\sum_{k=x}^{y}\sum_{t=x}^{k}\text{)} \\
&= \sum_{k=1}^{\infty}\alpha\delta_k^s\sum_{t=1}^{k}\rho_t\nabla\hat{v}(S_t,\mathbf{w}_t)\prod_{i=t+1}^{k}\gamma_i\lambda_i\rho_i,
\end{aligned}$$

which would be in the form of the sum of a backward-view TD update if the entire expression from the second sum left could be written and updated incrementally as an eligibility trace, which we now show can be done. That is, we show that if this expression was the trace at time k, then we could update it from its value at time $k-1$ by:

$$\begin{aligned}
\mathbf{z}_k &= \sum_{t=1}^{k}\rho_t\nabla\hat{v}(S_t,\mathbf{w}_t)\prod_{i=t+1}^{k}\gamma_i\lambda_i\rho_i \\
&= \sum_{t=1}^{k-1}\rho_t\nabla\hat{v}(S_t,\mathbf{w}_t)\prod_{i=t+1}^{k}\gamma_i\lambda_i\rho_i \quad + \quad \rho_k\nabla\hat{v}(S_k,\mathbf{w}_k) \\
&= \gamma_k\lambda_k\rho_k\underbrace{\sum_{t=1}^{k-1}\rho_t\nabla\hat{v}(S_t,\mathbf{w}_t)\prod_{i=t+1}^{k-1}\gamma_i\lambda_i\rho_i}_{\mathbf{z}_{k-1}} \quad + \quad \rho_k\nabla\hat{v}(S_k,\mathbf{w}_k) \\
&= \rho_k\big(\gamma_k\lambda_k\mathbf{z}_{k-1}+\nabla\hat{v}(S_k,\mathbf{w}_k)\big),
\end{aligned}$$

which, changing the index from k to t, is the general accumulating trace update for state values:

$$\mathbf{z}_t \doteq \rho_t\big(\gamma_t\lambda_t\mathbf{z}_{t-1}+\nabla\hat{v}(S_t,\mathbf{w}_t)\big), \tag{12.25}$$

This eligibility trace, together with the usual semi-gradient parameter-update rule for TD(λ) (12.7), forms a general TD(λ) algorithm that can be applied to either on-policy or off-policy data. In the on-policy case, the algorithm is exactly TD(λ) because ρ_t is alway 1 and (12.25) becomes the usual accumulating trace (12.5) (extended to variable λ and γ). In the off-policy case, the algorithm often works well but, as a semi-gradient method, is not guaranteed to be stable. In the next few sections we will consider extensions of it that do guarantee stability.

A very similar series of steps can be followed to derive the off-policy eligibility traces for *action*-value methods and corresponding general Sarsa(λ) algorithms. One could start with either recursive form for the general action-based λ-return, (12.19) or (12.20), but the latter (the Expected Sarsa form) works out to be simpler. We extend (12.20) to the

off-policy case after the model of (7.14) to produce

$$\begin{aligned} G_t^{\lambda a} &\doteq R_{t+1}+\gamma_{t+1}\Big((1-\lambda_{t+1})\bar{V}_t(S_{t+1})+\lambda_{t+1}\left[\rho_{t+1}G_{t+1}^{\lambda a} + \bar{V}_t(S_{t+1}) - \rho_{t+1}\hat{q}(S_{t+1},A_{t+1},\mathbf{w}_t)\right]\Big) \\ &= R_{t+1} + \gamma_{t+1}\Big(\bar{V}_t(S_{t+1}) + \lambda_{t+1}\rho_{t+1}\left[G_{t+1}^{\lambda a} - \hat{q}(S_{t+1},A_{t+1},\mathbf{w}_t)\right]\Big) \end{aligned} \tag{12.26}$$

where $\bar{V}_t(S_{t+1})$ is as given by (12.21). Again the λ-return can be written approximately as the sum of TD errors,

$$G_t^{\lambda a} \approx \hat{q}(S_t,A_t,\mathbf{w}_t) + \sum_{k=t}^{\infty}\delta_k^a \prod_{i=t+1}^{k}\gamma_i\lambda_i\rho_i, \tag{12.27}$$

using the expectation form of the action-based TD error:

$$\delta_t^a = R_{t+1} + \gamma_{t+1}\bar{V}_t(S_{t+1}) - \hat{q}(S_t,A_t,\mathbf{w}_t). \tag{12.28}$$

As before, the approximation becomes exact if the approximate value function does not change.

Exercise 12.10 Prove that (12.27) becomes exact if the value function does not change. To save writing, consider the case of $t = 0$, and use the notation $Q_k = \hat{q}(S_k,A_k,\mathbf{w})$. Hint: Start by writing out δ_0^a and $G_0^{\lambda a}$, then $G_0^{\lambda a} - Q_0$. □

Exercise 12.11 The truncated version of the general off-policy return is denoted $G_{t:h}^{\lambda a}$. Guess the correct equation for it, based on (12.27). □

Using steps entirely analogous to those for the state case, one can write a forward-view update based on (12.27), transform the sum of the updates using the summation rule, and finally derive the following form for the eligibility trace for action values:

$$\mathbf{z}_t \doteq \gamma_t\lambda_t\rho_t\mathbf{z}_{t-1} + \nabla\hat{q}(S_t,A_t,\mathbf{w}_t). \tag{12.29}$$

This eligibility trace, together with the expectation-based TD error (12.28) and the usual semi-gradient parameter-update rule (12.7), forms an elegant, efficient Expected Sarsa(λ) algorithm that can be applied to either on-policy or off-policy data. It is probably the best algorithm of this type at the current time (though of course it is not guaranteed to be stable until combined in some way with one of the methods presented in the following sections). In the on-policy case with constant λ and γ, and the usual state-action TD error (12.16), the algorithm would be identical to the Sarsa(λ) algorithm presented in Section 12.7.

Exercise 12.12 Show in detail the steps outlined above for deriving (12.29) from (12.27). Start with the update (12.15), substitute $G_t^{\lambda a}$ from (12.26) for G_t^{λ}, then follow similar steps as led to (12.25). □

At $\lambda = 1$, these algorithms become closely related to corresponding Monte Carlo algorithms. One might expect that an exact equivalence would hold for episodic problems and offline updating, but in fact the relationship is subtler and slightly weaker than that. Under these most favorable conditions still there is not an episode by episode equivalence of updates, only of their expectations. This should not be surprising as these methods

make irrevocable updates as a trajectory unfolds, whereas true Monte Carlo methods would make no update for a trajectory if any action within it has zero probability under the target policy. In particular, all of these methods, even at $\lambda = 1$, still bootstrap in the sense that their targets depend on the current value estimates—it's just that the dependence cancels out in expected value. Whether this is a good or bad property in practice is another question. Recently, methods have been proposed that do achieve an exact equivalence (Sutton, Mahmood, Precup and van Hasselt, 2014). These methods require an additional vector of "provisional weights" that keep track of updates which have been made but may need to be retracted (or emphasized) depending on the actions taken later. The state and state–action versions of these methods are called PTD(λ) and PQ(λ) respectively, where the 'P' stands for Provisional.

The practical consequences of all these new off-policy methods have not yet been established. Undoubtedly, issues of high variance will arise as they do in all off-policy methods using importance sampling (Section 11.9).

If $\lambda < 1$, then all these off-policy algorithms involve bootstrapping and the deadly triad applies (Section 11.3), meaning that they can be guaranteed stable only for the tabular case, for state aggregation, and for other limited forms of function approximation. For linear and more-general forms of function approximation the parameter vector may diverge to infinity as in the examples in Chapter 11. As we discussed there, the challenge of off-policy learning has two parts. Off-policy eligibility traces deal effectively with the first part of the challenge, correcting for the expected value of the targets, but not at all with the second part of the challenge, having to do with the distribution of updates. Algorithmic strategies for meeting the second part of the challenge of off-policy learning with eligibility traces are summarized in Section 12.11.

Exercise 12.13 What are the dutch-trace and replacing-trace versions of off-policy eligibility traces for state-value and action-value methods? □

12.10 Watkins's Q(λ) to Tree-Backup(λ)

Several methods have been proposed over the years to extend Q-learning to eligibility traces. The original is *Watkins's Q(λ)*, which decays its eligibility traces in the usual way as long as a greedy action was taken, then cuts the traces to zero after the first non-greedy action. The backup diagram for Watkins's Q(λ) is shown in Figure 12.12. In Chapter 6, we unified Q-learning and Expected Sarsa in the off-policy version of the latter, which includes Q-learning as a special case, and generalizes it to arbitrary target policies, and in the previous section of this chapter we completed our treatment of Expected Sarsa by generalizing it to off-policy eligibility traces. In Chapter 7, however, we distinguished n-step Expected Sarsa from n-step Tree Backup, where the latter retained the property of not using importance sampling. It remains then to present the eligibility trace version of Tree Backup, which we will call *Tree-Backup(λ)*, or *TB(λ)* for short. This is arguably the true successor to Q-learning because it retains its appealing absence of importance sampling even though it can be applied to off-policy data.

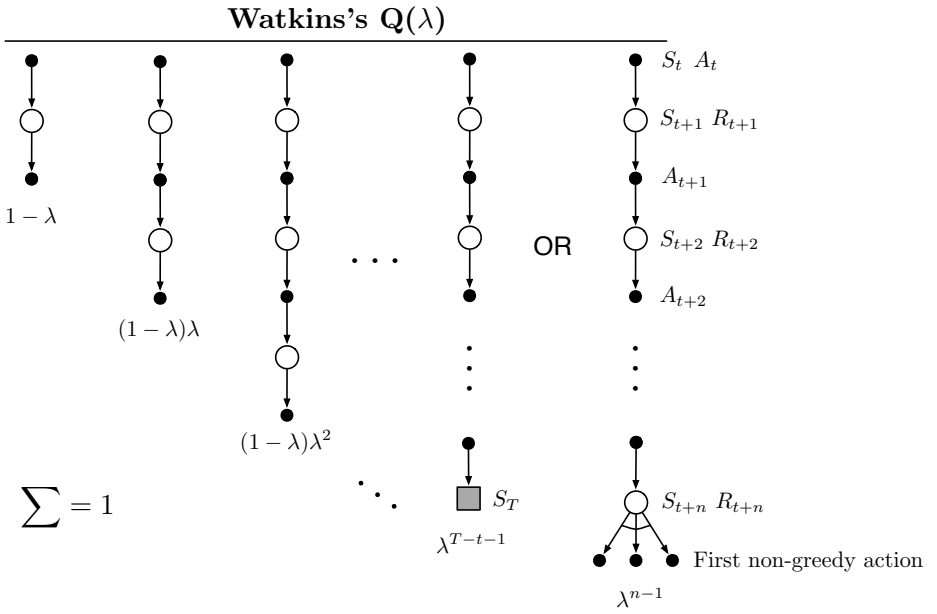


Figure 12.12: The backup diagram for Watkins's Q(λ). The series of component updates ends either with the end of the episode or with the first nongreedy action, whichever comes first.

The concept of TB(λ) is straightforward. As shown in its backup diagram in Figure 12.13, the tree-backup updates of each length (from Section 7.5) are weighted in the usual way dependent on the bootstrapping parameter λ. To get the detailed equations, with the right indices on the general bootstrapping and discounting parameters, it is best to start with a recursive form (12.20) for the λ-return using action values, and then expand the bootstrapping case of the target after the model of (7.16):

$$G_t^{\lambda a} \doteq R_{t+1} + \gamma_{t+1}\Big((1-\lambda_{t+1})\bar{V}_t(S_{t+1}) + \lambda_{t+1}\Big[\sum_{a \neq A_{t+1}} \pi(a|S_{t+1})\hat{q}(S_{t+1}, a, \mathbf{w}_t) + \pi(A_{t+1}|S_{t+1})G_{t+1}^{\lambda a}\Big]\Big)$$

$$= R_{t+1} + \gamma_{t+1}\Big(\bar{V}_t(S_{t+1}) + \lambda_{t+1}\pi(A_{t+1}|S_{t+1})\Big(G_{t+1}^{\lambda a} - \hat{q}(S_{t+1}, A_{t+1}, \mathbf{w}_t)\Big)\Big)$$

As per the usual pattern, it can also be written approximately (ignoring changes in the approximate value function) as a sum of TD errors,

$$G_t^{\lambda a} \approx \hat{q}(S_t, A_t, \mathbf{w}_t) + \sum_{k=t}^{\infty} \delta_k^a \prod_{i=t+1}^{k} \gamma_i \lambda_i \pi(A_i|S_i),$$

using the expectation form of the action-based TD error (12.28).

Following the same steps as in the previous section, we arrive at a special eligibility trace update involving the target-policy probabilities of the selected actions,

$$\mathbf{z}_t \doteq \gamma_t \lambda_t \pi(A_t|S_t)\mathbf{z}_{t-1} + \nabla \hat{q}(S_t, A_t, \mathbf{w}_t).$$

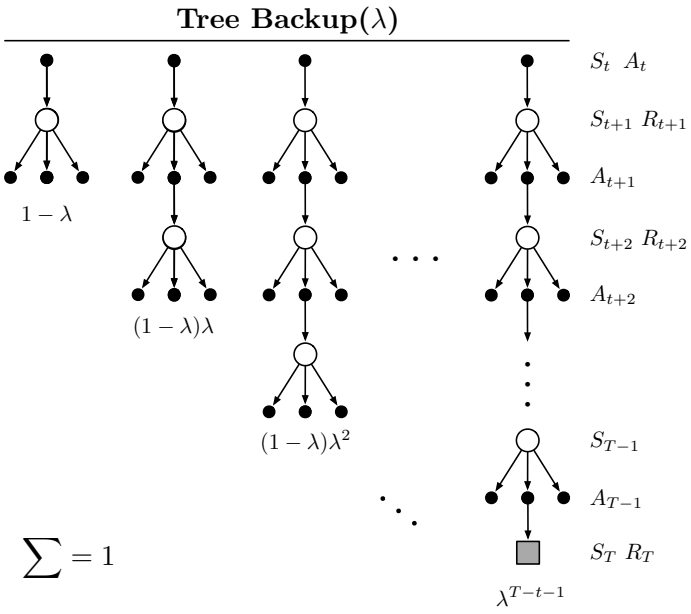


Figure 12.13: The backup diagram for the λ version of the Tree Backup algorithm.

This, together with the usual parameter-update rule (12.7), defines the TB(λ) algorithm. Like all semi-gradient algorithms, TB(λ) is not guaranteed to be stable when used with off-policy data and with a powerful function approximator. To obtain those assurances, TB(λ) would have to be combined with one of the methods presented in the next section.

**Exercise 12.14* How might Double Expected Sarsa be extended to eligibility traces? □

12.11 Stable Off-policy Methods with Traces

Several methods using eligibility traces have been proposed that achieve guarantees of stability under off-policy training, and here we present four of the most important using this book's standard notation, including general bootstrapping and discounting functions. All are based on either the Gradient-TD or the Emphatic-TD ideas presented in Sections 11.7 and 11.8. All the algorithms assume linear function approximation, though extensions to nonlinear function approximation can also be found in the literature.

GTD(λ) is the eligibility-trace algorithm analogous to TDC, the better of the two state-value Gradient-TD prediction algorithms discussed in Section 11.7. Its goal is to learn a parameter $\mathbf{w}_t$ such that $\hat{v}(s,\mathbf{w}) \doteq \mathbf{w}_t^\top \mathbf{x}(s) \approx v_\pi(s)$, even from data that is due to following another policy b. Its update is

$$\mathbf{w}_{t+1} \doteq \mathbf{w}_t + \alpha \delta_t^s \mathbf{z}_t - \alpha\gamma_{t+1}(1-\lambda_{t+1})\left(\mathbf{z}_t^\top \mathbf{v}_t\right)\mathbf{x}_{t+1},$$

with δ_t^s, $\mathbf{z}_t$, and ρ_t defined in the usual ways for state values (12.23) (12.25) (11.1), and

$$\mathbf{v}_{t+1} \doteq \mathbf{v}_t + \beta\delta_t^s \mathbf{z}_t - \beta\left(\mathbf{v}_t^\top \mathbf{x}_t\right)\mathbf{x}_t, \tag{12.30}$$

where, as in Section 11.7, $\mathbf{v} \in \mathbb{R}^d$ is a vector of the same dimension as $\mathbf{w}$, initialized to $\mathbf{v}_0 = \mathbf{0}$, and $\beta > 0$ is a second step-size parameter.

GQ(λ) is the Gradient-TD algorithm for action values with eligibility traces. Its goal is to learn a parameter $\mathbf{w}_t$ such that $\hat{q}(s, a, \mathbf{w}_t) \doteq \mathbf{w}_t^\top \mathbf{x}(s, a) \approx q_\pi(s, a)$ from off-policy data. If the target policy is ε-greedy, or otherwise biased toward the greedy policy for $\hat{q}$, then GQ(λ) can be used as a control algorithm. Its update is

$$\mathbf{w}_{t+1} \doteq \mathbf{w}_t + \alpha\delta_t^a \mathbf{z}_t - \alpha\gamma_{t+1}(1-\lambda_{t+1})\left(\mathbf{z}_t^\top \mathbf{v}_t\right)\bar{\mathbf{x}}_{t+1},$$

where $\bar{\mathbf{x}}_t$ is the average feature vector for S_t under the target policy,

$$\bar{\mathbf{x}}_t \doteq \sum_a \pi(a|S_t)\mathbf{x}(S_t, a),$$

δ_t^a is the expectation form of the TD error, which can be written

$$\delta_t^a \doteq R_{t+1} + \gamma_{t+1}\mathbf{w}_t^\top \bar{\mathbf{x}}_{t+1} - \mathbf{w}_t^\top \mathbf{x}_t,$$

$\mathbf{z}_t$ is defined in the usual way for action values (12.29), and the rest is as in GTD(λ), including the update for $\mathbf{v}_t$ (12.30).

HTD(λ) is a hybrid state-value algorithm combining aspects of GTD(λ) and TD(λ). Its most appealing feature is that it is a strict generalization of TD(λ) to off-policy learning, meaning that if the behavior policy happens to be the same as the target policy, then HTD(λ) becomes the same as TD(λ), which is not true for GTD(λ). This is appealing because TD(λ) is often faster than GTD(λ) when both algorithms converge, and TD(λ) requires setting only a single step size. HTD(λ) is defined by

$$\begin{aligned}
\mathbf{w}_{t+1} &\doteq \mathbf{w}_t + \alpha\delta_t^s \mathbf{z}_t + \alpha\left((\mathbf{z}_t - \mathbf{z}_t^b)^\top \mathbf{v}_t\right)(\mathbf{x}_t - \gamma_{t+1}\mathbf{x}_{t+1}),\\
\mathbf{v}_{t+1} &\doteq \mathbf{v}_t + \beta\delta_t^s \mathbf{z}_t - \beta\left(\mathbf{z}_t^{b\top} \mathbf{v}_t\right)(\mathbf{x}_t - \gamma_{t+1}\mathbf{x}_{t+1}), \qquad \text{with } \mathbf{v}_0 \doteq \mathbf{0},\\
\mathbf{z}_t &\doteq \rho_t\big(\gamma_t\lambda_t \mathbf{z}_{t-1} + \mathbf{x}_t\big), \qquad \text{with } \mathbf{z}_{-1} \doteq \mathbf{0},\\
\mathbf{z}_t^b &\doteq \gamma_t\lambda_t \mathbf{z}_{t-1}^b + \mathbf{x}_t, \qquad \text{with } \mathbf{z}_{-1}^b \doteq \mathbf{0},
\end{aligned}$$

where $\beta > 0$ again is a second step-size parameter. In addition to the second set of weights, $\mathbf{v}_t$, HTD(λ) also has a second set of eligibility traces, $\mathbf{z}_t^b$. These are conventional accumulating eligibility traces for the behavior policy and become equal to $\mathbf{z}_t$ if all the ρ_t are 1, which causes the last term in the $\mathbf{w}_t$ update to be zero and the overall update to reduce to TD(λ).

Emphatic TD(λ) is the extension of the one-step Emphatic-TD algorithm (Sections 9.11 and 11.8) to eligibility traces. The resultant algorithm retains strong off-policy convergence guarantees while enabling any degree of bootstrapping, albeit at the cost of

high variance and potentially slow convergence. Emphatic TD(λ) is defined by

$$\begin{aligned}
\mathbf{w}_{t+1} &\doteq \mathbf{w}_t + \alpha\delta_t\mathbf{z}_t \\
\delta_t &\doteq R_{t+1} + \gamma_{t+1}\mathbf{w}_t^\top\mathbf{x}_{t+1} - \mathbf{w}_t^\top\mathbf{x}_t \\
\mathbf{z}_t &\doteq \rho_t\big(\gamma_t\lambda_t\mathbf{z}_{t-1} + M_t\mathbf{x}_t\big), \qquad \text{with } \mathbf{z}_{-1} \doteq \mathbf{0}, \\
M_t &\doteq \lambda_t I_t + (1-\lambda_t)F_t \\
F_t &\doteq \rho_{t-1}\gamma_t F_{t-1} + I_t, \qquad \text{with } F_0 \doteq i(S_0),
\end{aligned}$$

where $M_t \geq 0$ is the general form of *emphasis*, $F_t \geq 0$ is termed the *followon trace*, and $I_t \geq 0$ is the *interest*, as described in Section 11.8. Note that M_t, like δ_t, is not really an additional memory variable. It can be removed from the algorithm by substituting its definition into the eligibility-trace equation. Pseudocode and software for the true online version of Emphatic-TD(λ) are available on the web (Sutton, 2015b).

In the on-policy case ($\rho_t = 1$, for all t), Emphatic-TD(λ) is similar to conventional TD(λ), but still significantly different. In fact, whereas Emphatic-TD(λ) is guaranteed to converge for all state-dependent λ functions, TD(λ) is not. TD(λ) is guaranteed convergent only for all constant λ. See Yu's counterexample (Ghiassian, Rafiee, and Sutton, 2016).

12.12 Implementation Issues

It might at first appear that tabular methods using eligibility traces are much more complex than one-step methods. A naive implementation would require every state (or state–action pair) to update both its value estimate and its eligibility trace on every time step. This would not be a problem for implementations on single-instruction, multiple-data, parallel computers or in plausible artificial neural network (ANN) implementations, but it is a problem for implementations on conventional serial computers. Fortunately, for typical values of λ and γ the eligibility traces of almost all states are almost always nearly zero; only those states that have recently been visited will have traces significantly greater than zero and only these few states need to be updated to closely approximate these algorithms.

In practice, then, implementations on conventional computers may keep track of and update only the few traces that are significantly greater than zero. Using this trick, the computational expense of using traces in tabular methods is typically just a few times that of a one-step method. The exact multiple of course depends on λ and γ and on the expense of the other computations. Note that the tabular case is in some sense the worst case for the computational complexity of eligibility traces. When function approximation is used, the computational advantages of not using traces generally decrease. For example, if ANNs and backpropagation are used, then eligibility traces generally cause only a doubling of the required memory and computation per step. Truncated λ-return methods (Section 12.3) can be computationally efficient on conventional computers though they always require some additional memory.

12.13 Conclusions

Eligibility traces in conjunction with TD errors provide an efficient, incremental way of shifting and choosing between Monte Carlo and TD methods. The n-step methods of Chapter 7 also enabled this, but eligibility trace methods are more general, often faster to learn, and offer different computational complexity tradeoffs. This chapter has offered an introduction to the elegant, emerging theoretical understanding of eligibility traces for on- and off-policy learning and for variable bootstrapping and discounting. One aspect of this elegant theory is true online methods, which exactly reproduce the behavior of expensive ideal methods while retaining the computational congeniality of conventional TD methods. Another aspect is the possibility of derivations that automatically convert from intuitive forward-view methods to more efficient incremental backward-view algorithms. We illustrated this general idea in a derivation that started with a classical, expensive Monte Carlo algorithm and ended with a cheap incremental non-TD implementation using the same novel eligibility trace used in true online TD methods.

As we mentioned in Chapter 5, Monte Carlo methods may have advantages in non-Markov tasks because they do not bootstrap. Because eligibility traces make TD methods more like Monte Carlo methods, they also can have advantages in these cases. If one wants to use TD methods because of their other advantages, but the task is at least partially non-Markov, then the use of an eligibility trace method is indicated. Eligibility traces are the first line of defense against both long-delayed rewards and non-Markov tasks.

By adjusting λ, we can place eligibility trace methods anywhere along a continuum from Monte Carlo to one-step TD methods. Where shall we place them? We do not yet have a good theoretical answer to this question, but a clear empirical answer appears to be emerging. On tasks with many steps per episode, or many steps within the half-life of discounting, it appears significantly better to use eligibility traces than not to (e.g., see Figure 12.14). On the other hand, if the traces are so long as to produce a pure Monte Carlo method, or nearly so, then performance degrades sharply. An intermediate mixture appears to be the best choice. Eligibility traces should be used to bring us toward Monte Carlo methods, but not all the way there. In the future it may be possible to more finely vary the trade-off between TD and Monte Carlo methods by using variable λ, but at present it is not clear how this can be done reliably and usefully.

Methods using eligibility traces require more computation than one-step methods, but in return they offer significantly faster learning, particularly when rewards are delayed by many steps. Thus it often makes sense to use eligibility traces when data are scarce and cannot be repeatedly processed, as is often the case in online applications. On the other hand, in offline applications in which data can be generated cheaply, perhaps from an inexpensive simulation, then it often does not pay to use eligibility traces. In these cases the objective is not to get more out of a limited amount of data, but simply to process as much data as possible as quickly as possible. In these cases the speedup per datum due to traces is typically not worth their computational cost, and one-step methods are favored.

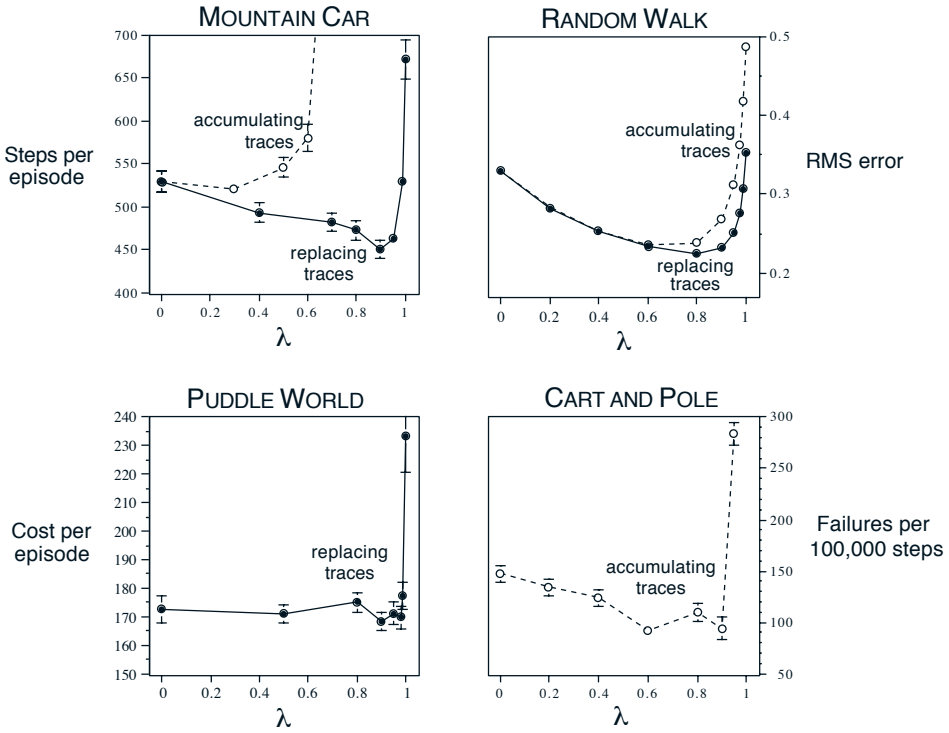


Figure 12.14: The effect of λ on reinforcement learning performance in four different test problems. In all cases, performance is generally best (a *lower* number in the graph) at an intermediate value of λ. The two left panels are applications to simple continuous-state control tasks using the Sarsa(λ) algorithm and tile coding, with either replacing or accumulating traces (Sutton, 1996). The upper-right panel is for policy evaluation on a random walk task using TD(λ) (Singh and Sutton, 1996). The lower right panel is unpublished data for the pole-balancing task (Example 3.4) from an earlier study (Sutton, 1984).

Bibliographical and Historical Remarks

Eligibility traces came into reinforcement learning via the fecund ideas of Klopf (1972). Our use of eligibility traces is based on Klopf's work (Sutton, 1978a, 1978b, 1978c; Barto and Sutton, 1981a, 1981b; Sutton and Barto, 1981a; Barto, Sutton, and Anderson, 1983; Sutton, 1984). We may have been the first to use the term "eligibility trace" (Sutton and Barto, 1981a). The idea that stimuli produce after effects in the nervous system that are important for learning is very old (see Chapter 14). Some of the earliest uses of eligibility traces were in the actor–critic methods discussed in Chapter 13 (Barto, Sutton, and Anderson, 1983; Sutton, 1984).

12.1 Compound updates were called "complex backups" in the first edition of this book.

The λ-return and its error-reduction properties were introduced by Watkins (1989) and further developed by Jaakkola, Jordan, and Singh (1994). The random walk results in this and subsequent sections are new to this text, as are the terms "forward view" and "backward view." The notion of a λ-return algorithm was introduced in the first edition of this text. The more refined treatment presented here was developed in conjunction with Harm van Seijen (e.g., van Seijen and Sutton, 2014).

12.2 TD(λ) with accumulating traces was introduced by Sutton (1988, 1984). Convergence in the mean was proved by Dayan (1992), and with probability 1 by many researchers, including Peng (1993), Dayan and Sejnowski (1994), Tsitsiklis (1994), and Gurvits, Lin, and Hanson (1994). The bound on the error of the asymptotic λ-dependent solution of linear TD(λ) is due to Tsitsiklis and Van Roy (1997).

12.3 Truncated TD methods were developed by Cichosz (1995) and van Seijen (2016).

12.4 The idea of redoing updates was extensively developed by van Seijen, originally under the name "best-match learning" (van Seijen, 2011; van Seijen, Whiteson, van Hasselt, and Weiring, 2011).

12.5 True online TD(λ) is primarily due to Harm van Seijen (van Seijen and Sutton, 2014; van Seijen et al., 2016) though some of its key ideas were discovered independently by Hado van Hasselt (personal communication). The name "dutch traces" is in recognition of the contributions of both scientists.

Replacing traces are due to Singh and Sutton (1996).

12.6 The material in this section is from van Hasselt and Sutton (2015).

12.7 Sarsa(λ) with accumulating traces was first explored as a control method by Rummery and Niranjan (1994; Rummery, 1995). True online Sarsa(λ) was introduced by van Seijen and Sutton (2014). The algorithm on page 307 was

adapted from van Seijen et al. (2016). The Mountain Car results were made for this text, except for Figure 12.11 which is adapted from van Seijen and Sutton (2014).

12.8 Perhaps the first published discussion of variable λ was by Watkins (1989), who pointed out that the cutting off of the update sequence (Figure 12.12) in his Q(λ) when a nongreedy action was selected could be implemented by temporarily setting λ to 0.

Variable λ was introduced in the first edition of this text. The roots of variable γ are in the work on options (Sutton, Precup, and Singh, 1999) and its precursors (Sutton, 1995a), becoming explicit in the GQ(λ) paper (Maei and Sutton, 2010), which also introduced some of these recursive forms for the λ-returns.

A different notion of variable λ has been developed by Yu (2012).

12.9 Off-policy eligibility traces were introduced by Precup et al. (2000, 2001), then further developed by Bertsekas and Yu (2009), Maei (2011; Maei and Sutton, 2010), Yu (2012), and by Sutton, Mahmood, Precup, and van Hasselt (2014). The last reference in particular gives a powerful forward view for off-policy TD methods with general state-dependent λ and γ. The presentation here seems to be new.

This section ends with an elegant Expected Sarsa(λ) algorithm. Although it is a natural algorithm, to our knowledge it has not previously been described or tested in the literature.

12.10 Watkins's Q(λ) is due to Watkins (1989). The tabular, episodic, offline version has been proven convergent by Munos, Stepleton, Harutyunyan, and Bellemare (2016). Alternative Q(λ) algorithms were proposed by Peng and Williams (1994, 1996) and by Sutton, Mahmood, Precup, and van Hasselt (2014). Tree Backup(λ) is due to Precup, Sutton, and Singh (2000).

12.11 GTD(λ) is due to Maei (2011). GQ(λ) is due to Maei and Sutton (2010). HTD(λ) is due to White and White (2016) based on the one-step HTD algorithm introduced by Hackman (2012). The latest developments in the theory of Gradient-TD methods are by Yu (2017). Emphatic TD(λ) was introduced by Sutton, Mahmood, and White (2016), who proved its stability. Yu (2015, 2016) proved its convergence, and the algorithm was developed further by Hallak et al. (2015, 2016).

Chapter 13

Policy Gradient Methods

In this chapter we consider something new. So far in this book almost all the methods have been *action-value methods*; they learned the values of actions and then selected actions based on their estimated action values[1]; their policies would not even exist without the action-value estimates. In this chapter we consider methods that instead learn a *parameterized policy* that can select actions without consulting a value function. A value function may still be used to *learn* the policy parameter, but is not required for action selection. We use the notation $\boldsymbol{\theta} \in \mathbb{R}^{d'}$ for the policy's parameter vector. Thus we write $\pi(a|s, \boldsymbol{\theta}) = \Pr\{A_t = a \mid S_t = s, \boldsymbol{\theta}_t = \boldsymbol{\theta}\}$ for the probability that action a is taken at time t given that the environment is in state s at time t with parameter $\boldsymbol{\theta}$. If a method uses a learned value function as well, then the value function's weight vector is denoted $\mathbf{w} \in \mathbb{R}^d$ as usual, as in $\hat{v}(s,\mathbf{w})$.

In this chapter we consider methods for learning the policy parameter based on the gradient of some scalar performance measure $J(\boldsymbol{\theta})$ with respect to the policy parameter. These methods seek to *maximize* performance, so their updates approximate gradient *ascent* in J:

$$\boldsymbol{\theta}_{t+1} = \boldsymbol{\theta}_t + \alpha \widehat{\nabla J(\boldsymbol{\theta}_t)}, \tag{13.1}$$

where $\widehat{\nabla J(\boldsymbol{\theta}_t)} \in \mathbb{R}^{d'}$ is a stochastic estimate whose expectation approximates the gradient of the performance measure with respect to its argument $\boldsymbol{\theta}_t$. All methods that follow this general schema we call *policy gradient methods*, whether or not they also learn an approximate value function. Methods that learn approximations to both policy and value functions are often called *actor–critic methods*, where 'actor' is a reference to the learned policy, and 'critic' refers to the learned value function, usually a state-value function. First we treat the episodic case, in which performance is defined as the value of the start state under the parameterized policy, before going on to consider the continuing case, in which performance is defined as the average reward rate, as in Section 10.3. In the end, we are able to express the algorithms for both cases in very similar terms.

[1] The lone exception is the gradient bandit algorithms of Section 2.8. In fact, that section goes through many of the same steps, in the single-state bandit case, as we go through here for full MDPs. Reviewing that section would be good preparation for fully understanding this chapter.

13.1 Policy Approximation and its Advantages

In policy gradient methods, the policy can be parameterized in any way, as long as $\pi(a|s,\boldsymbol{\theta})$ is differentiable with respect to its parameters, that is, as long as $\nabla\pi(a|s,\boldsymbol{\theta})$ (the column vector of partial derivatives of $\pi(a|s,\boldsymbol{\theta})$ with respect to the components of $\boldsymbol{\theta}$) exists and is finite for all $s \in \mathcal{S}$, $a \in \mathcal{A}(s)$, and $\boldsymbol{\theta} \in \mathbb{R}^{d'}$. In practice, to ensure exploration we generally require that the policy never becomes deterministic (i.e., that $\pi(a|s,\boldsymbol{\theta}) \in (0,1)$, for all $s, a, \boldsymbol{\theta}$). In this section we introduce the most common parameterization for discrete action spaces and point out the advantages it offers over action-value methods. Policy-based methods also offer useful ways of dealing with continuous action spaces, as we describe later in Section 13.7.

If the action space is discrete and not too large, then a natural and common kind of parameterization is to form parameterized numerical preferences $h(s,a,\boldsymbol{\theta}) \in \mathbb{R}$ for each state–action pair. The actions with the highest preferences in each state are given the highest probabilities of being selected, for example, according to an exponential soft-max distribution:

$$\pi(a|s,\boldsymbol{\theta}) \doteq \frac{e^{h(s,a,\boldsymbol{\theta})}}{\sum_b e^{h(s,b,\boldsymbol{\theta})}}, \tag{13.2}$$

where $e \approx 2.71828$ is the base of the natural logarithm. Note that the denominator here is just what is required so that the action probabilities in each state sum to one. We call this kind of policy parameterization *soft-max in action preferences*.

The action preferences themselves can be parameterized arbitrarily. For example, they might be computed by a deep artificial neural network (ANN), where $\boldsymbol{\theta}$ is the vector of all the connection weights of the network (as in the AlphaGo system described in Section 16.6). Or the preferences could simply be linear in features,

$$h(s,a,\boldsymbol{\theta}) = \boldsymbol{\theta}^\top \mathbf{x}(s,a), \tag{13.3}$$

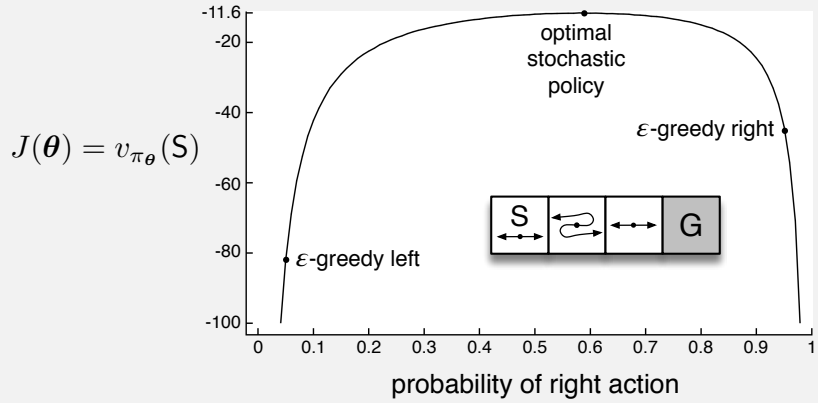
using feature vectors $\mathbf{x}(s,a) \in \mathbb{R}^{d'}$ constructed by any of the methods described in Chapter 9.

One advantage of parameterizing policies according to the soft-max in action preferences is that the approximate policy can approach a deterministic policy, whereas with ε-greedy action selection over action values there is always an ε probability of selecting a random action. Of course, one could select according to a soft-max distribution based on action values, but this alone would not allow the policy to approach a deterministic policy. Instead, the action-value estimates would converge to their corresponding true values, which would differ by a finite amount, translating to specific probabilities other than 0 and 1. If the soft-max distribution included a temperature parameter, then the temperature could be reduced over time to approach determinism, but in practice it would be difficult to choose the reduction schedule, or even the initial temperature, without more prior knowledge of the true action values than we would like to assume. Action preferences are different because they do not approach specific values; instead they are driven to produce the optimal stochastic policy. If the optimal policy is deterministic, then the preferences of the optimal actions will be driven infinitely higher than all suboptimal actions (if permitted by the parameterization).

A second advantage of parameterizing policies according to the soft-max in action preferences is that it enables the selection of actions with arbitrary probabilities. In problems with significant function approximation, the best approximate policy may be stochastic. For example, in card games with imperfect information the optimal play is often to do two different things with specific probabilities, such as when bluffing in Poker. Action-value methods have no natural way of finding stochastic optimal policies, whereas policy approximating methods can, as shown in Example 13.1.

Example 13.1 Short corridor with switched actions

Consider the small corridor gridworld shown inset in the graph below. The reward is -1 per step, as usual. In each of the three nonterminal states there are only two actions, right and left. These actions have their usual consequences in the first and third states (left causes no movement in the first state), but in the second state they are reversed, so that right moves to the left and left moves to the right. The problem is difficult because all the states appear identical under the function approximation. In particular, we define $\mathbf{x}(s, \text{right}) = [1, 0]^\top$ and $\mathbf{x}(s, \text{left}) = [0, 1]^\top$, for all s. An action-value method with ε-greedy action selection is forced to choose between just two policies: choosing right with high probability $1 - \varepsilon/2$ on all steps or choosing left with the same high probability on all time steps. If $\varepsilon = 0.1$, then these two policies achieve a value (at the start state) of less than -44 and -82, respectively, as shown in the graph. A method can do significantly better if it can learn a specific probability with which to select right. The best probability is about 0.59, which achieves a value of about -11.6.



Perhaps the simplest advantage that policy parameterization may have over action-value parameterization is that the policy may be a simpler function to approximate. Problems vary in the complexity of their policies and action-value functions. For some, the action-value function is simpler and thus easier to approximate. For others, the policy is simpler. In the latter case a policy-based method will typically learn faster and yield a superior asymptotic policy (as in Tetris; see Şimşek, Algórta, and Kothiyal, 2016).

Finally, we note that the choice of policy parameterization is sometimes a good way of injecting prior knowledge about the desired form of the policy into the reinforcement learning system. This is often the most important reason for using a policy-based learning method.

Exercise 13.1 Use your knowledge of the gridworld and its dynamics to determine an *exact* symbolic expression for the optimal probability of selecting the right action in Example 13.1. □

13.2 The Policy Gradient Theorem

In addition to the practical advantages of policy parameterization over ε-greedy action selection, there is also an important theoretical advantage. With continuous policy parameterization the action probabilities change smoothly as a function of the learned parameter, whereas in ε-greedy selection the action probabilities may change dramatically for an arbitrarily small change in the estimated action values, if that change results in a different action having the maximal value. Largely because of this stronger convergence guarantees are available for policy-gradient methods than for action-value methods. In particular, it is the continuity of the policy dependence on the parameters that enables policy-gradient methods to approximate gradient ascent (13.1).

The episodic and continuing cases define the performance measure, $J(\boldsymbol{\theta})$, differently and thus have to be treated separately to some extent. Nevertheless, we will try to present both cases uniformly, and we develop a notation so that the major theoretical results can be described with a single set of equations.

In this section we treat the episodic case, for which we define the performance measure as the value of the start state of the episode. We can simplify the notation without losing any meaningful generality by assuming that every episode starts in some particular (non-random) state s_0. Then, in the episodic case we define performance as

$$J(\boldsymbol{\theta}) \doteq v_{\pi_{\boldsymbol{\theta}}}(s_0), \tag{13.4}$$

where $v_{\pi_{\boldsymbol{\theta}}}$ is the true value function for $\pi_{\boldsymbol{\theta}}$, the policy determined by $\boldsymbol{\theta}$. From here on in our discussion we will assume no discounting ($\gamma = 1$) for the episodic case, although for completeness we do include the possibility of discounting in the boxed algorithms.

With function approximation, it may seem challenging to change the policy parameter in a way that ensures improvement. The problem is that performance depends on both the action selections and the distribution of states in which those selections are made, and that both of these are affected by the policy parameter. Given a state, the effect of the policy parameter on the actions, and thus on reward, can be computed in a relatively straightforward way from knowledge of the parameterization. But the effect of the policy on the state distribution is a function of the environment and is typically unknown. How can we estimate the performance gradient with respect to the policy parameter when the gradient depends on the unknown effect of policy changes on the state distribution?

Fortunately, there is an excellent theoretical answer to this challenge in the form of the *policy gradient theorem*, which provides an analytic expression for the gradient of

Proof of the Policy Gradient Theorem (episodic case)

With just elementary calculus and re-arranging of terms, we can prove the policy gradient theorem from first principles. To keep the notation simple, we leave it implicit in all cases that π is a function of $\boldsymbol{\theta}$, and all gradients are also implicitly with respect to $\boldsymbol{\theta}$. First note that the gradient of the state-value function can be written in terms of the action-value function as

$$
\begin{aligned}
\nabla v_\pi(s) &= \nabla \left[\sum_a \pi(a|s) q_\pi(s,a) \right], \quad \text{for all } s \in \mathcal{S} && \text{(Exercise 3.18)} \\
&= \sum_a \Big[\nabla \pi(a|s) q_\pi(s,a) + \pi(a|s) \nabla q_\pi(s,a) \Big] && \text{(product rule of calculus)} \\
&= \sum_a \Big[\nabla \pi(a|s) q_\pi(s,a) + \pi(a|s) \nabla \sum_{s',r} p(s',r\,|\,s,a) \big(r + v_\pi(s')\big) \Big] \\
& && \text{(Exercise 3.19 and Equation 3.2)} \\
&= \sum_a \Big[\nabla \pi(a|s) q_\pi(s,a) + \pi(a|s) \sum_{s'} p(s'\,|\,s,a) \nabla v_\pi(s') \Big] && \text{(Eq. 3.4)} \\
&= \sum_a \Big[\nabla \pi(a|s) q_\pi(s,a) + \pi(a|s) \sum_{s'} p(s'\,|\,s,a) && \text{(unrolling)} \\
&\qquad \sum_{a'} \big[\nabla \pi(a'|s') q_\pi(s',a') + \pi(a'|s') \sum_{s''} p(s''\,|\,s',a') \nabla v_\pi(s'') \big] \Big] \\
&= \sum_{x \in \mathcal{S}} \sum_{k=0}^{\infty} \Pr(s \to x, k, \pi) \sum_a \nabla \pi(a|x) q_\pi(x,a),
\end{aligned}
$$

after repeated unrolling, where $\Pr(s \to x, k, \pi)$ is the probability of transitioning from state s to state x in k steps under policy π. It is then immediate that

$$
\begin{aligned}
\nabla J(\boldsymbol{\theta}) &= \nabla v_\pi(s_0) \\
&= \sum_s \left(\sum_{k=0}^{\infty} \Pr(s_0 \to s, k, \pi) \right) \sum_a \nabla \pi(a|s) q_\pi(s,a) \\
&= \sum_s \eta(s) \sum_a \nabla \pi(a|s) q_\pi(s,a) && \text{(box page 199)} \\
&= \sum_{s'} \eta(s') \sum_s \frac{\eta(s)}{\sum_{s'} \eta(s')} \sum_a \nabla \pi(a|s) q_\pi(s,a) \\
&= \sum_{s'} \eta(s') \sum_s \mu(s) \sum_a \nabla \pi(a|s) q_\pi(s,a) && \text{(Eq. 9.3)} \\
&\propto \sum_s \mu(s) \sum_a \nabla \pi(a|s) q_\pi(s,a) && \text{(Q.E.D.)}
\end{aligned}
$$

performance with respect to the policy parameter (which is what we need to approximate for gradient ascent (13.1)) that does *not* involve the derivative of the state distribution. The policy gradient theorem for the episodic case establishes that

$$\nabla J(\boldsymbol{\theta}) \propto \sum_s \mu(s) \sum_a q_\pi(s,a) \nabla \pi(a|s,\boldsymbol{\theta}), \tag{13.5}$$

where the gradients are column vectors of partial derivatives with respect to the components of $\boldsymbol{\theta}$, and π denotes the policy corresponding to parameter vector $\boldsymbol{\theta}$. The symbol $\propto$ here means "proportional to". In the episodic case, the constant of proportionality is the average length of an episode, and in the continuing case it is 1, so that the relationship is actually an equality. The distribution μ here (as in Chapters 9 and 10) is the on-policy distribution under π (see page 199). The policy gradient theorem is proved for the episodic case in the box on the previous page.

13.3 REINFORCE: Monte Carlo Policy Gradient

We are now ready to derive our first policy-gradient learning algorithm. Recall our overall strategy of stochastic gradient ascent (13.1), which requires a way to obtain samples such that the expectation of the sample gradient is proportional to the actual gradient of the performance measure as a function of the parameter. The sample gradients need only be proportional to the gradient because any constant of proportionality can be absorbed into the step size α, which is otherwise arbitrary. The policy gradient theorem gives an exact expression proportional to the gradient; all that is needed is some way of sampling whose expectation equals or approximates this expression. Notice that the right-hand side of the policy gradient theorem is a sum over states weighted by how often the states occur under the target policy π; if π is followed, then states will be encountered in these proportions. Thus

$$\begin{aligned}\nabla J(\boldsymbol{\theta}) &\propto \sum_s \mu(s) \sum_a q_\pi(s,a) \nabla \pi(a|s,\boldsymbol{\theta}) \\ &= \mathbb{E}_\pi \left[\sum_a q_\pi(S_t,a) \nabla \pi(a|S_t,\boldsymbol{\theta}) \right].\end{aligned} \tag{13.6}$$

We could stop here and instantiate our stochastic gradient-ascent algorithm (13.1) as

$$\boldsymbol{\theta}_{t+1} \doteq \boldsymbol{\theta}_t + \alpha \sum_a \hat{q}(S_t,a,\mathbf{w}) \nabla \pi(a|S_t,\boldsymbol{\theta}), \tag{13.7}$$

where $\hat{q}$ is some learned approximation to q_π. This algorithm, which has been called an *all-actions* method because its update involves all of the actions, is promising and deserving of further study, but our current interest is the classical REINFORCE algorithm (Willams, 1992) whose update at time t involves just A_t, the one action actually taken at time t.

We continue our derivation of REINFORCE by introducing A_t in the same way as we introduced S_t in (13.6)—by replacing a sum over the random variable's possible values

by an expectation under π, and then sampling the expectation. Equation (13.6) involves an appropriate sum over actions, but each term is not weighted by $\pi(a|S_t,\boldsymbol{\theta})$ as is needed for an expectation under π. So we introduce such a weighting, without changing the equality, by multiplying and then dividing the summed terms by $\pi(a|S_t,\boldsymbol{\theta})$. Continuing from (13.6), we have

$$\begin{aligned}
\nabla J(\boldsymbol{\theta}) &= \mathbb{E}_\pi\left[\sum_a \pi(a|S_t,\boldsymbol{\theta}) q_\pi(S_t,a)\frac{\nabla\pi(a|S_t,\boldsymbol{\theta})}{\pi(a|S_t,\boldsymbol{\theta})}\right] \\
&= \mathbb{E}_\pi\left[q_\pi(S_t,A_t)\frac{\nabla\pi(A_t|S_t,\boldsymbol{\theta})}{\pi(A_t|S_t,\boldsymbol{\theta})}\right] && \text{(replacing } a \text{ by the sample } A_t\sim\pi) \\
&= \mathbb{E}_\pi\left[G_t\frac{\nabla\pi(A_t|S_t,\boldsymbol{\theta})}{\pi(A_t|S_t,\boldsymbol{\theta})}\right], && \text{(because } \mathbb{E}_\pi[G_t|S_t,A_t]=q_\pi(S_t,A_t))
\end{aligned}$$

where G_t is the return as usual. The final expression in brackets is exactly what is needed, a quantity that can be sampled on each time step whose expectation is equal to the gradient. Using this sample to instantiate our generic stochastic gradient ascent algorithm (13.1) yields the REINFORCE update:

$$\boldsymbol{\theta}_{t+1} \doteq \boldsymbol{\theta}_t + \alpha G_t \frac{\nabla\pi(A_t|S_t,\boldsymbol{\theta}_t)}{\pi(A_t|S_t,\boldsymbol{\theta}_t)}. \tag{13.8}$$

This update has an intuitive appeal. Each increment is proportional to the product of a return G_t and a vector, the gradient of the probability of taking the action actually taken divided by the probability of taking that action. The vector is the direction in parameter space that most increases the probability of repeating the action A_t on future visits to state S_t. The update increases the parameter vector in this direction proportional to the return, and inversely proportional to the action probability. The former makes sense because it causes the parameter to move most in the directions that favor actions that yield the highest return. The latter makes sense because otherwise actions that are selected frequently are at an advantage (the updates will be more often in their direction) and might win out even if they do not yield the highest return.

Note that REINFORCE uses the complete return from time t, which includes all future rewards up until the end of the episode. In this sense REINFORCE is a Monte Carlo algorithm and is well defined only for the episodic case with all updates made in retrospect after the episode is completed (like the Monte Carlo algorithms in Chapter 5). This is shown explicitly in the boxed on the next page.

Notice that the update in the last line of pseudocode appears rather different from the REINFORCE update rule (13.8). One difference is that the pseudocode uses the compact expression $\nabla \ln \pi(A_t|S_t,\boldsymbol{\theta}_t)$ for the fractional vector $\frac{\nabla\pi(A_t|S_t,\boldsymbol{\theta}_t)}{\pi(A_t|S_t,\boldsymbol{\theta}_t)}$ in (13.8). That these two expressions for the vector are equivalent follows from the identity $\nabla \ln x = \frac{\nabla x}{x}$. This vector has been given several names and notations in the literature; we will refer to it simply as the *eligibility vector*. Note that it is the only place that the policy parameterization appears in the algorithm.

REINFORCE: Monte-Carlo Policy-Gradient Control (episodic) for π_*

Input: a differentiable policy parameterization $\pi(a|s,\boldsymbol{\theta})$
Algorithm parameter: step size $\alpha > 0$
Initialize policy parameter $\boldsymbol{\theta} \in \mathbb{R}^{d'}$ (e.g., to $\mathbf{0}$)

Loop forever (for each episode):
  Generate an episode $S_0, A_0, R_1, \ldots, S_{T-1}, A_{T-1}, R_T$, following $\pi(\cdot|\cdot,\boldsymbol{\theta})$
  Loop for each step of the episode $t = 0, 1, \ldots, T-1$:
    $G \leftarrow \sum_{k=t+1}^{T} \gamma^{k-t-1} R_k$ $\qquad (G_t)$
    $\boldsymbol{\theta} \leftarrow \boldsymbol{\theta} + \alpha \gamma^t G \nabla \ln \pi(A_t|S_t,\boldsymbol{\theta})$

The second difference between the pseudocode update and the REINFORCE update equation (13.8) is that the former includes a factor of γ^t. This is because, as mentioned earlier, in the text we are treating the non-discounted case ($\gamma = 1$) while in the boxed algorithms we are giving the algorithms for the general discounted case. All of the ideas go through in the discounted case with appropriate adjustments (including to the box on page 199) but involve additional complexity that distracts from the main ideas.

**Exercise 13.2* Generalize the box on page 199, the policy gradient theorem (13.5), the proof of the policy gradient theorem (page 325), and the steps leading to the REINFORCE update equation (13.8), so that (13.8) ends up with a factor of γ^t and thus aligns with the general algorithm given in the pseudocode. □

Figure 13.1 shows the performance of REINFORCE on the short-corridor gridworld from Example 13.1.

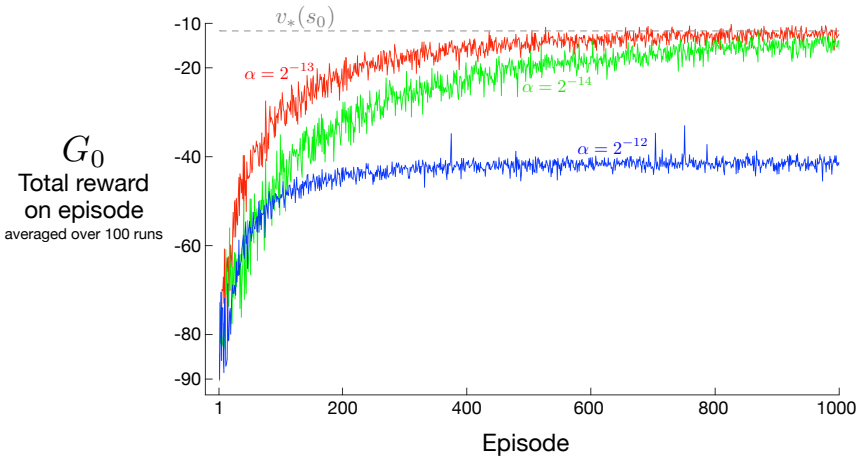


Figure 13.1: REINFORCE on the short-corridor gridworld (Example 13.1). With a good step size, the total reward per episode approaches the optimal value of the start state.

As a stochastic gradient method, REINFORCE has good theoretical convergence properties. By construction, the expected update over an episode is in the same direction as the performance gradient. This assures an improvement in expected performance for sufficiently small α, and convergence to a local optimum under standard stochastic approximation conditions for decreasing α. However, as a Monte Carlo method REINFORCE may be of high variance and thus produce slow learning.

Exercise 13.3 In Section 13.1 we considered policy parameterizations using the soft-max in action preferences (13.2) with linear action preferences (13.3). For this parameterization, prove that the eligibility vector is

$$\nabla \ln \pi(a|s,\boldsymbol{\theta}) = \mathbf{x}(s,a) - \sum_b \pi(b|s,\boldsymbol{\theta})\mathbf{x}(s,b), \tag{13.9}$$

using the definitions and elementary calculus. □

13.4 REINFORCE with Baseline

The policy gradient theorem (13.5) can be generalized to include a comparison of the action value to an arbitrary *baseline* $b(s)$:

$$\nabla J(\boldsymbol{\theta}) \propto \sum_s \mu(s) \sum_a \Big(q_\pi(s,a) - b(s)\Big) \nabla \pi(a|s,\boldsymbol{\theta}). \tag{13.10}$$

The baseline can be any function, even a random variable, as long as it does not vary with a; the equation remains valid because the subtracted quantity is zero:

$$\sum_a b(s) \nabla \pi(a|s,\boldsymbol{\theta}) \;=\; b(s) \nabla \sum_a \pi(a|s,\boldsymbol{\theta}) \;=\; b(s) \nabla 1 \;=\; 0.$$

The policy gradient theorem with baseline (13.10) can be used to derive an update rule using similar steps as in the previous section. The update rule that we end up with is a new version of REINFORCE that includes a general baseline:

$$\boldsymbol{\theta}_{t+1} \doteq \boldsymbol{\theta}_t + \alpha\Big(G_t - b(S_t)\Big) \frac{\nabla \pi(A_t|S_t,\boldsymbol{\theta}_t)}{\pi(A_t|S_t,\boldsymbol{\theta}_t)}. \tag{13.11}$$

Because the baseline could be uniformly zero, this update is a strict generalization of REINFORCE. In general, the baseline leaves the expected value of the update unchanged, but it can have a large effect on its variance. For example, we saw in Section 2.8 that an analogous baseline can significantly reduce the variance (and thus speed the learning) of gradient bandit algorithms. In the bandit algorithms the baseline was just a number (the average of the rewards seen so far), but for MDPs the baseline should vary with state. In some states all actions have high values and we need a high baseline to differentiate the higher valued actions from the less highly valued ones; in other states all actions will have low values and a low baseline is appropriate.

One natural choice for the baseline is an estimate of the state value, $\hat{v}(S_t,\mathbf{w})$, where $\mathbf{w} \in \mathbb{R}^m$ is a weight vector learned by one of the methods presented in previous chapters.

Because REINFORCE is a Monte Carlo method for learning the policy parameter, $\boldsymbol{\theta}$, it seems natural to also use a Monte Carlo method to learn the state-value weights, $\mathbf{w}$. A complete pseudocode algorithm for REINFORCE with baseline using such a learned state-value function as the baseline is given in the box below.

REINFORCE with Baseline (episodic), for estimating $\pi_{\boldsymbol{\theta}} \approx \pi_*$

Input: a differentiable policy parameterization $\pi(a|s,\boldsymbol{\theta})$
Input: a differentiable state-value function parameterization $\hat{v}(s,\mathbf{w})$
Algorithm parameters: step sizes $\alpha^{\boldsymbol{\theta}} > 0$, $\alpha^{\mathbf{w}} > 0$
Initialize policy parameter $\boldsymbol{\theta} \in \mathbb{R}^{d'}$ and state-value weights $\mathbf{w} \in \mathbb{R}^{d}$ (e.g., to $\mathbf{0}$)

Loop forever (for each episode):
  Generate an episode $S_0, A_0, R_1, \ldots, S_{T-1}, A_{T-1}, R_T$, following $\pi(\cdot|\cdot,\boldsymbol{\theta})$
  Loop for each step of the episode $t = 0, 1, \ldots, T-1$:
    $G \leftarrow \sum_{k=t+1}^{T} \gamma^{k-t-1} R_k$ $\qquad (G_t)$
    $\delta \leftarrow G - \hat{v}(S_t,\mathbf{w})$
    $\mathbf{w} \leftarrow \mathbf{w} + \alpha^{\mathbf{w}} \delta \nabla \hat{v}(S_t,\mathbf{w})$
    $\boldsymbol{\theta} \leftarrow \boldsymbol{\theta} + \alpha^{\boldsymbol{\theta}} \gamma^t \delta \nabla \ln \pi(A_t|S_t,\boldsymbol{\theta})$

This algorithm has two step sizes, denoted $\alpha^{\boldsymbol{\theta}}$ and $\alpha^{\mathbf{w}}$ (where $\alpha^{\boldsymbol{\theta}}$ is the α in (13.11)). Choosing the step size for values (here $\alpha^{\mathbf{w}}$) is relatively easy; in the linear case we have rules of thumb for setting it, such as $\alpha^{\mathbf{w}} = 0.1/\mathbb{E}\big[\|\nabla \hat{v}(S_t,\mathbf{w})\|^2_{\mu}\big]$ (see Section 9.6). It is much less clear how to set the step size for the policy parameters, $\alpha^{\boldsymbol{\theta}}$, whose best value depends on the range of variation of the rewards and on the policy parameterization.

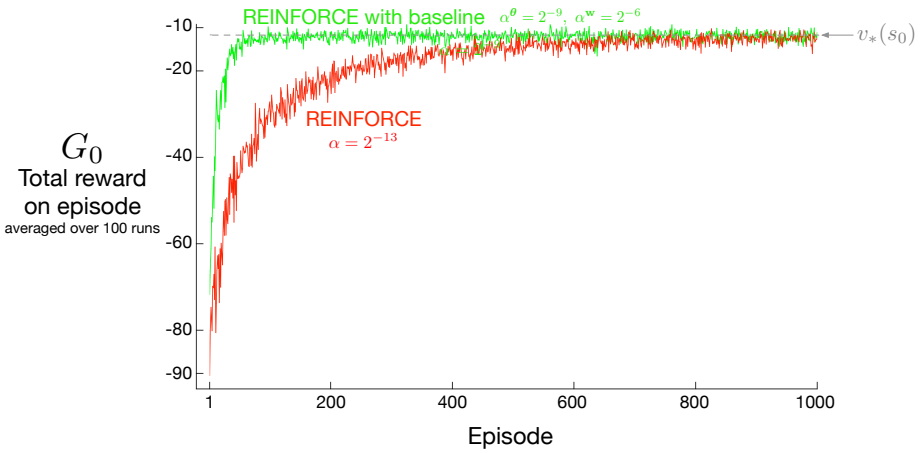


Figure 13.2: Adding a baseline to REINFORCE can make it learn much faster, as illustrated here on the short-corridor gridworld (Example 13.1). The step size used here for plain REINFORCE is that at which it performs best (to the nearest power of two; see Figure 13.1).

Figure 13.2 compares the behavior of REINFORCE with and without a baseline on the short-corridor gridword (Example 13.1). Here the approximate state-value function used in the baseline is $\hat{v}(s,\mathbf{w}) = w$. That is, $\mathbf{w}$ is a single component, w.

13.5 Actor–Critic Methods

Although the REINFORCE-with-baseline method learns both a policy and a state-value function, we do not consider it to be an actor–critic method because its state-value function is used only as a baseline, not as a critic. That is, it is not used for bootstrapping (updating the value estimate for a state from the estimated values of subsequent states), but only as a baseline for the state whose estimate is being updated. This is a useful distinction, for only through bootstrapping do we introduce bias and an asymptotic dependence on the quality of the function approximation. As we have seen, the bias introduced through bootstrapping and reliance on the state representation is often beneficial because it reduces variance and accelerates learning. REINFORCE with baseline is unbiased and will converge asymptotically to a local minimum, but like all Monte Carlo methods it tends to learn slowly (produce estimates of high variance) and to be inconvenient to implement online or for continuing problems. As we have seen earlier in this book, with temporal-difference methods we can eliminate these inconveniences, and through multi-step methods we can flexibly choose the degree of bootstrapping. In order to gain these advantages in the case of policy gradient methods we use actor–critic methods with a bootstrapping critic.

First consider one-step actor–critic methods, the analog of the TD methods introduced in Chapter 6 such as TD(0), Sarsa(0), and Q-learning. The main appeal of one-step methods is that they are fully online and incremental, yet avoid the complexities of eligibility traces. They are a special case of the eligibility trace methods, and not as general, but easier to understand. One-step actor–critic methods replace the full return of REINFORCE (13.11) with the one-step return (and use a learned state-value function as the baseline) as follows:

$$\boldsymbol{\theta}_{t+1} \doteq \boldsymbol{\theta}_t + \alpha\Big(G_{t:t+1} - \hat{v}(S_t,\mathbf{w})\Big)\frac{\nabla\pi(A_t|S_t,\boldsymbol{\theta}_t)}{\pi(A_t|S_t,\boldsymbol{\theta}_t)} \tag{13.12}$$

$$= \boldsymbol{\theta}_t + \alpha\Big(R_{t+1} + \gamma\hat{v}(S_{t+1},\mathbf{w}) - \hat{v}(S_t,\mathbf{w})\Big)\frac{\nabla\pi(A_t|S_t,\boldsymbol{\theta}_t)}{\pi(A_t|S_t,\boldsymbol{\theta}_t)} \tag{13.13}$$

$$= \boldsymbol{\theta}_t + \alpha\delta_t\frac{\nabla\pi(A_t|S_t,\boldsymbol{\theta}_t)}{\pi(A_t|S_t,\boldsymbol{\theta}_t)}. \tag{13.14}$$

The natural state-value-function learning method to pair with this is semi-gradient TD(0). Pseudocode for the complete algorithm is given in the box at the top of the next page. Note that it is now a fully online, incremental algorithm, with states, actions, and rewards processed as they occur and then never revisited.

One-step Actor–Critic (episodic), for estimating $\pi_{\boldsymbol{\theta}} \approx \pi_*$

Input: a differentiable policy parameterization $\pi(a|s,\boldsymbol{\theta})$
Input: a differentiable state-value function parameterization $\hat{v}(s,\mathbf{w})$
Parameters: step sizes $\alpha^{\boldsymbol{\theta}} > 0$, $\alpha^{\mathbf{w}} > 0$
Initialize policy parameter $\boldsymbol{\theta} \in \mathbb{R}^{d'}$ and state-value weights $\mathbf{w} \in \mathbb{R}^d$ (e.g., to $\mathbf{0}$)
Loop forever (for each episode):
  Initialize S (first state of episode)
  $I \leftarrow 1$
  Loop while S is not terminal (for each time step):
    $A \sim \pi(\cdot|S,\boldsymbol{\theta})$
    Take action A, observe S', R
    $\delta \leftarrow R + \gamma\hat{v}(S',\mathbf{w}) - \hat{v}(S,\mathbf{w})$ (if S' is terminal, then $\hat{v}(S',\mathbf{w}) \doteq 0$)
    $\mathbf{w} \leftarrow \mathbf{w} + \alpha^{\mathbf{w}}\,\delta\,\nabla\hat{v}(S,\mathbf{w})$
    $\boldsymbol{\theta} \leftarrow \boldsymbol{\theta} + \alpha^{\boldsymbol{\theta}}\,I\,\delta\,\nabla \ln\pi(A|S,\boldsymbol{\theta})$
    $I \leftarrow \gamma I$
    $S \leftarrow S'$

The generalizations to the forward view of n-step methods and then to a λ-return algorithm are straightforward. The one-step return in (13.12) is merely replaced by $G_{t:t+n}$ or G_t^λ respectively. The backward view of the λ-return algorithm is also straightforward, using separate eligibility traces for the actor and critic, each after the patterns in Chapter 12. Pseudocode for the complete algorithm is given in the box below.

Actor–Critic with Eligibility Traces (episodic), for estimating $\pi_{\boldsymbol{\theta}} \approx \pi_*$

Input: a differentiable policy parameterization $\pi(a|s,\boldsymbol{\theta})$
Input: a differentiable state-value function parameterization $\hat{v}(s,\mathbf{w})$
Parameters: trace-decay rates $\lambda^{\boldsymbol{\theta}} \in [0,1]$, $\lambda^{\mathbf{w}} \in [0,1]$; step sizes $\alpha^{\boldsymbol{\theta}} > 0$, $\alpha^{\mathbf{w}} > 0$
Initialize policy parameter $\boldsymbol{\theta} \in \mathbb{R}^{d'}$ and state-value weights $\mathbf{w} \in \mathbb{R}^d$ (e.g., to $\mathbf{0}$)
Loop forever (for each episode):
  Initialize S (first state of episode)
  $\mathbf{z}^{\boldsymbol{\theta}} \leftarrow \mathbf{0}$ (d'-component eligibility trace vector)
  $\mathbf{z}^{\mathbf{w}} \leftarrow \mathbf{0}$ (d-component eligibility trace vector)
  $I \leftarrow 1$
  Loop while S is not terminal (for each time step):
    $A \sim \pi(\cdot|S,\boldsymbol{\theta})$
    Take action A, observe S', R
    $\delta \leftarrow R + \gamma\hat{v}(S',\mathbf{w}) - \hat{v}(S,\mathbf{w})$ (if S' is terminal, then $\hat{v}(S',\mathbf{w}) \doteq 0$)
    $\mathbf{z}^{\mathbf{w}} \leftarrow \gamma\lambda^{\mathbf{w}}\mathbf{z}^{\mathbf{w}} + \nabla\hat{v}(S,\mathbf{w})$
    $\mathbf{z}^{\boldsymbol{\theta}} \leftarrow \gamma\lambda^{\boldsymbol{\theta}}\mathbf{z}^{\boldsymbol{\theta}} + I\,\nabla \ln\pi(A|S,\boldsymbol{\theta})$
    $\mathbf{w} \leftarrow \mathbf{w} + \alpha^{\mathbf{w}}\,\delta\,\mathbf{z}^{\mathbf{w}}$
    $\boldsymbol{\theta} \leftarrow \boldsymbol{\theta} + \alpha^{\boldsymbol{\theta}}\,\delta\,\mathbf{z}^{\boldsymbol{\theta}}$
    $I \leftarrow \gamma I$
    $S \leftarrow S'$

13.6 Policy Gradient for Continuing Problems

As discussed in Section 10.3, for continuing problems without episode boundaries we need to define performance in terms of the average rate of reward per time step:

$$\begin{aligned} J(\boldsymbol{\theta}) \doteq r(\pi) &\doteq \lim_{h\to\infty} \frac{1}{h} \sum_{t=1}^{h} \mathbb{E}[R_t \mid S_0, A_{0:t-1} \sim \pi] \\ &= \lim_{t\to\infty} \mathbb{E}[R_t \mid S_0, A_{0:t-1} \sim \pi] \\ &= \sum_s \mu(s) \sum_a \pi(a|s) \sum_{s',r} p(s', r \,|\, s, a) r, \end{aligned} \tag{13.15}$$

where μ is the steady-state distribution under π, $\mu(s) \doteq \lim_{t\to\infty} \Pr\{S_t = s | A_{0:t} \sim \pi\}$, which is assumed to exist and to be independent of S_0 (an ergodicity assumption). Remember that this is the special distribution under which, if you select actions according to π, you remain in the same distribution:

$$\sum_s \mu(s) \sum_a \pi(a|s, \boldsymbol{\theta}) p(s' \,|\, s, a) = \mu(s'), \text{ for all } s' \in \mathcal{S}. \tag{13.16}$$

Complete pseudocode for the actor–critic algorithm in the continuing case (backward view) is given in the box below.

Actor–Critic with Eligibility Traces (continuing), for estimating $\pi_{\boldsymbol{\theta}} \approx \pi_*$

Input: a differentiable policy parameterization $\pi(a|s, \boldsymbol{\theta})$
Input: a differentiable state-value function parameterization $\hat{v}(s, \mathbf{w})$
Algorithm parameters: $\lambda^{\mathbf{w}} \in [0, 1]$, $\lambda^{\boldsymbol{\theta}} \in [0, 1]$, $\alpha^{\mathbf{w}} > 0$, $\alpha^{\boldsymbol{\theta}} > 0$, $\alpha^{\bar{R}} > 0$
Initialize $\bar{R} \in \mathbb{R}$ (e.g., to 0)
Initialize state-value weights $\mathbf{w} \in \mathbb{R}^d$ and policy parameter $\boldsymbol{\theta} \in \mathbb{R}^{d'}$ (e.g., to $\mathbf{0}$)
Initialize $S \in \mathcal{S}$ (e.g., to s_0)

$\mathbf{z}^{\mathbf{w}} \leftarrow \mathbf{0}$ (d-component eligibility trace vector)
$\mathbf{z}^{\boldsymbol{\theta}} \leftarrow \mathbf{0}$ (d'-component eligibility trace vector)
Loop forever (for each time step):
  $A \sim \pi(\cdot|S, \boldsymbol{\theta})$
  Take action A, observe S', R
  $\delta \leftarrow R - \bar{R} + \hat{v}(S', \mathbf{w}) - \hat{v}(S, \mathbf{w})$
  $\bar{R} \leftarrow \bar{R} + \alpha^{\bar{R}} \delta$
  $\mathbf{z}^{\mathbf{w}} \leftarrow \lambda^{\mathbf{w}} \mathbf{z}^{\mathbf{w}} + \nabla \hat{v}(S, \mathbf{w})$
  $\mathbf{z}^{\boldsymbol{\theta}} \leftarrow \lambda^{\boldsymbol{\theta}} \mathbf{z}^{\boldsymbol{\theta}} + \nabla \ln \pi(A|S, \boldsymbol{\theta})$
  $\mathbf{w} \leftarrow \mathbf{w} + \alpha^{\mathbf{w}} \delta \mathbf{z}^{\mathbf{w}}$
  $\boldsymbol{\theta} \leftarrow \boldsymbol{\theta} + \alpha^{\boldsymbol{\theta}} \delta \mathbf{z}^{\boldsymbol{\theta}}$
  $S \leftarrow S'$

Naturally, in the continuing case, we define values, $v_\pi(s) \doteq \mathbb{E}_\pi[G_t|S_t=s]$ and $q_\pi(s,a) \doteq \mathbb{E}_\pi[G_t|S_t=s, A_t=a]$, with respect to the differential return:

$$G_t \doteq R_{t+1} - r(\pi) + R_{t+2} - r(\pi) + R_{t+3} - r(\pi) + \cdots. \tag{13.17}$$

With these alternate definitions, the policy gradient theorem as given for the episodic case (13.5) remains true for the continuing case. A proof is given in the box on the next page. The forward and backward view equations also remain the same.

Proof of the Policy Gradient Theorem (continuing case)

The proof of the policy gradient theorem for the continuing case begins similarly to the episodic case. Again we leave it implicit in all cases that π is a function of $\boldsymbol{\theta}$ and that the gradients are with respect to $\boldsymbol{\theta}$. Recall that in the continuing case $J(\boldsymbol{\theta}) = r(\pi)$ (13.15) and that v_π and q_π denote values with respect to the differential return (13.17). The gradient of the state-value function can be written, for any $s \in \mathcal{S}$, as

$$\begin{aligned}
\nabla v_\pi(s) &= \nabla \left[\sum_a \pi(a|s) q_\pi(s,a)\right], \quad \text{for all } s \in \mathcal{S} && \text{(Exercise 3.18)} \\
&= \sum_a \Big[\nabla\pi(a|s) q_\pi(s,a) + \pi(a|s)\nabla q_\pi(s,a)\Big] && \text{(product rule of calculus)} \\
&= \sum_a \Big[\nabla\pi(a|s) q_\pi(s,a) + \pi(a|s)\nabla \sum_{s',r} p(s',r\,|\,s,a)\big(r - r(\boldsymbol{\theta}) + v_\pi(s')\big)\Big] \\
&= \sum_a \Big[\nabla\pi(a|s) q_\pi(s,a) + \pi(a|s)\big[-\nabla r(\boldsymbol{\theta}) + \sum_{s'} p(s'\,|\,s,a)\nabla v_\pi(s')\big]\Big].
\end{aligned}$$

After re-arranging terms, we obtain

$$\nabla r(\boldsymbol{\theta}) = \sum_a \Big[\nabla\pi(a|s) q_\pi(s,a) + \pi(a|s)\sum_{s'} p(s'\,|\,s,a)\nabla v_\pi(s')\Big] - \nabla v_\pi(s).$$

Notice that the left-hand side can be written $\nabla J(\boldsymbol{\theta})$, and that it does not depend on s. Thus the right-hand side does not depend on s either, and we can safely sum it over all $s \in \mathcal{S}$, weighted by $\mu(s)$, without changing it (because $\sum_s \mu(s) = 1$):

$$\begin{aligned}
\nabla J(\boldsymbol{\theta}) &= \sum_s \mu(s) \left(\sum_a \Big[\nabla\pi(a|s) q_\pi(s,a) + \pi(a|s)\sum_{s'} p(s'\,|\,s,a)\nabla v_\pi(s')\Big] - \nabla v_\pi(s)\right) \\
&= \sum_s \mu(s) \sum_a \nabla\pi(a|s) q_\pi(s,a) \\
&\quad + \sum_s \mu(s) \sum_a \pi(a|s) \sum_{s'} p(s'\,|\,s,a)\nabla v_\pi(s') - \sum_s \mu(s)\nabla v_\pi(s)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_s \mu(s) \sum_a \nabla \pi(a|s) q_\pi(s,a) \\
&\quad + \sum_{s'} \underbrace{\sum_s \mu(s) \sum_a \pi(a|s) p(s'|s,a)}_{\mu(s') \text{ (13.16)}} \nabla v_\pi(s') - \sum_s \mu(s) \nabla v_\pi(s) \\
&= \sum_s \mu(s) \sum_a \nabla \pi(a|s) q_\pi(s,a) + \sum_{s'} \mu(s') \nabla v_\pi(s') - \sum_s \mu(s) \nabla v_\pi(s) \\
&= \sum_s \mu(s) \sum_a \nabla \pi(a|s) q_\pi(s,a). \qquad \text{Q.E.D.}
\end{aligned}
$$

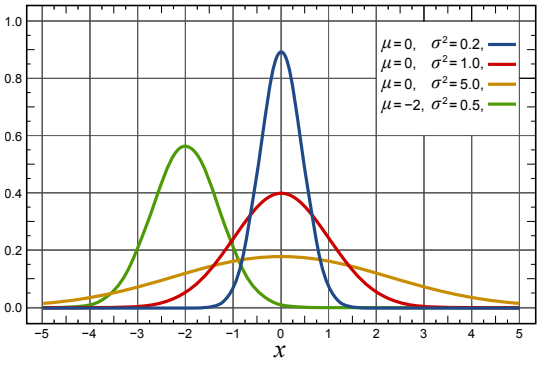
13.7 Policy Parameterization for Continuous Actions

Policy-based methods offer practical ways of dealing with large actions spaces, even continuous spaces with an infinite number of actions. Instead of computing learned probabilities for each of the many actions, we instead learn statistics of the probability distribution. For example, the action set might be the real numbers, with actions chosen from a normal (Gaussian) distribution.

The probability density function for the normal distribution is conventionally written

$$
p(x) \doteq \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \tag{13.18}
$$

where μ and σ here are the mean and standard deviation of the normal distribution, and of course π here is just the number $\pi \approx 3.14159$. The probability density functions for several different means and standard deviations are shown to the right. The value $p(x)$ is the *density* of the probability at x, not the probability. It can be greater than 1; it is the total area under $p(x)$ that must sum to 1. In general, one can take the integral under $p(x)$ for any range of x values to get the probability of x falling within that range.



To produce a policy parameterization, the policy can be defined as the normal probability density over a real-valued scalar action, with mean and standard deviation given by parametric function approximators that depend on the state. That is,

$$
\pi(a|s,\boldsymbol{\theta}) \doteq \frac{1}{\sigma(s,\boldsymbol{\theta})\sqrt{2\pi}} \exp\left(-\frac{(a-\mu(s,\boldsymbol{\theta}))^2}{2\sigma(s,\boldsymbol{\theta})^2}\right), \tag{13.19}
$$

where $\mu : \mathcal{S} \times \mathbb{R}^{d'} \to \mathbb{R}$ and $\sigma : \mathcal{S} \times \mathbb{R}^{d'} \to \mathbb{R}^+$ are two parameterized function approximators.

To complete the example we need only give a form for these approximators. For this we divide the policy's parameter vector into two parts, $\boldsymbol{\theta} = [\boldsymbol{\theta}_\mu, \boldsymbol{\theta}_\sigma]^\top$, one part to be used for the approximation of the mean and one part for the approximation of the standard deviation. The mean can be approximated as a linear function. The standard deviation must always be positive and is better approximated as the exponential of a linear function. Thus

$$\mu(s, \boldsymbol{\theta}) \doteq \boldsymbol{\theta}_\mu{}^\top \mathbf{x}_\mu(s) \quad \text{and} \quad \sigma(s, \boldsymbol{\theta}) \doteq \exp\Big(\boldsymbol{\theta}_\sigma{}^\top \mathbf{x}_\sigma(s)\Big), \tag{13.20}$$

where $\mathbf{x}_\mu(s)$ and $\mathbf{x}_\sigma(s)$ are state feature vectors perhaps constructed by one of the methods described in Chapter 9. With these definitions, all the algorithms described in the rest of this chapter can be applied to learn to select real-valued actions.

Exercise 13.4 Show that for the gaussian policy parameterization (13.19) the eligibility vector has the following two parts:

$$\nabla \ln \pi(a|s, \boldsymbol{\theta}_\mu) = \frac{\nabla \pi(a|s, \boldsymbol{\theta}_\mu)}{\pi(a|s, \boldsymbol{\theta})} = \frac{1}{\sigma(s, \boldsymbol{\theta})^2}\big(a - \mu(s, \boldsymbol{\theta})\big)\mathbf{x}_\mu(s), \text{ and}$$

$$\nabla \ln \pi(a|s, \boldsymbol{\theta}_\sigma) = \frac{\nabla \pi(a|s, \boldsymbol{\theta}_\sigma)}{\pi(a|s, \boldsymbol{\theta})} = \left(\frac{\big(a - \mu(s, \boldsymbol{\theta})\big)^2}{\sigma(s, \boldsymbol{\theta})^2} - 1\right)\mathbf{x}_\sigma(s). \qquad \square$$

Exercise 13.5 A *Bernoulli-logistic unit* is a stochastic neuron-like unit used in some ANNs (Section 9.6). Its input at time t is a feature vector $\mathbf{x}(S_t)$; its output, A_t, is a random variable having two values, 0 and 1, with $\Pr\{A_t = 1\} = P_t$ and $\Pr\{A_t = 0\} = 1 - P_t$ (the Bernoulli distribution). Let $h(s, 0, \boldsymbol{\theta})$ and $h(s, 1, \boldsymbol{\theta})$ be the preferences in state s for the unit's two actions given policy parameter $\boldsymbol{\theta}$. Assume that the difference between the action preferences is given by a weighted sum of the unit's input vector, that is, assume that $h(s, 1, \boldsymbol{\theta}) - h(s, 0, \boldsymbol{\theta}) = \boldsymbol{\theta}^\top \mathbf{x}(s)$, where $\boldsymbol{\theta}$ is the unit's weight vector.

(a) Show that if the exponential soft-max distribution (13.2) is used to convert action preferences to policies, then $P_t = \pi(1|S_t, \boldsymbol{\theta}_t) = 1/(1 + \exp(-\boldsymbol{\theta}_t^\top \mathbf{x}(S_t)))$ (the logistic function).

(b) What is the Monte-Carlo REINFORCE update of $\boldsymbol{\theta}_t$ to $\boldsymbol{\theta}_{t+1}$ upon receipt of return G_t?

(c) Express the eligibility $\nabla \ln \pi(a|s, \boldsymbol{\theta})$ for a Bernoulli-logistic unit, in terms of a, $\mathbf{x}(s)$, and $\pi(a|s, \boldsymbol{\theta})$ by calculating the gradient.

Hint: separately for each action compute the derivative of the logarithm first with respect to $P_t = \pi(a|s, \boldsymbol{\theta}_t)$, combine the two results into one expression that depends on a and P_t, and then use the chain rule, noting that the derivative of the logistic function $f(x)$ is $f(x)(1 - f(x))$. $\square$

13.8 Summary

Prior to this chapter, this book focused on *action-value methods*—meaning methods that learn action values and then use them to determine action selections. In this chapter, on the other hand, we considered methods that learn a parameterized policy that enables actions to be taken without consulting action-value estimates. In particular, we have considered *policy-gradient methods*—meaning methods that update the policy parameter on each step in the direction of an estimate of the gradient of performance with respect to the policy parameter.

Methods that learn and store a policy parameter have many advantages. They can learn specific probabilities for taking the actions. They can learn appropriate levels of exploration and approach deterministic policies asymptotically. They can naturally handle continuous action spaces. All these things are easy for policy-based methods but awkward or impossible for ε-greedy methods and for action-value methods in general. In addition, on some problems the policy is just simpler to represent parametrically than the value function; these problems are more suited to parameterized policy methods.

Parameterized policy methods also have an important theoretical advantage over action-value methods in the form of the *policy gradient theorem*, which gives an exact formula for how performance is affected by the policy parameter that does not involve derivatives of the state distribution. This theorem provides a theoretical foundation for all policy gradient methods.

The REINFORCE method follows directly from the policy gradient theorem. Adding a state-value function as a *baseline* reduces REINFORCE's variance without introducing bias. Using the state-value function for bootstrapping introduces bias but is often desirable for the same reason that bootstrapping TD methods are often superior to Monte Carlo methods (substantially reduced variance). The state-value function assigns credit to—critizes—the policy's action selections, and accordingly the former is termed the *critic* and the latter the *actor*, and these overall methods are termed *actor–critic* methods.

Overall, policy-gradient methods provide a significantly different set of strengths and weaknesses than action-value methods. Today they are less well understood in some respects, but a subject of excitement and ongoing research.

Bibliographical and Historical Remarks

Methods that we now see as related to policy gradients were actually some of the earliest to be studied in reinforcement learning (Witten, 1977; Barto, Sutton, and Anderson, 1983; Sutton, 1984; Williams, 1987, 1992) and in predecessor fields (Phansalkar and Thathachar, 1995). They were largely supplanted in the 1990s by the action-value methods that are the focus of the other chapters of this book. In recent years, however, attention has returned to actor–critic methods and to policy-gradient methods in general. Among the further developments beyond what we cover here are natural-gradient methods (Amari, 1998; Kakade, 2002, Peters, Vijayakumar and Schaal, 2005; Peters and Schaal, 2008; Park, Kim and Kang, 2005; Bhatnagar, Sutton, Ghavamzadeh and Lee, 2009; see Grondman, Busoniu, Lopes and Babuska, 2012), deterministic policy gradient methods (Silver et al.,

2014), off-policy policy-gradient methods (Degris, White, and Sutton, 2012; Maei, 2018), and entropy regularization (see Schulman, Chen, and Abbeel, 2017). Major applications include acrobatic helicopter autopilots and AlphaGo (Section 16.6).

Our presentation in this chapter is based primarily on that by Sutton, McAllester, Singh, and Mansour (2000), who introduced the term "policy gradient methods." A useful overview is provided by Bhatnagar et al. (2009). One of the earliest related works is by Aleksandrov, Sysoyev, and Shemeneva (1968). Thomas (2014) first realized that the factor of γ^t, as specified in the boxed algorithms of this chapter, was needed in the case of discounted episodic problems.

13.1 Example 13.1 and the results with it in this chapter were developed with Eric Graves.

13.2 The policy gradient theorem here and on page 334 was first obtained by Marbach and Tsitsiklis (1998, 2001) and then independently by Sutton et al. (2000). A similar expression was obtained by Cao and Chen (1997). Other early results are due to Konda and Tsitsiklis (2000, 2003), Baxter and Bartlett (2001), and Baxter, Bartlett, and Weaver (2001). Some additional results are developed by Sutton, Singh, and McAllester (2000).

13.3 REINFORCE is due to Williams (1987, 1992). Phansalkar and Thathachar (1995) proved both local and global convergence theorems for modified versions of REINFORCE algorithms.

The all-actions algorithm was first presented in an unpublished but widely circulated incomplete paper (Sutton, Singh, and McAllester, 2000) and then developed further by Ciosek and Whiteson (2017, 2018), who termed it "expected policy gradients," and by Asadi, Allen, Roderick, Mohamed, Konidaris, and Littman (2017) who called it "mean actor critic."

13.4 The baseline was introduced in Williams's (1987, 1992) original work. Greensmith, Bartlett, and Baxter (2004) analyzed an arguably better baseline (see Dick, 2015). Thomas and Brunskill (2017) argue that an action-dependent baseline can be used without incurring bias.

13.5–6 Actor–critic methods were among the earliest to be investigated in reinforcement learning (Witten, 1977; Barto, Sutton, and Anderson, 1983; Sutton, 1984). The algorithms presented here are based on the work of Degris, White, and Sutton (2012). Actor–critic methods are sometimes referred to as advantage actor–critic methods in the literature.

13.7 The first to show how continuous actions could be handled this way appears to have been Williams (1987, 1992). The figure on page 335 is adapted from Wikipedia.