# Euclidean Spaces

## Ch.10

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# Objects in Euclidean Spaces

## Objects in $n$-dimensional Euclidean Spaces

| Dimension | Object | Representation |
|---|---|---|
| 0 | point | $\emptyset$ |
| 1 | line | $x_1 \in \mathbb{R}^1$ |
| 2 | plane | $(x_1, x_2) \in \mathbb{R}^2$ |
| 3 | 3d space | $(x_1, x_2, x_3) \in \mathbb{R}^3$ |
| $n$ | $n$d space | $(x_1, \cdots, x_n) \in \mathbb{R}^n$ |

# Vector

## Definition ((Euclidean) Vector, displacement)

*$n$-tuples of real numbers $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ are (Euclidean) Vectors that represents displacement in $\mathbb{R}^n$ space (or Cartesian coordinate system)*
*Let coordination of $\mathbf{p} = (p_1, \cdots, p_n)$, $\mathbf{q} = (q_1, \cdots, q_n)$. Then the displacement from $\mathbf{p}$ to $\mathbf{q}$ is defined as $\overrightarrow{\mathbf{pq}} := (q_1 - p_1, \cdots, q_n - p_n)$. In this definition, $\mathbf{p}$ is an origin, and $\mathbf{q}$ is a destination.*

Note: Any vector $\mathbf{p} = (p_1, \cdots, p_n)$ can be interpreted as a location ($(p_1, \cdots, p_n)$) or, displacement with origin $\mathbf{0} := (0, \cdots, 0)$ (more explicit notation: $\begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix}$)

# Addition and Subtraction

Let $\mathbf{u}, \mathbf{v}$ be the vectors in $\mathbb{R}^n$ space and $u_i, v_i \in \mathbb{R}^1$ be their $i$-th element.

**Definition ($\pm$ of vectors)**

$$(\mathbf{u} \pm \mathbf{v})_i := u_i \pm v_i \quad \forall i$$

*or,*

$$\mathbf{u} \pm \mathbf{v} := (u_1 \pm v_1, \cdots, u_n \pm v_n)$$

Geographically, addition of vectors means a sequence of displacements.

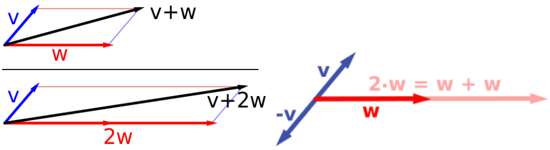


Figure: Geographical Meaning: Addition, Subtraction, and Scalar Multiplication

# Scalar Multiplication

Let $r, s$ be scalars (or real numbers). $i.e.$, $r, s \in \mathbb{R}^1$.

## Definition (Scalar Multiplication)

$$(r\mathbf{u})_i := ru_i \quad \forall i$$

Geographically, scalar multiplcation means stretching or shrinking.

## Albebraic Properties of Vector Operation

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u} \qquad \text{(Commutative Law)}$$

$$(r + s)\mathbf{u} = r\mathbf{u} + s\mathbf{u} \qquad \text{(Destributive Law 1)}$$

$$r(\mathbf{u} + \mathbf{v}) = r\mathbf{u} + r\mathbf{v} \qquad \text{(Destributive Law 2)}$$

In fact, any set of objects with a vector addition and scalar multiplication satisfying above laws is called vector space and vector is defined as the element of vector space. (Vector is defined by operations and their laws)

# Length and Direction

**Definition (Length of Vector $||\overrightarrow{\mathbf{pq}}||$)**

$$||\overrightarrow{pq}|| := \sqrt{\sum_{i}^{n}(q_i - p_i)^2}$$

**Theorem (10.1)**

$$||r\mathbf{v}|| = |r| \cdot ||\mathbf{v}|| \quad \forall r \in \mathbb{R} \wedge \forall \mathbf{v} \in \mathbb{R}^n$$

Any vector has two informations: (1) length, and (2) direction.

**Definition (Unit Vector (Direction of a vector))**

$$\textit{Unit vector of } \mathbf{v} := \frac{\mathbf{v}}{||\mathbf{v}||}$$

# The Inner Product

## Definition (Euclidean Inner Product (or dot product))

$$\mathbf{u} \bullet \mathbf{v} := \sum_{i}^{n} u_i v_i \in \mathbb{R}^1$$

## Theorem (10.2: Properties of Inner Product)

1. $\mathbf{u} \bullet \mathbf{v} = \mathbf{v} \bullet \mathbf{u}$
2. $\mathbf{u} \bullet (\mathbf{v} + \mathbf{w}) = u \bullet v + u \bullet w$
3. $\mathbf{u} \bullet (r\mathbf{v}) = r(\mathbf{u} \bullet \mathbf{v}) = (r\mathbf{u}) \bullet \mathbf{v}$
4. $\mathbf{u} \bullet \mathbf{u} \leq 0$
5. $\mathbf{u} \bullet \mathbf{u} = 0 \quad \iff \quad \mathbf{u} = \mathbf{0}$
6. $(u + v) \bullet (u + v) = u \bullet u + 2u \bullet v + v \bullet v$

# Inner Product and Angle between Two Vectors

**Theorem (10.3)**

*Let $\theta$ be the angle between $\mathbf{u}, \mathbf{v}$. Then,*

$$\mathbf{u} \bullet \mathbf{v} = ||\mathbf{u}|| \cdot ||\mathbf{v}|| \cos\theta$$

**Theorem (10.4)**

1. *$\theta$ is acute if $\mathbf{u} \bullet \mathbf{v} > 0$*
2. *$\theta$ is obtuse if $\mathbf{u} \bullet \mathbf{v} < 0$*
3. *$\theta$ is right if $\mathbf{u} \bullet \mathbf{v} = 0$*

**Theorem (10.5: Triangle Inequality)**

$$||\mathbf{u} + \mathbf{v}|| \leq ||\mathbf{u}|| + ||\mathbf{v}||$$

$$\Big| ||\mathbf{u}|| - ||\mathbf{v}|| \Big| \leq ||\mathbf{u} - \mathbf{v}||$$

# Norms

## Definition (Norms)

*Any operation* $(X)$ *of a vector to a real number satisfying below three properties is* *norm*. *Length of vector is a norm.*

1. $X(\mathbf{u}) \geq 0 \land X(\mathbf{u}) = 0$ *only when* $\mathbf{u} = \mathbf{0}$
2. $X(r\mathbf{u}) = |r|X(\mathbf{u})$
3. $X(\mathbf{u}+\mathbf{v}) \leq X(\mathbf{u}) + X(\mathbf{v})$

Norm is set of distance measures between two vectors.

# 1D,2D Objects in $\mathbb{R}^n$ Spaces

## Lines:One dimensional objects in $\mathbb{R}^n$ Spaces

$\mathbf{x}$ representing a line passing $\overline{\mathbf{x_0}}$ with direction $\overline{\mathbf{v}}$ is:

$$\mathbf{x} = \overline{\mathbf{x_0}} + t\overline{\mathbf{v}} \quad \forall t \in \mathbb{R} \qquad \text{(Parametric Representation)}$$

$\mathbf{x}$ representing a line passing $\overline{\mathbf{x_0}}, \overline{\mathbf{x_1}}$ is:

$$\mathbf{x} = (1-t)\overline{\mathbf{x_0}} + t\overline{\mathbf{x_1}} \quad \forall t \in \mathbb{R} \qquad \text{(Parametric Representation)}$$

## Planes: Two dimensional objects in $\mathbb{R}^n$ Spaces

$\mathbf{x}$ representing a plane passing $\overline{\mathbf{x_0}}$ with direction $\overline{\mathbf{v_1}}$ and $\overline{\mathbf{v_2}}$ is:

$$\mathbf{x} = \overline{\mathbf{x_0}} + t_1\overline{\mathbf{v_1}} + t_2\overline{\mathbf{v_2}} \quad \forall t_i \in \mathbb{R} \qquad \text{(Parametric Representation)}$$

$\mathbf{x}$ representing a plane containing $\overline{\mathbf{x_0}}, \overline{\mathbf{x_1}}, \overline{\mathbf{x_2}}$ is:

$$\mathbf{x} = (1-t_1-t_2)\overline{\mathbf{x_0}} + t_1\overline{\mathbf{x_1}} + t_2\overline{\mathbf{x_2}} \quad \forall t_i \in \mathbb{R} \quad \text{(Parametric Representation)}$$

# Nonparametric Equations

## Definition (Normal Vector)

*A normal vector* $\bar{\mathbf{n}}$ *of a* $n$*-dimensional object* $\mathbf{X}$ *is a vector which is perpendicular to any vectors in the object. Suppose* $\mathbf{x}$, $\bar{\mathbf{p}}$ *are location vectors in the object. Then,*

$$\bar{\mathbf{n}} \bullet (\mathbf{x} - \bar{\mathbf{p}}) = 0 \quad \forall \mathbf{x}, \mathbf{p} \in \mathbf{X}$$

## Finding Normal Vector

1. For linearly independent vectors $\mathbf{u_i}$, solve below systems of equations satisfying
$$\mathbf{n} \bullet \mathbf{u_i} = 0 \quad \forall i$$
2. $\mathbf{n} = \mathbf{u} \times \mathbf{v}$ (Only for $\mathbb{R}^3$ space)
$$\mathbf{u} \times \mathbf{v} := \left( \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right)$$

# Hyperplanes

## Definition (Hyperplane)

$\mathbf{x} \in \mathbb{R}^{n-1}$ *object in* $\mathbb{R}^n$ *space is a* *hyperplane* *with normal vector* $\bar{\mathbf{a}}$ *if:*

$$\bar{a}_1 x_1 + \bar{a}_2 x_2 + \cdots \bar{a}_n x_n = \bar{d}$$

*or,*

$$\bar{\mathbf{a}} \bullet \mathbf{x} = \bar{d}$$

# Budget Constraint

## Definition (Commotity Bundle, Price Vector, Budget Set)

*Let* $x_i \geq 0$ *be the quantity of* $i$*th commodity.*

$$\mathbf{x} := (x_1, \ldots, x_n), \quad x_i \geq 0 \quad \forall i \tag{Commodity Bundle}$$

*Let* $p_i \geq 0$ *be the preice of* $i$*th commodity.*

$$\mathbf{p} := (p_1, \cdots, p_n), \quad p_i \geq 0 \quad \forall i \tag{Price Vector}$$

*Then budget set* $\mathbf{x}$ *can be defined with given budget* $\bar{I}$*:*

$$\bar{\mathbf{p}} \bullet \mathbf{x} \leq \bar{I} \tag{Budget Constraint}$$