

# Functions of Several Variables

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## 1 Functions Between Euclidean Spaces

### Definitions

**Definition 1** (Function, Domain, Target, Image: General Definitions). *Function*  $f : A \rightarrow B$  is a rule that assigns each object of  $A$  (domain) to one object in  $B$  (target space). Image of  $f$  is  $\{f(\mathbf{x}) | \mathbf{x} \in A\} \subset B$

**Examples:**  $f : \mathbb{R}^n \rightarrow \mathbb{R}$

$$f(\mathbf{x}) = \bar{\mathbf{a}} \bullet \mathbf{x} \quad (\text{Linear})$$

$$f(\mathbf{x}) = \bar{k} \prod_i x_i^{\bar{b}_i} \quad (\text{Cobb-Douglas})$$

$$f(\mathbf{x}) = \bar{k} \left( \sum_i \bar{c}_i x_i^{-\bar{a}} \right)^{-\bar{b}/\bar{a}} \quad (\text{CES})$$

$$\mathbf{f} : \mathbb{R}^k \rightarrow \mathbb{R}^m$$

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Let  $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$ . Then  $\mathbf{f} : \mathbb{R}^k \rightarrow \mathbb{R}^m$  can be represented by  $f_i$  ( $i = 1, 2, \dots, m$ )

$$\mathbf{f}(\mathbf{x}) := (f_1(\mathbf{x}), \dots, f_m(\mathbf{x}))$$

### Examples

- Production function with  $k$  input factors and  $m$  products
- Utility mapping  $\mathbf{u} : \mathbb{R}^{km} \rightarrow \mathbb{R}^m$ 
  - $u_i : \mathbb{R}^k \rightarrow \mathbb{R}$ : Individual utility function of customer  $i$
  - $k$ : # of goods
  - $m$ : # of consumers
  - $\mathbf{x}_i$ : consumption of customer  $i$

$$\mathbf{u}(\mathbf{x}_1, \dots, \mathbf{x}_m) = (u_1(\mathbf{x}_1), \dots, u_m(\mathbf{x}_m))$$

$$\mathbf{f} : \mathbb{R} \rightarrow \mathbb{R}^m$$

$$\mathbf{f}(t)$$

$$\mathbf{f}(t) := (f_1(t), \dots, f_m(t))$$

Geometrically,  $\mathbf{f}(t)$  is a parametric curve on  $\mathbb{R}^m$  space (cf. parametric line)

## 2 Geometric Representation of Functions

### Level Curves

#### Level Curves

Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^1$ . Then level curves of  $f$  are curves on domain space with same  $f(\mathbf{x})$ . *I.e.*,

$$\{\mathbf{x} | f(\mathbf{x}) = \bar{c}\}$$

- Isoquant: level curve of production function
- Indifference curve: level curve of utility function
- Generally, when  $f : \mathbb{R}^k \rightarrow \mathbb{R}$  it is called level set and this is  $k$  dimensional nonlinear object

## 3 Special Kinds of Functions

### Linear Functions on $\mathbb{R}^k$

**Definition 2** (Linear Function from  $\mathbb{R}^k$  to  $\mathbb{R}^m$ ).  $\mathbf{f}$  is a linear function when

1.  $\mathbf{f}(\mathbf{x}_1 + \mathbf{x}_2) = \mathbf{f}(\mathbf{x}_1) + \mathbf{f}(\mathbf{x}_2)$
2.  $\mathbf{f}(r\mathbf{x}) = r\mathbf{f}(\mathbf{x})$

**Theorem 1** (13.1,2). •  $f : \mathbb{R}^k \rightarrow \mathbb{R}$  is a linear function  $\Rightarrow f(\mathbf{x}) = \bar{\mathbf{a}} \bullet \mathbf{x}$ ,  $\mathbf{a} \in \mathbb{R}^k$

- $\mathbf{f} : \mathbb{R}^k \rightarrow \mathbb{R}^m$  is a linear function  $\Rightarrow \mathbf{f}(\mathbf{x}) = \bar{A} \bullet \mathbf{x}$ ,  $A : m \times k$  matrix

### Quadratic Forms

**Definition 3** (Quadratic Form on  $\mathbb{R}^k$ ).  $f : \mathbb{R}^k \rightarrow \mathbb{R}$  is of the quadratic form if:

$$f(\mathbf{x}) = \sum_{i,j}^k \bar{a}_{ij} x_i x_j$$

more elegantly,

$$f(\mathbf{x}) = \mathbf{x}^T \bar{A} \mathbf{x}$$

In this case,  $A$  can be always symmetric.

### Monomial

**Definition 4** (Monomials on  $\mathbb{R}^k$ ).  $f : \mathbb{R}^k \rightarrow \mathbb{R}$  is a monomial if:

$$f(\mathbf{x}) = \bar{c} \prod_i^k x_i^{\bar{a}_i}, \quad a_i \in \mathbb{N} \cup \{0\}$$

The degree of above monomial is  $\sum_i^k a_i$

## 4 Continuous Functions

### Continuous

**Definition 5** (Continuous Function on  $\mathbb{R}^k$ ).  $\mathbf{f} = (f_1, \dots, f_m)$  is continuous at  $\mathbf{x}$  iff all  $f_i$  are continuous at  $\mathbf{x}$