

One-Variable Calculus: Exponents and Logarithms

CH5

`mailto:econMath.namun+2016f@gmail.com` (한신대)

`mailto:ku.econ205+2016f@gmail.com` (고려대)

2016년 9월 21일

Table of Contents

- 1 Exponential Functions
- 2 The Number e
- 3 Logarithms
- 4 Properties of Exp and Log
- 5 Derivatives of Exp and Log
- 6 Applications

Exponential Functions

Definition (Exponential Function)

$f : \mathbb{R} \rightarrow \mathbb{R}$ is exponential function if $f(x) = \bar{a}\bar{b}^x$, $\bar{b} > 0$

- $x \in \mathbb{N} \Rightarrow f(x) := \bar{a} \prod_{i=1}^x \bar{b}$
- $f(0) := \bar{a}$
- $f(1/n) := \bar{a} \sqrt[n]{\bar{b}}$
- $f(m/n) := \bar{a} \sqrt[n]{\bar{b}^m}$
- $x < 0 \Rightarrow f(x) = \bar{a}(1/\bar{b})^{|x|}$
- Graph: convex, monotonic increasing ($\bar{b} > 1$) or decreasing ($\bar{b} \in (0, 1)$) function

Growth of an Account with Interest rate r

Saving Account at $t = \bar{T}$ with Interest rate \bar{r} , Initial Endowment \bar{A}

$$A_t = \bar{A} (1 + \bar{r})^{\bar{T}}$$

Compound Interest

If interest is compounded n times per time unit,

$$A_t = \bar{A} \left(1 + \frac{\bar{r}}{n}\right)^{n\bar{T}}$$

Continuous Compounding

Compound Interest with $n \rightarrow \infty$

$$A_t = \lim_{n \rightarrow \infty} \bar{A} \left(1 + \frac{\bar{r}}{n}\right)^{n\bar{T}} = \bar{A} e^{\bar{r}\bar{T}}$$

Number e

Definition (The Number e)

$$e := \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = \sum_{n=0}^{\infty} \frac{1}{n!} \approx 2.718281693 \dots$$

e is irrational number.

Theorem (5.1)

$$\lim_{n \rightarrow \infty} A \left(1 + \frac{r}{n}\right)^{nt} = Ae^{rt}$$

In general, an initial quantity a_0 with growth rate r (per time unit) become $a_0 e^{rt}$ at time t (time unit)

Logarithm

Definition (Base b Logarithm)

Base b logarithm is an inverse of exponential function with base b

$$f = b^x \quad \Leftrightarrow \quad x = \log_b f$$

- $a^{\log_a z} = z$
- $\log_a a^y = y$
- Graph: concave, monotonic increasing ($b > 1$) or convex, monotonic decreasing ($b \in (0, 1)$)

Natural Logarithm

Definition (Natural Logarithm)

Base e logarithm is natural logarithm

$$\ln x := \log_e x$$

$$\ln x = y \quad \Leftrightarrow \quad e^y = x$$

$$e^{\ln x} = x$$

$$\ln e^x = x$$

Basic Properties of Exponential functions

$\forall r, s \in \mathbb{R},$

① $a^r a^s = a^{r+s}$

② $a^{-r} := 1/a^r$

③ $a^r / a^s = a^{r-s}$

④ $(a^r)^s = a^{rs}$

⑤ $a^0 := 1$

Basic Properties of Logarithmic functions

$$\forall r, s, a, b, c > 0 \wedge a, c \neq 1,$$

$$\textcircled{1} \log(rs) = \log r + \log s$$

$$\textcircled{2} \log(1/s) = -\log s$$

$$\textcircled{3} \log(r/s) = \log r - \log s$$

$$\textcircled{4} \log r^s = s \log r$$

$$\textcircled{5} \log 1 = 0$$

$$\textcircled{6} \log_a b = \frac{\log_c b}{\log_c a} = \frac{\ln b}{\ln a}$$

(Ex5.4) Rule of 70 (or 69)

Derivatives of Exp and Log functions

Theorem (5.2)

$$(e^x)' = e^x$$

$$(\ln x)' = \frac{1}{x}$$

if $u \in \mathbf{C}^1$, from chain rule,

$$(e^u)' = (e^u) u'$$

$$(\ln u)' = \frac{u'}{u} \quad (u > 0)$$

Present Value

Present Value (PV)

After time T , A (at $t = 0$) grow to B (at $t = T$)

$$B = Ae^{rT}$$

A is the present value (PV) of B at $t = T$

$$A = Be^{-rT}$$

- PV of annuity

Logarithmic Derivative

(Ex5.10) $(x^x)' = ?$

Elasticity of f is the Slope in log-log Graph of f

$$\epsilon := \frac{\frac{df}{f}}{\frac{dx}{x}} = \frac{d \ln f}{d \ln x}$$