

# Eigenvalues and Eigenvectors (1)

## Ch.23

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# Terms

## State

In each period  $t$ , the system is in one and only one of  $k$  states  $S_1, \dots, S_k$ .

## Definition (Stochastic Process)

*A stochastic process is a rule which gives the probability of the state  $i$  at the period  $t = n + 1$  given the probabilities of all previous states ( $t = 1, 2, \dots, n$ )*

Note:  $\mathbf{x}_t = (x_{1,t}, \dots, x_{k,t})$  is the probabilities of all  $k$  possible states at time  $t$

## Definition (Markov Process)

*A stochastic process that the probability of state  $i$  at  $t = n + 1$  depends only on what state the system was in at  $t = n$  is a Markov process.*

Note: Markov processes are memoryless.

## Definition (Transition Matrix)

$M$  is a transition matrix for stochastic process  $\mathbf{x}_t$  if:

$$\mathbf{x}_{t+1} = M\mathbf{x}_t$$

If  $\sum_i M_{ij} = 1 \quad \forall j$  (i.e., all column sums are 1), this process is a Markov process. Here, nonnegative scalar  $M_{ij}$  is transition probabilities that the process will be in state  $i$  at  $t = n + 1$  if it is in state  $j$  at  $t = n$

If  $M_{ij}$ , transition probabilities are fixed and independent of time indices  $t$ , this process is time-homogeneous or that  $M_{ij}$  are stationary.

# Regular Markov Matrix

## Definition (Regular Markov Matrix)

$M$  is a regular Markov matrix if:

- ①  $\sum_i M_{ij} = 1 \quad \forall j$
- ②  $M_{ij} \geq 0 \quad \forall i, j$
- ③  $\exists r \in \mathbb{N}$  s.t.  $M^r > 0 \quad \forall i, j$
- ④ Condition 3 hold when  $r = 1$

## Theorem (23.15)

Let  $M$  be a regular Markov matrix. Then,

- ① 1 is an eigenvalue of  $M$  of multiplicity 1 (i.e., 1 is not a repeated root)
- ② For every other eigenvalue  $r$  of  $M$ ,  $|r| < 1$
- ③  $\mathbf{w}_1$ , Eigenvector for eigenvalue 1 has strict positive components
- ④ If  $\mathbf{v}_1 = \mathbf{w}_1 / \|\mathbf{w}_1\|$ ,  $\mathbf{v}_1$  is a probability vector and if  $\mathbf{x}_{t+1} = M\mathbf{x}_t$ ,

$$\lim_{n \rightarrow \infty} \mathbf{x}_n = \mathbf{v}_1$$

Note: example of non-regular Markov process. If  $\exists i$  s.t.  $M_{ii} = 1$ , This state  $i$  is absorbing state. I.e., once process reach state  $i$ , this state does not change forever. Therefore, this process will eventually reach one of these states  $i$  and then stay there forever.

# Symmetric Matrices

## Example of Symmetric Matrices in Economics

- (Bordered) Hessians in optimization problem
- Variance-covariance matrices in statistics

Fortunately, symmetric matrices do not have repeated or complex eigenvalues.

## Definition (Orthogonal Matrix)

*A matrix  $P$  satisfies the condition  $P^{-1} = P^T$ , (i.e.,  $P^T P = I$ ) is orthogonal matrix.*

We can find uncoupled system when  $A$  is symmetric.

# Properties of Symmetric Matrices

## Theorem (23.16)

Let  $A \in M_k$  and  $A^T = A$ . Then,

- All  $k$  roots of  $\det A - rI = 0$  are real numbers.
- All corresponding eigenvectors  $\mathbf{w}_i$  are orthogonal
- $\exists P$  staisfying:
  - $\mathbf{w}_i$ s are normalized eigenvectors for each eigenvalues  $r_i$ :  $\|\mathbf{w}_i\| = 1 \quad \forall i$
  - Matrix  $[\mathbf{w}_1 \quad \cdots \quad \mathbf{w}_k]$  is nonsingular
  - $\mathbf{w}_i \mathbf{w}_j = 0 \quad \forall i \neq j$  (orthogonal to each other)
  - $P^{-1} = P^T$
  -

$$P^{-1}AP = P^TAP = \begin{pmatrix} r_1 & 0 & \cdots & 0 \\ 0 & r_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r_k \end{pmatrix}$$



# Quadratic Forms

## Quadratic Forms

Every quadratic form  $Q(\mathbf{x})$  can be represented by symmetric matrix  $A$ :

$$Q(\mathbf{x}) = \mathbf{x}^T A \mathbf{x} \quad \wedge \quad A^T = A$$

Always, we can find uncoupled system by taking

$P^T \mathbf{x} = [\mathbf{w}_1 \ \cdots \ \mathbf{w}_k]^T \mathbf{x}$  when  $\mathbf{w}_i$  are corresponding normalized eigenvalues  $r_1, \dots, r_k$ . Let the transformed uncoupled system be  $\mathbf{y} = P^T \mathbf{x}$ . Then,

$$Q(\mathbf{x}) = Q(P\mathbf{y}) = (P\mathbf{y})^T A (P\mathbf{y}) = \mathbf{y}^T (P^T A P) \mathbf{y}$$

Note:  $\mathbf{y}$  is a linear change of coordinates from  $\mathbf{x}$ .

# Definiteness and Eigenvalues

## Theorem (23.17)

Let  $A^T = A \in M_k$  and  $r_1, \dots, r_k$  are eigenvalues of  $A$ . Then,

- ①  $A$  is PD  $\iff r_i > 0 \quad \forall i$
- ②  $A$  is ND  $\iff r_i < 0 \quad \forall i$
- ③  $A$  is PSD  $\iff r_i \geq 0 \quad \forall i$
- ④  $A$  is NSD  $\iff r_i \leq 0 \quad \forall i$
- ⑤  $A$  is ID  $\iff \exists i, j \quad \text{s.t.} \quad r_i < 0 \wedge r_j > 0$

## Theorem (23.18)

Let  $A^T = A \in M_k$ . Then the below statements are equivalent:

- ①  $A$  is PD
- ②  $\exists B \quad \text{s.t.} \quad A = B^T B \wedge \exists B^{-1}$
- ③  $\exists Q \quad \text{s.t.} \quad Q^T A Q = I \wedge \exists Q^{-1}$