Linear Independence

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LI: Definition

Definition (Linear Combinations, Span, Linear (in)dependence)

$\mathcal{L}$ is spanned set generated by linear combination of k vectors $\mathbf{v_1}, \cdots, \mathbf{v_k}$

$$\mathcal{L}[\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_k} \in \mathbb{R}^{\mathbf{n}}] := \left\{ \sum_i^k r_i \mathbf{v}_i : \forall r_i \in \mathbb{R} \right\}$$

$\mathbf{v_1}, \cdots, \mathbf{v_k}$ *are linearly independent iff:*

$$\mathcal{L}[\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_k} \in \mathbb{R}^{\mathbf{n}}] \subset \mathbb{R}^k$$

Otherwise, $\mathbf{v_1}, \cdots, \mathbf{v_k}$ *are linearly dependent*

Note: Different vectors can span the same space. Canonical basis $\mathbf{e_i}$ can be the representative vector (basis) for $\mathbb{R}^n$ space.

LI: Alternative Definition

Definition (Linear (in)dependence: Alternative definition)

$\mathbf{v_1}, \cdots, \mathbf{v_k} \in \mathbb{R}^\mathbf{n}$ *are* *linearly dependent* *iff:*

$$\exists c_1, \cdots, c_k \neq 0 \quad s.t. \quad \sum_i^k c_i \mathbf{v_i} = \mathbf{0}$$

$\mathbf{v_1}, \cdots, \mathbf{v_k} \in \mathbb{R}^\mathbf{n}$ *are* *linearly independent* *iff:*

$$\sum_i^k c_i \mathbf{v_i} = \mathbf{0} \quad \Rightarrow \quad c_1 = \cdots = c_k = 0$$

Theorems

Theorem (11.1)

$\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_k} \in \mathbb{R}^{\mathbf{n}}$ *are linearly dependent iff*

$$A \begin{pmatrix} c_1 \\ \vdots \\ c_k \end{pmatrix} = \mathbf{0}$$

has nonzero solution $\mathbf{c}$*, where* A *is* $n \times k$ *matrix whose* $C_i = \mathbf{v_i}$*.* $i.e.,$

$$A = \begin{pmatrix} \mathbf{v_1} & \mathbf{v_2} & \cdots & \mathbf{v_k} \end{pmatrix}$$

Theorem (11.2)

$\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_n} \in \mathbb{R}^{\mathbf{n}}$ *are linearly independent iff*

$$\det \begin{pmatrix} \mathbf{v_1} & \mathbf{v_2} & \cdots & \mathbf{v_n} \end{pmatrix} \neq 0$$

Checking LI

Procedure: Checking Linear (in)dependence

1. Stack $\mathbf{v_i}$ to make $k \times n$ matrix, A.
2. Calculate $rank(A)$: this is the dimension of $\mathcal{L}[\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_k} \in \mathbb{R}^{\mathbf{n}}]$
3. $\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_k} \in \mathbb{R}^{\mathbf{n}}$ are linearly independent iff $rank(A) = k$. Otherwise, they are linearly dependent

Note: $rank(A^T) = rank(A)$

Theorem (11.3)

If $k > n$, any set of k vectors in $\mathbb{R}^n$ is linearly dependent.

Basis

Definition (Basis)

Let $V = \mathcal{L}[\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_k} \in \mathbb{R}^\mathbf{n}]$. *If* $\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_k} \in \mathbb{R}^\mathbf{n}$ *are linearly independent,* $\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_k} \in \mathbb{R}^\mathbf{n}$ *is called a basis of* V. *More generally,* $\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_k} \in \mathbb{R}^\mathbf{n}$ *forms a basis of* V *if:*

1. $\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_k} \in \mathbb{R}^\mathbf{n}$ *span* V
2. $\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_k} \in \mathbb{R}^\mathbf{n}$ *are linearly independent*

Theorem (11.7)

Every basis of $\mathbb{R}^n$ *contains* n *vectors.*

Linear Independence and Basis

Theorem (11.8)

Let $\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_n} \in \mathbb{R}^{\mathbf{n}}$ *be a collection of* n *vectors in* $\mathbb{R}^n$*. And let* $n \times n$ *matrix* $A = \begin{pmatrix} \mathbf{v_1} & \cdots & \mathbf{v_n} \end{pmatrix}$*. Then the following statements are equivalent:*

1. $\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_n} \in \mathbb{R}^{\mathbf{n}}$ *are linearly independent.*
2. $\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_n} \in \mathbb{R}^{\mathbf{n}}$ *span* $\mathbb{R}^n$
3. $\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_n} \in \mathbb{R}^{\mathbf{n}}$ *form a basis of* $\mathbb{R}^n$
4. $\det(A) \neq 0$