Matrix Algebra

Ch.8

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1 Matrix Algebra

Matrix

Definition 1 (Matrix). *Matrix is a rectangular array of numbers (scalars)*

Let $a_{ij} \in \mathbb{R}$ or $A_{ij} \in \mathbb{R}$ be the ith row and jth column element of matrix A

Definition 2 (Equal).

$$A = B \quad \iff \begin{cases} \text{same size} \\ a_{ij} = b_{ij} \quad \forall i, j \end{cases}$$

Addition, Subtraction

Let A, B be $n \times k$ matrices and $r \in \mathbb{R}$

Definition 3 (Addition).

$$(A + B)_{ij} := a_{ij} + b_{ij} \quad \forall i, j$$

Important note: the first + and the second + are not same operators

Definition 4 (Subtraction).

$$(A - B)_{ij} := a_{ij} - b_{ij} \quad \forall i, j$$

Multiplications of Matrices

Definition 5 (Scalar Multiplication).

$$(rA)_{ij} := rA_{ij} \quad \forall i, j$$

Let A be $n \times k$ matrix and B be $k \times m$ matrix. Then AB is $n \times m$ matrix.

Definition 6 (Matrix Multiplication)**.**

$$(AB)_{ij} := A_{i1}B_{1j} + A_{i2}B_{2j} + \cdots + A_{ik}B_{kj} = \sum_{r=1}^{k} A_{ir}B_{rj}$$

For $n \times n$ matrices, identity matrix I_n is a multiplicative identity.

$$AI = IA = A$$

Laws of Matrix Algebra

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$$(A + B) + C = A + (B + C) \qquad \text{(Associative Law for Addition)}$$

$$(AB)C = A(BC) \qquad \text{(Associative Law for Multiplication)}$$

$$A + B = B + A \qquad \text{(Commutative Law for Addition)}$$

$$A(B + C) = AB + AC \qquad \text{(Distributive Law)}$$

$$(A + B)C = AC + BC \qquad \text{(Distributive Law)}$$

Important Note: $AB \neq BA$

Transpose

Definition 7 (Transpose)**.** *$A^\intercal$ $(n \times m)$ is a transpose of A $(m \times n)$ if:*

$$(A^\intercal)_{ij} := A_{ji} \quad \forall i, j$$

$$(A \pm B)^\intercal = A^\intercal \pm B^\intercal$$

$$(A^\intercal)^\intercal = A$$

$$(rA)^\intercal = rA^\intercal$$

$$(AB)^\intercal = B^\intercal A^\intercal \qquad \text{(Theorem 8.1)}$$

2 Special Kinds of Matrices

Special Kinds of Matrices (1)

Suppose A is $k \times n$ matrix. Then,

Definition 8 (Special Kinds of Matrices (1)).

- *A is a <u>square matrix</u> if $k = n$*
- *A is a <u>column matrix</u> if $n = 1$*
- *A is a <u>row matrix</u> if $k = 1$*
- *A is a <u>diagonal matrix</u> if $k = n$ and $a_{ij} = 0 \quad \forall i \neq j$*
- *A is a <u>scalar matrix</u> if $A = tI_n$*
- *A is an <u>upper-triangular matrix</u> if $a_{ij} = 0 \quad \forall i > j$*
- *A is a <u>lower-triangular matrix</u> if $a_{ij} = 0 \quad \forall i < j$*

Special Kinds of Matrices (2)

Definition 9 (Special Kinds of Matrices (2)).

- *A is a <u>symmetric matrix</u> if A is square matrix and $a_{ij} = a_{ji} \quad \forall i, j$. Or, $A^\intercal = A$*
- *A is an <u>Idempotent matrix</u> if $AA = A$*
- *A is a <u>permutation matrix</u> if A is the result of I_n with ERO_1 (row exchange)*
- *A is a <u>nonsingular matrix</u> if $rankA = \#row = \#column$*

If a coefficient matrix of a system of linear equations is <u>nonsingular</u>, this system has only one solution $\mathbf{x} = A^{-1}\mathbf{b}$

3 Elementary Matrices

Elementary Matrix

Let E be an elementary matrix of some EROs. Then,

Theorem 1 (8.3). *ERO with a matrix A is equivalent to EA*

Theorem 2 (8.2).

- *Let $E1_{ij}$ be the permutation matrix with interchanging R_i and R_j of I_n, then $E1_{ij}$ is equivalent to $ERO_1(i, j)$*
- *Let $E2_{k,j,i}$ be the result of $ERO_2(k, j, i)$ from I_n, then $E2_{k,j,i}$ is equivalent to $ERO_2(k, j, i)$*
- *Let $E3_{k,i}$ be the result of $ERO_3(k, i)$ from I_n, then $E3_{k,i}$ is equivalent to $ERO_3(k, i)$*

Elementary Matrix

Definition 10 (Elementary Matrix)**.** *$E1, E2, E3$ are elementary matrices corresponding to their EROs*

Theorem 3 (8.4)**.** *Let $A \in M_n$ (set of $n \times n$ matrices), $E_i \in EM$ (set of elementary matrices), and $(R)REFM$ be the set of (R)REF matrices. Then:*

$$\exists E_i \quad i = 1, 2, \cdots, m \quad s.t. \quad \prod_{i=m}^{1} E_i A \in (R)REFM$$

or

$$E_m E_{m-1} \cdots E_2 E_1 A \in (R)REFM$$

4 Algebra of Square Matrices

Inverse of Matrices

Suppose $A, B \in M_n$

Definition 11 (Inverse, Invertible)**.** *B is (left, or right) inverse for A if:*

$$\underbrace{AB}_{B\text{: Right inverse}} = \overbrace{BA}^{B\text{: Left inverse}} = I$$

A is invertible if $\exists B$

Notation: $B = A^{-1}$

Theorem 4 (8.5:Uniquenes of Inverse)**.** *$A \in M_n$ can have at most one inverse. (left inverse = right inverse)*

Inverse Matrices and the Solution of Linear Systems

Theorem 5 (8.6)**.** *For $A \in M_n$,*

$$\exists A^{-1} \quad \Rightarrow \quad \begin{cases} A \text{ is nonsingular} \\ \text{Unique solution of } A\mathbf{x} = \mathbf{b} \quad \Rightarrow \quad \mathbf{x} = A^{-1}\mathbf{b} \end{cases}$$

Proof: easy

Theorem 6 (8.7: inverse of Th8.6)**.**

$$A \in M_n \text{ is nonsingular} \quad \Rightarrow \quad \exists A^{-1}$$

Proof: difficult

Calculation of Inverse Matrix

Calculation of Inverse Matrix

$$[A|I] \xrightarrow{EROs} [I|A^{-1}]$$

If RREF is not I_n, $\nexists A^{-1}$

Theorem 7 (8.8)**.** *Let* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2$. *$A$ is nonsingular iff* $ad - bc \neq 0$

For general case ($A \in M_n$), see Ch.9

Equivalent statements

Theorem 8 (8.9)**.** *For $A \in M_n$, the following statements are equivalent*

1. $\exists A^{-1}$
2. *A has right inverse*
3. *A has left inverse*
4. $A\mathbf{x} = \mathbf{b}$ *has at least one solution for every* $\mathbf{b}$
5. $A\mathbf{x} = \mathbf{b}$ *has at most one solution for every* $\mathbf{b}$
6. *A is nonsingular*
7. $rankA = n$

Properties of Inverse Matrices and Their Exponentials

Theorem 9 (8.10)**.** *If $A, B \in M_n$ and $\exists A^{-1}, B^{-1}$,*

1. $(A^{-1})^{-1} = A$
2. $(A^\intercal)^{-1} = (A^{-1})^\intercal$
3. $\exists (AB)^{-1} \wedge (AB)^{-1} = B^{-1}A^{-1}$

Definition 12 (Matrix Exponential)**.**

$$A^m := \prod_{i=1}^{m} A$$

$$A^{-m} := (A^{-1})^m$$

Expoenetial Properties of Invertible Matrices

Theorem 10 (8.11)**.**

$$\exists A^{-1} \quad \Rightarrow \quad \begin{cases} \exists A^{-m} \quad \forall m \in \mathbb{N} \\ A^r A^s = A^{r+s} \quad \forall r, s \in \mathbb{N} \\ \forall r \in \mathbb{R} - \{0\}, \quad \exists (rA)^{-1} \wedge (rA)^{-1} = \frac{1}{r} A^{-1} \end{cases}$$

Important Note: $(AB)^k \neq A^k B^k$

5 Partitioned Matrices

Partitioned Matrices

Somtimes, matrix of matrices can be more convenient.

Definition 13 (Submatrix, Partitioned matrix)**.**

- *A submatrix of matrix A is a matrix obtained by deleting some R_i or C_j*
- *A partitioned matrix is a matrix partitioned into submatrices by horizontal and/or vertical lines which extended along entire rows or columns of a matrix A*

Partitioned Matrices

Theorem 11 (8.15)**.** *Let A be a square matrix partitioned as*

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

and $A_{11}, A_{22} \in M_n$. Then,

$$\exists A_{22}^{-1} \wedge \exists D^{-1} \wedge D = A_{11} - A_{12} A_{22}^{-1} A_{21}$$
$$\Rightarrow \quad A^{-1} = \begin{pmatrix} D^{-1} & -D^{-1} A_{12} A_{22}^{-1} \\ -A_{22}^{-1} A_{21} D^{-1} & A_{22}^{-1}(I + A_{21} D^{-1} A_{12} A_{22}^{-1}) \end{pmatrix}$$