

# Calculus of Several Variables

## Ch.14

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# Partial Derivative

Let  $f : D \subset \mathbb{R}^n \rightarrow \mathbb{R}$  and  $\mathbf{e}_i$  be a vector whose  $i$ th element is 1 and others are 0.

$$\mathbf{e}_i := (\overbrace{0, 0, \dots, 0}^i, 1, 0, \dots, 0)$$

## Definition (Partial Derivative)

Partial derivative at  $\bar{\mathbf{x}}_0 \in D$  is

$$\frac{\partial f}{\partial x_i} := \lim_{h \rightarrow 0} \frac{f(\bar{\mathbf{x}}_0 + h\mathbf{e}_i) - f(\bar{\mathbf{x}}_0)}{h}$$

When  $n = 1$ , partial derivative is equivalent to derivative of one variable function.

## Calculation Procedure

- Treat  $x_i$  as the only variable in  $f$
- Treat  $x_{-i}$  as constant

# Marginal Products

## Production Function, Marginal Product of Labor [or Capital]

Let  $Q$  be the production function of a firm. If the firm's resources for production are  $\mathbf{x} = (L, \mathbf{K}) = (L, K_1, K_2, \dots, K_N)$ ,

$$MP_L := \frac{\partial Q}{\partial L}, \quad MP_{K_i} := \frac{\partial Q}{\partial K_i}$$

Interpretation: Small change  $\Delta K_i$  (*ceteris paribus*) in  $K_i$  can cause output change  $\Delta Q$  around  $(L^*, \mathbf{K}^*)$

$$\Delta Q \approx \frac{\partial Q}{\partial K_i}(L^*, \mathbf{K}^*) \Delta K_i$$

## Marginal Utility

Let  $U(\mathbf{x})$  be the utility function with respect to commodity bundle  $\mathbf{x}$ . Then  $\frac{\partial U}{\partial x_i}$  is marginal utility of commodity  $i$  at  $\mathbf{x}^*$

## Elasticity: Multi variable version

$x_i$  elasticity of  $Q(\mathbf{x})$  around  $(\mathbf{x}^*, Q^*)$  is:

$$\epsilon_i := \frac{\frac{\partial Q}{\partial x_i}}{\frac{Q^*}{x_i^*}} = \frac{x_i^*}{Q^*} \frac{\partial Q}{\partial x_i}(\mathbf{x}^*)$$

In general, elasticity is ratio of rates of changes. When the sign of elasticity is not important,  $|\epsilon|$  can be used.

# Partial Derivative: Geometric Interpretation

$$f : \mathbb{R}^2 \rightarrow \mathbb{R}$$

- Think of  $f(\mathbf{x}) = x_1^2 + x_2^2$ .
- If  $x_2 = \bar{x}_2$ ,  $f(x_1, \bar{x}_2)$  is equivalent to one variable function  $\tilde{f}(x_1) = x_1^2 + \bar{x}_2^2$ .
- Graph of  $\tilde{f}$  is intersection of the graph of  $f$  with the slice  $x_2 = \bar{x}_2$ .
- $\frac{\partial f}{\partial x_1}(\bar{x}_1, \bar{x}_2) = \frac{\partial \tilde{f}}{\partial x_1}(\bar{x}_1)$  is the slope of  $\tilde{f}$  on  $\bar{x}_1$ , slope of the tangent line to the curve  $\tilde{f}$  (on the plane  $x_2 = \bar{x}_2$ )

# Example

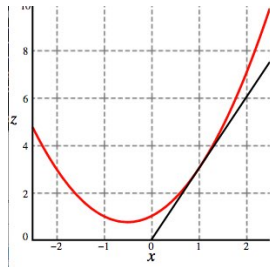
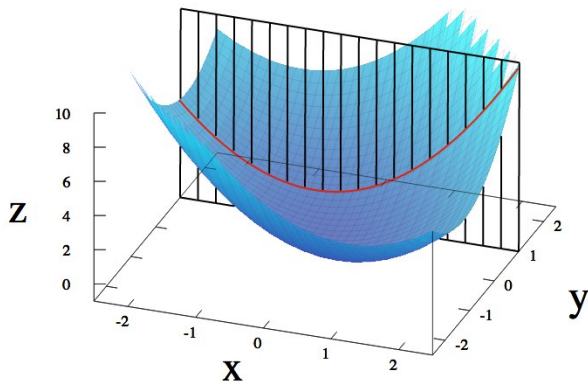


Figure: Graph of  $z = x^2 + xy + y^2$  with intersection  $y = 1$

## Finding Tangent Plane

Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  is differentiable. When finding a tangent plane on  $(\bar{x}_1, \bar{x}_2)$ , we need to get at least two independent vectors:

$\left(1, 0, \frac{\partial f}{\partial x_1}(\bar{x}_1, \bar{x}_2)\right)$  (slice  $x_2 = \bar{x}_2$ ), and  $\left(0, 1, \frac{\partial f}{\partial x_2}(\bar{x}_1, \bar{x}_2)\right)$  (slice  $x_1 = \bar{x}_1$ )

Then the tangent plane with two parameters  $\Delta x_1, \Delta x_2$  is:

$$\begin{aligned} & (\bar{x}_1, \bar{x}_2, f(\bar{\mathbf{x}})) + \Delta x_1 \left(1, 0, \frac{\partial f}{\partial x_1}(\bar{x}_1, \bar{x}_2)\right) + \Delta x_2 \left(0, 1, \frac{\partial f}{\partial x_2}(\bar{x}_1, \bar{x}_2)\right) \\ &= \left(\bar{x}_1 + \Delta x_1, \bar{x}_2 + \Delta x_2, f(\bar{\mathbf{x}}) + \frac{\partial f}{\partial x_1}(\bar{x}_1, \bar{x}_2)\Delta x_1 + \frac{\partial f}{\partial x_2}(\bar{x}_1, \bar{x}_2)\Delta x_2\right) \end{aligned}$$

This interpretation can be extended to  $n$  dimension.



# The Total Derivative

Changes in All Direction:  $f : \mathbb{R}^n \rightarrow \mathbb{R}^1$

Let  $d\mathbf{x} = (dx_1, \dots, dx_n)$  and  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ , differentiable. Then small change of  $d\mathbf{x}$  will cause small change of  $df = f(\bar{\mathbf{x}} + d\bar{\mathbf{x}}) - f(\bar{\mathbf{x}}) \in \mathbb{R}$  and

$$df = f(\bar{\mathbf{x}} + d\bar{\mathbf{x}}) - f(\bar{\mathbf{x}}) = \frac{\partial f}{\partial x_1}(\bar{\mathbf{x}})dx_1 + \dots + \frac{\partial f}{\partial x_n}(\bar{\mathbf{x}})dx_n = Df_{\mathbf{x}}d\mathbf{x}$$

And  $Df_{\mathbf{x}} := \left( \frac{\partial f}{\partial x_1}(\bar{\mathbf{x}}) \quad \frac{\partial f}{\partial x_2}(\bar{\mathbf{x}}) \quad \dots \quad \frac{\partial f}{\partial x_n}(\bar{\mathbf{x}}) \right)$ : (Jacobian) derivative of  $f$  at  $\bar{\mathbf{x}}$  or The linear approximation of  $f$  at  $\bar{\mathbf{x}}$ , or Gradient vector  $\nabla f$

Note: In this case,  $Df_{\mathbf{x}}$  is a vector or  $1 \times n$  matrix.

# More General Case

Changes in All Direction:  $\mathbf{f} : \mathbb{R}^n \rightarrow \mathbb{R}^m$

Let  $d\mathbf{x} = (dx_1, \dots, dx_n)$  and  $\mathbf{f} : \mathbb{R}^n \rightarrow \mathbb{R}^m$ , differentiable. Then small change of  $d\mathbf{x}$  will cause small change of  $d\mathbf{f} = \mathbf{f}(\bar{\mathbf{x}} + d\bar{\mathbf{x}}) - \mathbf{f}(\bar{\mathbf{x}}) \in \mathbb{R}^m$  and

$$d\mathbf{f} = \mathbf{f}(\bar{\mathbf{x}} + d\bar{\mathbf{x}}) - \mathbf{f}(\bar{\mathbf{x}}) = \frac{\partial \mathbf{f}}{\partial x_1}(\bar{\mathbf{x}})dx_1 + \dots + \frac{\partial \mathbf{f}}{\partial x_n}(\bar{\mathbf{x}})dx_n = D\mathbf{f}_{\mathbf{x}}d\mathbf{x}$$

Note: In this case,  $D\mathbf{f}_{\mathbf{x}}$  is  $m \times n$  matrix

# Curve in $\mathbb{R}^n$

## Definition (curve)

A curve in  $\mathbb{R}^n$  is  $n$ -tuple of continuous one variable functions

$$\mathbf{x}(t) = (x_1(t), \dots, x_n(t))$$

$x_i$ : coordination function,  $t$ : parameter

## Velocity (or Tangent) Vector

$\mathbf{x}'$  is the velocity (tangent) vector of the curve at  $t$

$$\mathbf{x}' := \lim_{h_j \rightarrow 0} \frac{\mathbf{x}(t + h_j) - \mathbf{x}(t)}{h_j} = (x'_1(t), \dots, x'_n(t))$$

Geometrically, The velocity (tangent) vector is a limit of secant vector

# Regular, cusp

## Definition (regular)

A curve  $\mathbf{x}(t)$  is regular iff  $x'_i(t)$  is continuous and  $\mathbf{x}'(t) \neq \mathbf{0} \quad \forall t$

When  $\mathbf{x}'(\bar{t}) = \mathbf{0}$ , this curve has cusp at  $\mathbf{x}(\bar{t})$

Geometrically, regular curve means smooth curve

## Definition (continuously differentiable)

$f : \mathbb{R}^n \rightarrow \mathbb{R}$  is continuously differentiable (or  $C^1$ ) on an open set  $D \subset \mathbb{R}^n$   
iff

$$\forall \mathbf{x} \in D, \forall i, \quad \exists \frac{\partial f}{\partial x_i}(\mathbf{x}) \quad \wedge \quad \text{continuous}$$

# Chain Rule I

## Chain Rule I

Let  $g(t) = f(\mathbf{x}(t))$ ,  $g : \mathbb{R} \rightarrow \mathbb{R}$ ,  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $\mathbf{x} : \mathbb{R} \rightarrow \mathbb{R}^n$ . Then,

$$\frac{dg}{dt} = Df_{\mathbf{x}(\mathbf{x})} \frac{d\mathbf{x}(t)}{dt} = \frac{\partial f}{\partial x_1} \frac{dx_1}{dt} + \cdots + \frac{\partial f}{\partial x_n} \frac{dx_n}{dt}$$

## Chain Rule II

Let  $g(\mathbf{t}) = f(\mathbf{x}(\mathbf{t}))$ ,  $g : \mathbb{R}^s \rightarrow \mathbb{R}$ ,  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $\mathbf{x} : \mathbb{R}^s \rightarrow \mathbb{R}^n$ . Then,

$$Dg_{\mathbf{t}} = Df_{\mathbf{x}(\mathbf{x})} D\mathbf{x}_{\mathbf{t}}(\mathbf{t})$$

$$\frac{\partial g}{\partial t_i} = \left( \frac{\partial f}{\partial x_1} \quad \cdots \quad \frac{\partial f}{\partial x_n} \right) \begin{pmatrix} \frac{\partial x_1}{\partial t_i} \\ \vdots \\ \frac{\partial x_n}{\partial t_i} \end{pmatrix}$$

# Directional Derivatives and Gradients

## Directional Derivative

Let  $\mathbf{x} = \bar{\mathbf{x}} + t\bar{\mathbf{v}}$ : line passing  $\bar{\mathbf{x}}$  with direction  $\bar{\mathbf{v}}$  and  $g(t) = f(\bar{\mathbf{x}} + t\bar{\mathbf{v}})$ .  
From chain rule I,

$$\left. \frac{dg}{dt} \right|_{t=0} = \left. \frac{df}{dt} \right|_{t=0} = Df_{\mathbf{x}}(\bar{\mathbf{x}}) \cdot \frac{d\mathbf{x}}{dt} = Df_{\mathbf{x}}(\bar{\mathbf{x}}) \cdot \bar{\mathbf{v}}$$

This is the derivative of  $f$  at  $\bar{\mathbf{x}}$  in the direction  $\bar{\mathbf{v}}$ , and other notations are  $\frac{\partial f}{\partial \mathbf{v}}(\bar{\mathbf{x}})$  and  $D_{\mathbf{v}}f(\bar{\mathbf{x}})$

## Theorem (14.2)

*At any point  $\mathbf{x} \in D$  and  $\nabla f \neq 0$ ,  $\nabla f(\mathbf{x})$  points at  $x$  into the direction in which  $f$  increases most rapidly*

This theorem will be used for finding normal vector of tangent hyperplane to level set.

# Chain Rule III

## Chain Rule III

Let  $\mathbf{f}(\mathbf{x}) : \mathbb{R}^n \rightarrow \mathbb{R}^m$ ,  $\mathbf{a}(t) : \mathbb{R} \rightarrow \mathbb{R}^n$ , and  $\mathbf{g}(t) = \mathbf{f} \circ \mathbf{a}(t)$ . Then,

$$\frac{d\mathbf{g}}{dt} = D\mathbf{f}_{\mathbf{a}}(\mathbf{a}(t)) \cdot \mathbf{a}'(t)$$

## Chain Rule IV

Let  $\mathbf{f}(\mathbf{x}) : \mathbb{R}^n \rightarrow \mathbb{R}^m$ ,  $\mathbf{a}(\mathbf{t}) : \mathbb{R}^s \rightarrow \mathbb{R}^n$ , and  $\mathbf{g}(\mathbf{t}) = \mathbf{f} \circ \mathbf{a}(\mathbf{t})$ . Then,

$$D\mathbf{g}_{\mathbf{t}} = D\mathbf{f}_{\mathbf{a}}(\mathbf{a}(\mathbf{t})) \cdot D\mathbf{a}_{\mathbf{t}}$$

## Definition

*Hessian matrix*

$$D^2 f_{\mathbf{x}} = D(Df)_{\mathbf{x}} := \begin{pmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \frac{\partial^2 f}{\partial x_2 \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_1} \\ \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_2 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_1 \partial x_n} & \frac{\partial^2 f}{\partial x_2 \partial x_n} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n} \end{pmatrix}$$

## Theorem (14.5: Young's theorem)

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i} \quad \forall i, j$$

This means hessian is symmetric.