

Determinants: An Overview

Ch.9

<mailto:eyeofyou@korea.ac.kr>

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1 The Determinant of A Matrix

1.1 Defining the Determinant

Defining the Determinant

We need to define a function $\det(A) : M_n \rightarrow \mathbb{R}$ satisfying

$$\det A = \det(A) = |A| \begin{cases} = 0 & \text{if } A \text{ is singular} \\ \neq 0 & \text{if } A \text{ is nonsingular} \end{cases}$$

This chapter is an easy overview for determinant. For full version, see Ch.26.

Big Picture

Definition of $\det(A)$ when A is

1. 1×1 matrices,
2. 2×2 matrices, and
3. $n \times n$ matrices

$\det(A)$ **when** $A \in M_1$ **and** $A \in M_2$

Definition 1 ($\det(A)$ when $A \in M_1$).

$$\det(A) := A$$

When $A \in M_1$, A is just a scalar. *i.e.*, $M_1 = \mathbf{R}$

Definition 2 ($\det(A)$ when $A \in M_2$).

$$\det(A) := a_{11} \det(a_{22}) - a_{12} \det(a_{21}) = a_{11}a_{22} - a_{12}a_{21}$$

For M_n , above definition is extended recursively

Minor and Cofactor of Matrices

Let $A \in M_n$

Definition 3 (Minor, Cofactor). Let $A_{ij} \in M_{n-1}$ be a submatrix obtained by deleting R_i and C_j . Then i, j th minor M_{ij} is defined as follows:

$$M_{ij} := \det A_{ij}$$

And C_{ij} , the i, j th cofactor of A is defined as:

$$C_{ij} := (-1)^{i+j} M_{ij}$$

Determinant of M_n Matrices

Definition 4 (Determinant of M_n Matrices).

$$\det A := \sum_i^n a_{i\bar{j}} C_{i\bar{j}} = \sum_j^n a_{i\bar{j}} C_{i\bar{j}}$$

Calculation Procedure for General $A \in M_n$

STEP 1: Select one R_i or C_j

STEP 2: Calculate $\det A$ from deleting each element in R_i or C_j from STEP 1

STEP 3: For all M_{ij} in STEP2, follow STEP1-2 recursively

Example: M_3 Matrix

$$\begin{aligned} \det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} &= a_{11}(-1)^{1+1} \begin{vmatrix} e & f \\ h & i \end{vmatrix} + a_{12}(-1)^{1+2} \begin{vmatrix} d & f \\ g & i \end{vmatrix} + a_{13}(-1)^{1+3} \begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ &= a(ei - fh) - b(di - fg) + c(dh - eg) \end{aligned}$$

Important Note: Rule of Sarrus (https://en.wikipedia.org/wiki/Rule_of_Sarrus) only for 3×3 matrices, NOT FOR $n \times n$ matrices!!!

1.2 Computing the Determinant

Computing the Determinant of Special Form

In general, calculation of determinant is very complex except for some cases:

Theorem 1 (9.1). *If $A \in M_n$ is (1) lower-triangular, (2) upper-triangular, or (3) diagonal matrix,*

$$\det(A) = \prod_i^n A_{ii}$$

Determinant of REF

Theorem 2 (9.2). *Let A_{REF} be the REF of $A \in M_n$ by only using ERO_1, ERO_2 . Then,*

$$\det A = \pm \det A_{REF}$$

If only ERO_2 is used to make A_{REF} ,

$$\det A = \det A_{REF}$$

Proof Sketch: $\det(EM_1(R_i \leftrightarrow R_j)) = -1, \det(EM_2(R_i \leftarrow R_i + kR_j)) = 1, \det(EM_3(R_i \leftarrow kR_i)) = k$ and use Theorem 9.5(b)

Theorem 3 (9.3: Main Property of Determinant). *$A \in M_n$ is nonsingular iff $\det A \neq 0$*

2 Uses of The Determinant

Determinant and Inverse Matrix

Definition 5 (Adjoint). *Let $C_{ij} \in \mathbb{R}$ be the i, j th cofactor of $A \in M_n$. Then,*

$$(adj A)_{ij} := C_{ji} = (-1)^{j+i} M_{ji} \quad \forall i, j$$

Theorem 4 (9.4 (a)).

$$A^{-1} = \frac{1}{\det A} adj A$$

Cramer's rule

For any nonsingular matrix $A \in M_n$,

Theorem 5 (9.4 (b):Cramer's Rule). *The unique solution $\mathbf{x} = (x_1, \dots, x_n)$ of $A\mathbf{x} = \mathbf{b}$ is:*

$$x_i = \frac{\det B_i}{\det A} \quad \forall i = 1, \dots, n$$

When

$$B_i := (A_1 \quad \mathbf{b} \quad A_n),$$

$$A_1 := \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1,i-1} \\ a_{21} & a_{22} & \cdots & a_{2,i-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{n,i-1} \end{pmatrix}, A_n := \begin{pmatrix} a_{1,i+1} & a_{1,i+2} & \cdots & a_{1,n} \\ a_{2,i+1} & a_{2,i+2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,i+1} & a_{n,i+2} & \cdots & a_{n,n} \end{pmatrix}$$

Algebraic Properties of the Determinant Function

Theorem 6 (9.5). For $A \in M_n$,

(a) $\det A^T = \det A$

(b) $\det(AB) = \det A \det B$

(c) $\det(A + B) \neq \det A + \det B$ in general

G-J Elimination ($\sim n^3/3$) is far, far, far more efficient than Cramer's rule ($\sim (n-1)((n+1)!)$) when n is large

3 IS-LM Analysis via Cramer's Rule

IS-LM Analysis (Revisited)

$$\begin{aligned} sY + ar &= I^0 + G \\ mY - hr &= M_s - M^0 \end{aligned}$$

Let's solve above IS-LM model:

$$Y = \frac{\begin{vmatrix} I^0 + G & a \\ M_s - M^0 & -h \end{vmatrix}}{\begin{vmatrix} s & a \\ m & -h \end{vmatrix}}, \quad r = \frac{\begin{vmatrix} s & I^0 + G \\ m & M_s - M^0 \end{vmatrix}}{\begin{vmatrix} s & a \\ m & -h \end{vmatrix}}$$

IS-LM Analysis

- Expansionary fiscal policy: $\Delta G > 0$
- Contractionary fiscal policy: $\Delta G < 0$
- Expansionary monetary policy: $\Delta M_s > 0$
- Contractionary monetary policy: $\Delta M_s < 0$