

# One-Variable Calculus: Applications

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## 1 Using the First Derivatives for Graphing

### Positive Derivative implies Increasing Function

Suppose  $f$  is continuously differentiable at  $x_0$  (i.e.,  $\exists f' \wedge f'$  is continuous)

**Theorem 1** (3.1). 1.  $f'(x_0) > 0 \Rightarrow \exists \bar{\alpha}, \bar{\beta} \in \mathbb{R}$  s.t.  $x_0 \in (\bar{\alpha}, \bar{\beta}) \wedge f$  is increasing on  $(\bar{\alpha}, \bar{\beta})$

2.  $f'(x_0) < 0 \Rightarrow \exists \bar{\alpha}, \bar{\beta} \in \mathbb{R}$  s.t.  $x_0 \in (\bar{\alpha}, \bar{\beta}) \wedge f$  is decreasing on  $(\bar{\alpha}, \bar{\beta})$

**Theorem 2** (3.2). 1.  $f' > 0$  on  $(\bar{a}, \bar{b}) \subset D \Rightarrow f$  is increasing on  $(\bar{a}, \bar{b})$

2.  $f' < 0$  on  $(\bar{a}, \bar{b}) \subset D \Rightarrow f$  is decreasing on  $(\bar{a}, \bar{b})$

3.  $f$  is increasing on  $(\bar{a}, \bar{b}) \Rightarrow f' \geq 0$  on  $(\bar{a}, \bar{b})$

4.  $f$  is decreasing on  $(\bar{a}, \bar{b}) \Rightarrow f' \leq 0$  on  $(\bar{a}, \bar{b})$

### Graph Sketching using First Derivatives

#### Procedure

(STEP 1) Find all  $x_i^*$  s.t.  $f'(x_i^*) = 0$  (critical points), boundary points, and around undefined points

(STEP 2) Calculate  $f(x_i^*)$

(STEP 3) Make table for graph sketch

- $f' > 0 \Rightarrow \nearrow$
- $f' < 0 \Rightarrow \searrow$

$$f(x) = x^4 - 8x^3 + 18x^2 - 11 \quad (\text{Ex3.1})$$

## 2 Second Derivatives and Convexity

### Convexity and Concavity

**Definition 1** (Convex (Concave up), Concave (Concave down)). *f* is convex on  $(\bar{\alpha}, \bar{\beta})$  iff:

$$f((1-t)\bar{a} + t\bar{b}) \leq (1-t)f(\bar{a}) + tf(\bar{b}), \quad \forall t \in [0, 1] \quad \forall \bar{a}, \bar{b} \in [\alpha, \beta]$$

*f* is concave on  $(\bar{\alpha}, \bar{\beta})$  iff:

$$f((1-t)\bar{a} + t\bar{b}) \geq (1-t)f(\bar{a}) + tf(\bar{b}), \quad \forall t \in [0, 1] \quad \forall \bar{a}, \bar{b} \in [\alpha, \beta]$$

	$f' > 0$	$f' < 0$
$f'' > 0$		
$f'' < 0$		

### Using Second Derivative for Graph Sketch

#### Procedure

(STEP 1) Find all  $x_i^*$  s.t.  $f'(x_i^*) = 0$  (critical points),  $f''(x_i^*) = 0$ , boundary points, and around undefined points

(STEP 2) Calculate  $f(x_i^*)$

(STEP 3) Make table for graph sketch

$$f(x) = x^4 - 8x^3 + 18x^2 - 11 \quad (\text{Ex3.1})$$

## 3 Graphing Rational Functions

### Graphing Rational Function

#### Procedure

(STEP 1) Find all  $x_i^*$  s.t.  $f'(x_i^*) = 0$  (critical points),  $f''(x_i^*) = 0$ , boundary points, convergence toward undefined points, and tail (i.e., convergence toward  $\pm\infty$ )

(STEP 2) Calculate  $f(x_i^*)$

(STEP 3) Make table for graph sketch

$$f(x) = \frac{16(x+1)}{(x-2)^2} \quad (\text{Ex3.6})$$

## 4 Tails and Horizontal Asymptotes

### Tail

#### Tails of Polynomial

Only two cases: diverge to  $\pm\infty$

#### Tails of Rational Function

$$g(x) = \frac{\bar{a}_0 x^{\bar{k}} + \bar{a}_1 x^{\bar{k}-1} + \dots + \bar{a}_{\bar{k}}}{\bar{b}_0 x^{\bar{m}} + \dots + \bar{b}_{\bar{m}}}$$

Tails of  $g(x)$  is determined by  $\frac{\bar{a}_0}{\bar{b}_0} \frac{x^{\bar{k}}}{x^{\bar{m}}}$

- $k > m$ : Same as polynomials with degree  $k - m$
- $k = m$ : Converges to  $\frac{a_0}{b_0}$  (Horizontal asymptote)
- $k < m$ : converges to 0 (Horizontal asymptote)

## 5 Maxima and Minima

### Boundary Max and Interior Max

**Theorem 3** (3.3: First Order Condition (FOC)).  $x_0$  is an interior max or min of  $f \Rightarrow x_0$  is a critical point of  $f$ . i.e.,  $f'(x_0) = 0$  (Inverse is not always true)

**Theorem 4** (3.4: Second Order Condition (SOC)). 1.  $f'(x_0) = 0 \wedge f''(x_0) < 0 \Rightarrow x_0$  is local max of  $f$

2.  $f'(x_0) = 0 \wedge f''(x_0) > 0 \Rightarrow x_0$  is local min of  $f$

3.  $f'(x_0) = 0 \wedge f''(x_0) = 0 \Rightarrow x_0$  can be max, min, or neither

## Global Maxima and Minima

- Finding global max (or min) is not easy problem
- These cases guarantee the existence of global max (or min)
  - (1) Domain of  $f$  is an interval  $\wedge$  (2)  $f$  has only one critical point  $\wedge$  (3)  $f'' > 0$  (g.min)  
 $\vee f'' < 0$  (g.max) in domain of  $f$
  - Domain of  $f$  is compact (closed and bounded) ( $\exists$  global max, global min)
- Below case guarantees the nonexistence of global max (or min)
  - Strictly increasing (or decreasing) functions with open domain

## 6 Applications to Economics

### Producer's Problem in Perfect Competitive Market

#### Producer's Problem in perfect competitive market

$$\arg \max_x \Pi(x)$$

$$x = f(L) \quad (\text{Production Function})$$

#### Exogenous (Given) variables

- $\bar{w}$ : unit price of labor
- $\bar{p}$ : unit price of end product

#### Assumptions

#### Assumptions

- $f : D \rightarrow \mathbb{R} \in \mathbf{C}^2$
- $f$  is increasing:  $f'(L) > 0 \forall L \in D$
- $\exists \bar{a} \geq 0$  s.t. (1)  $f''(L) > 0 \forall L \in [0, \bar{a})$  (*i.e.*, convex on  $[0, \bar{a})$ ) and (2)  $f''(L) < 0 \forall L \in (\bar{a}, \infty)$  (*i.e.*, concave on  $(\bar{a}, \infty)$ )
- Quantity of input (labor)  $L$  is the only factor for production

## Cost Functions

### Big Picture for problem solving

Production Function  $\rightarrow$  Cost Function (in terms of  $x$ )  $\rightarrow$  Profit Function  $\Pi(x) \rightarrow$  Finding  $x^*$  maximizing  $\Pi(x)$

**Definition 2** (Total Cost, Marginal Cost, and Average Cost). •  $TC(x)$ : Total cost for producing  $x$

- $MC(x) := TC'(x)$

- $AC(x) := \frac{TC(x)}{x}$

**Theorem 5** (3.7c). At interior minimum of  $AC$  (i.e.,  $AC' = 0$ ),  $AC = MC$

## Revenue and Profit Functions

**Definition 3** (Total Revenue, Marginal Revenue).

$$TR(x) := \bar{p}x$$

$$MR(x) := TR'(x)$$

**Definition 4** (Profit Function).

$$\Pi(x) := TR - TC$$

## Producer's problem in Monopoly Case

In monopoly,  $p$  is endogenous

### Producer's Problem in Monopoly

$$\arg \max_{p,x} \Pi(x)$$

However, firm is facing demand directly in monopoly

$$x = D(p) \quad (\text{Demand Function})$$

## Elasticity

**Definition 5** (A Elasticity of B).

$$\epsilon_{B,A} := \frac{\frac{dB}{B}}{\frac{dA}{A}} = \frac{A}{B} \frac{dB}{dA}$$

- $\frac{\Delta x}{x}$ : rate of change
- Elasticity: ratio of rate of change
  - $|\epsilon| < 1$ : inelastic
  - $|\epsilon| > 1$ : elastic
  - $|\epsilon| = 1$ : unit elastic

### Functions with Constant Demand

- Elasticity of linear demand function is not constant (not realistic)

$$x = D(p) = \bar{a} - \bar{b}p, \quad \bar{a}, \bar{b} > 0$$

- Example of constant elasticity demand function (more realistic)

$$x = D(p) = \bar{k}p^{-\bar{r}}, \quad \bar{k}, \bar{r} > 0$$