Quadratic Forms and Definite Matrices

Ch.16

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Quadratic Forms

Definition (Quadratic Form)

A quadratic form on $\mathbb{R}^n$ *is a real-valued function of the form*

$$Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}, \quad \mathbf{x} \in \mathbb{R}^n, \quad A^T = A$$

For more detailed description, see Ch13 (section 3).

Definiteness

Definiteness: Overview

When $Q = \mathbf{x}^T A \mathbf{x}$ and A is a diagonal matrix

$$A = \begin{pmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{pmatrix}$$

- Positive Definite (PD): $a_{ii} > 0 \quad \forall i$
- Positive Semi Definite (PSD): $a_{ii} \geq 0 \quad \forall i$
- Negative Definite (ND): $a_{ii} < 0 \quad \forall i$
- Negative Semi Definite (NSD): $a_{ii} \leq 0 \quad \forall i$
- Indefinite (ID): $a_{ii} < 0$ for some i, and $a_{ii} > 0$ for some i

Definite Symmetric Matrices

Definition (PD,PSD,ND,NSD,ID)

Let A *be an* $n \times n$ *symmetric matrix and* $Q = \mathbf{x}^T A\mathbf{x}$*, then* A *is:*

1. *PD if* $Q > 0 \ \forall \mathbf{x} \neq \mathbf{0}$ *in* $\mathbb{R}^n$
2. *PSD if* $Q \geq 0 \ \forall \mathbf{x} \neq \mathbf{0}$ *in* $\mathbb{R}^n$
3. *ND if* $Q < 0 \ \forall \mathbf{x} \neq \mathbf{0}$ *in* $\mathbb{R}^n$
4. *NSD if* $Q \leq 0 \ \forall \mathbf{x} \neq \mathbf{0}$ *in* $\mathbb{R}^n$
5. *ID if* $Q > 0$ *for some* $\mathbf{x} \in \mathbb{R}^n$ *and* $Q < 0$ *for some* $\mathbf{x} \in \mathbb{R}^n$

Principle Minors of a Matrix

Definition (Principal Submatrix, Principal Minor)

Let A be an $n \times n$ symmetric matrix. kth order principal submatrix of A is $k \times k$ submatrix of A obtained by deleting $n-k$ columns $C_1, \cdots, C_{n-k}$ and same $n-k$ rows $R_1, \cdots, R_{n-k}$.
kth order principal minor of A is the determinant of kth order prinicipal submatrix.

Note: the number of kth order principal submatrix can be nCk.