Functions of Several Variables

Ch.13

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1 Functions Between Euclidean Spaces

Definitions

Definition 1 (Function, Domain, Target, Image: General Definitions)**.** *Function $f : A \to B$ is a rule that assign each object of A (domain) to one object in B (target space). Image of f is* $\{f(\mathbf{x})|\mathbf{x} \in A\} \in B$

Examples: $f : \mathbb{R}^n \to \mathbb{R}$

$$f(\mathbf{x}) = \bar{\mathbf{a}} \bullet \mathbf{x} \tag{Linear}$$

$$f(\mathbf{x}) = \bar{k} \prod_i x_i^{\bar{b}_i} \tag{Cobb-Douglas}$$

$$f(\mathbf{x}) = \bar{k} \left(\sum_i \bar{c}_i x_i^{-\bar{a}} \right)^{-\bar{b}/\bar{a}} \tag{CES}$$

$\mathbf{f} : \mathbb{R}^k \to \mathbb{R}^m$

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Let $f_i : \mathbb{R}^n \to \mathbb{R}$. Then $\mathbf{f} : \mathbb{R}^k \to \mathbb{R}^m$ can be represented by f_i $(i = 1, 2, \cdots, m)$

$$\mathbf{f}(\mathbf{x}) := (f_1(\mathbf{x}), \cdots, f_m(\mathbf{x}))$$

Examples

- Production function with k input factors and m products
- Utility mapping $\mathbf{u} : \mathbb{R}^{km} \to \mathbb{R}^m$
  - $u_i : \mathbb{R}^k \to \mathbb{R}$: Individual utility function of customer i
  - k: # of goods
  - m: # of consumers
  - $\mathbf{x_i}$: consumption of customer i

$$\mathbf{u}(\mathbf{x_1}, \cdots, \mathbf{x_m}) = (u_1(\mathbf{x_1}), \cdots, u_m(\mathbf{x_m}))$$

$\mathbf{f}:\mathbb{R}\rightarrow\mathbb{R}^m$

$\mathbf{f}(t)$

$$\mathbf{f}(t):=(f_1(t),\cdots,f_m(t))$$

Geometrically, $\mathbf{f}(t)$ is a parametric curve on $\mathbb{R}^m$ space (cf. parametric line)

2 Geometric Representation of Functions

Level Curves

Level Curves
Let $f:\mathbb{R}^2\rightarrow\mathbb{R}^1$. Then level curves of f are curves on domain space with same $f(\mathbf{x})$. *I.e.,*

$$\{\mathbf{x}|f(\mathbf{x})=\bar{c}\}$$

- Isoquant: level curve of production function
- Indifference curve: level curve of utility function
- Generally, when $f:\mathbb{R}^k\rightarrow\mathbb{R}$ it is called level set and this is k dimensioal nonlinear object

3 Special Kinds of Functions

Linear Functions on $\mathbb{R}^k$

Definition 2 (Linear Function from $\mathbf{R}^k$ to $\mathbf{R}^m$)**.** $\mathbf{f}$ *is a linear function when*

1. $\mathbf{f}(\mathbf{x_1}+\mathbf{x_2})=\mathbf{f}(\mathbf{x_1})+\mathbf{f}(\mathbf{x_2})$
2. $\mathbf{f}(r\mathbf{x})=r\mathbf{f}(\mathbf{x})$

Theorem 1 (13.1,2)**.**
- $f:\mathbb{R}^k\rightarrow\mathbb{R}$ *is a linear function* $\Rightarrow f(\mathbf{x})=\bar{\mathbf{a}}\bullet\mathbf{x}$, $\mathbf{a}\in\mathbb{R}^k$
- $\mathbf{f}:\mathbb{R}^k\rightarrow\mathbb{R}^m$ *is a linear function* $\Rightarrow \mathbf{f}(\mathbf{x})=\bar{A}\bullet\mathbf{x}$, $A:m\times k$ *matrix*

Quadaratic Forms

Definition 3 (Quadratic Form on $\mathbb{R}^k$)**.** $f:\mathbb{R}^k\rightarrow\mathbb{R}$ *is of the quadratic form if:*

$$f(\mathbf{x})=\sum_{i,j}^{k}\bar{a}_{ij}x_ix_j$$

more elegantly,

$$f(\mathbf{x})=\mathbf{x}^T\bar{A}\mathbf{x}$$

In this case, A can be always symmetric.

Monomial

Definition 4 (Monomials on $\mathbb{R}^k$)**.** $f:\mathbb{R}^k\rightarrow\mathbb{R}$ *is a monomial if:*

$$f(\mathbf{x})=\bar{c}\prod_{i}^{k}x_i^{\bar{a}_i},\quad a_i\in\mathbb{N}+\{0\}$$

The degree of above monomial is $\sum_i^k a_i$

4 Continuous Functions

Continuous

Definition 5 (Continuous Function on $\mathbb{R}^k$). $\mathbf{f} = (f_1, \cdots, f_m)$ *is* <u>*continuous*</u> *at* $\mathbf{x}$ *iff all* f_i *are continuous at* $\mathbf{x}$