# Systems of Linear Equations

CH7

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# General Linear System

## General Linear System

$$
\begin{aligned}
\bar{a}_{11}x_1 + \bar{a}_{12}x_2 + \cdots + \bar{a}_{1n}x_n &= \bar{b}_1 \\
\bar{a}_{21}x_1 + \bar{a}_{22}x_2 + \cdots + \bar{a}_{2n}x_n &= \bar{b}_2 \\
&\vdots \\
\bar{a}_{m1}x_1 + \bar{a}_{m2}x_2 + \cdots + \bar{a}_{mn}x_n &= \bar{b}_m
\end{aligned}
$$

More elegantly,

$$\bar{A}\mathbf{x} = \bar{\mathbf{b}}$$

(Compare with one-var version: $\bar{a}x = \bar{b}$)

## Solution of Linear System: $\mathbf{x} = (x_1, x_2, \cdots, x_n)$

- Solution of linear system is $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ satisfies all equations in equation system
- Main considerations:
  - Existance of Solution
  - $\#$ of Solutions
  - Efficient deriving methods
    - Substitution
    - Elimination of variables
    - Matrix methods

# Elementary Row Operations (EROs)

## Solving Procedure: Big Picture

$$(A|\mathbf{b}) \xrightarrow{EROs} (A_{REF}|\mathbf{b}^*) \xrightarrow{EROs} (A_{RREF}|\mathbf{b}^{**}) \xrightarrow{EROs} (I|\mathbf{b}^{***})$$

Solution: $\mathbf{x} = \mathbf{b}^{***}$

## General linear system

We should solve

$$\begin{aligned}
\bar{a}_{11}x_1 + \bar{a}_{12}x_2 + \cdots + \bar{a}_{1n}x_n &= \bar{b}_1 \\
\bar{a}_{21}x_1 + \bar{a}_{22}x_2 + \cdots + \bar{a}_{2n}x_n &= \bar{b}_2 \\
&\vdots \\
\bar{a}_{m1}x_1 + \bar{a}_{m2}x_2 + \cdots + \bar{a}_{mn}x_n &= \bar{b}_m
\end{aligned}$$

# Coefficient Matrix, Augment Matrix

## Definition (Coefficient Matrix)

*$A$ is a <u>coefficient matrix</u> for general linear system*

$$A := \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

## Definition (Augment Matrix)

*$\hat{A}$ is a <u>augment matrix</u> for general linear system*

$$\hat{A} := (A|\mathbf{b})$$

# Elementary Row Operations (EROs)

**Definition**

*Let $R_i$ be the $i$th row matrix of $A$, then,*

$$R_i \leftrightarrow R_j \qquad (ERO_1(i,j))$$

$$R_i \leftarrow \bar{k}R_j + R_i \qquad (ERO_2(k,j,i))$$

$$R_i \leftarrow \bar{k}R_i \qquad (ERO_3(k,i))$$

*Above operations do not change solution*

- Notations
  - $R_i$: $i$th row (of $A$)
  - $C_i$: $i$th column (of $A$)
  - $a_{ij}$: Element of $i$th row, $j$th column (of $A$)

# Row Echelon Form

**Definition ($k$-leading zeros, pivot, Row Echelon Form (REF))**

*$R_i$ has $k$-leading zeros if:*

$$a_{ij} = \begin{cases} 0, & \forall j \leq k \\ \textit{not } 0, & j = k+1 \end{cases}$$

*Above $a_{i,k+1}$ is a pivot of $R_i$*

*Let $k_i$ be # of leading zeros of $R_i$. Then $\hat{A}_{REF}$ is row echelon form (REF) of $\hat{A}$ if:*

- $k_i > k_j \quad \forall i > j$
- *Zero rows are placed on the bottom of $A$*

# Reduced Row Echelon Form (RREF)

## Definition (Reduced Row Echelon Form (RREF))

*$A_{RREF}$ is a reduced row echelon form (RREF) of $A$ if:*

- *$A_{RREF}$ is REF*
- *If $C_i$ has pivot, only pivot is non-zero element in $C_i$*
- *Pivot = 1*

## Solving Procedure: Gauss-Jordan Elimination

$$(A|\mathbf{b}) \xrightarrow{EROs} (A_{REF}|\mathbf{b}^*) \xrightarrow{EROs} (A_{RREF}|\mathbf{b}^{**}) \xrightarrow{EROs} (I|\mathbf{b}^{***})$$

Solution: If the linear system $Ax = \mathbf{b}$ has an unique solution ($A$ is $m \times n$ matrix), $A_{RREF} = I$ $(m = n)$ or, $R_i = I_i$ $(i \leq n)$ and $R_i = \mathbf{O}$ $(i > n)$

# Systems with No Solution

During EROs for $\hat{A} = A|\mathbf{b}$, if you encounter with $\mathbf{O}|k \neq 0$ , *i.e.*, $(000\cdots0|k \neq 0)$, this system has <u>no solution</u>

Meaning:

$$0x_1 + 0x_2 + \cdots + 0x_n = 0 = k \neq 0 \qquad \text{(contradiction!)}$$

No $\mathbf{x}$ can satisfy this equation

**Theorem (Fact7.2)**

*$\hat{A}$ has a solution iff*

$$rank\hat{A} = rankA$$

# Systems with Many Solutions

Here, $w$ stands for independent zero or non-zero element (can have any values) and $*$ is nonzero pivot.

## REF of systems with many solutions

Systems with many solutions have following form of RREF (blank means zero)

$$\left(\begin{array}{cccccccc|c} * & w & w & w & w & w & w & w & w \\ & & & * & w & w & w & w & w \\ & & & & & * & w & w & w \\ & & & & & & * & w & w \\ & & & & & & & * & w \end{array}\right) \qquad \text{(REF)}$$

# RREF of Systems with many solutions

Important note: This is just an example of systems with many solutions.

## RREF of systems with many solutions

Systems with many solutions have following form of RREF (blank means zero)

$$\left(\begin{array}{cccccccc|c} 1 & w & w & 0 & w & 0 & 0 & 0 & w \\ & & & 1 & w & 0 & 0 & 0 & w \\ & & & & & 1 & 0 & 0 & w \\ & & & & & & 1 & 0 & w \\ & & & & & & & 1 & w \end{array}\right) \qquad \text{(RREF)}$$

In systems with many solutions, there exist $C_i$ with no pivot (in our example, $C_2, C_3, C_5$)

# Solutions of linear systems with many solutions

- Two types of solutions
  1. Basic variables are dependent on free variables or, fixed value:
     - Dependent on free variables: $x_1, x_4$ (in our example)
     - Fixed value: $x_6, x_7, x_8$ (in our example)
  2. Free variables can have any value: $x_2, x_3, x_5$ (in our example)
     - If $C_i$ has no pivot, $x_i$ is free variable
- In solving systems with many solutions, above types should be addressed explicitly

# Rank

## Definition (Rank)

*The Rank of a matrix is # of the nonzero rows in its REF (or RREF)*

## Property of Rank

Rank = # of pivots = # of nonzero rows of REF (or RREF)

- Implication: Rank means the number of effective (meaningful) equations

## Theorem (Fact7.11)

*A system of $m$ equations and $n$ unknowns (its coefficient matrix is $m \times n$ matrix)*

$$A\mathbf{x} = \mathbf{b}$$

| $m < n$ | # of solutions |
|---|---|
| $\mathbf{b} = \mathbf{0}$ | $\infty$ |
| $\forall \mathbf{b}$ | $0 \vee \infty$ |
| $rank A = m$ | $\infty(\forall \mathbf{b})$ |
| $m > n$ | |
| $\mathbf{b} = \mathbf{0}$ | $1 \vee \infty$ |
| $\forall \mathbf{b}$ | $0 \vee 1 \vee \infty$ |
| $rank A = n$ | $0 \vee 1(\forall \mathbf{b})$ |
| $m = n$ | |
| $\mathbf{b} = \mathbf{0}$ | $1 \vee \infty$ |
| $\forall \mathbf{b}$ | $0 \vee 1 \vee \infty$ |
| $rank A = n = m$ | $1(\forall \mathbf{b})$ |

## Example: IS-LM model

### IS-LM Model

$$sY + ar = I^0 + G \quad \text{(IS)}$$

$$mY - hr = M_s + M^0 \quad \text{(LM)}$$

- Endogenous variables: $Y, r$
- Exogenous parameters: $s, a, m, h, I^0, G, M_s, M^0$
  - Policy variables: $G, M_s$
  - Behavioral variables: $s, a, m, h, I^0, M^0$
- Coefficient matrix:

$$A = \begin{pmatrix} s & a \\ m & -h \end{pmatrix}$$

# Linear Implicit Function Theorem

**Theorem (Linear Implicit Function Theorem)**

*A general linear model with $m$ equations and $n$ unknowns can have an unique solution (and suppose $x_i$ are arranged by endogeneity) iff:*

- $x_1, \cdots, x_k$ *are endogenous variables*
- $x_{k+1}, \cdots, x_n$ *are exogenous variables ($i.e.$, constant in this system)*
- $k = m$
- $rank A = k$

# Linear IFT

Implication of Linear IFT: treat exogenous variables as constant.

$$\bar{a}_{11}x_1 + \bar{a}_{12}x_2 + \cdots + \bar{a}_{1n}x_n = \bar{b}_1$$
$$\vdots$$
$$\bar{a}_{m1}x_1 + \bar{a}_{m2}x_2 + \cdots + \bar{a}_{mn}x_n = \bar{b}_m$$

Above system can be solvable iff $rank\tilde{A} = k = m$

$$\tilde{A}\mathbf{x} = \tilde{\mathbf{b}}$$

$$\tilde{A} := \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kk} \end{pmatrix}, \tilde{\mathbf{b}} = \begin{pmatrix} b_1 - a_{1,k+1}x_{k+1} - \cdots - a_{1n}x_n \\ b_2 - a_{2,k+1}x_{k+1} - \cdots - a_{2n}x_n \\ \vdots \\ b_k - a_{k,k+1}x_{k+1} - \cdots - a_{kn}x_n \end{pmatrix}$$