One-Variable Calculus: Exponents and Logarithms

CH5

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1 Exponential Functions

Exponential Functions

Definition 1 (Exponential Function). $f : \mathbb{R} \to \mathbb{R}$ *is exponential function if* $f(x) = \bar{a}\bar{b}^x, \quad \bar{b} > 0$

- $x \in \mathbb{N} \Rightarrow f(x) := \bar{a} \prod_{i=1}^{x} \bar{b}$
- $f(0) := \bar{a}$
- $f(1/n) := \bar{a} \sqrt[n]{\bar{b}}$
- $f(m/n) := \bar{a} \sqrt[n]{\bar{b}^m}$
- $x < 0 \Rightarrow f(x) = \bar{a}(1/\bar{b})^{|x|}$
- Graph: convex, monotonic increasing ($b > 1$) or decreasing ($b \in (0, 1)$) function

2 The Number e

Growth of an Account with Interest rate r

Saving Account at $t = \bar{T}$ with Interest rate $\bar{r}$, Initial Endowment $\bar{A}$

$$A_t = \bar{A}\,(1 + \bar{r})^{\bar{T}}$$

Compounded Interest

If interest is compounded n times per time unit,

$$A_t = \bar{A}\left(1 + \frac{\bar{r}}{n}\right)^{n\bar{T}}$$

Continuous Compounding

Compounded Interest with $n \to \infty$

$$A_t = \lim_{n\to\infty} \bar{A}\left(1+\frac{\bar{r}}{n}\right)^{n\bar{T}} = \bar{A}e^{\bar{r}\bar{T}}$$

Number e

Definition 2 (The Number e)**.**

$$e := \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \sum_{n=0}^{\infty}\frac{1}{n!} \approx 2.718281693\cdots$$

e is irrational number.

Theorem 1 (5.1)**.**

$$\lim_{n\to\infty} A\left(1+\frac{r}{n}\right)^{nt} = Ae^{rt}$$

In general, an initial quantity a_0 with growth rate r (per time unit) become a_0e^{rt} at time t (time unit)

3 Logarithms

Logarithm

Definition 3 (Base b Logarithm)**.** *Base b logarithm is an inverse of exponential function with base b*

$$f = b^x \quad \Leftrightarrow \quad x = \log_b f$$

- $a^{\log_a z} = z$
- $\log_a a^y = y$
- Graph: concave, monotonic increasing ($b > 1$) or convex, monotonic decreasing ($b \in (0,1)$)

Natural Logarithm

Definition 4 (Natural Logarithm)**.** *Base e logarithm is natural logarithm*

$$\ln x := \log_e x$$

$$\ln x = y \quad \Leftrightarrow \quad e^y = x$$

$$e^{\ln x} = x$$

$$\ln e^x = x$$

4 Properties of Exp and Log

Basic Properties of Exponential functions

$\forall r, s \in \mathbb{R}$,

1. $a^r a^s = a^{r+s}$
2. $a^{-r} := 1/a^r$
3. $a^r/a^s = a^{r-s}$
4. $(a^r)^s = a^{rs}$
5. $a^0 := 1$

Basic Properties of Logarithmic functions

$\forall r, s, a, b, c > 0 \wedge a, c \neq 1$,

1. $\log(rs) = \log r + \log s$
2. $\log(1/s) = -\log s$
3. $\log(r/s) = \log r - \log s$
4. $\log r^s = s \log r$
5. $\log 1 = 0$
6. $\log_a b = \frac{\log_c b}{\log_c a} = \frac{\ln b}{\ln a}$

(Ex5.4) Rule of 70 (or 69)

5 Derivatives of Exp and Log

Derivatives of Exp and Log functions

Theorem 2 (5.2)**.**

$$(e^x)' = e^x$$
$$(\ln x)' = \frac{1}{x}$$

if $u \in \mathbf{C}^1$, *from chain rule,*

$$(e^u)' = (e^u)\, u'$$
$$(\ln u)' = \frac{u'}{u} \qquad (u > 0)$$

6 Applications

Present Value

Present Value (PV)

After time T, A (at $t = 0$) grow to B (at $t = T$)

$$B = Ae^{rT}$$

A is the present value (PV) of B at $t = T$

$$A = Be^{-rT}$$

- PV of annuity

Logarithmic Derivative

(Ex5.10) $(x^x)' = ?$

Elasticity of f is the Slope in log-log Graph of f

$$\epsilon := \frac{\frac{df}{f}}{\frac{dx}{x}} = \frac{d \ln f}{d \ln x}$$