

Matrix Algebra

Ch.8

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1 Matrix Algebra

Matrix

Definition 1 (Matrix). *Matrix is a rectangular array of numbers (scalars)*

Let $a_{ij} \in \mathbb{R}$ or $A_{ij} \in \mathbb{R}$ be the i th row and j th column element of matrix A

Definition 2 (Equal).

$$A = B \iff \begin{cases} \text{same size} \\ a_{ij} = b_{ij} \quad \forall i, j \end{cases}$$

Addition, Subtraction

Let A, B be $n \times k$ matrices and $r \in \mathbb{R}$

Definition 3 (Addition).

$$(A + B)_{ij} := a_{ij} + b_{ij} \quad \forall i, j$$

Important note: the first $+$ and the second $+$ are not same operators

Definition 4 (Subtraction).

$$(A - B)_{ij} := a_{ij} - b_{ij} \quad \forall i, j$$

Multiplications of Matrices

Definition 5 (Scalar Multiplication).

$$(rA)_{ij} := rA_{ij} \quad \forall i, j$$

Let A be $n \times k$ matrix and B be $k \times m$ matrix. Then AB is $n \times m$ matrix.

Definition 6 (Matrix Multiplication).

$$(AB)_{ij} := A_{i1}B_{1j} + A_{i2}B_{2j} + \cdots + A_{ik}B_{kj} = \sum_{r=1}^k A_{ir}B_{rj}$$

For $n \times n$ matrices, identity matrix I_n is a multiplicative identity.

$$AI = IA = A$$

Laws of Matrix Algebra

Laws of Matrix Algebra

$$(A + B) + C = A + (B + C) \quad (\text{Associative Law for Addition})$$

$$(AB)C = A(BC) \quad (\text{Associative Law for Multiplication})$$

$$A + B = B + A \quad (\text{Commutative Law for Addition})$$

$$A(B + C) = AB + AC \quad (\text{Distributive Law})$$

$$(A + B)C = AC + BC \quad (\text{Distributive Law})$$

Important Note: $AB \neq BA$

Transpose

Definition 7 (Transpose). A^\top ($n \times m$) is a transpose of A ($m \times n$) if:

$$(A^\top)_{ij} := A_{ji} \quad \forall i, j$$

$$(A \pm B)^\top = A^\top \pm B^\top$$

$$(A^\top)^\top = A$$

$$(rA)^\top = rA^\top$$

$$(AB)^\top = B^\top A^\top \quad (\text{Theorem 8.1})$$

2 Special Kinds of Matrices

Special Kinds of Matrices (1)

Suppose A is $k \times n$ matrix. Then,

Definition 8 (Special Kinds of Matrices (1)). • A is a square matrix if $k = n$

- A is a column matrix if $n = 1$
- A is a row matrix if $k = 1$
- A is a diagonal matrix if $k = n$ and $a_{ij} = 0 \quad \forall i \neq j$
- A is a scalar matrix if $A = tI_n$
- A is an upper-triangular matrix if $a_{ij} = 0 \quad \forall i > j$
- A is a lower-triangular matrix if $a_{ij} = 0 \quad \forall i < j$

Special Kinds of Matrices (2)

Definition 9 (Special Kinds of Matrices (2)). • A is a symmetric matrix if A is square matrix and $a_{ij} = a_{ji} \quad \forall i, j$. Or, $A^T = A$

- A is an Idempotent matrix if $AA = A$
- A is a permutation matrix if A is the result of I_n with ERO_1 (row exchange)
- A is a nonsingular matrix if $\text{rank}A = \#row = \#column$

If a coefficient matrix of a system of linear equations is nonsingular, this system has only one solution $\mathbf{x} = A^{-1}\mathbf{b}$

3 Elementary Matrices

Elementary Matrix

Let E be an elementary matrix of some ERO s. Then,

Theorem 1 (8.3). ERO with a matrix A is equivalent to EA

Theorem 2 (8.2). • Let $E1_{ij}$ be the permutation matrix with interchanging R_i and R_j of I_n , then $E1_{ij}$ is equivalent to $ERO_1(i, j)$

- Let $E2_{k,j,i}$ be the result of $ERO_2(k, j, i)$ from I_n , then $E2_{k,j,i}$ is equivalent to $ERO_2(k, j, i)$
- Let $E3_{k,i}$ be the result of $ERO_3(k, i)$ from I_n , then $E3_{k,i}$ is equivalent to $ERO_3(k, i)$

Elementary Matrix

Definition 10 (Elementary Matrix). E_1, E_2, E_3 are elementary matrices corresponding to their EROs

Theorem 3 (8.4). Let $A \in M_n$ (set of $n \times n$ matrices), $E_i \in EM$ (set of elementary matrices), and $(R)REFM$ be the set of $(R)REF$ matrices. Then:

$$\exists E_i \quad i = 1, 2, \dots, m \quad \text{s.t.} \quad \prod_{i=m}^1 E_i A \in (R)REFM$$

or

$$E_m E_{m-1} \cdots E_2 E_1 A \in (R)REFM$$

4 Algebra of Square Matrices

Inverse of Matrices

Suppose $A, B \in M_n$

Definition 11 (Inverse, Invertible). B is (left, or right) inverse for A if:

$$\underbrace{AB}_{B: \text{Right inverse}} = \underbrace{BA}_{B: \text{Left inverse}} = I$$

A is invertible if $\exists B$

Notation: $B = A^{-1}$

Theorem 4 (8.5: Uniqueness of Inverse). $A \in M_n$ can have at most one inverse. (left inverse = right inverse)

Inverse Matrices and the Solution of Linear Systems

Theorem 5 (8.6). For $A \in M_n$,

$$\exists A^{-1} \Rightarrow \begin{cases} A \text{ is nonsingular} \\ \text{Unique solution of } A\mathbf{x} = \mathbf{b} \Rightarrow \mathbf{x} = A^{-1}\mathbf{b} \end{cases}$$

Proof: easy

Theorem 6 (8.7: inverse of Th8.6).

$$A \in M_n \text{ is nonsingular} \Rightarrow \exists A^{-1}$$

Proof: difficult

Calculation of Inverse Matrix

Calculation of Inverse Matrix

$$[A|I] \xrightarrow{EROs} [I|A^{-1}]$$

If RREF is not I_n , $\nexists A^{-1}$

Theorem 7 (8.8). Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2$. A is nonsingular iff $ad - bc \neq 0$

For general case ($A \in M_n$), see Ch.9

Equivalent statements

Theorem 8 (8.9). For $A \in M_n$, the following statements are equivalent

1. $\exists A^{-1}$
2. A has right inverse
3. A has left inverse
4. $A\mathbf{x} = \mathbf{b}$ has at least one solution for every \mathbf{b}
5. $A\mathbf{x} = \mathbf{b}$ has at most one solution for every \mathbf{b}
6. A is nonsingular
7. $\text{rank} A = n$

Properties of Inverse Matrices and Their Exponentials

Theorem 9 (8.10). If $A, B \in M_n$ and $\exists A^{-1}, B^{-1}$,

1. $(A^{-1})^{-1} = A$
2. $(A^T)^{-1} = (A^{-1})^T$
3. $\exists (AB)^{-1} \wedge (AB)^{-1} = B^{-1}A^{-1}$

Definition 12 (Matrix Exponential).

$$A^m := \prod_{i=1}^m A$$

$$A^{-m} := (A^{-1})^m$$

Exponential Properties of Invertible Matrices

Theorem 10 (8.11).

$$\exists A^{-1} \Rightarrow \begin{cases} \exists A^{-m} & \forall m \in \mathbb{N} \\ A^r A^s = A^{r+s} & \forall r, s \in \mathbb{N} \\ \forall r \in \mathbb{R} - \{0\}, \quad \exists (rA)^{-1} \wedge (rA)^{-1} = \frac{1}{r} A^{-1} \end{cases}$$

Important Note: $(AB)^k \neq A^k B^k$

5 Partitioned Matrices

Partitioned Matrices

Sometimes, matrix of matrices can be more convenient.

Definition 13 (Submatrix, Partitioned matrix). • A submatrix of matrix A is a matrix obtained by deleting some R_i or C_j

- A partitioned matrix is a matrix partitioned into submatrices by horizontal and/or vertical lines which extended along entire rows or columns of a matrix A

Partitioned Matrices

Theorem 11 (8.15). Let A be a square matrix partitioned as

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

and $A_{11}, A_{22} \in M_n$. Then,

$$\exists A_{22}^{-1} \wedge \exists D^{-1} \wedge D = A_{11} - A_{12} A_{22}^{-1} A_{21}$$

$$\Rightarrow A^{-1} = \begin{pmatrix} D^{-1} & -D^{-1} A_{12} A_{22}^{-1} \\ -A_{22}^{-1} A_{21} D^{-1} & A_{22}^{-1} (I + A_{21} D^{-1} A_{12} A_{22}^{-1}) \end{pmatrix}$$