Euclidean Spaces

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July 2, 2020

1 Points and Vectors in Euclidean Space

Objects in Euclidean Spaces

Objects in n-dimensional Euclidean Spaces

Dimension	Object	Representation
0	point	$\emptyset$
1	line	$x_1 \in \mathbb{R}^1$
2	plane	$(x_1, x_2) \in \mathbb{R}^2$
3	3d space	$(x_1, x_2, x_3) \in \mathbb{R}^3$
n	nd space	$(x_1, \cdots, x_n) \in \mathbb{R}^n$

2 Vectors

Vector

Definition 1 ((Euclidean) Vector, displacement). *n-tuples of real numbers $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ are (Euclidean) Vectors that represent displacement in $\mathbb{R}^n$ space (or Cartesian coordinate system)*

Let coordination of $\mathbf{p} = (p_1, \cdots, p_n)$, $\mathbf{q} = (q_1, \cdots, q_n)$. Then the displacement from $\mathbf{p}$ to $\mathbf{q}$ is defined as $\overrightarrow{\mathbf{pq}} := (q_1 - p_1, \cdots, q_n - p_n)$. In this definition, $\mathbf{p}$ is an origin, and $\mathbf{q}$ is a destination.

Note: Any vector $\mathbf{p} = (p_1, \cdots, p_n)$ can be interpreted as a location $((p_1, \cdots, p_n))$ or, displacement with origin $\mathbf{0} := (0, \cdots, 0)$ (more explicit notation: $\begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix}$)

3 The Algebra of Vectors

Addition and Subtraction

Let $\mathbf{u}, \mathbf{v}$ be the vectors in $\mathbb{R}^n$ space and $u_i, v_i \in \mathbb{R}^1$ be their i-th element.

Definition 2 ($\pm$ of vectors).

$$(\mathbf{u} \pm \mathbf{v})_i := u_i \pm v_i \quad \forall i$$

or,

$$\mathbf{u} \pm \mathbf{v} := (u_1 \pm v_1, \cdots, u_n \pm v_n)$$

Geometrically, addition of vectors means a sequence of displacements.

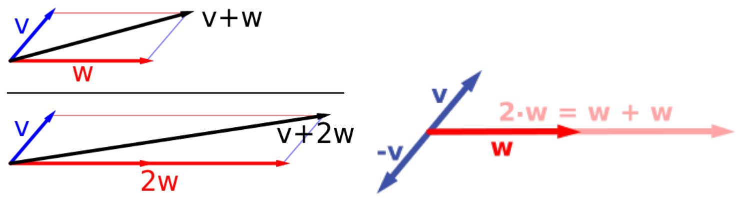

Figure 1: Geometrical Meaning: Addition, Subtraction, and Scalar Multiplication

Scalar Multiplication

Let r, s be scalars (or real numbers). *i.e.*, $r, s \in \mathbb{R}^1$.

Definition 3 (Scalar Multiplication)**.**

$$(r\mathbf{u})_i := ru_i \quad \forall i$$

Geometrically, scalar multiplcation means stretching or shrinking.

Algebraic Properties of Vector Operation

$$\mathbf{u}+\mathbf{v} = \mathbf{v}+\mathbf{u} \tag{Commutative Law}$$
$$(r+s)\mathbf{u} = r\mathbf{u}+s\mathbf{u} \tag{Distributive Law 1}$$
$$r(\mathbf{u}+\mathbf{v}) = r\mathbf{u}+r\mathbf{v} \tag{Distributive Law 2}$$

In fact, any set of objects with a vector addition and scalar multiplication satisfying above laws is called vector space and vector is defined as the element of vector space. (Vector is defined by operations and their laws)

4 Length and Inner Product in $\mathbb{R}^n$

Length and Direction

Definition 4 (Length of Vector $||\overrightarrow{\mathbf{pq}}||$)**.**

$$||\overrightarrow{pq}|| := \sqrt{\sum_i^n (q_i - p_i)^2}$$

Theorem 1 (10.1)**.**

$$||r\mathbf{v}|| = |r| \cdot ||\mathbf{v}|| \quad \forall r \in \mathbb{R} \wedge \forall \mathbf{v} \in \mathbb{R}^n$$

Any vector has two kinds of information: (1) length, and (2) direction.

Definition 5 (Unit Vector (Direction of a vector))**.**

$$\textit{Unit vector of } \mathbf{v} := \frac{\mathbf{v}}{||\mathbf{v}||}$$

The Inner Product

Definition 6 (Euclidean Inner Product (or dot product))**.**

$$\mathbf{u} \bullet \mathbf{v} := \sum_i^n u_i v_i \in \mathbb{R}^1$$

Theorem 2 (10.2: Properties of Inner Product)**.** *1.* $\mathbf{u} \bullet \mathbf{v} = \mathbf{v} \bullet \mathbf{u}$

2. $\mathbf{u} \bullet (\mathbf{v} + \mathbf{w}) = \mathbf{u} \bullet \mathbf{v} + \mathbf{u} \bullet \mathbf{w}$

3. $\mathbf{u} \bullet (r\mathbf{v}) = r(\mathbf{u} \bullet \mathbf{v}) = (r\mathbf{u}) \bullet \mathbf{v}$

4. $\mathbf{u} \bullet \mathbf{u} \geq 0$

5. $\mathbf{u} \bullet \mathbf{u} = 0 \quad \iff \quad \mathbf{u} = \mathbf{0}$

6. $(\mathbf{u} + \mathbf{v}) \bullet (\mathbf{u} + \mathbf{v}) = \mathbf{u} \bullet \mathbf{u} + \mathbf{2u} \bullet \mathbf{v} + \mathbf{v} \bullet \mathbf{v}$

Inner Product and Angle between Two Vectors

Theorem 3 (10.3)**.** *Let θ be the angle between $\mathbf{u}, \mathbf{v}$. Then,*

$$\mathbf{u} \bullet \mathbf{v} = ||\mathbf{u}|| \cdot ||\mathbf{v}|| \cos\theta$$

Theorem 4 (10.4)**.** *1. θ is acute if $\mathbf{u} \bullet \mathbf{v} > 0$*

2. θ is obtuse if $\mathbf{u} \bullet \mathbf{v} < 0$

3. θ is right if $\mathbf{u} \bullet \mathbf{v} = 0$

Theorem 5 (10.5: Triangle Inequality)**.**

$$||\mathbf{u} + \mathbf{v}|| \leq ||\mathbf{u}|| + ||\mathbf{v}||$$

$$\big| ||\mathbf{u}|| - ||\mathbf{v}|| \big| \leq ||\mathbf{u} - \mathbf{v}||$$

Norms

Definition 7 (Norms)**.** *Any operation (X) of a vector to a real number satisfying below three properties is <u>norm</u>. Length of vector is a norm.*

1. $X(\mathbf{u}) \geq 0 \land X(\mathbf{u}) = 0$ *only when* $\mathbf{u} = \mathbf{0}$

2. $X(r\mathbf{u}) = |r| X(\mathbf{u})$

3. $X(\mathbf{u} + \mathbf{v}) \leq X(\mathbf{u}) + X(\mathbf{v})$

Norm is set of distance measures between two vectors.

5 Lines and Planes

1D,2D Objects in $\mathbb{R}^n$ Spaces

Lines:One dimensional objects in $\mathbb{R}^n$ Spaces
$\mathbf{x}$ representing a line passing $\overline{\mathbf{x_0}}$ with direction $\overline{\mathbf{v}}$ is:

$$\mathbf{x} = \overline{\mathbf{x_0}} + t\overline{\mathbf{v}} \quad \forall t \in \mathbb{R} \qquad \text{(Parametric Representation)}$$

$\mathbf{x}$ representing a line passing $\overline{\mathbf{x_0}}, \overline{\mathbf{x_1}}$ is:

$$\mathbf{x} = (1-t)\overline{\mathbf{x_0}} + t\overline{\mathbf{x_1}} \quad \forall t \in \mathbb{R} \qquad \text{(Parametric Representation)}$$

Planes: Two dimensional objects in $\mathbb{R}^n$ Spaces
$\mathbf{x}$ representing a plane passing $\overline{\mathbf{x_0}}$ with direction $\overline{\mathbf{v_1}}$ and $\overline{\mathbf{v_2}}$ is:

$$\mathbf{x} = \overline{\mathbf{x_0}} + t_1\overline{\mathbf{v_1}} + t_2\overline{\mathbf{v_2}} \quad \forall t_i \in \mathbb{R} \qquad \text{(Parametric Representation)}$$

$\mathbf{x}$ representing a plane containing $\overline{\mathbf{x_0}}, \overline{\mathbf{x_1}}, \overline{\mathbf{x_2}}$ is:

$$\mathbf{x} = (1-t_1-t_2)\overline{\mathbf{x_0}} + t_1\overline{\mathbf{x_1}} + t_2\overline{\mathbf{x_2}} \quad \forall t_i \in \mathbb{R} \qquad \text{(Parametric Representation)}$$

Nonparametric Equations

Definition 8 (Normal Vector)**.** *A normal vector $\bar{\mathbf{n}}$ of a n-dimensional object $\mathbf{X}$ is a vector which is perpendicular to any vectors in the object. Suppose $\mathbf{x}$, $\bar{\mathbf{p}}$ are location vectors in the object. Then,*

$$\bar{\mathbf{n}} \bullet (\mathbf{x} - \bar{\mathbf{p}}) = 0 \quad \forall \mathbf{x}, \mathbf{p} \in \mathbf{X}$$

Finding Normal Vector

1. For linearly independent vectors $\mathbf{u_i}$, solve below systems of equations satisfying

$$\mathbf{n} \bullet \mathbf{u_i} = 0 \quad \forall i$$

2. $\mathbf{n} = \mathbf{u} \times \mathbf{v}$ (Only for $\mathbb{R}^3$ space)

$$\mathbf{u} \times \mathbf{v} := \left(\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right)$$

Hyperplanes

Definition 9 (Hyperplane)**.** *$n-1$ dimensional object in $\mathbb{R}^n$ space is a hyperplane with normal vector $\bar{\mathbf{a}}$ if:*

$$\bar{a}_1 x_1 + \bar{a}_2 x_2 + \cdots \bar{a}_n x_n = \bar{d}$$

or,

$$\bar{\mathbf{a}} \bullet \mathbf{x} = \bar{d}$$

6 Economic Applications

Budget Constraint

Definition 10 (Commodity Bundle, Price Vector, Budget Set)**.** *Let $x_i \geq 0$ be the quantity of ith commodity.*

$$\mathbf{x} := (x_1, \ldots, x_n), \quad x_i \geq 0 \quad \forall i \tag{Commodity Bundle}$$

Let $p_i \geq 0$ be the price of ith commodity.

$$\mathbf{p} := (p_1, \cdots, p_n), \quad p_i \geq 0 \quad \forall i \tag{Price Vector}$$

Then budget set $\mathbf{x}$ can be defined with given budget $\bar{I}$:

$$\bar{\mathbf{p}} \bullet \mathbf{x} \leq \bar{I} \tag{Budget Constraint}$$