# Functions of Several Variables

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## 1 Functions Between Euclidean Spaces

### Definitions

**Definition 1** (Function, Domain, Target, Image: General Definitions)**.** *Function $f : A \to B$ is a rule that assigns each object of $A$ (domain) to one object in $B$ (target space). Image of $f$ is* $\{f(\mathbf{x})|\mathbf{x} \in A\} \subset B$

**Examples:** $f : \mathbb{R}^n \to \mathbb{R}$

$$f(\mathbf{x}) = \bar{\mathbf{a}} \bullet \mathbf{x} \tag{Linear}$$

$$f(\mathbf{x}) = \bar{k} \prod_i x_i^{\bar{b}_i} \tag{Cobb-Douglas}$$

$$f(\mathbf{x}) = \bar{k} \left( \sum_i \bar{c}_i x_i^{-\bar{a}} \right)^{-\bar{b}/\bar{a}} \tag{CES}$$

$\mathbf{f} : \mathbb{R}^k \to \mathbb{R}^m$

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Let $f_i : \mathbb{R}^k \to \mathbb{R}$. Then $\mathbf{f} : \mathbb{R}^k \to \mathbb{R}^m$ can be represented by $f_i$ $(i = 1, 2, \cdots, m)$

$$\mathbf{f}(\mathbf{x}) := (f_1(\mathbf{x}), \cdots, f_m(\mathbf{x}))$$

### Examples

- Production function with $k$ input factors and $m$ products
- Utility mapping $\mathbf{u} : \mathbb{R}^{km} \to \mathbb{R}^m$
  - $u_i : \mathbb{R}^k \to \mathbb{R}$: Individual utility function of customer $i$
  - $k$: # of goods
  - $m$: # of consumers
  - $\mathbf{x_i}$: consumption of customer $i$

$$\mathbf{u}(\mathbf{x_1}, \cdots, \mathbf{x_m}) = (u_1(\mathbf{x_1}), \cdots, u_m(\mathbf{x_m}))$$

$\mathbf{f} : \mathbb{R} \to \mathbb{R}^m$

$\mathbf{f}(t)$

$$\mathbf{f}(t) := (f_1(t), \cdots, f_m(t))$$

Geometrically, $\mathbf{f}(t)$ is a parametric curve on $\mathbb{R}^m$ space (cf. parametric line)

## 2 Geometric Representation of Functions

**Level Curves**

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Let $f : \mathbb{R}^2 \to \mathbb{R}^1$. Then level curves of $f$ are curves on domain space with same $f(\mathbf{x})$. *I.e.,*

$$\{\mathbf{x} | f(\mathbf{x}) = \bar{c}\}$$

- Isoquant: level curve of production function
- Indifference curve: level curve of utility function
- Generally, when $f : \mathbb{R}^k \to \mathbb{R}$ it is called level set and this is $k$ dimensional nonlinear object

## 3 Special Kinds of Functions

**Linear Functions on $\mathbb{R}^k$**

**Definition 2** (Linear Function from $\mathbf{R}^k$ to $\mathbf{R}^m$)**.** $\mathbf{f}$ *is a linear function when*

1. $\mathbf{f}(\mathbf{x_1} + \mathbf{x_2}) = \mathbf{f}(\mathbf{x_1}) + \mathbf{f}(\mathbf{x_2})$
2. $\mathbf{f}(r\mathbf{x}) = r\mathbf{f}(\mathbf{x})$

**Theorem 1** (13.1,2)**.**
- $f : \mathbb{R}^k \to \mathbb{R}$ *is a linear function* $\Rightarrow f(\mathbf{x}) = \bar{\mathbf{a}} \bullet \mathbf{x}$, $\mathbf{a} \in \mathbb{R}^k$
- $\mathbf{f} : \mathbb{R}^k \to \mathbb{R}^m$ *is a linear function* $\Rightarrow \mathbf{f}(\mathbf{x}) = \bar{A} \bullet \mathbf{x}$, $A : m \times k$ *matrix*

**Quadratic Forms**

**Definition 3** (Quadratic Form on $\mathbb{R}^k$)**.** $f : \mathbb{R}^k \to \mathbb{R}$ *is of the quadratic form if:*

$$f(\mathbf{x}) = \sum_{i,j}^{k} \bar{a}_{ij} x_i x_j$$

*more elegantly,*

$$f(\mathbf{x}) = \mathbf{x}^T \bar{A} \mathbf{x}$$

*In this case, $A$ can be always symmetric.*

**Monomial**

**Definition 4** (Monomials on $\mathbb{R}^k$)**.** $f : \mathbb{R}^k \to \mathbb{R}$ *is a monomial if:*

$$f(\mathbf{x}) = \bar{c} \prod_{i}^{k} x_i^{\bar{a}_i}, \quad a_i \in \mathbb{N} \cup \{0\}$$

*The degree of above monomial is* $\sum_i^k a_i$

# 4 Continuous Functions

### Continuous

**Definition 5** (Continuous Function on $\mathbb{R}^k$)**.** $\mathbf{f} = (f_1, \cdots, f_m)$ *is continuous at* $\mathbf{x}$ *iff all* $f_i$ *are continuous at* $\mathbf{x}$