

Eigenvalues and Eigenvectors (2)

Ch.23

econMath.namun@gmail.com

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Terms

State

In each period t , the system is in one and only one of k states S_1, \dots, S_k .

Definition (Stochastic Process)

A stochastic process is a rule which gives the probability of the state i at the period $t = n + 1$ given the probabilities of all previous states ($t = 1, 2, \dots, n$)

Note: $\mathbf{x}_t = (x_{1,t}, \dots, x_{k,t})$ is the probabilities of all k possible states at time t

Definition (Markov Process)

A stochastic process that the probability of state i at $t = n + 1$ depends only on what state the system was in at $t = n$ is a Markov process.

Note: Markov processes are memoryless.

Definition (Transition Matrix)

M is a transition matrix for stochastic process \mathbf{x}_t if:

$$\mathbf{x}_{t+1} = M\mathbf{x}_t$$

If $\sum_i M_{ij} = 1 \quad \forall j$ (i.e., all column sums are 1), this process is a Markov process. Here, nonnegative scalar M_{ij} is transition probabilities that the process will be in state i at $t = n + 1$ if it is in state j at $t = n$

If M_{ij} , transition probabilities are fixed and independent of time indices t , this process is time-homogeneous or that M_{ij} are stationary.

Regular Markov Matrix

Definition (Regular Markov Matrix)

M is a regular Markov matrix if:

- ① $\sum_i M_{ij} = 1 \quad \forall j$
- ② $M_{ij} \geq 0 \quad \forall i, j$
- ③ $\exists r \in \mathbb{N}$ s.t. $M^r > 0 \quad \forall i, j$
- ④ Condition 3 hold when $r = 1$

Theorem (23.15)

Let M be a regular Markov matrix. Then,

- ① 1 is an eigenvalue of M of multiplicity 1 (i.e., 1 is not a repeated root)
- ② For every other eigenvalue r of M , $|r| < 1$
- ③ \mathbf{w}_1 , Eigenvector for eigenvalue 1 has strict positive components
- ④ If $\mathbf{v}_1 = \mathbf{w}_1 / \|\mathbf{w}_1\|$, \mathbf{v}_1 is a probability vector and if $\mathbf{x}_{t+1} = M\mathbf{x}_t$,

$$\lim_{n \rightarrow \infty} \mathbf{x}_n = \mathbf{v}_1$$

Note: example of non-regular Markov process. If $\exists i$ s.t. $M_{ii} = 1$, This state i is absorbing state. I.e., once process reach state i , this state does not change forever. Therefore, this process will eventually reach one of these states i and then stay there forever.

Symmetric Matrices

Example of Symmetric Matrices in Economics

- (Bordered) Hessians in optimization problem
- Variance-covariance matrices in statistics

Fortunately, symmetric matrices do not have complex eigenvalues.

Definition (Orthogonal Matrix)

A matrix P satisfies the condition $P^{-1} = P^T$, (i.e., $P^T P = I$) is orthogonal matrix.

We can find uncoupled system when A is symmetric.

Properties of Symmetric Matrices

Theorem (23.16)

Let $A \in M_k$ and $A^T = A$. Then,

- All k roots of $\det A - rI = 0$ are real numbers.
- All corresponding eigenvectors \mathbf{w}_i are orthogonal
- $\exists P$ satisfying:
 - \mathbf{w}_i s are normalized eigenvectors for each eigenvalues r_i : $\|\mathbf{w}_i\| = 1 \quad \forall i$
 - Matrix $[\mathbf{w}_1 \quad \cdots \quad \mathbf{w}_k]$ is nonsingular
 - $\mathbf{w}_i \mathbf{w}_j^T = 0 \quad \forall i \neq j$ (orthogonal to each other)
 - $P^{-1} = P^T$
 -

$$P^{-1}AP = P^TAP = \begin{pmatrix} r_1 & 0 & \cdots & 0 \\ 0 & r_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r_k \end{pmatrix}$$

Quadratic Forms

Quadratic Forms

Every quadratic form $Q(\mathbf{x})$ can be represented by symmetric matrix A :

$$Q(\mathbf{x}) = \mathbf{x}^T A \mathbf{x} \quad \wedge \quad A^T = A$$

Always, we can find uncoupled system by taking

$P^T \mathbf{x} = [\mathbf{w}_1 \ \cdots \ \mathbf{w}_k]^T \mathbf{x}$ when \mathbf{w}_i are corresponding normalized eigenvalues r_1, \dots, r_k . Let the transformed uncoupled system be $\mathbf{y} = P^T \mathbf{x}$. Then,

$$Q(\mathbf{x}) = Q(P\mathbf{y}) = (P\mathbf{y})^T A (P\mathbf{y}) = \mathbf{y}^T (P^T A P) \mathbf{y}$$

Note: \mathbf{y} is a linear change of coordinates from \mathbf{x} .

Definiteness and Eigenvalues

Theorem (23.17)

Let $A^T = A \in M_k$ and r_1, \dots, r_k are eigenvalues of A . Then,

- ① A is PD $\iff r_i > 0 \quad \forall i$
- ② A is ND $\iff r_i < 0 \quad \forall i$
- ③ A is PSD $\iff r_i \geq 0 \quad \forall i$
- ④ A is NSD $\iff r_i \leq 0 \quad \forall i$
- ⑤ A is ID $\iff \exists i, j \quad \text{s.t.} \quad r_i < 0 \wedge r_j > 0$

Theorem (23.18)

Let $A^T = A \in M_k$. Then the below statements are equivalent:

- ① A is PD
- ② $\exists B \quad \text{s.t.} \quad A = B^T B \wedge \exists B^{-1}$
- ③ $\exists Q \quad \text{s.t.} \quad Q^T A Q = I \wedge \exists Q^{-1}$