Determinants: An Overview

Ch.9

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Defining the Determinant

We need to define a function $\det(A) : M_n \to \mathbb{R}$ satisfying

$$\det A = \det(A) = |A| \begin{cases} = 0 \text{ if } A \text{ is singular} \\ \neq 0 \text{ if } A \text{ is nonsingular} \end{cases}$$

This chapter is an easy overview for determinant. For full version, see Ch.26.

Big Picture

Definition of $\det(A)$ when A is

1. 1×1 matrices,
2. 2×2 matrices, and
3. $n \times n$ matrices

$\det(A)$ when $A \in M_1$ and $A \in M_2$

Definition ($\det(A)$ when $A \in M_1$)

$$\det(A) := A$$

When $A \in M_1$, A is just a scalar. $i.e.$, $M_1 = \mathbf{R}$

Definition ($\det(A)$ when $A \in M_2$)

$$\det(A) := a_{11} \det(a_{22}) - a_{12} \det(a_{21}) = a_{11}a_{22} - a_{12}a_{21}$$

For M_n, above definition is extended recursively

Minor and Cofactor of Matrices

Let $A \in M_n$

Definition (Minor, Cofactor)

Let $A_{ij} \in M_{n-1}$ be a submatrix obtained by deleting R_i and C_j. Then i, jth minor M_{ij} is defined as follows:

$$M_{ij} := \det A_{ij}$$

And C_{ij}, the i, jth cofactor of A is defined as:

$$C_{ij} := (-1)^{i+j} M_{ij}$$

Determinant of M_n Matrices

Definition (Determinant of M_n Matrices)

$$\det A := \sum_{i}^{n} a_{i\bar{j}} C_{i\bar{j}} = \sum_{j}^{n} a_{\bar{i}j} C_{\bar{i}j}$$

Calculation Procedure for General $A \in M_n$

STEP 1: Select one R_i or C_j

STEP 2: Calculate $\det A$ from deleting each element in R_i or C_j from STEP 1

STEP 3: For all M_{ij} in STEP2, follow STEP1-2 recursively

Example: M_3 Matrix

$$\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

$$= a_{11}(-1)^{1+1}\begin{vmatrix} e & f \\ h & i \end{vmatrix} + a_{12}(-1)^{1+2}\begin{vmatrix} d & f \\ g & i \end{vmatrix} + a_{13}(-1)^{1+3}\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$= a(ei - fh) - b(di - fg) + c(dh - eg)$$

Important Note: Rule of Sarrus

(`https://en.wikipedia.org/wiki/Rule_of_Sarrus`) only for 3×3

matrices, NOT FOR $n \times n$ matrices!!!

Computing the Determinant of Special Form

In general, calculation of determinant is very complex except for some cases:

Theorem (9.1)

If $A \in M_n$ is (1) lower-trangular, (2) upper-trangular, or (3) diagonal matrix,

$$\det(A) = \prod_{i}^{n} A_{ii}$$

Determinant of $REFM$

Theorem (9.2)

Let A_{REF} be the REFM of $A \in M_n$ by only using ERO_1, ERO_2. Then,

$$\det A = \pm \det A_{REF}$$

If only ERO_2 is used to make A_{REF},

$$\det A = \det A_{REF}$$

Proof Sketch: $\det(EM_1(R_i \leftrightarrow R_j)) = -1, \det(EM_2(R_i \leftarrow R_i + kR_j)) = 1, \det(EM_3(R_i \leftarrow kR_i)) = k$ and use Theorem 9.5(b)

Theorem (9.3: Main Property of Determinant)

$A \in M_n$ is nonsingular iff $\det A \neq 0$

Determinant and Inverse Matrix

Definition (Adjoint)

Let $C_{ij} \in \mathbb{R}$ *be the* i, j*th cofactor of* $A \in M_n$. *Then,*

$$(adj\, A)_{ij} := C_{ji} = (-1)^{j+i} M_{ji} \quad \forall i, j$$

Theorem (9.4 (a))

$$A^{-1} = \frac{1}{\det A} adj\, A$$

Cramer's rule

For any nonsingular matrix $A \in M_n$,

Theorem (9.4 (b):Cramer's Rule)

The unique solution $\mathbf{x} = (x_1, \cdots, x_n)$ *of* $A\mathbf{x} = \mathbf{b}$ *is:*

$$x_i = \frac{\det B_i}{\det A} \quad \forall i = 1, \cdots, n$$

When

$$B_i := \begin{pmatrix} A_1 & \mathbf{b} & A_2 \end{pmatrix},$$

$$A_1 := \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1,i-1} \\ a_{21} & a_{22} & \cdots & a_{2,i-1} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{n,i-1} \end{pmatrix}, A_2 := \begin{pmatrix} a_{1,i+1} & a_{1,i+2} & \cdots & a_{1,n} \\ a_{2,i+1} & a_{2,i+2} & \cdots & a_{2,n} \\ \vdots & & & \\ a_{n,i+1} & a_{n,i+2} & \cdots & a_{n,n} \end{pmatrix}$$

Algebraic Properties of the Determinant Function

Theorem (9.5)

For $A \in M_n$,

a. $\det A^T = \det A$
b. $\det(AB) = \det A \det B$
c. $\det(A+B) \neq \det A + \det B$ *in general*

G-J Elimination ($\sim n^3/3$) is far, far, far more efficient than Cramer's rule ($\sim (n-1)((n+1)!)$) when n is large

IS-LM Analysis (Revisited)

$$sY + ar = I^0 + G$$

$$mY - hr = M_s - M^0$$

Let's solve above IS-LM model:

$$Y = \frac{\begin{vmatrix} I^0 + G & a \\ M_s - M^0 & -h \end{vmatrix}}{\begin{vmatrix} s & a \\ m & -h \end{vmatrix}}, \quad r = \frac{\begin{vmatrix} s & I^0 + G \\ m & M_s - M^0 \end{vmatrix}}{\begin{vmatrix} s & a \\ m & -h \end{vmatrix}}$$

IS-LM Analysis

- Expansionary fiscal policy: $\Delta G > 0$
- Contractionary fiscal policy: $\Delta G < 0$
- Expansionary monetary policy: $\Delta M_s > 0$
- Contractionary monetary policy: $\Delta M_s < 0$