Implicit Functions and Their Derivatives

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Explicit Function, Implicit Function

Explicit Function

$$x_{n+1} = x_{n+1}(\mathbf{x})$$

In explicit functions, all input $\mathbf{x} = (x_1, \cdots, x_n)$ are free (or exogenous) variables. In this form, exogenous variables $(x_1, \cdots, x_n)$ and endogenous variable (x_{n+1}) can be distinguished easily.

Implicit Function

Let $x_{n+1} = x_{n+1}(\mathbf{x})$. Then, we can find alternative representation

$$G = G(\mathbf{x}, x_{n+1}) = 0$$

G is not a function but an equation (implicit equation). In this representation, x_{n+1} is an implicit function of the exogeneous variables $\mathbf{x} = (x_1, \cdots, x_n)$. In this form, we can not distinguish easily between exogenous and endogenous variables.

Implicit Functions: Example

Representing Implicit Function by Explicit Function(s)

$$G(x, y) = x^2 + y^2 - 1 = 0$$

y can be an implicit function of x. On the other hand, x also can be an implicit function of y.

$$y = \begin{cases} \sqrt{1 - x^2}, & y \geq 0 \\ -\sqrt{1 - x^2}, & y < 0 \end{cases}$$

$$x = \begin{cases} \sqrt{1 - y^2}, & x \geq 0 \\ -\sqrt{1 - y^2}, & x < 0 \end{cases}$$

We cannot find well-defined functional relationship on the boundary of these explicit functions.

The Implicit Function Theorem (IFT) for $\mathbb{R}^2$

Main Question

1. Does $G(x, y) = \overline{c}$ determine y as a well-defined continuous function of x for around $\overline{x}_0$ and $\overline{y}_0$?
2. If (1) is true, $y' = \frac{\partial y}{\partial x} = ?$

We can get IFT on $\mathbb{R}^2$ by differentiating $G(x, y(x)) = \overline{c}$ with regard to x at $\overline{x}_0$ (Use Chain Rule I: Th14.1)

Chain Rule I

Let $g(t) = f(\mathbf{x}(t))$, $g : \mathbb{R} \to \mathbb{R}$, $f : \mathbb{R}^n \to \mathbb{R}$, $\mathbf{x} : \mathbb{R} \to \mathbb{R}^n$. Then,

$$\frac{dg}{dt} = Df_{\mathbf{x}}(\mathbf{x})\frac{\mathbf{x}(t)}{dt} = \frac{\partial f}{\partial x_1}\frac{dx_1}{dt} + \cdots + \frac{\partial f}{\partial x_n}\frac{dx_n}{dt}$$

IFT ($\mathbb{R}^2$)

Theorem (15.1 (IFT))

Let $G(x,y)$ *be a* C^1 *function on* $B_\epsilon(\overline{x}_0, \overline{y}_0)$ *in* $\mathbb{R}^2$. *Suppose that* $G(\overline{x}_0, \overline{y}_0) = \overline{c}$ *and consider the implicit equation*

$$G(x,y) = \overline{c}$$

If $\frac{\partial G}{\partial y}(\overline{x}_0, \overline{y}_0) \neq 0$, *(i.e., tangent line is not vertical) then* $\exists y = y(x) \in C^1$ *on* $I = I_\epsilon(\overline{x}_0)$ *s.t.,*

1. $G(x, y(x)) \equiv \overline{c} \quad \forall x \in I$
2. $y(\overline{x}_0) = \overline{y}_0$
3. *and*

$$y'(\overline{x}_0) = -\frac{\frac{\partial G}{\partial x}(\overline{x}_0, \overline{y}_0)}{\frac{\partial G}{\partial y}(\overline{x}_0, \overline{y}_0)}$$

We can extend IFT on $\mathbb{R}^n$

IFT on $\mathbb{R}^n$

Theorem (15.2)

Let $G(\mathbf{x}, f)$ be a C^1 function on $B_\epsilon(\overline{\mathbf{x}_0}, \overline{f}_0)$ in $\mathbb{R}^n$. Suppose that $G(\overline{\mathbf{x}_0}, \overline{f}_0) = \overline{c}$ and consider the implicit equation

$$G(\mathbf{x}, f) = \overline{c}$$

If $\frac{\partial G}{\partial f}(\overline{\mathbf{x}_0}, \overline{f}_0) \neq 0$ (i.e., tangent hyperplane is not vertical), then $\exists f = f(\mathbf{x}) \in C^1$ on $B = B_\epsilon(\overline{\mathbf{x}_0})$ s.t.,

1. $G(\mathbf{x}, f(\mathbf{x})) \equiv \overline{c} \quad \forall \mathbf{x} \in B$
2. $f(\overline{\mathbf{x}_0}) = \overline{f}_0$
3. *and*

$$\frac{\partial f}{\partial x_i}(\overline{\mathbf{x}_0}) = -\frac{\frac{\partial G}{\partial x_i}(\overline{\mathbf{x}_0}, \overline{f}_0)}{\frac{\partial G}{\partial f}(\overline{\mathbf{x}_0}, \overline{f}_0)} \quad \forall i$$

IFT: Geometric Implication

Theorem (15.3)

Let (x_0, y_0) *is on the* $G(x, y) = \bar{c}$ *in the plane and* $G \in C^1$.

- Case 1 *If* $\frac{\partial G}{\partial y}(x_0, y_0) \neq 0$, $\exists y = y(x) \in C^1$ *around* $x = x_0$ *with slope*

$$-\frac{\frac{\partial G}{\partial x}(\overline{x}_0, \overline{y}_0)}{\frac{\partial G}{\partial y}(\overline{x}_0, \overline{y}_0)}$$

- Case 2 *If* $\frac{\partial G}{\partial y}(x_0, y_0) = 0$,
  - Case 2-1 *If* $\frac{\partial G}{\partial x}(x_0, y_0) \neq 0$, $\exists x = x(y) \in C^1$ *around* $y = y_0$ *with slope*

$$-\frac{\frac{\partial G}{\partial y}(\overline{x}_0, \overline{y}_0)}{\frac{\partial G}{\partial x}(\overline{x}_0, \overline{y}_0)}$$

  - Case 2-2 *If* $\frac{\partial G}{\partial x}(x_0, y_0) = 0$, *there is no well-defined function around* (x_0, y_0) *(irregular point)*

Regular on $\mathbb{R}^2$

Definition (Regular Point)

(x_0, y_0) *is a* *regular point* *of the* $G(x,y) \in C^1$ *if:*

$$DG_{(x,y)}(x_0, y_0) = \left(\frac{\partial G}{\partial x}(x_0, y_0), \frac{\partial G}{\partial y}(x_0, y_0) \right) \neq \mathbf{0} = (0,0)$$

We can find well-defined explicit function form around regular point.
Geometrically, this implies smooth curve (or 1d manifold, 1d object) in $\mathbb{R}^2$

Theorem (15.4)

Let $G \in C^1$ *around* (x_0, y_0) *and this point is regular. Then,* $\nabla G(x_0, y_0)$ *is perpendicular to the level set of G at* (x_0, y_0)

$$\nabla G(x_0, y_0) \bullet \left(1, -\frac{\frac{\partial G}{\partial x}(\overline{x}_0, \overline{y}_0)}{\frac{\partial G}{\partial y}(\overline{x}_0, \overline{y}_0)} \right) = 0$$

Extention to $\mathbb{R}^n$ Space

Definition (Regular Point on $\mathbb{R}^n$)

$\mathbf{x}_0$ *is a* <u>*regular point*</u> *of the* $G(\mathbf{x}) \in C^1$ *if:*

$$\nabla G(\mathbf{x}_0) = DG_{\mathbf{x}}(\mathbf{x}_0) \neq \mathbf{0}$$

We can find well-defined explicit function form around regular point. Geometrically, this implies smooth hypersurface (or $n-1$ dimensional manifold, $n-1$ dimensional object) in $\mathbb{R}^n$

Theorem (15.6)

If $f : \mathbb{R}^n \to \mathbb{R} \in C^1$, $\mathbf{x}^* \in \mathbb{R}^n$, *and* $\nabla f(\mathbf{x}^*) \neq \mathbf{0}$, *Then:*

1. *The level set of* f *through* $\mathbf{x}^*$,

$$\mathcal{F}_{f(\mathbf{x}^*)} \equiv \{\mathbf{x} : f(\mathbf{x}) = f(\mathbf{x}^*)\}$$

   can be viewed as the graph of real-valued C^1 *function of* $(n-1)$ *variables in a neighborhood of* $\mathbf{x}^*$

2. $\nabla f(\mathbf{x}^*)$ *is perpendicular to the tangent hyperplane of* $\mathcal{F}_{f(\mathbf{x}^*)}$ *at* $\mathbf{x}^*$
3. $\mathbf{v}$ *is a tangent vector of* $\mathcal{F}_{f(\mathbf{x}^*)}$ *at* $\mathbf{x}^*$ *iff* $Df_{\mathbf{x}}(\mathbf{x}^*) \bullet \mathbf{v} = 0$

Systems of Implicit Functions

Definition (System of implicit functions)

A set of m equations in $m+n$ unknowns

$$\mathbf{f}(x_1, \cdots, x_{m+n}) = \mathbf{c} \in \mathbb{R}^m$$

is called a <u>*system of implicit functions*</u> *if there is a partition of the variables into n exogenous variables and m endogenous variables, so that if exogenous variables are given, the resulting system can be solved uniquely.*

By linearization, we can solve $df_1, \cdots, df_m$ from given $dx_1, \cdots, dx_n$ around $(\mathbf{f}, \mathbf{x}) = (\mathbf{f}^*, \mathbf{x}^*)$

Linearization

Linearized System

We can get a linearized system from nonlinear system

$$\begin{aligned} F_1(f_1, \cdots, f_m, x_1, \cdots, x_n) &= \bar{c}_1 \\ F_2(f_1, \cdots, f_m, x_1, \cdots, x_n) &= \bar{c}_2 \\ &\cdots \\ F_m(f_1, \cdots, f_m, x_1, \cdots, x_n) &= \bar{c}_m \end{aligned}$$

by taking derivative on a given point $(\mathbf{f}, \mathbf{x}) = (\mathbf{f}^*, \mathbf{x}^*)$,

$$\begin{aligned} \frac{\partial F_1}{\partial f_1} df_1 + \cdots + \frac{\partial F_1}{\partial f_m} df_m + \frac{\partial F_1}{\partial x_1} dx_1 + \cdots + \frac{\partial F_1}{\partial x_n} dx_n &= 0 \\ \vdots \qquad\qquad\qquad\qquad\qquad\qquad &\vdots \\ \frac{\partial F_m}{\partial f_1} df_1 + \cdots + \frac{\partial F_m}{\partial f_m} df_m + \frac{\partial F_m}{\partial x_1} dx_1 + \cdots + \frac{\partial F_m}{\partial x_n} dx_n &= 0 \end{aligned}$$

Solving Linearized System

Solving Prodecure

$$\begin{aligned}
\frac{\partial F_1}{\partial f_1}df_1 + \cdots + \frac{\partial F_1}{\partial f_m}df_m &= -\left(\frac{\partial F_1}{\partial x_1}dx_1 + \cdots + \frac{\partial F_1}{\partial x_n}dx_n\right) \\
&\vdots \\
\frac{\partial F_m}{\partial f_1}df_1 + \cdots + \frac{\partial F_m}{\partial f_m}df_m &= -\left(\frac{\partial F_m}{\partial x_1}dx_1 + \cdots + \frac{\partial F_m}{\partial x_n}dx_n\right)
\end{aligned}$$

In this system, $d\mathbf{f}$ is unknown and others are given explicitly. Therefore,

$$d\mathbf{f} = -(D\mathbf{F}_{\mathbf{f}}(\mathbf{f}^*, \mathbf{x}^*))^{-1} \cdot D\mathbf{F}_{\mathbf{x}}(\mathbf{f}^*, \mathbf{x}^*)d\mathbf{x}$$

and when $d\mathbf{x} = d\mathbf{x}^*$, $\mathbf{f} = \mathbf{f}^* + d\mathbf{f}$