

Quadratic Forms and Definite Matrices

Ch.16

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Quadratic Forms

Definition (Quadratic Form)

A quadratic form on \mathbb{R}^n is a real-valued function of the form

$$Q(\mathbf{x}) = \mathbf{x}^T A \mathbf{x}, \quad \mathbf{x} \in \mathbb{R}^n, \quad A^T = A$$

For more detailed description, see Ch13 (section 3).

Definiteness: Overview

When $Q = \mathbf{x}^T A \mathbf{x}$ and A is a diagonal matrix

$$A = \begin{pmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{pmatrix}$$

- Positive Definite (PD): $a_{ii} > 0 \quad \forall i$
- Positive Semi Definite (PSD): $a_{ii} \geq 0 \quad \forall i$
- Negative Definite (ND): $a_{ii} < 0 \quad \forall i$
- Negative Semi Definite (NSD): $a_{ii} \leq 0 \quad \forall i$
- Indefinite (ID): $a_{ii} < 0$ for some i , and $a_{ii} > 0$ for some i

Definite Symmetric Matrices

Definition (PD,PSD,ND,NSD,ID)

Let A be an $n \times n$ symmetric matrix and $Q = \mathbf{x}^T A \mathbf{x}$, then A is:

- 1 PD if $Q > 0 \forall \mathbf{x} \neq \mathbf{0}$ in \mathbb{R}^n
- 2 PSD if $Q \geq 0 \forall \mathbf{x} \neq \mathbf{0}$ in \mathbb{R}^n
- 3 ND if $Q < 0 \forall \mathbf{x} \neq \mathbf{0}$ in \mathbb{R}^n
- 4 NSD if $Q \leq 0 \forall \mathbf{x} \neq \mathbf{0}$ in \mathbb{R}^n
- 5 ID if $Q > 0$ for some $\mathbf{x} \in \mathbb{R}^n$ and $Q < 0$ for some $\mathbf{x} \in \mathbb{R}^n$

Principal Minors of a Matrix

Definition (Principal Submatrix (PS), Principal Minor (PM))

Let A be an $n \times n$ symmetric matrix. k th order principal submatrix of A is $k \times k$ submatrix of A obtained by deleting $n - k$ columns C_1, \dots, C_{n-k} and same $n - k$ rows R_1, \dots, R_{n-k} .

k th order principal minor of A is the determinant of k th order principal submatrix.

Note: the number of k th order principal submatrix can be nCk . We will denote k th order principal minor by $PM_k(A)$.

Definition (Leading PS, Leading PM)

k th order leading principal submatrix of A is an unique k th order submatrix obtained by deleting the last $n - k$ rows and columns from A .

k th order leading principal minor of A ($LPM_k(A)$ or $|A_k|$) is the determinant of k th order leading principal submatrix of A

Test for Definiteness

Theorem (16.1,2)

Let A be an $n \times n$ symmetric matrix. Then,

- ① A is PD iff $LPM_k(A) > 0 \ \forall k$
- ② A is PSD iff $PM_k(A) \geq 0 \ \forall k$
- ③ A is ND iff $\text{sign}(LPM_k(A)) = \text{sign}((-1)^k) \ \forall k$
- ④ A is NSD iff $\text{sign}(PM_k(A)) = \text{sign}((-1)^k) \ \forall PM_k(A) \neq 0$
- ⑤ Otherwise, A is ID

Note: We can find more elegant criteria using eigenvalues (Ch.23). To check all PM , $\sum_i^n nCi$ determinants should be calculated.

Finding global max/min of $Q(x_1, x_2)$ with one linear constraint

$$Q(x_1, x_2) = \mathbf{x}^T H \mathbf{x} = (x_1 \ x_2) \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = ax_1^2 + 2bx_1x_2 + cx_2^2$$

on

$$(A \ B) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = G\mathbf{x} = 0$$

Substitute x_1 to $-Bx_2/A$ and we can get one variable function \tilde{Q} in terms of x_2

$$\tilde{Q}(x_2) = Q(-Bx_2/A, x_2) = \frac{aB^2 - 2bAB + cA^2}{A^2} x_2^2$$

$$aB^2 - 2bAB + cA^2 = -\det \begin{pmatrix} 0 & A & B \\ A & a & b \\ B & b & c \end{pmatrix} = -\det \begin{pmatrix} 0 & G \\ G^T & H \end{pmatrix}$$

Definiteness of Bordered Matrix

Theorem (16.3)

$Q(\mathbf{x})$ is PD[ND] on the constraint set $G\mathbf{x} = 0$ iff

$$\det \begin{pmatrix} 0 & A & B \\ A & a & b \\ B & b & c \end{pmatrix} = \det \begin{pmatrix} 0 & G \\ G^T & H \end{pmatrix}$$

is negative[positive]

Note: sign of determinant is dependent on both n (size of \mathbf{x}) and m (number of restriction)

General Bordered Matrix

Consider a general quadratic form

$$Q(\mathbf{x}) = \mathbf{x}^T A \mathbf{x} = \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix} \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

with linear constraint set

$$B\mathbf{x} = \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \cdots & b_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

We can make $(n + m) \times (n + m)$ symmetric matrix (general bordered matrix)

$$H = \begin{pmatrix} \mathbf{0} & B \\ B^T & A \end{pmatrix}$$

Definiteness of General Bordered Matrix

Theorem (16.4)

To determine the definiteness of general bordered matrix, Check the signs of the last $n - m$ LPMs of H , starting with $LPM_{n+m}(H)$ (i.e., the determinant of H itself). This means you should check the sign of

$$\underbrace{LPM_{n+m}(H), LPM_{n+m-1}(H), \dots, LPM_{n+m-(n-m-1)}(H)}_{n-m \text{ LPMs}}$$

- Ⓐ If $\text{sign}(\det H) = \text{sign}((-1)^m)$ and all $n - m$ LPMs have same sign, Q is PD on the constraint set $B\mathbf{x} = \mathbf{0}$ and $\mathbf{x} = \mathbf{0}$ is strict global min of Q on the constraint set
- Ⓑ If $\text{sign}(\det H) = \text{sign}((-1)^n)$ and following $n - m$ LPMs alternates in sign, Q is ND on the constraint set $B\mathbf{x} = \mathbf{0}$ and $\mathbf{x} = \mathbf{0}$ is strict global max of Q on the constraint set

Definiteness of General Bordered Matrix

Continued

- (c) if (a),(b) is violated by nonzero LPMs, Q is ID on the constraint set $B\mathbf{x} = \mathbf{0}$ and $\mathbf{x} = \mathbf{0}$ is neither a max nor a min of Q on the constraint set

Note: Test for NSD, PSD is much more tedious and trivial in economics
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