

# Limits and Open Sets

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## 1 Sequences of Real Numbers

### (Sub)Sequences of Real Number: Definition

**Definition 1** (Sequence of Real Number).  $\{x_n\}_{n=1}^{\infty}$  is a sequence of real number if:

$$x : \mathbb{N} \rightarrow \mathbb{R}, \quad x(i) = x_i$$

*I.e., sequence of real number is just a real function whose domain is  $\mathbb{N}$  (the set of (all) natural numbers, or the set of (all) positive integers)*

**Definition 2** (Subsequence). Let  $M = \{n_i\}_{i=1}^{\infty}$  be any infinite subset of  $\mathbb{N}$  and  $n_i > n_j \forall i > j$ . (I.e., increasing sequence of natural numbers). A sequence  $\{y_n\}_{n=1}^{\infty}$  is a subsequence of  $\{x_n\}_{n=1}^{\infty}$  if:

$$y_j = x_{n_j}, \quad j \in \mathbb{N}$$

### Limit and Convergence: Definition

**Definition 3** (Limit of a Sequence, Convergence).  $\bar{r} \in \mathbb{R}$  is the limit of a sequence of  $\{x_n\}_{n=1}^{\infty}$  if:

$$\forall \epsilon > 0, \quad \exists \bar{N} \in \mathbb{N} \quad \text{s.t.} \quad \forall n \geq \bar{N} \quad |x_n - \bar{r}| < \epsilon$$

$$\text{Then, } \lim x_n = \bar{r} \quad \text{or} \quad \lim_{n \rightarrow \infty} x_n = \bar{r} \quad \text{or} \quad x_n \rightarrow \bar{r} \quad (x_n \text{ converges to } \bar{r})$$

Note 1: Sometimes,  $\epsilon \in (0, \bar{\alpha})$  is used (for all small positive real numbers)

Note 2:  $|x_n - \bar{r}| < \epsilon$  has alternative notation:  $\epsilon$ -interval:  $x_n \in I_{\epsilon}(\bar{r})$

**Definition 4** (Limit of a Real Function ( $\lim_{x \rightarrow \bar{x}_0} f(x) = \bar{r}$ )).

$$\forall \epsilon > 0, \exists \delta > 0 \quad \text{s.t.} \quad x \in D \wedge 0 < |x - \bar{x}_0| < \delta \Rightarrow |f(x) - \bar{r}| < \epsilon$$

### Algebraic Properties of Limits

**Theorem 1** (12.1). A sequence can have at most one limit.

**Theorem 2** (12.2).

$$\text{If } x_n \rightarrow \bar{x} \quad \wedge \quad y_n \rightarrow \bar{y},$$

$$1. \quad x_n \pm y_n \rightarrow \bar{x} \pm \bar{y} \quad (\text{Th 12.2})$$

$$2. \quad x_n y_n \rightarrow \bar{x} \bar{y} \quad (\text{Th 12.3})$$

$$3. \quad x_n / y_n \rightarrow \bar{x} / \bar{y}$$

**Theorem 3** (12.4).

$$x_n \rightarrow \bar{x} \quad \wedge \quad x_n \leq [\geq] \bar{b} \quad \forall n \Rightarrow \bar{x} \leq [\geq] \bar{b}$$

## 2 Sequences in $\mathbb{R}^m$

### Convergence in $\mathbb{R}^m$ Space

**Definition 5** (Sequence of Vector).  $\{\mathbf{x}_n\}_{n=1}^{\infty}$  is a sequence of vector if:

$$\mathbf{x} : \mathbb{N} \rightarrow \mathbb{R}^m, \quad \mathbf{x}(i) = \mathbf{x}_i$$

**Definition 6** ( $\epsilon$ -ball about  $\bar{\mathbf{r}}$ ).  $B_\epsilon(\mathbf{r})$ ,  $\epsilon$ -ball about  $\mathbf{r}$  is defined as:

$$B_\epsilon(\mathbf{r}) := \{\mathbf{x} \in \mathbb{R}^m : \|\mathbf{x} - \mathbf{r}\| < \epsilon\}$$

Note: Geometrically,  $\epsilon$ -ball is hyperball in  $m$  dimensions, or bounded by an  $m - 1$  sphere

**Definition 7** (Limit of a Sequence of Vector).

$$\mathbf{x}_n \rightarrow \mathbf{x} \quad \text{if} \quad \forall \epsilon > 0, \quad \exists \bar{N} \quad \text{s.t.} \quad \forall n \geq \bar{N}, \quad \mathbf{x}_n \in B_\epsilon(\mathbf{x})$$

### Convergence of Vectors

**Theorem 4** (12.5). Let  $\mathbf{x}_n = (x_{1n}, \dots, x_{mn})$ .  $\mathbf{x}_n$  converges iff:

$$x_{in} \rightarrow \bar{x}_{in} \quad \forall i$$

**Theorem 5** (12.6). If  $\mathbf{x}_n \rightarrow \mathbf{x}^*$ ,  $\mathbf{y}_n \rightarrow \mathbf{y}^*$ , and  $c_n \rightarrow c^*$ , then

$$c_n \mathbf{x}_n + \mathbf{y}_n \rightarrow c^* \mathbf{x}^* + \mathbf{y}^*$$

## 3 Open Sets

### Open: Definition

**Definition 8** (Open). A set  $S \in \mathbb{R}^m$  is open if

$$\forall \mathbf{x} \in S \quad \Rightarrow \quad \exists \epsilon > 0 \quad \text{s.t.} \quad B_\epsilon(\mathbf{x}) \subset S$$

Geometrically, open set has no boundary.

**Theorem 6** (12.7). Open balls are open sets

**Theorem 7** (12.8). 1. Any union of open set is open

2. The finite intersection of open sets is open

### Interior

**Definition 9** (Interior).  $\text{int}S$ , or Interior of  $S$  is union of all open sets contained in  $S$

Note: Interior is the largest open subset of  $S$

### Open and Closed

|            | Open | Not Open |
|------------|------|----------|
| Closed     |      |          |
| Not Closed |      |          |

## 4 Closed Sets

### Closed: Definition

**Definition 10** (Closed). A set  $S \in \mathbb{R}^m$  is closed if, the limits of all convergent sequence  $\{\mathbf{x}_n\}_{n=1}^{\infty} \in S$  are contained in  $S$

Note: Closed set must contain all its boundary points.

**Theorem 8** (12.9).  $S \in \mathbb{R}^m$  is closed iff  $S^c = \mathbb{R}^m - S$  is open

**Theorem 9** (12.10). 1. Any intersection of closed sets is closed

2. The finite union of closed sets is closed

### Closure, Boundary

**Definition 11** (Closure).  $clS$  or  $\bar{S}$  is closure of  $S$  if It is the intersection of all closed sets containing  $S$

Intuitively, closure is the smallest closed set contains  $S$

**Definition 12** (Boundary).  $\mathbf{x}$  is in the boundary of a set  $S$  if

$$\forall \epsilon > 0, \quad B_{\epsilon}(\mathbf{x}) \cap S \neq \emptyset \quad \wedge \quad B_{\epsilon}(\mathbf{x}) \cap S^c \neq \emptyset$$

**Theorem 10** (12.12). Boundary of  $S = clS \cap clS^c$

## 5 Compact Sets

### Bounded, Compact

**Definition 13** (bounded).  $S \in \mathbb{R}^n$  is bounded if:

$$\exists b \in \mathbb{R} \quad s.t. \quad ||\mathbf{x}|| \leq b \quad \forall \mathbf{x} \in S$$

**Definition 14** (Compact).  $S \in \mathbb{R}^n$  is compact iff  $S$  is closed and bounded

**Theorem 11** (12.13-14). • Any sequence contained in the compact set  $[0, 1]$  has a convergent subsequence (Th 12.13)

- Any sequence contained in the compact set  $C \in \mathbb{R}^n$  has a convergent subsequence whose limit lies in  $C$  (Bolzano-Weierstrass Theorem)