# Calculus of Several Variables

## Ch.14

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# Table of Contents

# Partial Derivative

Let $f : D \in \mathbb{R}^n \to \mathbb{R}$ and $\mathbf{e_i}$ be a vector whose $i$ th element is 1 and others are 0.

$$\mathbf{e_i} := (\overbrace{0, 0, \cdots, 0, 1}^{i}, 0, \cdots, 0)$$

## Definition (Partial Derivative)

*Partial derivative at $\bar{\mathbf{x_0}} \in D$ is*

$$\frac{\partial f}{\partial x_i} := \lim_{h \to 0} \frac{f(\overline{\mathbf{x_0}} + h\mathbf{e_i}) - f(\overline{\mathbf{x_0}})}{h}$$

When $n = 1$, partial derivative is equivalent to derivative of one variable function.

## Calculation Procedure

- Treat $x_i$ as the only variable in $f$
- Treat $x_{-i}$ as constant

# Marginal Products

## Production Function, Marginal Product of Labor [or Capital]

Let $Q$ be the production function of a firm. If the firm's resources for production are $\mathbf{x} = (L, \mathbf{K}) = (L, K_1, K_2, \cdots, K_N)$,

$$MP_L := \frac{\partial Q}{\partial L}, \quad MP_{K_i} := \frac{\partial Q}{\partial K_i}$$

Interpretation: Small change $\Delta K_i$ (*ceteris paribus*) in $K_i$ can cause output change $\Delta Q$ around $(L^*, \mathbf{K}^*)$

$$\Delta Q \approx \frac{\partial Q}{\partial K_i}(L^*, \mathbf{K}^*) \Delta K_i$$

## Marginal Utility

Let $U(\mathbf{x})$ be the utility function with respect to commodity bundle $\mathbf{x}$. Then $\frac{\partial U}{\partial x_i}$ is <u>marginal utility</u> of commodity $i$ at $\mathbf{x}^*$

# Elasticity

## Elasticity: Multi variable version

$x_i$ elasticity of $Q(\mathbf{x})$ around $(\mathbf{x}^*, Q^*)$ is:

$$\epsilon_i := \frac{\frac{\partial Q}{Q^*}}{\frac{\partial x_i}{x_i^*}} = \frac{x_i^*}{Q^*} \frac{\partial Q}{\partial x_i}(\mathbf{x}^*)$$

In general, elasticity is ratio of rates of changes. When the sign of elasticity is not important, $|\epsilon|$ can be used.

# Partial Derivative: Geometric Interpretation

## $f : \mathbb{R}^2 \to \mathbb{R}$

- Think of $f(\mathbf{x}) = x_1^2 + x_2^2$.
- If $x_2 = \bar{x}_2$, $f(x_1, \bar{x}_2)$ is equivalent to one variable function $\tilde{f}(x_1) = x_1^2 + \bar{x}_2^2$.
- Graph of $\tilde{f}$ is intersection of the graph of $f$ with the slice $x_2 = \bar{x}_2$.
- $\frac{\partial f}{\partial x_1}(\bar{x}_1, \bar{x}_2) = \frac{\partial \tilde{f}}{\partial x_1}(\bar{x}_1)$ is the slope of $\tilde{f}$ on $\bar{x}_1$, slope of the tangent line to the curve $\tilde{f}$ (on the plane $x_2 = \bar{x}_2$)

# Example

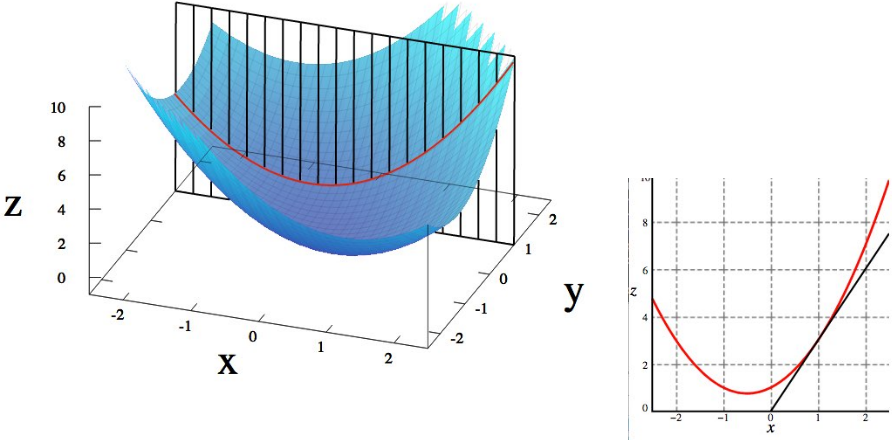


Figure: Graph of $z = x^2 + xy + y^2$ with intersection $y = 1$

## Finding Tangent Plane

Let $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable. When finding a tangent plane on $(\bar{x}_1, \bar{x}_2)$, we need to get at least two independent vectors: $\left(1, 0, \frac{\partial f}{\partial x_1}(\bar{x}_1, \bar{x}_2)\right)$ (slice $x_2 = 0$), and $\left(0, 1, \frac{\partial f}{\partial x_2}(\bar{x}_1, \bar{x}_2)\right)$ (slice $x_1 = 0$)
Then the tangent plane with two parameters $\Delta x_1, \Delta x_2$ is:

$$(\bar{x_1}, \bar{x_2}, f(\bar{\mathbf{x}})) + \Delta x_1 \left(1, 0, \frac{\partial f}{\partial x_1}(\bar{x}_1, \bar{x}_2)\right) + \Delta x_2 \left(0, 1, \frac{\partial f}{\partial x_2}(\bar{x}_1, \bar{x}_2)\right)$$

$$= \left(\bar{x_1} + \Delta x_1, \bar{x_2} + \Delta x_2, f(\bar{\mathbf{x}}) + \frac{\partial f}{\partial x_1}(\bar{x}_1, \bar{x}_2)\Delta x_1 + \frac{\partial f}{\partial x_2}(\bar{x}_1, \bar{x}_2)\Delta x_2\right)$$

This interpretation can be extended to $n$ dimension.

# The Total Derivative

## Changes in All Direction: $f : \mathbb{R}^n \to \mathbb{R}^1$

Let $d\mathbf{x} = (dx_1, \cdots, dx_n)$ and $f : \mathbb{R}^n \to \mathbb{R}$, differentiable. Then small change of $d\mathbf{x}$ will cause small change of $df = f(\bar{\mathbf{x}} + d\bar{\mathbf{x}}) - f(\bar{\mathbf{x}}) \in \mathbb{R}$ and

$$df = f(\bar{\mathbf{x}} + d\bar{\mathbf{x}}) - f(\bar{\mathbf{x}}) = \frac{\partial f}{\partial x_1}(\bar{\mathbf{x}})dx_1 + \cdots + \frac{\partial f}{\partial x_n}(\bar{\mathbf{x}})dx_n = Df_{\mathbf{x}}d\mathbf{x}$$

And $Df_{\mathbf{x}} := \begin{pmatrix} \frac{\partial f}{\partial x_1}(\bar{\mathbf{x}}) & \frac{\partial f}{\partial x_2}(\bar{\mathbf{x}}) & \cdots & \frac{\partial f}{\partial x_n}(\bar{\mathbf{x}}) \end{pmatrix}$: (Jacobian) derivative of $f$ at $\bar{\mathbf{x}}$ or The linear approximation of $f$ at $\bar{\mathbf{x}}$, or Gradient vector $\nabla f$

Note: In this case, $Df_{\mathbf{x}}$ is a vector or $1 \times n$ matrix.

# More General Case

## Changes in All Direction: $\mathbf{f}:\mathbb{R}^n \to \mathbb{R}^m$

Let $d\mathbf{x} = (dx_1, \cdots, dx_n)$ and $\mathbf{f}:\mathbb{R}^n \to \mathbb{R}^m$, differentiable. Then small change of $d\mathbf{x}$ will cause small change of $d\mathbf{f} = \mathbf{f}(\bar{\mathbf{x}} + d\bar{\mathbf{x}}) - \mathbf{f}(\bar{\mathbf{x}}) \in \mathbb{R}^m$ and

$$d\mathbf{f} = \mathbf{f}(\bar{\mathbf{x}} + d\bar{\mathbf{x}}) - \mathbf{f}(\bar{\mathbf{x}}) = \frac{\partial \mathbf{f}}{\partial x_1}(\bar{\mathbf{x}})dx_1 + \cdots + \frac{\partial \mathbf{f}}{\partial x_n}(\bar{\mathbf{x}})dx_n = D\mathbf{f}_{\mathbf{x}}d\mathbf{x}$$

Note: In this case, $D\mathbf{f}_{\mathbf{x}}$ is $m \times n$ matrix

# Curve in $\mathbb{R}^n$

## Definition (curve)

*A curve in $\mathbb{R}^n$ is $n$-tuple of continuous one variable functions*

$$\mathbf{x}(t) = (x_1(t), \cdots, x_n(t))$$

$x_i$: *coordination function,* $t$: *parameter*

## Velocity (or Tangent) Vector

$\mathbf{x}'$ is the velocity (tangent) vector of the curve at $t$

$$\mathbf{x}' := \lim_{h_j \to 0} \frac{\mathbf{x}(t+h_j) - \mathbf{x}(t)}{h_j} = (x_1'(t), \cdots, x_n'(t))$$

Geometrically, The velocity (tangent) vector is a limit of secant vector

# Regular, cusp

**Definition (regular)**

*A curve* $\mathbf{x}(t)$ *is regular iff* $x_i'(t)$ *is continuous and* $\mathbf{x}'(t) \neq \mathbf{0} \quad \forall t$
*When* $\mathbf{x}'(\bar{t}) = \mathbf{0}$, *this curve has cusp at* $\mathbf{x}(\bar{t})$

Geometrically, regular curve means smooth curve

**Definition (continuously differentiable)**

$f : \mathbb{R}^n \to \mathbb{R}$ *is continuously differentiable (or* $C^1$*) on an open set* $D \subset \mathbb{R}^n$ *iff*

$$\forall \mathbf{x} \in D, \forall i, \quad \exists \frac{\partial f}{\partial x_i}(\mathbf{x}) \quad \wedge \quad continuous$$

# Chain Rule I

## Chain Rule I

Let $g(t) = f(\mathbf{x}(t))$, $g : \mathbb{R} \to \mathbb{R}$, $f : \mathbb{R}^n \to \mathbb{R}$, $\mathbf{x} : \mathbb{R} \to \mathbb{R}^n$. Then,

$$\frac{dg}{dt} = Df_{\mathbf{x}}(\mathbf{x})\frac{d\mathbf{x}(t)}{dt} = \frac{\partial f}{\partial x_1}\frac{dx_1}{dt} + \cdots + \frac{\partial f}{\partial x_n}\frac{dx_n}{dt}$$

## Chain Rule II

Let $g(\mathbf{t}) = f(\mathbf{x}(\mathbf{t}))$, $g : \mathbb{R}^s \to \mathbb{R}$, $f : \mathbb{R}^n \to \mathbb{R}$, $\mathbf{x} : \mathbb{R}^s \to \mathbb{R}^n$. Then,

$$Dg_{\mathbf{t}} = Df_{\mathbf{x}}(\mathbf{x})D\mathbf{x}_{\mathbf{t}}(\mathbf{t})$$

$$\frac{\partial g}{\partial t_i} = \begin{pmatrix} \frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_n} \end{pmatrix} \begin{pmatrix} \frac{\partial x_1}{\partial t_i} \\ \vdots \\ \frac{\partial x_n}{\partial t_i} \end{pmatrix}$$

# Directional Derivatives and Gradients

## Directional Derivative

Let $\mathbf{x} = \overline{\mathbf{x}} + t\overline{\mathbf{v}}$: line passing $\overline{\mathbf{x}}$ with direction $\overline{\mathbf{v}}$ and $g(t) = f(\overline{\mathbf{x}} + t\overline{\mathbf{v}})$. From chain rule I,

$$\left.\frac{dg}{dt}\right|_{t=0} = \left.\frac{df}{dt}\right|_{t=0} = Df_{\mathbf{x}}(\overline{\mathbf{x}}) \cdot \frac{d\mathbf{x}}{dt} = Df_{\mathbf{x}}(\overline{\mathbf{x}}) \cdot \overline{\mathbf{v}}$$

This is the derivative of $f$ at $\overline{\mathbf{x}}$ in the direction $\overline{\mathbf{v}}$, and other notations are $\frac{\partial f}{\partial \mathbf{v}}(\overline{\mathbf{x}})$ and $D_{\mathbf{v}}f(\overline{\mathbf{x}})$

## Theorem (14.2)

*At any point $\mathbf{x} \in D$ and $\nabla f \neq 0$, $\nabla f(\mathbf{x})$ points at $x$ into the direction in which $f$ increases most rapidly*

This theorem will be used for finding normal vector of tangent hyperplane to level set.

# Chain Rule III

## Chain Rule III

Let $\mathbf{f}(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}^m$, $\mathbf{a}(t) : \mathbb{R} \to \mathbb{R}^n$, and $\mathbf{g}(t) = \mathbf{f} \circ \mathbf{a}(t)$. Then,

$$\frac{dg}{dt} = Df_{\mathbf{a}}(\mathbf{a}(t)) \cdot \mathbf{a}'(t)$$

## Chain Rule IV

Let $\mathbf{f}(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}^m$, $\mathbf{a}(\mathbf{t}) : \mathbb{R}^s \to \mathbb{R}^n$, and $\mathbf{g}(\mathbf{t}) = \mathbf{f} \circ \mathbf{a}(\mathbf{t})$. Then,

$$D\mathbf{g}_{\mathbf{t}} = Df_{\mathbf{a}}(\mathbf{a}(\mathbf{t})) \cdot D\mathbf{a}_{\mathbf{t}}$$

# Hessian

## Definition

*Hessian matrix*

$$D^2 f_{\mathbf{x}} = D(Df)_{\mathbf{x}} := \begin{pmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \frac{\partial^2 f}{\partial x_2 \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_1} \\ \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_2 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_2} \\ \vdots & \vdots & & \vdots \\ \frac{\partial^2 f}{\partial x_1 \partial x_n} & \frac{\partial^2 f}{\partial x_2 \partial x_n} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n} \end{pmatrix}$$

## Theorem (14.5:Young's theorem)

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i} \quad \forall i, j$$

This means hessian is symmetric.