

# Implicit Functions and Their Derivatives

## Ch.15

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# Explicit Function, Implicit Function

## Explicit Function

$$x_{n+1} = x_{n+1}(\mathbf{x})$$

In explicit functions, all input  $\mathbf{x} = (x_1, \dots, x_n)$  are free (or exogenous) variables. In this form, endogenous variables and exogenous variable ( $x_{n+1}$ ) can be distinguished easily.

## Implicit Function

Let  $x_{n+1} = x_{n+1}(\mathbf{x})$ . Then, we can find alternative representation

$$G = G(\mathbf{x}, x_{n+1}) = 0$$

$G$  is not a function but an equation (implicit equation). In this representation,  $x_{n+1}$  is an implicit function of the exogeneous variables  $\mathbf{x} = (x_1, \dots, x_n)$ . In this form, we can not distinguish easily between exogenous and endogenous variables.

# Implicit Functions: Example

## Representing Implicit Function by Explicit Function(s)

$$G(x, y) = x^2 + y^2 - 1 = 0$$

$y$  can be an implicit function of  $x$ . On the other hand,  $x$  also can be an implicit function of  $y$ .

$$y = \begin{cases} \sqrt{1-x^2}, & y \geq 0 \\ -\sqrt{1-x^2}, & y < 0 \end{cases}$$

$$x = \begin{cases} \sqrt{1-y^2}, & x \geq 0 \\ -\sqrt{1-y^2}, & x < 0 \end{cases}$$

We cannot find well-defined functional relationship on the boundary of these explicit functions.

# The Implicit Function Theorem (IFT) for $\mathbb{R}^2$

## Main Question

- 1 Does  $G(x, y) = \bar{c}$  determine  $y$  as a well-defined continuous function of  $x$  for around  $\bar{x}_0$  and  $\bar{y}_0$ ?
- 2 If (1) is true,  $y' = \frac{\partial y}{\partial x} = ?$

We can get IFT on  $\mathbb{R}^2$  by differentiating  $G(x, y(x)) = \bar{c}$  with regard to  $x$  at  $\bar{x}_0$  (Use Chain Rule I: Th14.1)

## Chain Rule I

Let  $g(t) = f(\mathbf{x}(t))$ ,  $g : \mathbb{R} \rightarrow \mathbb{R}$ ,  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $\mathbf{x} : \mathbb{R} \rightarrow \mathbb{R}^n$ . Then,

$$\frac{dg}{dt} = Df_{\mathbf{x}(\mathbf{x})} \frac{\mathbf{x}(t)}{dt} = \frac{\partial f}{\partial x_1} \frac{dx_1}{dt} + \cdots + \frac{\partial f}{\partial x_n} \frac{dx_n}{dt}$$

## Theorem (15.1 (IFT))

Let  $G(x, y)$  be a  $C^1$  function on  $B_\epsilon(\bar{x}_0, \bar{y}_0)$  in  $\mathbb{R}^2$ . Suppose that  $G(\bar{x}_0, \bar{y}_0) = \bar{c}$  and consider the implicit equation

$$G(x, y) = \bar{c}$$

If  $\frac{\partial G}{\partial y}(\bar{x}_0, \bar{y}_0) \neq 0$ , (i.e., tangent line is not vertical) then  $\exists y = y(x) \in C^1$  on  $I = I_\epsilon(\bar{x}_0)$  s.t.,

①  $G(x, y(x)) \equiv \bar{c} \quad \forall x \in I$

②  $y(\bar{x}_0) = \bar{y}_0$

③ and

$$y'(\bar{x}_0) = -\frac{\frac{\partial G}{\partial x}(\bar{x}_0, \bar{y}_0)}{\frac{\partial G}{\partial y}(\bar{x}_0, \bar{y}_0)}$$

We can extend IFT on  $\mathbb{R}^n$

## Theorem (15.2)

Let  $G(\mathbf{x}, f)$  be a  $C^1$  function on  $B_\epsilon(\overline{\mathbf{x}}_0, \overline{f}_0)$  in  $\mathbb{R}^n$ . Suppose that  $G(\overline{\mathbf{x}}_0, \overline{f}_0) = \overline{c}$  and consider the implicit equation

$$G(\mathbf{x}, f) = \overline{c}$$

If  $\frac{\partial G}{\partial f}(\overline{\mathbf{x}}_0, \overline{f}_0) \neq 0$  (i.e., tangent hyperplane is not vertical), then  $\exists f = f(\mathbf{x}) \in C^1$  on  $B = B_\epsilon(\overline{\mathbf{x}}_0)$  s.t.,

①  $G(\mathbf{x}, f(\mathbf{x})) \equiv \overline{c} \quad \forall \mathbf{x} \in B$

②  $f(\overline{\mathbf{x}}_0) = \overline{f}_0$

③ and

$$\frac{\partial f}{\partial x_i}(\overline{\mathbf{x}}_0) = -\frac{\frac{\partial G}{\partial x_i}(\overline{\mathbf{x}}_0, \overline{f}_0)}{\frac{\partial G}{\partial f}(\overline{\mathbf{x}}_0, \overline{f}_0)} \quad \forall i$$

## Theorem (15.3)

Let  $(x_0, y_0)$  is on the  $G(x, y) = \bar{c}$  in the plane and  $G \in C^1$ .

Case 1 If  $\frac{\partial G}{\partial y}(x_0, y_0) \neq 0$ ,  $\exists y = y(x) \in C^1$  around  $x = x_0$  with slope

$$-\frac{\frac{\partial G}{\partial x}(\bar{x}_0, \bar{y}_0)}{\frac{\partial G}{\partial y}(\bar{x}_0, \bar{y}_0)}$$

Case 2 If  $\frac{\partial G}{\partial y}(x_0, y_0) = 0$ ,

Case 2-1 If  $\frac{\partial G}{\partial x}(x_0, y_0) \neq 0$ ,  $\exists x = x(y) \in C^1$  around  $y = y_0$  with slope

$$-\frac{\frac{\partial G}{\partial y}(\bar{x}_0, \bar{y}_0)}{\frac{\partial G}{\partial x}(\bar{x}_0, \bar{y}_0)}$$

Case 2-2 If  $\frac{\partial G}{\partial x}(x_0, y_0) = 0$ , there is no well-defined function around  $(x_0, y_0)$  (irregular point)



## Definition (Regular Point)

$(x_0, y_0)$  is a regular point of the  $G(x, y) \in C^1$  if:

$$DG_{(x,y)}(x_0, y_0) = \left( \frac{\partial G}{\partial x}(x_0, y_0), \frac{\partial G}{\partial y}(x_0, y_0) \right) \neq \mathbf{0} = (0, 0)$$

We can find well-defined explicit function form around regular point.  
Geometrically, this implies smooth curve (or 1d manifold, 1d object) in  $\mathbb{R}^2$

## Theorem (15.4)

Let  $G \in C^1$  around  $(x_0, y_0)$  and this point is regular. Then,  $\nabla G(x_0, y_0)$  is perpendicular to the level set of  $G$  at  $(x_0, y_0)$

$$\nabla G(x_0, y_0) \bullet \left( 1, -\frac{\frac{\partial G}{\partial x}(\bar{x}_0, \bar{y}_0)}{\frac{\partial G}{\partial y}(\bar{x}_0, \bar{y}_0)} \right) = 0$$

# Extention to $\mathbb{R}^n$ Space

## Definition (Regular Point on $\mathbb{R}^n$ )

$\mathbf{x}_0$  is a regular point of the  $G(\mathbf{x}) \in C^1$  if:

$$\nabla G(\mathbf{x}_0) = DG_{\mathbf{x}}(\mathbf{x}_0) \neq \mathbf{0}$$

We can find well-defined explicit function form around regular point. Geometrically, this implies smooth hypersurface (or  $n - 1$  dimensional manifold,  $n - 1$  dimensional object) in  $\mathbb{R}^n$

## Theorem (15.6)

If  $f : \mathbb{R}^n \rightarrow \mathbb{R} \in C^1$ ,  $\mathbf{x}^* \in \mathbb{R}^n$ , and  $\nabla f(\mathbf{x}^*) \neq \mathbf{0}$ , Then:

- 1 The level set of  $f$  through  $\mathbf{x}^*$ ,

$$\mathcal{F}_{f(\mathbf{x}^*)} \equiv \{\mathbf{x} : f(\mathbf{x}) = f(\mathbf{x}^*)\}$$

can be viewed as the graph of real-valued  $C^1$  function of  $(n - 1)$  variables in a neighborhood of  $\mathbf{x}^*$

- 2  $\nabla f(\mathbf{x}^*)$  is perpendicular to the tangent hyperplane of  $\mathcal{F}_{f(\mathbf{x}^*)}$  at  $\mathbf{x}^*$
- 3  $\mathbf{v}$  is a tangent vector of  $\mathcal{F}_{f(\mathbf{x}^*)}$  at  $\mathbf{x}^*$  iff  $Df_{\mathbf{x}}(\mathbf{x}^*) \bullet \mathbf{v} = 0$

# Systems of Implicit Functions

## Definition (System of implicit functions)

*A set of  $m$  equations in  $m + n$  unknowns*

$$\mathbf{f}(x_1, \dots, x_{m+n}) = \mathbf{c} \in \mathbb{R}^m$$

*is called a system of implicit functions if there is a partition of the variables into  $n$  exogenous variables and  $m$  endogenous variables, so that if exogenous variables are given, the resulting system can be solved uniquely.*

By linearization, we can solve  $df_1, \dots, df_m$  from given  $dx_1, \dots, dx_n$  around  $(\mathbf{f}, \mathbf{x}) = (\mathbf{f}^*, \mathbf{x}^*)$

## Linearized System

We can get a linearized system from nonlinear system

$$F_1(f_1, \dots, f_m, x_1, \dots, x_n) = \bar{c}_1$$

$$F_2(f_1, \dots, f_m, x_1, \dots, x_n) = \bar{c}_2$$

...

$$F_m(f_1, \dots, f_m, x_1, \dots, x_n) = \bar{c}_m$$

by taking derivative on a given point  $(\mathbf{f}, \mathbf{x}) = (\mathbf{f}^*, \mathbf{x}^*)$ ,

$$\frac{\partial F_1}{\partial f_1} df_1 + \dots + \frac{\partial F_1}{\partial f_m} df_m + \frac{\partial F_1}{\partial x_1} dx_1 + \dots + \frac{\partial F_1}{\partial x_n} dx_n = 0$$

$\vdots$

$\vdots$

$$\frac{\partial F_m}{\partial f_1} df_1 + \dots + \frac{\partial F_m}{\partial f_m} df_m + \frac{\partial F_m}{\partial x_1} dx_1 + \dots + \frac{\partial F_m}{\partial x_n} dx_n = 0$$

# Solving Linearized System

## Solving Procedure

$$\begin{aligned}\frac{\partial F_1}{\partial f_1}df_1 + \cdots + \frac{\partial F_1}{\partial f_m}df_m &= - \left( \frac{\partial F_1}{\partial x_1}dx_1 + \cdots + \frac{\partial F_1}{\partial x_n}dx_n \right) \\ \vdots \\ \frac{\partial F_m}{\partial f_1}df_1 + \cdots + \frac{\partial F_m}{\partial f_m}df_m &= - \left( \frac{\partial F_m}{\partial x_1}dx_1 + \cdots + \frac{\partial F_m}{\partial x_n}dx_n \right)\end{aligned}$$

In this system,  $d\mathbf{f}$  is unknown and others are given explicitly. Therefore,

$$d\mathbf{f} = -(D\mathbf{F}_{\mathbf{f}}(\mathbf{f}^*, \mathbf{x}^*))^{-1} \cdot D\mathbf{F}_{\mathbf{x}}(\mathbf{f}^*, \mathbf{x}^*)d\mathbf{x}$$

and when  $d\mathbf{x} = d\mathbf{x}^*$ ,  $\mathbf{f} = \mathbf{f}^* + d\mathbf{f}$