Calculus of Several Variables

Ch.14

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1 Definitions and Examples

Partial Derivative

Let $f : D \in \mathbb{R}^n \to \mathbb{R}$ and $\mathbf{e_i}$ be a vector whose i th element is 1 and others are 0.

$$\mathbf{e_i} := (\overbrace{0, 0, \cdots, 0, 1}^{i}, 0, \cdots, 0)$$

Definition 1 (Partial Derivative)**.** *Partial derivative at* $\bar{\mathbf{x_0}} \in D$ *is*

$$\frac{\partial f}{\partial x_i} := \lim_{h \to 0} \frac{f(\overline{\mathbf{x_0}} + h\mathbf{e_i}) - f(\overline{\mathbf{x_0}})}{h}$$

When $n = 1$, partial derivative is equivalent to derivative of one variable function.

Calculation Procedure

- Treat x_i as the only variable in f
- Treat x_{-i} as constant

2 Economic Interpretation

2.1 Marginal

Marginal Products

Production Function, Marginal Product of Labor [or Capital]

Let Q be the production function of a firm. If the firm's resources for production are $\mathbf{x} = (L, \mathbf{K}) = (L, K_1, K_2, \cdots, K_N)$,

$$MP_L := \frac{\partial Q}{\partial L}, \quad MP_{K_i} := \frac{\partial Q}{\partial K_i}$$

Interpretation: Small change ΔK_i (*ceteris paribus*) in K_i can cause output change ΔQ around $(L^*, \mathbf{K}^*)$

$$\Delta Q \approx \frac{\partial Q}{\partial K_i}(L^*, \mathbf{K}^*)\Delta K_i$$

Marginal Utility

Let $U(\mathbf{x})$ be the utility function with respect to commodity bundle $\mathbf{x}$. Then $\frac{\partial U}{\partial x_i}$ is marginal utility of commodity i at $\mathbf{x}^*$

2.2 Elasticity

Elasticity

Elasticity: Multi variable version
x_i elasticity of $Q(\mathbf{x})$ around $(\mathbf{x}^*, Q^*)$ is:

$$\epsilon_i := \frac{\frac{\partial Q}{Q^*}}{\frac{\partial x_i}{x_i^*}} = \frac{x_i^*}{Q^*}\frac{\partial Q}{\partial x_i}(\mathbf{x}^*)$$

In general, elasticity is ratio of rates of changes. When the sign of elasticity is not important, $|\epsilon|$ can be used.

3 Geometric Interpretation

Partial Derivative: Geometric Interpretation

$f : \mathbb{R}^2 \to \mathbb{R}$

- Think of $f(\mathbf{x}) = x_1^2 + x_2^2$.
- If $x_2 = \bar{x}_2$, $f(x_1, \bar{x}_2)$ is equivalent to one variable function $\tilde{f}(x_1) = x_1^2 + \bar{x}_2^2$.
- Graph of $\tilde{f}$ is intersection of the graph of f with the slice $x_2 = \bar{x}_2$.
- $\frac{\partial f}{\partial x_1}(\bar{x}_1, \bar{x}_2) = \frac{\partial \tilde{f}}{\partial x_1}(\bar{x}_1)$ is the slope of $\tilde{f}$ on $\bar{x}_1$, slope of the tangent line to the curve $\tilde{f}$ (on the plane $x_2 = \bar{x}_2$)

Example

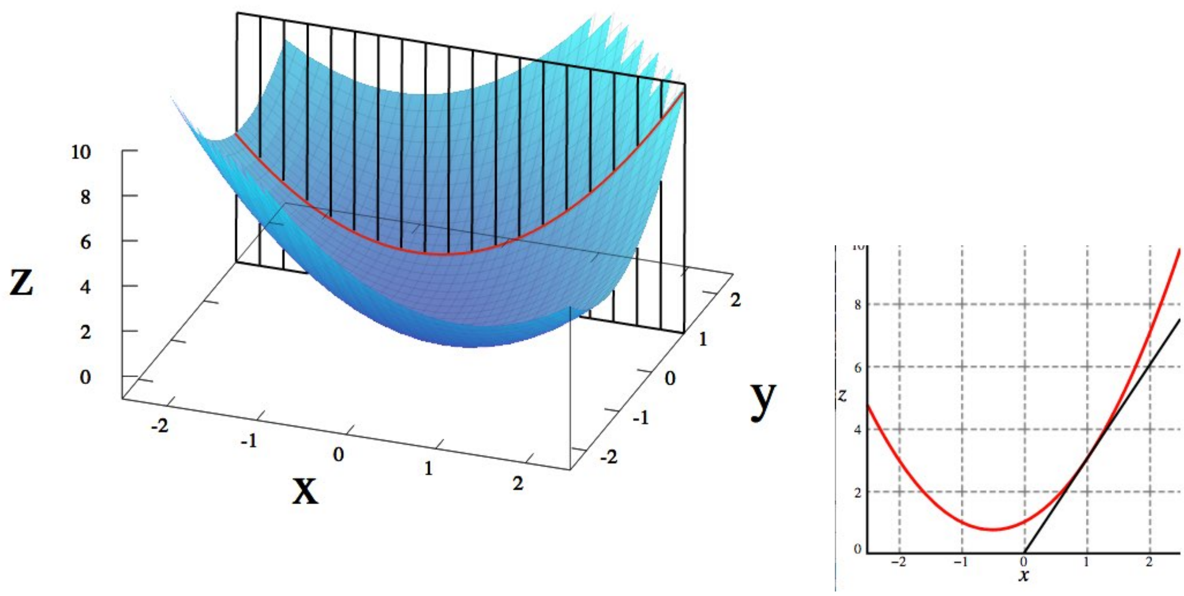

Figure 1: Graph of $z = x^2 + xy + y^2$ with intersection $y = 1$

4 The Total Derivative

Geometrical Approach

Finding Tangent Plane
Let $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable. When finding a tangent plane on $(\bar{x}_1, \bar{x}_2)$, we need to get at least two independent vectors: $\left(1, 0, \frac{\partial f}{\partial x_1}(\bar{x}_1, \bar{x}_2)\right)$ (slice $x_2 = \bar{x}_2$), and $\left(0, 1, \frac{\partial f}{\partial x_2}(\bar{x}_1, \bar{x}_2)\right)$ (slice $x_1 = \bar{x}_1$)

Then the tangent plane with two parameters $\Delta x_1, \Delta x_2$ is:

$$(\bar{x_1}, \bar{x_2}, f(\bar{\mathbf{x}})) + \Delta x_1 \left(1, 0, \frac{\partial f}{\partial x_1}(\bar{x}_1, \bar{x}_2)\right) + \Delta x_2 \left(0, 1, \frac{\partial f}{\partial x_2}(\bar{x}_1, \bar{x}_2)\right)$$
$$= \left(\bar{x_1} + \Delta x_1, \bar{x_2} + \Delta x_2, f(\bar{\mathbf{x}}) + \frac{\partial f}{\partial x_1}(\bar{x}_1, \bar{x}_2)\Delta x_1 + \frac{\partial f}{\partial x_2}(\bar{x}_1, \bar{x}_2)\Delta x_2\right)$$

This interpretation can be extended to n dimension.

The Total Derivative

Changes in All Direction: $f : \mathbb{R}^n \to \mathbb{R}^1$
Let $d\mathbf{x} = (dx_1, \cdots, dx_n)$ and $f : \mathbb{R}^n \to \mathbb{R}$, differentiable. Then small change of $d\mathbf{x}$ will cause small change of $df = f(\bar{\mathbf{x}} + d\bar{\mathbf{x}}) - f(\bar{\mathbf{x}}) \in \mathbb{R}$ and

$$df = f(\bar{\mathbf{x}} + d\bar{\mathbf{x}}) - f(\bar{\mathbf{x}}) = \frac{\partial f}{\partial x_1}(\bar{\mathbf{x}})dx_1 + \cdots + \frac{\partial f}{\partial x_n}(\bar{\mathbf{x}})dx_n = Df_{\mathbf{x}}d\mathbf{x}$$

And $Df_{\mathbf{x}} := \begin{pmatrix} \frac{\partial f}{\partial x_1}(\bar{\mathbf{x}}) & \frac{\partial f}{\partial x_2}(\bar{\mathbf{x}}) & \cdots & \frac{\partial f}{\partial x_n}(\bar{\mathbf{x}}) \end{pmatrix}$: (Jacobian) derivative of f at $\bar{\mathbf{x}}$ or The linear approximation of f at $\bar{\mathbf{x}}$, or Gradient vector ∇f

Note: In this case, $Df_{\mathbf{x}}$ is a vector or $1 \times n$ matrix.

More General Case

Changes in All Direction: $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$
Let $d\mathbf{x} = (dx_1, \cdots, dx_n)$ and $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$, differentiable. Then small change of $d\mathbf{x}$ will cause small change of $d\mathbf{f} = \mathbf{f}(\bar{\mathbf{x}} + d\bar{\mathbf{x}}) - \mathbf{f}(\bar{\mathbf{x}}) \in \mathbb{R}^m$ and

$$d\mathbf{f} = \mathbf{f}(\bar{\mathbf{x}} + d\bar{\mathbf{x}}) - \mathbf{f}(\bar{\mathbf{x}}) = \frac{\partial \mathbf{f}}{\partial x_1}(\bar{\mathbf{x}})dx_1 + \cdots + \frac{\partial \mathbf{f}}{\partial x_n}(\bar{\mathbf{x}})dx_n = D\mathbf{f}_{\mathbf{x}}d\mathbf{x}$$

Note: In this case, $D\mathbf{f}_{\mathbf{x}}$ is $m \times n$ matrix

5 The Chain Rule

Curve in $\mathbb{R}^n$

Definition 2 (curve)**.** *A <u>curve</u> in $\mathbb{R}^n$ is n-tuple of continuous one variable functions*

$$\mathbf{x}(t) = (x_1(t), \cdots, x_n(t))$$

x_i: coordination function, t: parameter

Velocity (or Tangent) Vector
$\mathbf{x}'$ is the velocity (tangent) vector of the curve at t

$$\mathbf{x}' := \lim_{h_j \to 0} \frac{\mathbf{x}(t+h_j) - \mathbf{x}(t)}{h_j} = (x_1'(t), \cdots, x_n'(t))$$

Geometrically, The velocity (tangent) vector is a limit of secant vector

Regular, cusp

Definition 3 (regular)**.** *A curve $\mathbf{x}(t)$ is regular iff $x_i'(t)$ is continuous and $\mathbf{x}'(t) \neq \mathbf{0} \quad \forall t$*
When $\mathbf{x}'(\bar{t}) = \mathbf{0}$, this curve has cusp at $\mathbf{x}(\bar{t})$

Geometrically, regular curve means smooth curve

Definition 4 (continuously differentiable)**.** *$f : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable (or C^1) on an open set $D \subset \mathbb{R}^n$ iff*

$$\forall \mathbf{x} \in D, \forall i, \quad \exists \frac{\partial f}{\partial x_i}(\mathbf{x}) \quad \wedge \quad \textit{continuous}$$

Chain Rule I

Chain Rule I
Let $g(t) = f(\mathbf{x}(t))$, $g : \mathbb{R} \to \mathbb{R}$, $f : \mathbb{R}^n \to \mathbb{R}$, $\mathbf{x} : \mathbb{R} \to \mathbb{R}^n$. Then,

$$\frac{dg}{dt} = Df_{\mathbf{x}}(\mathbf{x}) \frac{d\mathbf{x}(t)}{dt} = \frac{\partial f}{\partial x_1}\frac{dx_1}{dt} + \cdots + \frac{\partial f}{\partial x_n}\frac{dx_n}{dt}$$

Chain Rule II
Let $g(\mathbf{t}) = f(\mathbf{x}(\mathbf{t}))$, $g : \mathbb{R}^s \to \mathbb{R}$, $f : \mathbb{R}^n \to \mathbb{R}$, $\mathbf{x} : \mathbb{R}^s \to \mathbb{R}^n$. Then,

$$Dg_{\mathbf{t}} = Df_{\mathbf{x}}(\mathbf{x}) D\mathbf{x}_{\mathbf{t}}(\mathbf{t})$$

$$\frac{\partial g}{\partial t_i} = \begin{pmatrix} \frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_n} \end{pmatrix} \begin{pmatrix} \frac{\partial x_1}{\partial t_1} \\ \vdots \\ \frac{\partial x_n}{\partial t_i} \end{pmatrix}$$

6 Directional Derivatives and Gradients

Directional Derivatives and Gradients

Directional Derivative
Let $\mathbf{x} = \overline{\mathbf{x}} + t\overline{\mathbf{v}}$: line passing $\overline{\mathbf{x}}$ with direction $\overline{\mathbf{v}}$ and $g(t) = f(\overline{\mathbf{x}} + t\overline{\mathbf{v}})$. From chain rule I,

$$\left.\frac{dg}{dt}\right|_{t=0} = \left.\frac{df}{dt}\right|_{t=0} = Df_{\mathbf{x}}(\overline{\mathbf{x}}) \cdot \frac{d\mathbf{x}}{dt} = Df_{\mathbf{x}}(\overline{\mathbf{x}}) \cdot \overline{\mathbf{v}}$$

This is the derivative of f at $\overline{\mathbf{x}}$ in the direction $\overline{\mathbf{v}}$, and other notations are $\frac{\partial f}{\partial \mathbf{v}}(\overline{\mathbf{x}})$ and $D_{\mathbf{v}} f(\overline{\mathbf{x}})$

Theorem 1 (14.2)**.** *At any point $\mathbf{x} \in D$ and $\nabla f \neq 0$, $\nabla f(\mathbf{x})$ points at x into the direction in which f increases most rapidly*

This theorem will be used for finding normal vector of tangent hyperplane to level set.

7 Explicit Functions from $\mathbb{R}^n$ to $\mathbb{R}^m$

Chain Rule III

Chain Rule III

Let $\mathbf{f}(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}^m$, $\mathbf{a}(t) : \mathbb{R} \to \mathbb{R}^n$, and $\mathbf{g}(t) = \mathbf{f} \circ \mathbf{a}(t)$. Then,

$$\frac{d\mathbf{g}}{dt} = Df_{\mathbf{a}}(\mathbf{a}(t)) \cdot \mathbf{a}'(t)$$

Chain Rule IV

Let $\mathbf{f}(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}^m$, $\mathbf{a}(\mathbf{t}) : \mathbb{R}^s \to \mathbb{R}^n$, and $\mathbf{g}(\mathbf{t}) = \mathbf{f} \circ \mathbf{a}(\mathbf{t})$. Then,

$$D\mathbf{g}_{\mathbf{t}} = D\mathbf{f}_{\mathbf{a}}(\mathbf{a}(\mathbf{t})) \cdot D\mathbf{a}_{\mathbf{t}}$$

8 Higher-order Derivatives

Hessian

Definition 5. *Hessian matrix*

$$D^2 f_{\mathbf{x}} = D(Df)_{\mathbf{x}} := \begin{pmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \frac{\partial^2 f}{\partial x_2 \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_1} \\ \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_2 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_2} \\ \vdots & \vdots & & \vdots \\ \frac{\partial^2 f}{\partial x_1 \partial x_n} & \frac{\partial^2 f}{\partial x_2 \partial x_n} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n} \end{pmatrix}$$

Theorem 2 (14.5:Young's theorem)**.**

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i} \quad \forall i, j$$

This means hessian is symmetric.