

# Constrained Optimization (II)

## Ch.19

`econMath.namun+2016su@gmail.com`

December 12, 2016

# Table of Contents

1 The Meaning of The Multiplier

2 Envelope Theorems

3 Second Order Conditions

## Three Aspects of the Lagrangian Approach

- 1 Sensitivity of the solution to changes in the parameters
- 2 SOCs
- 3 CQs

Note:

- To analyze the sensitivity, we should (1) get  $\mathbf{x}^*$ , the solution of the optimization problem when parameters  $\mathbf{a}$  are given. And then, we should (2) see the solution  $\mathbf{x}^*$  as the function of the parameters.
- You should be able to distinguish between variables (*e.g.*,  $\mathbf{x}$ ,  $\mu$ ) and parameters (*e.g.*,  $\mathbf{a}$ ). To distinguish explicitly, we denote a function  $f$  of variables  $\mathbf{x}$  and parameters  $\mathbf{a}$  by:

$$f(\mathbf{x}; \mathbf{a})$$

# Multiplier $\mu$

Think about the solution of the optimization problem in terms of a parameter  $a$ , the level of the equality constraint.

## Theorem (19.1)

Let  $f, h$  be  $C^1$  functions of 2 vars. For any fixed  $a$  (parameter), let  $(x_1^*(a), x_2^*(a))$  be the solution of max(min)imization problem  $\arg \max_{\mathbf{x}} f(\mathbf{x})$  s.t.  $h(\mathbf{x}) = a$  with corresponding multiplier  $\mu^*(a)$ . Suppose (1)  $\mathbf{x}^*, \mu^*$  are  $C^1$  functions of  $a$  and (2) NDCQ holds at  $(\mathbf{x}^*(a), \mu^*(a))$ . Then,

$$\mu^*(a) = \frac{d}{da} f(\mathbf{x}^*(a); a) = \frac{d}{da} f^*(a)$$

In this view,  $f^*(a) := f(\mathbf{x}^*(a); a)$ , i.e., the function of optimized value with regard to  $a$ , and  $\mu^*(a)$  is the slope of the  $f^*$ : changes in maximum value ( $f^*(a)$ ) in terms of  $a$

## Theorem (19.2)

Let  $f, \mathbf{H}$  be  $C^1$  functions of  $n$  vars. For any fixed  $\mathbf{a} \in \mathbb{R}^m$  (parameters), let  $(\mathbf{x}^*(\mathbf{a}))$  be the solution of max(min)imization problem

$$\arg \max_{\mathbf{x}} f(\mathbf{x}) \quad s.t. \quad \mathbf{H}(\mathbf{x}) = \mathbf{a}$$

with corresponding multiplier  $\mu^*(\mathbf{a}) = (\mu_1^*(\mathbf{a}), \dots, \mu_m^*(\mathbf{a}))$ . Suppose (1)  $\mathbf{x}^*, \mu^*$  are  $C^1$  functions of  $\mathbf{a}$  and (2) NDCQ holds at  $(\mathbf{x}^*(\mathbf{a}), \mu^*(\mathbf{a}))$ . Then,

$$\mu^*(\mathbf{a}) = Df_{\mathbf{a}}(\mathbf{x}^*(\mathbf{a}); \mathbf{a}) = Df_{\mathbf{a}}^*(\mathbf{a})$$

Geographical Explanation:

$$\arg \max_{\mathbf{x}} (-x_1^2 - x_2^2) \quad s.t. \quad x_1 + x_2 = a$$

# Inequality Constraints

## Theorem (19.3)

Let  $\mathbf{a}^* \in \mathbb{R}^k$ . Consider the *max(min)imization* problem

$$\arg \max_{\mathbf{x}} f(\mathbf{x}) \quad s.t. \quad \mathbf{G}(\mathbf{x}) \leq \mathbf{a}$$

Let  $\mathbf{x}^*(\mathbf{a}^*)$  be the solution of above problem, and let  $\lambda^*(\mathbf{a}^*) = (\lambda_1^*(\mathbf{a}^*), \dots, \lambda_k^*(\mathbf{a}^*))$  be the corresponding Lagrange multipliers. Suppose  $n + k$  functions  $\mathbf{x}^*(\mathbf{a})$  and  $\lambda^*(\mathbf{a})$  are differentiable around  $\mathbf{a}^*$  and NDCQ holds at  $\mathbf{a}^*$ . Then,

$$\lambda(\mathbf{a}^*)^* = Df_{\mathbf{a}}(\mathbf{x}(\mathbf{a}^*); \mathbf{a}^*) = Df_{\mathbf{a}}^*(\mathbf{a}^*)$$

When  $\mathbf{x}^*$  is interior solution,  $\lambda^*(\mathbf{a}^*) = Df_{\mathbf{a}}^*(\mathbf{a}^*) = 0$ . When  $f$  is a profit function,  $\lambda_j^*(\mathbf{a})$  can be interpreted as the shadow price of input  $j$ .

Geographical Explanation:

$$\arg \max_{\mathbf{x}} (-x_1^2 - x_2^2) \quad s.t. \quad x_1 + x_2 \leq a$$

# Envelope Theorems: Unconstrained Problems

## Theorem (19.4)

Let  $f(\mathbf{x}; a)$  be a  $C^1$  function of  $\mathbf{x} \in \mathbb{R}^n$  and scalar  $a$ . For a given parameter  $a$ , consider the unconstrained max(min)imization problem

$$\arg \max_{\mathbf{x}} f(\mathbf{x}; a)$$

And let  $\mathbf{x}^*(a)$  be a solution of above problem. Suppose that  $\mathbf{x}^*(a)$  is a  $C^1$  function of  $a$ . Then,

$$\frac{d}{da} f(\mathbf{x}^*(a); a) = \frac{\partial}{\partial a} f(\mathbf{x}^*(a); a) = \frac{\partial}{\partial a} f^*(a)$$

Proof: From chain rule and FOC of unconstrained optimization problem

$$\frac{d}{da} f(\mathbf{x}^*(a); a) = \cancel{Df_{\mathbf{x}}(\mathbf{x}^*(a); a) \frac{d\mathbf{x}^*(a)}{da}} \overset{\text{FOC}}{=} \frac{\partial f}{\partial a}(\mathbf{x}^*(a); a) \frac{da}{da} \overset{1}{\uparrow}$$

# Envelope Theorems: Constrained Problems

## Theorem (19.5)

Let  $f, \mathbf{H} = (H_1, \dots, H_k)$  be  $C^1$  functions of  $\mathbf{x} \in \mathbb{R}^n$ . Let  $\mathbf{x}^*(a)$  denote the solution of the max(min)imization problem for any fixed parameter  $a$ :

$$\arg \max_{\mathbf{x}} f(\mathbf{x}; a) \quad \text{s.t.} \quad \mathbf{H}(\mathbf{x}; a) = 0$$

Suppose that  $\mathbf{x}^*(a)$  and the Lagrange multipliers  $\mu^*(a)$  are  $C^1$  functions of  $a$  and that the NDCQ holds. Then,

$$\frac{d}{da} f(\mathbf{x}^*(a); a) = \frac{\partial}{\partial a} L(\mathbf{x}^*(a), \mu^*(a); a) = \frac{\partial}{\partial a} L^*(a)$$



# Strict Local Equality Constrained Max(Min)

## Theorem (19.6)

Let  $f, \mathbf{H}$  be  $C^2$  functions on  $\mathbf{x} \in \mathbb{R}^n$ . Consider the equality constrained max(min)imization problem

$$\arg \max_{\mathbf{x}} f(\mathbf{x}) \quad s.t. \quad \mathbf{H}(\mathbf{x}) = \mathbf{c}$$

Let  $L := f(\mathbf{x}) + \mu(\mathbf{c} - \mathbf{H})$  and suppose that

- ①  $\mathbf{H}(\mathbf{x}^*) = \mathbf{c}$  (Satisfies constraint)
- ②  $DL_{\mathbf{x}, \mu}(\mathbf{x}^*, \mu^*) = \mathbf{0}$  (Satisfies FOC)
- ③ Hession of  $L = D_{\mathbf{x}}^2 L(\mathbf{x}^*, \mu^*)$  is ND on the linear constraint set  $\{\mathbf{v} : D\mathbf{H}(\mathbf{x}^*)\mathbf{v} = \mathbf{0}\}$  (Satisfies Sufficient SOC)

Then,  $\mathbf{x}^*$  is a strict local constrained max(min) of  $f$  on the constrained set

$\mathbf{v}$  is the tangent vector on the constraint set around  $\mathbf{x}^*$ . Proof: See Ch.30

## Sufficient SOC: Calculation Procedure

Suppose  $\mathbf{x} \in \mathbb{R}^n$ ,  $\mathbf{H} \in \mathbb{R}^m$ , and NCDQ holds.

- 1 Form a Lagrangian function  $L$

$$L := f(\mathbf{x}) + \mu(\mathbf{c} - \mathbf{H})$$

- 2 Get points  $\mathbf{x}^*, \mu^*$  satisfy FOCs
- 3 Make a bordered Hessian

$$H := D^2 L_{\mu, \mathbf{x}}(\mathbf{x}^*, \mu^*) = \begin{pmatrix} \mathbf{0} & D\mathbf{H}_{\mathbf{x}}(\mathbf{x}^*) \\ D\mathbf{H}_{\mathbf{x}}(\mathbf{x}^*)^T & D^2 L_{\mathbf{x}}(\mathbf{x}^*, \mu^*) \end{pmatrix}$$

- 4 If  $H$  is PD, then  $\mathbf{x}^*$  is strict local min. If ND,  $\mathbf{x}^*$  is strict local max.
  - (a) If  $\text{sign}(\det H) = \text{sign}((-1)^m)$  and all  $n - m$  LPMs have same sign,  $H$  is PD on the constraint set
  - (b) If  $\text{sign}(\det H) = \text{sign}((-1)^n)$  and following  $n - m$  LPMs alternates in sign,  $H$  is ND on the constraint set

## Theorem (19.8)

Let  $f, \mathbf{H} \in \mathbb{R}^m, \mathbf{G} \in \mathbb{R}^k$  be  $C^2$  functions on  $\mathbf{x} \in \mathbb{R}^n$ . Consider the mixed constrained max(min)imization problem

$$\arg \max_{\mathbf{x}} f(\mathbf{x}) \quad \text{s.t.} \quad \mathbf{H}(\mathbf{x}) = \mathbf{c} \wedge \mathbf{G}(\mathbf{x}) \leq \mathbf{b}.$$

- 1 Form the Lagrangian function  $L$

$$L := f(\mathbf{x}) + \mu(\mathbf{c} - \mathbf{H}(\mathbf{x})) + \lambda(\mathbf{b} - \mathbf{G}(\mathbf{x}))$$

- 2 Suppose  $\exists \mathbf{x}^*, \mu^*, \lambda^*$  satisfying FOCs (Theorem 18.5)
- 3 For convenience, suppose  $\mathbf{G}_E := (G_1, \dots, G_e)$  are binding at  $\mathbf{x}^*$  and the others  $\mathbf{G}_{-E} := (G_{e+1}, \dots, G_k)$  are not binding. Let  $\lambda_E$  be the corresponding multiplier of  $\mathbf{G}_E$ . Then if the bordered Hessian  $D^2 L_{\lambda_E, \mu, \mathbf{x}}(\lambda_E^*, \mu^*, \mathbf{x}^*)$  is PD(ND),  $\mathbf{x}^*$  is a strict local mixed constrained max(min) of  $f$ .

## Bordered Hessian (Mixed Constraints)

$$H = D^2 L_{\lambda_{\mathbf{E}}, \mu, \mathbf{x}}(\lambda_{\mathbf{E}}^*, \mu^*, \mathbf{x}^*) = \left( \begin{array}{ccc} \mathbf{0} & \mathbf{0} & D\mathbf{G}_{\mathbf{E}\mathbf{x}} \\ \mathbf{0} & \mathbf{0} & D\mathbf{H}_{\mathbf{x}} \\ D\mathbf{G}_{\mathbf{E}\mathbf{x}}^T & D\mathbf{H}_{\mathbf{x}}^T & D^2 L_{\mathbf{x}} \end{array} \right) \Big|_{(\lambda_{\mathbf{E}}, \mu, \mathbf{x}) = (\lambda_{\mathbf{E}}^*, \mu^*, \mathbf{x}^*)}$$

Let  $F_{ab} := \frac{\partial}{\partial a} \frac{\partial}{\partial b} F$ . Then

$$H = \left( \begin{array}{ccc} \cancel{L_{\lambda_{\mathbf{E}} \lambda_{\mathbf{E}}}} \rightarrow \mathbf{0} & \cancel{L_{\lambda_{\mathbf{E}} \mu}} \rightarrow \mathbf{0} & \cancel{L_{\lambda_{\mathbf{E}} \mathbf{x}}} \rightarrow D\mathbf{G}_{\mathbf{E}\mathbf{x}} \\ \cancel{L_{\mu \lambda_{\mathbf{E}}}} \rightarrow \mathbf{0} & \cancel{L_{\mu \mu}} \rightarrow \mathbf{0} & \cancel{L_{\mu \mathbf{x}}} \rightarrow D\mathbf{H}_{\mathbf{x}} \\ \cancel{L_{\mathbf{x} \lambda_{\mathbf{E}}}} \rightarrow D\mathbf{G}_{\mathbf{E}\mathbf{x}}^T & \cancel{L_{\mathbf{x} \mu}} \rightarrow D\mathbf{H}_{\mathbf{x}}^T & \cancel{L_{\mathbf{x} \mathbf{x}}} \rightarrow D^2 L_{\mathbf{x}} \end{array} \right) \Big|_{(\lambda_{\mathbf{E}}, \mu, \mathbf{x}) = (\lambda_{\mathbf{E}}^*, \mu^*, \mathbf{x}^*)}$$

# Determining Definiteness of Bordered Hessian (Mixed Constraints)

## Determining Definiteness

- (a) If  $\text{sign}(\det H) = \text{sign}((-1)^{(m+e)})$  and all  $n - (m + e)$  LPMs have same sign,  $H$  is PD on the constraint set
- (b) If  $\text{sign}(\det H) = \text{sign}((-1)^n)$  and following  $n - (m + e)$  LPMs alternates in sign,  $H$  is ND on the constraint set