Quadratic Forms and Definite Matrices

Ch.16

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1 Quadratic Forms

Quadratic Forms

Definition 1 (Quadratic Form). *A quadratic form on $\mathbb{R}^n$ is a real-valued function of the form*

$$Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}, \quad \mathbf{x} \in \mathbb{R}^n, \quad A^T = A$$

For more detailed description, see Ch13 (section 3).

2 Definiteness of Quadratic Forms

Definiteness

Definiteness: Overview

When $Q = \mathbf{x}^T A\mathbf{x}$ and A is a diagonal matrix

$$A = \begin{pmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{pmatrix}$$

- Positive Definite (PD): $a_{ii} > 0 \quad \forall i$
- Positive Semi Definite (PSD): $a_{ii} \geq 0 \quad \forall i$
- Negative Definite (ND): $a_{ii} < 0 \quad \forall i$
- Negative Semi Definite (NSD): $a_{ii} \leq 0 \quad \forall i$
- Indefinite (ID): $a_{ii} < 0$ for some i, and $a_{ii} > 0$ for some i

Definite Symmetric Matrices

Definition 2 (PD,PSD,ND,NSD,ID). *Let A be an $n \times n$ symmetric matrix and $Q = \mathbf{x}^T A\mathbf{x}$, then A is:*

1. *PD if $Q > 0$ $\forall \mathbf{x} \neq \mathbf{0}$ in $\mathbb{R}^n$*
2. *PSD if $Q \geq 0$ $\forall \mathbf{x} \neq \mathbf{0}$ in $\mathbb{R}^n$*
3. *ND if $Q < 0$ $\forall \mathbf{x} \neq \mathbf{0}$ in $\mathbb{R}^n$*
4. *NSD if $Q \leq 0$ $\forall \mathbf{x} \neq \mathbf{0}$ in $\mathbb{R}^n$*
5. *ID if $Q > 0$ for some $\mathbf{x} \in \mathbb{R}^n$ and $Q < 0$ for some $\mathbf{x} \in \mathbb{R}^n$*

Principal Minors of a Matrix

Definition 3 (Principal Submatrix (PS), Principal Minor (PM))**.** *Let A be an $n \times n$ symmetric matrix. kth order principal submatrix of A is $k \times k$ submatrix of A obtained by deleting $n-k$ columns $C_1, \cdots, C_{n-k}$ and same $n-k$ rows $R_1, \cdots, R_{n-k}$.*

kth order principal minor of A is the determinant of kth order prinicipal submatrix.

Note: the number of kth order principal submatrix can be nCk. We will denote kth order principal minor by $PM_k(A)$.

Definition 4 (Leading PS, Leading PM)**.** *kth order leading principal submatrix of A is an unique kth order submatrix obtained by deleting the last $n-k$ rows and columns from A. kth order leading principal minor of A ($LPM_k(A)$ or $|A_k|$) is the determinant of kth order leading principal submatrix of A*

Test for Definiteness

Theorem 1 (16.1,2)**.** *Let A be an $n \times n$ symmetric matrix. Then,*

1. *A is PD iff $LPM_k(A) > 0 \ \forall k$*
2. *A is PSD iff $PM_k(A) \geq 0 \ \forall k$*
3. *A is ND iff $\operatorname{sign}(LPM_k(A)) = \operatorname{sign}((-1)^k) \quad \forall k$*
4. *A is NSD iff $\operatorname{sign}(PM_k(A)) = \operatorname{sign}((-1)^k) \quad \forall PM_k(A) \neq 0$*
5. *Otherwise, A is ID*

Note: We can find more elegant criteria using eigenvalues (Ch.23). To check all PM, $\sum_i^n nCi$ determinants should be calculated.

3 Linear Constraints and Bordered Matrices

Bordered Matrix

Finding global max/min of $Q(x_1, x_2)$ with one linear constraint

$$Q(x_1,x_2) = \mathbf{x}^T H \mathbf{x} = \begin{pmatrix} x_1 x_2 \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = ax_1^2 + 2bx_1x_2 + cx_2^2$$

on

$$\begin{pmatrix} A & B \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = G\mathbf{x} = 0$$

Substitute x_1 to $-Bx_2/A$ and we can get one variable function $\tilde{Q}$ in terms of x_2

$$\tilde{Q}(x_2) = Q(-Bx_2/A, x_2) = \frac{aB^2 - 2bAB + cA^2}{A^2} x_2^2$$

$$aB^2 - 2bAB + cA^2 = -\det \begin{pmatrix} 0 & A & B \\ A & a & b \\ B & b & c \end{pmatrix} = -\det \begin{pmatrix} 0 & G \\ G^T & H \end{pmatrix}$$

Definiteness of Bordered Matrix

Theorem 2 (16.3)**.** *$Q(\mathbf{x})$ is PD[ND] on the constraint set $G\mathbf{x} = 0$ iff*

$$\det \begin{pmatrix} 0 & A & B \\ A & a & b \\ B & b & c \end{pmatrix} = \det \begin{pmatrix} 0 & G \\ G^T & H \end{pmatrix}$$

is negative[positive]

Note: sign of determinant is dependent on both n (size of $\mathbf{x}$) and m (number of restriction)

General Bordered Matrix

Consider a general quadratic form

$$Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} = \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix} \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

with linear constraint set

$$B\mathbf{x} = \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \vdots & \vdots \\ b_{m1} & \cdots & b_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

We can make $(n + m) \times (n + m)$ symmetric matrix (general bordered matrix)

$$H = \begin{pmatrix} \mathbf{0} & B \\ B^T & A \end{pmatrix}$$

Definiteness of General Bordered Matrix

Theorem 3 (16.4)**.** *To determine the definiteness of general bordered matrix, Check the signs of the last $n - m$ LPMs of H, starting with $LPM_{n+m}(H)$ (i.e., the determinant of H itself). This means you should check the sign of*

$$\underbrace{LPM_{n+m}(H), LPM_{n+m-1}(H), \cdots, LPM_{n+m-(n-m-1)}(H)}_{n-m \ LPMs}$$

(a) If $\operatorname{sign}(\det H) = \operatorname{sign}((-1)^m)$ and all $n-m$ LPMs have same sign, Q is PD on the constraint set $B\mathbf{x} = \mathbf{0}$ and $\mathbf{x} = \mathbf{0}$ is strict global min of Q on the costraint set

(b) If $\operatorname{sign}(\det H) = \operatorname{sign}((-1)^n)$ and following $n - m$ LPMs alternates in sign, Q is ND on the constraint set $B\mathbf{x} = \mathbf{0}$ and $\mathbf{x} = \mathbf{0}$ is strict global max of Q on the costraint set

Definiteness of General Bordered Matrix

Continued

(c) if (a),(b) is violated by nonzero LPMs, Q is ID on the constraint set $B\mathbf{x} = \mathbf{0}$ and $\mathbf{x} = \mathbf{0}$ is neither a max nor a min of Q on the constraint set

Note: Test for NSD, PSD is much more tedious and trivial in economics $\rightarrow$ SKIP