Systems of Linear Equations

CH7

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1 Gaussian and Gauss-Jordan Elimination

General Linear System

General Linear System

$$
\begin{aligned}
\bar{a}_{11}x_1 + \bar{a}_{12}x_2 + \cdots + \bar{a}_{1n}x_n &= \bar{b}_1 \\
\bar{a}_{21}x_1 + \bar{a}_{22}x_2 + \cdots + \bar{a}_{2n}x_n &= \bar{b}_2 \\
&\vdots \\
\bar{a}_{m1}x_1 + \bar{a}_{m2}x_2 + \cdots + \bar{a}_{mn}x_n &= \bar{b}_m
\end{aligned}
$$

More elegantly,

$$\bar{A}\mathbf{x} = \bar{\mathbf{b}}$$

(Compare with one-var version: $\bar{a}x = \bar{b}$)

Solution of Linear System: $\mathbf{x} = (x_1, x_2, \cdots, x_n)$

- Solution of linear system is $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ satisfies all equations in equation system
- Main considerations:
  - Existance of Solution
  - # of Solutions
  - Efficient deriving methods
    - Substitution
    - Elimination of variables
    - Matrix methods

2 Elementary Row Operations (EROs)

Elementary Row Operations (EROs)

Solving Procedure: Big Picture

$$(A|\mathbf{b}) \xrightarrow{EROs} (A_{REF}|\mathbf{b}^*) \xrightarrow{EROs} (A_{RREF}|\mathbf{b}^{**}) \xrightarrow{EROs} (I|\mathbf{b}^{***})$$

Solution: $\mathbf{x} = \mathbf{b}^{***}$

General linear system

We should solve

$$\begin{aligned} \bar{a}_{11}x_1 + \bar{a}_{12}x_2 + \cdots + \bar{a}_{1n}x_n &= \bar{b}_1 \\ \bar{a}_{21}x_1 + \bar{a}_{22}x_2 + \cdots + \bar{a}_{2n}x_n &= \bar{b}_2 \\ &\vdots \\ \bar{a}_{m1}x_1 + \bar{a}_{m2}x_2 + \cdots + \bar{a}_{mn}x_n &= \bar{b}_m \end{aligned}$$

Coefficient Matrix, Augment Matrix

Definition 1 (Coefficient Matrix)**.** *A is a <u>coefficient matrix</u> for general linear system*

$$A := \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{mn} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

Definition 2 (Augment Matrix)**.** *$\hat{A}$ is a <u>augment matrix</u> for general linear system*

$$\hat{A} := (A|\mathbf{b})$$

Elementary Row Operations (EROs)

Definition 3. *Let R_i be the ith row matrix of A, then,*

$$R_i \leftrightarrow R_j \qquad (ERO_1(i,j))$$

$$R_i \leftarrow \bar{k}R_j + R_i \qquad (ERO_2(k,j,i))$$

$$R_i \leftarrow \bar{k}R_i \qquad (ERO_3(k,i))$$

Above operations do not change solution

- Notations

- R_i: ith row (of A)
- C_i: ith column (of A)
- a_{ij}: Element of ith row, jth column (of A)

Row Echelon Form

Definition 4 (k-leading zeros, pivot, Row Echelon Form (REF))**.** *R_i has <u>k-leading zeros</u> if:*

$$a_{ij} = \begin{cases} 0, & \forall j \leq k \\ \textit{not } 0, & j = k+1 \end{cases}$$

Above $a_{i,k+1}$ is a <u>pivot</u> of R_i

Let k_i be # of leading zeros of R_i. Then $\hat{A}_{REF}$ is <u>row echelon form</u> (REF) of $\hat{A}$ if:

- $k_i > k_j \quad \forall i > j$
- *Zero rows are placed on the bottom of A*

Reduced Row Echelon Form (RREF)

Definition 5 (Reduced Row Echelon Form (RREF))**.** *A_{RREF} is a reduced row echelon form (RREF) of A if:*

- *A_{RREF} is REF*
- *If C_i has pivot, only pivot is non-zero element in C_i*
- *Pivot = 1*

Solving Procedure: Gauss-Jordan Elimination

$$(A|\mathbf{b}) \xrightarrow{EROs} (A_{REF}|\mathbf{b}^*) \xrightarrow{EROs} (A_{RREF}|\mathbf{b}^{**}) \xrightarrow{EROs} (I|\mathbf{b}^{***})$$

Solution: If the liear system $Ax = \mathbf{b}$ has an unique solution (A is $m \times n$ matrix), $A_{RREF} = I$ ($m = n$) or, $R_i = I_i$ ($i \leq n$) and $R_i = \mathbf{O}$ ($i > n$)

3 Systems with Many or No Solutions

3.1 Systems with No Solution

Systems with No Solution

During EROs for $\hat{A} = A|\mathbf{b}$, if you encounter with $\mathbf{O}|k \neq 0$, *i.e.*, $(000\cdots 0|k \neq 0)$, this system has <u>no solution</u>

Meaning:

$$0x_1 + 0x_2 + \cdots + 0x_n = 0 = k \neq 0 \qquad \text{(contradiction!)}$$

No $\mathbf{x}$ can satisfy this equation

Theorem 1 (Fact7.2)**.** *$\hat{A}$ has a solution iff*

$$rank\hat{A} = rankA$$

3.2 Systems with Many Solutions

Systems with Many Solutions

Here, w stands for <u>independent</u> zero or non-zero element (can have any values) and $*$ is nonzero pivot.

REF of systems with many solutions

Systems with many solutions have following form of RREF (blank means zero)

$$\begin{pmatrix} * & w & w & w & w & w & w & w & | & w \\ & & & * & w & w & w & w & | & w \\ & & & & & * & w & w & | & w \\ & & & & & & * & w & | & w \\ & & & & & & & * & | & w \end{pmatrix} \qquad \text{(REF)}$$

RREF of Systems with many solutions

Important note: This is just an example of systems with many solutions.

RREF of systems with many solutions

Systems with many solutions have following form of RREF (blank means zero)

$$\begin{pmatrix} 1 & w & w & 0 & w & 0 & 0 & 0 & | & w \\ & & & 1 & w & 0 & 0 & 0 & | & w \\ & & & & & 1 & 0 & 0 & | & w \\ & & & & & & 1 & 0 & | & w \\ & & & & & & & 1 & | & w \end{pmatrix} \qquad \text{(RREF)}$$

In systems with many solutions, there exist C_i with no pivot (in our example, C_2, C_3, C_5)

Solutions of linear systems with many solutions

- Two types of solutions
    1. Basic variables are dependent on free variables or, fixed value:
        - Dependent on free variables: x_1, x_4 (in our example)
        - Fixed value: x_6, x_7, x_8 (in our example)
    2. Free variables can have any value: x_2, x_3, x_5 (in our example)
        - If C_i has no pivot, x_i is free variable
- In solving systems with many solutions, above types should be addressed explicitly

4 Rank - The Fundamental Criterion

Rank

Definition 6 (Rank)**.** *The Rank of a matrix is # of the nonzero rows in its REF (or RREF)*

Property of Rank

Rank = # of pivots = # of nonzero rows of REF (or RREF)

- Implication: Rank means the number of effective (meaningful) equations

Theorem 2 (Fact7.11)**.** *A system of m equations and n unknowns (its coefficient matrix is $m \times n$ matrix)*

$$A\mathbf{x} = \mathbf{b}$$

$m < n$	*# of solutions*
$\mathbf{b} = \mathbf{0}$	∞
$\forall \mathbf{b}$	$0 \vee \infty$
$rankA = m$	$\infty (\forall \mathbf{b})$
$m > n$	
$\mathbf{b} = \mathbf{0}$	$1 \vee \infty$
$\forall \mathbf{b}$	$0 \vee 1 \vee \infty$
$rankA = n$	$0 \vee 1 (\forall \mathbf{b})$
$m = n$	
$\mathbf{b} = \mathbf{0}$	$1 \vee \infty$
$\forall \mathbf{b}$	$0 \vee 1 \vee \infty$
$rankA = n = m$	$1 (\forall \mathbf{b})$

5 The Linear Implicit Function Theorem

Example: IS-LM model

IS-LM Model

$$sY + ar = I^0 + G \tag{IS}$$
$$mY - hr = M_s + M^0 \tag{LM}$$

- Endogenous variables: Y, r
- Exogenous parameters: $s, a, m, h, I^0, G, M_s, M^0$
  - Policy variables: G, M_s
  - Behavioral variables: s, a, m, h, I^0, M^0
- Coefficient matrix:

$$A = \begin{pmatrix} s & a \\ m & -h \end{pmatrix}$$

Linear Implicit Function Theorem

Theorem 3 (Linear Implicit Function Theorem)**.** *A general linear model with m equations and n knowns can have an unique solution (and suppose x_i are arranged by endogeneity) iff:*

- $x_1, \cdots, x_k$ *are endogenous variables*
- $x_{k+1}, \cdots, x_n$ *are exogenous variables (i.e., constant in this system)*
- $k = m$
- $rank A = k$

Linear IFT

Implication of Linear IFT: treat exogenous variables as constant.

$$\bar{a}_{11}x_1 + \bar{a}_{12}x_2 + \cdots + \bar{a}_{1n}x_n = \bar{b}_1$$
$$\vdots$$
$$\bar{a}_{m1}x_1 + \bar{a}_{m2}x_2 + \cdots + \bar{a}_{mn}x_n = \bar{b}_m$$

Above system can be solvable iff $rank\tilde{A} = k = m$

$$\tilde{A}\mathbf{x} = \tilde{\mathbf{b}}$$

$$\tilde{A} := \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kk} \end{pmatrix}, \tilde{\mathbf{b}} = \begin{pmatrix} b_1 - a_{1,k+1}x_{k+1} - \cdots - a_{1n}x_n \\ b_2 - a_{2,k+1}x_{k+1} - \cdots - a_{2n}x_n \\ \vdots \\ b_k - a_{k,k+1}x_{k+1} - \cdots - a_{kn}x_n \end{pmatrix}$$