

# One-Variable Calculus: Exponents and Logarithms

## CH5

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# Exponential Functions

## Definition (Exponential Function)

$f : \mathbb{R} \rightarrow \mathbb{R}$  is exponential function if  $f(x) = \bar{a}\bar{b}^x$ ,  $\bar{b} > 0$

- $x \in \mathbb{N} \Rightarrow f(x) := \bar{a} \prod_{i=1}^x \bar{b}$
- $f(0) := \bar{a}$
- $f(1/n) := \bar{a} \sqrt[n]{\bar{b}}$
- $f(m/n) := \bar{a} \sqrt[n]{\bar{b}^m}$
- $x < 0 \Rightarrow f(x) = \bar{a}(1/\bar{b})^{|x|}$
- Graph: convex, monotonic increasing ( $\bar{b} > 1$ ) or decreasing ( $\bar{b} \in (0, 1)$ ) function

# Growth of an Account with Interest rate $r$

Saving Account at  $t = \bar{T}$  with Interest rate  $\bar{r}$ , Initial Endowment  $\bar{A}$

$$A_t = \bar{A} (1 + \bar{r})^{\bar{T}}$$

## Compound Interest

If interest is compounded  $n$  times per time unit,

$$A_t = \bar{A} \left(1 + \frac{\bar{r}}{n}\right)^{n\bar{T}}$$

## Continuous Compounding

Compound Interest with  $n \rightarrow \infty$

$$A_t = \lim_{n \rightarrow \infty} \bar{A} \left(1 + \frac{\bar{r}}{n}\right)^{n\bar{T}} = \bar{A} e^{\bar{r}\bar{T}}$$

# Number $e$

## Definition (The Number $e$ )

$$e := \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = \sum_{n=0}^{\infty} \frac{1}{n!} \approx 2.718281693 \dots$$

$e$  is irrational number.

## Theorem (5.1)

$$\lim_{n \rightarrow \infty} A \left(1 + \frac{r}{n}\right)^{nt} = Ae^{rt}$$

In general, an initial quantity  $a_0$  with growth rate  $r$  (per time unit) become  $a_0 e^{rt}$  at time  $t$  (time unit)

# Logarithm

## Definition (Base $b$ Logarithm)

*Base  $b$  logarithm is an inverse of exponential function with base  $b$*

$$f = b^x \quad \Leftrightarrow \quad x = \log_b f$$

- $a^{\log_a z} = z$
- $\log_a a^y = y$
- Graph: concave, monotonic increasing ( $b > 1$ ) or convex, monotonic decreasing ( $b \in (0, 1)$ )

# Natural Logarithm

## Definition (Natural Logarithm)

Base  $e$  logarithm is natural logarithm

$$\ln x := \log_e x$$

$$\ln x = y \quad \Leftrightarrow \quad e^y = x$$

$$e^{\ln x} = x$$

$$\ln e^x = x$$

# Basic Properties of Exponential functions

$\forall r, s \in \mathbb{R},$

①  $a^r a^s = a^{r+s}$

②  $a^{-r} := 1/a^r$

③  $a^r / a^s = a^{r-s}$

④  $(a^r)^s = a^{rs}$

⑤  $a^0 := 1$



# Basic Properties of Logarithmic functions

$$\forall r, s, a, b, c > 0 \wedge a, c \neq 1,$$

$$\textcircled{1} \log(rs) = \log r + \log s$$

$$\textcircled{2} \log(1/s) = -\log s$$

$$\textcircled{3} \log(r/s) = \log r - \log s$$

$$\textcircled{4} \log r^s = s \log r$$

$$\textcircled{5} \log 1 = 0$$

$$\textcircled{6} \log_a b = \frac{\log_c b}{\log_c a} = \frac{\ln b}{\ln a}$$

(Ex5.4) Rule of 70 (or 69)

# Derivatives of Exp and Log functions

## Theorem (5.2)

$$(e^x)' = e^x$$

$$(\ln x)' = \frac{1}{x}$$

if  $u \in \mathbf{C}^1$ , from chain rule,

$$(e^u)' = (e^u) u'$$

$$(\ln u)' = \frac{u'}{u} \quad (u > 0)$$

# Present Value

## Present Value (PV)

After time  $T$ ,  $A$  (at  $t = 0$ ) grow to  $B$  (at  $t = T$ )

$$B = Ae^{rT}$$

$A$  is the present value (PV) of  $B$  at  $t = T$

$$A = Be^{-rT}$$

- PV of annuity

# Logarithmic Derivative

(Ex5.10)  $(x^x)' = ?$

Elasticity of  $f$  is the Slope in log-log Graph of  $f$

$$\epsilon := \frac{\frac{df}{f}}{\frac{dx}{x}} = \frac{d \ln f}{d \ln x}$$