

Linear Independence

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1 Linear Independence

LI: Definition

Definition 1 (Linear Combinations, Span, Linear (in)dependence). \mathcal{L} is spanned set generated by linear combination of k vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$

$$\mathcal{L}[\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k \in \mathbb{R}^n] := \left\{ \sum_i^k r_i \mathbf{v}_i : \forall r_i \in \mathbb{R} \right\}$$

$\mathbf{v}_1, \dots, \mathbf{v}_k$ are linearly independent iff:

$$\mathcal{L}[\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k \in \mathbb{R}^n] \subset \mathbb{R}^k$$

Otherwise, $\mathbf{v}_1, \dots, \mathbf{v}_k$ are linearly dependent

Note: Different vectors can span the same space. Canonical basis \mathbf{e}_i can be the representative vector (basis) for \mathbb{R}^n space.

LI: Alternative Definition

Definition 2 (Linear (in)dependence: Alternative definition). $\mathbf{v}_1, \dots, \mathbf{v}_k \in \mathbb{R}^n$ are linearly dependent iff:

$$\exists c_1, \dots, c_k \neq 0 \quad s.t. \quad \sum_i^k c_i \mathbf{v}_i = \mathbf{0}$$

$\mathbf{v}_1, \dots, \mathbf{v}_k \in \mathbb{R}^n$ are linearly independent iff:

$$\sum_i^k c_i \mathbf{v}_i = \mathbf{0} \quad \Rightarrow \quad c_1 = \dots = c_k = 0$$

Theorems

Theorem 1 (11.1). $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k \in \mathbb{R}^n$ are linearly dependent iff

$$A \begin{pmatrix} c_1 \\ \vdots \\ c_k \end{pmatrix} = \mathbf{0}$$

has nonzero solution \mathbf{c} , where A is $n \times k$ matrix whose $C_i = \mathbf{v}_i$. i.e.,

$$A = (\mathbf{v}_1 \quad \mathbf{v}_2 \quad \dots \quad \mathbf{v}_k)$$

Theorem 2 (11.2). $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n \in \mathbb{R}^n$ are linearly independent iff

$$\det(\mathbf{v}_1 \ \mathbf{v}_2 \ \cdots \ \mathbf{v}_n) \neq 0$$

Checking LI

Procedure: Checking Linear (in)dependence

1. Stack \mathbf{v}_i to make $k \times n$ matrix, A .
2. Calculate $\text{rank}(A)$: this is the dimension of $\mathcal{L}[\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k \in \mathbb{R}^n]$
3. $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k \in \mathbb{R}^n$ are linearly independent iff $\text{rank}(A) = k$. Otherwise, they are linearly dependent

Note: $\text{rank}(A^T) = \text{rank}(A)$

Theorem 3 (11.3). If $k > n$, any set of k vectors in \mathbb{R}^n is linearly dependent.

2 Spanning Sets

3 Basis and Dimension in \mathbb{R}^n

Basis

Definition 3 (Basis). Let $V = \mathcal{L}[\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k \in \mathbb{R}^n]$. If $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k \in \mathbb{R}^n$ are linearly independent, $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k \in \mathbb{R}^n$ is called a basis of V . More generally, $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k \in \mathbb{R}^n$ forms a basis of V if:

1. $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k \in \mathbb{R}^n$ span V
2. $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k \in \mathbb{R}^n$ are linearly independent

Theorem 4 (11.7). Every basis of \mathbb{R}^n contains n vectors.

Linear Independence and Basis

Theorem 5 (11.8). Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n \in \mathbb{R}^n$ be a collection of n vectors in \mathbb{R}^n . And let $n \times n$ matrix $A = (\mathbf{v}_1 \ \cdots \ \mathbf{v}_n)$. Then the following statements are equivalent:

1. $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n \in \mathbb{R}^n$ are linearly independent.
2. $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n \in \mathbb{R}^n$ span \mathbb{R}^n
3. $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n \in \mathbb{R}^n$ form a basis of \mathbb{R}^n
4. $\det(A) \neq 0$