

# Quadratic Forms and Definite Matrices

## Ch.16

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# Quadratic Forms

## Definition (Quadratic Form)

A quadratic form on  $\mathbb{R}^n$  is a real-valued function of the form

$$Q(\mathbf{x}) = \mathbf{x}^T A \mathbf{x}, \quad \mathbf{x} \in \mathbb{R}^n, \quad A^T = A$$

For more detailed description, see Ch13 (section 3).

## Definiteness: Overview

When  $Q = \mathbf{x}^T A \mathbf{x}$  and  $A$  is a diagonal matrix

$$A = \begin{pmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{pmatrix}$$

- Positive Definite (PD):  $a_{ii} > 0 \quad \forall i$
- Positive Semi Definite (PSD):  $a_{ii} \geq 0 \quad \forall i$
- Negative Definite (ND):  $a_{ii} < 0 \quad \forall i$
- Negative Semi Definite (NSD):  $a_{ii} \leq 0 \quad \forall i$
- Indefinite (ID):  $a_{ii} < 0$  for some  $i$ , and  $a_{ii} > 0$  for some  $i$

# Definite Symmetric Matrices

## Definition (PD,PSD,ND,NSD,ID)

Let  $A$  be an  $n \times n$  symmetric matrix and  $Q = \mathbf{x}^T A \mathbf{x}$ , then  $A$  is:

- ① PD if  $Q > 0 \ \forall \mathbf{x} \neq \mathbf{0}$  in  $\mathbb{R}^n$
- ② PSD if  $Q \geq 0 \ \forall \mathbf{x} \neq \mathbf{0}$  in  $\mathbb{R}^n$
- ③ ND if  $Q < 0 \ \forall \mathbf{x} \neq \mathbf{0}$  in  $\mathbb{R}^n$
- ④ NSD if  $Q \leq 0 \ \forall \mathbf{x} \neq \mathbf{0}$  in  $\mathbb{R}^n$
- ⑤ ID if  $Q > 0$  for some  $\mathbf{x} \in \mathbb{R}^n$  and  $Q < 0$  for some  $\mathbf{x} \in \mathbb{R}^n$

# Principal Minors of a Matrix

## Definition (Principal Submatrix (PS), Principal Minor (PM))

Let  $A$  be an  $n \times n$  symmetric matrix.  $k$ th order principal submatrix of  $A$  is  $k \times k$  submatrix of  $A$  obtained by deleting  $n - k$  columns  $C_1, \dots, C_{n-k}$  and same  $n - k$  rows  $R_1, \dots, R_{n-k}$ .

$k$ th order principal minor of  $A$  is the determinant of  $k$ th order principal submatrix.

Note: the number of  $k$ th order principal submatrix can be  $nCk$ . We will denote  $k$ th order principal minor by  $PM_k(A)$ .

## Definition (Leading PS, Leading PM)

$k$ th order leading principal submatrix of  $A$  is an unique  $k$ th order submatrix obtained by deleting the last  $n - k$  rows and columns from  $A$ .

$k$ th order leading principal minor of  $A$  ( $LPM_k(A)$  or  $|A_k|$ ) is the determinant of  $k$ th order leading principal submatrix of  $A$

# Test for Definiteness

## Theorem (16.1,2)

Let  $A$  be an  $n \times n$  symmetric matrix. Then,

- ①  $A$  is PD iff  $LPM_k(A) > 0 \ \forall k$
- ②  $A$  is PSD iff  $PM_k(A) \geq 0 \ \forall k$
- ③  $A$  is ND iff  $\text{sign}(LPM_k(A)) = \text{sign}((-1)^k) \ \forall k$
- ④  $A$  is NSD iff  $\text{sign}(PM_k(A)) = \text{sign}((-1)^k) \ \forall PM_k(A) \neq 0$
- ⑤ Otherwise,  $A$  is ID

Note: We can find more elegant criteria using eigenvalues (Ch.23). To check all  $PM$ ,  $\sum_i^n nCi$  determinants should be calculated.

## Finding global max/min of $Q(x_1, x_2)$ with one linear constraint

$$Q(x_1, x_2) = \mathbf{x}^T H \mathbf{x} = (x_1 \ x_2) \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = ax_1^2 + 2bx_1x_2 + cx_2^2$$

on

$$(A \ B) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = G\mathbf{x} = 0$$

Substitute  $x_1$  to  $-Bx_2/A$  and we can get one variable function  $\tilde{Q}$  in terms of  $x_2$

$$\tilde{Q}(x_2) = Q(-Bx_2/A, x_2) = \frac{aB^2 - 2bAB + cA^2}{A^2} x_2^2$$

$$aB^2 - 2bAB + cA^2 = -\det \begin{pmatrix} 0 & A & B \\ A & a & b \\ B & b & c \end{pmatrix} = -\det \begin{pmatrix} 0 & G \\ G^T & H \end{pmatrix}$$



# Definiteness of Bordered Matrix

## Theorem (16.3)

$Q(\mathbf{x})$  is  $PD[ND]$  on the constraint set  $G\mathbf{x} = 0$  iff

$$\det \begin{pmatrix} 0 & A & B \\ A & a & b \\ B & b & c \end{pmatrix} = \det \begin{pmatrix} 0 & G \\ G^T & H \end{pmatrix}$$

is negative[positive]

Note: sign of determinant is dependent on both  $n$  (size of  $\mathbf{x}$ ) and  $m$  (number of restriction)

# General Bordered Matrix

Consider a general quadratic form

$$Q(\mathbf{x}) = \mathbf{x}^T A \mathbf{x} = \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix} \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

with linear constraint set

$$B\mathbf{x} = \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \vdots & \vdots \\ b_{m1} & \cdots & b_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

We can make  $(n + m) \times (n + m)$  symmetric matrix (general bordered matrix)

$$H = \begin{pmatrix} \mathbf{0} & B \\ B^T & A \end{pmatrix}$$

# Definiteness of General Bordered Matrix

## Theorem (16.4)

*To determine the definiteness of general bordered matrix, Check the signs of the last  $n - m$  LPMs of  $H$ , starting with  $LPM_{n+m}(H)$  (i.e., the determinant of  $H$  itself). This means you should check the sign of*

$$\underbrace{LPM_{n+m}(H), LPM_{n+m-1}(H), \dots, LPM_{n+m-(n-m-1)}(H)}_{n-m \text{ LPMs}}$$

- (a) *If  $\text{sign}(\det H) = \text{sign}((-1)^m)$  and all  $n - m$  LPMs have same sign,  $Q$  is PD on the constraint set  $B\mathbf{x} = \mathbf{0}$  and  $\mathbf{x} = \mathbf{0}$  is strict global min of  $Q$  on the constraint set*
- (b) *If  $\text{sign}(\det H) = \text{sign}((-1)^n)$  and following  $n - m$  LPMs alternates in sign,  $Q$  is ND on the constraint set  $B\mathbf{x} = \mathbf{0}$  and  $\mathbf{x} = \mathbf{0}$  is strict global max of  $Q$  on the constraint set*

# Definiteness of General Bordered Matrix

## Continued

- (c) if (a),(b) is violated by nonzero LPMs,  $Q$  is ID on the constraint set  $B\mathbf{x} = \mathbf{0}$  and  $\mathbf{x} = \mathbf{0}$  is neither a max nor a min of  $Q$  on the constraint set

Note: Test for NSD, PSD is much more tedious and trivial in economics  
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