

Euclidean Spaces

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1 Points and Vectors in Euclidean Space

Objects in Euclidean Spaces

Objects in n -dimensional Euclidean Spaces

Dimension	Object	Representation
0	point	\emptyset
1	line	$x_1 \in \mathbb{R}^1$
2	plane	$(x_1, x_2) \in \mathbb{R}^2$
3	3d space	$(x_1, x_2, x_3) \in \mathbb{R}^3$
n	n d space	$(x_1, \dots, x_n) \in \mathbb{R}^n$

2 Vectors

Vector

Definition 1 ((Euclidean) Vector, displacement). n -tuples of real numbers $\mathbf{x} = (x_1, x_2, \dots, x_n)$ are (Euclidean) Vectors that represent displacement in \mathbb{R}^n space (or Cartesian coordinate system)

Let coordination of $\mathbf{p} = (p_1, \dots, p_n)$, $\mathbf{q} = (q_1, \dots, q_n)$. Then the displacement from \mathbf{p} to \mathbf{q} is defined as $\overrightarrow{\mathbf{pq}} := (q_1 - p_1, \dots, q_n - p_n)$. In this definition, \mathbf{p} is an origin, and \mathbf{q} is a destination.

Note: Any vector $\mathbf{p} = (p_1, \dots, p_n)$ can be interpreted as a location $((p_1, \dots, p_n))$ or, displacement with origin $\mathbf{0} := (0, \dots, 0)$ (more explicit notation: $\begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix}$)

3 The Algebra of Vectors

Addition and Subtraction

Let \mathbf{u}, \mathbf{v} be the vectors in \mathbb{R}^n space and $u_i, v_i \in \mathbb{R}^1$ be their i -th element.

Definition 2 (\pm of vectors).

$$(\mathbf{u} \pm \mathbf{v})_i := u_i \pm v_i \quad \forall i$$

or,

$$\mathbf{u} \pm \mathbf{v} := (u_1 \pm v_1, \dots, u_n \pm v_n)$$

Geometrically, addition of vectors means a sequence of displacements.

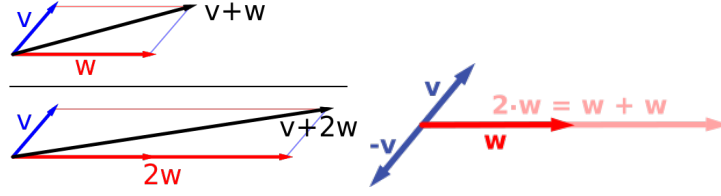


Figure 1: Geometrical Meaning: Addition, Subtraction, and Scalar Multiplication

Scalar Multiplication

Let r, s be scalars (or real numbers). *i.e.*, $r, s \in \mathbb{R}^1$.

Definition 3 (Scalar Multiplication).

$$(r\mathbf{u})_i := ru_i \quad \forall i$$

Geometrically, scalar multiplication means stretching or shrinking.

Algebraic Properties of Vector Operation

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u} \quad (\text{Commutative Law})$$

$$(r + s)\mathbf{u} = r\mathbf{u} + s\mathbf{u} \quad (\text{Distributive Law 1})$$

$$r(\mathbf{u} + \mathbf{v}) = r\mathbf{u} + r\mathbf{v} \quad (\text{Distributive Law 2})$$

In fact, any set of objects with a vector addition and scalar multiplication satisfying above laws is called vector space and vector is defined as the element of vector space. (Vector is defined by operations and their laws)

4 Length and Inner Product in \mathbb{R}^n

Length and Direction

Definition 4 (Length of Vector $\|\vec{pq}\|$).

$$\|\vec{pq}\| := \sqrt{\sum_i^n (q_i - p_i)^2}$$

Theorem 1 (10.1).

$$\|r\mathbf{v}\| = |r| \cdot \|\mathbf{v}\| \quad \forall r \in \mathbb{R} \wedge \forall \mathbf{v} \in \mathbb{R}^n$$

Any vector has two kinds of information: (1) length, and (2) direction.

Definition 5 (Unit Vector (Direction of a vector)).

$$\text{Unit vector of } \mathbf{v} := \frac{\mathbf{v}}{\|\mathbf{v}\|}$$

The Inner Product

Definition 6 (Euclidean Inner Product (or dot product)).

$$\mathbf{u} \bullet \mathbf{v} := \sum_i^n u_i v_i \in \mathbb{R}^1$$

Theorem 2 (10.2: Properties of Inner Product). 1. $\mathbf{u} \bullet \mathbf{v} = \mathbf{v} \bullet \mathbf{u}$

2. $\mathbf{u} \bullet (\mathbf{v} + \mathbf{w}) = \mathbf{u} \bullet \mathbf{v} + \mathbf{u} \bullet \mathbf{w}$

3. $\mathbf{u} \bullet (r\mathbf{v}) = r(\mathbf{u} \bullet \mathbf{v}) = (r\mathbf{u}) \bullet \mathbf{v}$

4. $\mathbf{u} \bullet \mathbf{u} \geq 0$

5. $\mathbf{u} \bullet \mathbf{u} = 0 \iff \mathbf{u} = \mathbf{0}$

6. $(\mathbf{u} + \mathbf{v}) \bullet (\mathbf{u} + \mathbf{v}) = \mathbf{u} \bullet \mathbf{u} + 2\mathbf{u} \bullet \mathbf{v} + \mathbf{v} \bullet \mathbf{v}$

Inner Product and Angle between Two Vectors

Theorem 3 (10.3). *Let θ be the angle between \mathbf{u}, \mathbf{v} . Then,*

$$\mathbf{u} \bullet \mathbf{v} = \|\mathbf{u}\| \cdot \|\mathbf{v}\| \cos \theta$$

Theorem 4 (10.4). 1. θ is acute if $\mathbf{u} \bullet \mathbf{v} > 0$

2. θ is obtuse if $\mathbf{u} \bullet \mathbf{v} < 0$

3. θ is right if $\mathbf{u} \bullet \mathbf{v} = 0$

Theorem 5 (10.5: Triangle Inequality).

$$\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$$

$$|\|\mathbf{u}\| - \|\mathbf{v}\|| \leq \|\mathbf{u} - \mathbf{v}\|$$

Norms

Definition 7 (Norms). *Any operation (X) of a vector to a real number satisfying below three properties is norm. Length of vector is a norm.*

1. $X(\mathbf{u}) \geq 0 \wedge X(\mathbf{u}) = 0$ only when $\mathbf{u} = \mathbf{0}$

2. $X(r\mathbf{u}) = |r|X(\mathbf{u})$

3. $X(\mathbf{u} + \mathbf{v}) \leq X(\mathbf{u}) + X(\mathbf{v})$

Norm is set of distance measures between two vectors.

5 Lines and Planes

1D,2D Objects in \mathbb{R}^n Spaces

Lines: One dimensional objects in \mathbb{R}^n Spaces

\mathbf{x} representing a line passing $\overline{\mathbf{x}_0}$ with direction $\overline{\mathbf{v}}$ is:

$$\mathbf{x} = \overline{\mathbf{x}_0} + t\overline{\mathbf{v}} \quad \forall t \in \mathbb{R} \quad (\text{Parametric Representation})$$

\mathbf{x} representing a line passing $\overline{\mathbf{x}_0}, \overline{\mathbf{x}_1}$ is:

$$\mathbf{x} = (1 - t)\overline{\mathbf{x}_0} + t\overline{\mathbf{x}_1} \quad \forall t \in \mathbb{R} \quad (\text{Parametric Representation})$$

Planes: Two dimensional objects in \mathbb{R}^n Spaces

\mathbf{x} representing a plane passing $\overline{\mathbf{x}_0}$ with direction $\overline{\mathbf{v}_1}$ and $\overline{\mathbf{v}_2}$ is:

$$\mathbf{x} = \overline{\mathbf{x}_0} + t_1\overline{\mathbf{v}_1} + t_2\overline{\mathbf{v}_2} \quad \forall t_i \in \mathbb{R} \quad (\text{Parametric Representation})$$

\mathbf{x} representing a plane containing $\overline{\mathbf{x}_0}, \overline{\mathbf{x}_1}, \overline{\mathbf{x}_2}$ is:

$$\mathbf{x} = (1 - t_1 - t_2)\overline{\mathbf{x}_0} + t_1\overline{\mathbf{x}_1} + t_2\overline{\mathbf{x}_2} \quad \forall t_i \in \mathbb{R} \quad (\text{Parametric Representation})$$

Nonparametric Equations

Definition 8 (Normal Vector). A *normal vector* $\bar{\mathbf{n}}$ of a n -dimensional object \mathbf{X} is a vector which is perpendicular to any vectors in the object. Suppose $\mathbf{x}, \bar{\mathbf{p}}$ are location vectors in the object. Then,

$$\bar{\mathbf{n}} \bullet (\mathbf{x} - \bar{\mathbf{p}}) = 0 \quad \forall \mathbf{x}, \bar{\mathbf{p}} \in \mathbf{X}$$

Finding Normal Vector

1. For linearly independent vectors \mathbf{u}_i , solve below systems of equations satisfying

$$\bar{\mathbf{n}} \bullet \mathbf{u}_i = 0 \quad \forall i$$

2. $\bar{\mathbf{n}} = \mathbf{u} \times \mathbf{v}$ (Only for \mathbb{R}^3 space)

$$\mathbf{u} \times \mathbf{v} := \left(\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right)$$

Hyperplanes

Definition 9 (Hyperplane). $n - 1$ dimensional object in \mathbb{R}^n space is a hyperplane with normal vector $\bar{\mathbf{a}}$ if:

$$\bar{a}_1x_1 + \bar{a}_2x_2 + \cdots + \bar{a}_nx_n = \bar{d}$$

or,

$$\bar{\mathbf{a}} \bullet \mathbf{x} = \bar{d}$$

6 Economic Applications

Budget Constraint

Definition 10 (Commodity Bundle, Price Vector, Budget Set). *Let $x_i \geq 0$ be the quantity of i th commodity.*

$$\mathbf{x} := (x_1, \dots, x_n), \quad x_i \geq 0 \quad \forall i \quad (\text{Commodity Bundle})$$

Let $p_i \geq 0$ be the price of i th commodity.

$$\mathbf{p} := (p_1, \dots, p_n), \quad p_i \geq 0 \quad \forall i \quad (\text{Price Vector})$$

Then budget set \mathbf{x} can be defined with given budget \bar{I} :

$$\bar{\mathbf{p}} \bullet \mathbf{x} \leq \bar{I} \quad (\text{Budget Constraint})$$