Eigenvalues and Eigenvectors (1)

Ch.23

econMath.namun+2016su@gmail.com

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1 Definitions and Examples

Eigenvalues

Definition 1 (Eigenvalue)**.** *Let $A \in M_n$. A scalar r is an eigenevalue of A iff:*

$$\det(A - rI) = 0$$

Theorem 1 (23.1)**.** *The diagonal entries a_{ii} of a diagonal matrix A are eigenvalues of A.*

Theorem 2 (23.2)**.** *A matrix $A \in M_n$ is singular iff 0 is an eigenvalue of A.*

Characteristic Polynomial

Definition 2 (Characteristic Polynomial)**.** *An $P_A(r)$, the nth order polynomial of variable r is an polynomial of $A \in M_n$ when:*

$$P_A(r) = \det(A - rI)$$

r is eigenvalue of A if $P_A(r) = 0$

For general 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$,

$$P_A(r) = \det \begin{pmatrix} a - r & b \\ c & d - r \end{pmatrix} = r^2 - (a + d)r + ad - bc$$

$n \times n$ matrices can have at most n eigenvalues

Eigenvectors

Definition 3 (Eigenvectors)**.** *$\mathbf{v}$ is an eigenvector of A if*

$$\det(A - rI) = 0 \quad \wedge \quad (A - rI)\mathbf{v} = \mathbf{0}$$

or,

$$\det(A - rI) = 0 \quad \wedge \quad A\mathbf{v} = r\mathbf{v}$$

Note1: Get the simplest nonzero vector from eigenspace of A with respect to each eigenvalue
Note2: $A - rI \in M_n$ is singular iff $\exists \mathbf{v} \neq \mathbf{0}$ s.t. $(A - rI)\mathbf{v} = \mathbf{0}$ (See Ch.8)

Th23.3

Theorem 3 (23.3)**.** *Let $A \in M_n$, and $r \in \mathbb{R}$. Then, following statements are equivalent:*

1. $A - rI$ *is singular*
2. $\det(A - rI) = 0$
3. $\exists \mathbf{v} \neq \mathbf{0}$ *s.t.* $(A - rI)\mathbf{v} - \mathbf{0}$
4. $A\mathbf{v} = r\mathbf{v}$ *and* $\mathbf{v} \neq \mathbf{0}$

Examples

Ex 23.6
Find the eigenvalues and eigenvectors of

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 5 & 0 \\ 3 & 0 & 2 \end{pmatrix}$$

Step 1) Get eigenvalues from characteristic polynomials

$$\det(A - rI) = 0$$

Step 2) Get eigenvectors from corresponding eigenvalues $r = 5, 4, -1$ by solving $(A - rI)\mathbf{v} = 0$

- $r = 5$
- $r = 4$
- $r = -1$

2 Solving Linear Difference Equations

One Dimensional Linear Difference Equations

One-Dimensional Equations

$$y_{t+1} = \bar{a} y_t, \quad t \in \mathbb{N} + \{0\}$$
$$\Rightarrow \quad y_n = \bar{a}^n \overline{y_0}$$

Note: The simplest dynamic – time dependent – model (cf. static model is time-invariant). In general, dynamic model is more difficult to solve.

Above system can extend to general n-dimensional linear difference equations

$$\mathbf{z}_{t+1} = A\mathbf{z}_t, \quad \mathbf{z}_t \in \mathbb{R}^n, \quad A \in M_n$$

However, solution is similar only if system is uncoupled. If the system is coupled, transform it to uncoupled system using eigenvalues and eigenvectors.

Two Dimensional Linear Difference Equations

Two-Dimensional Equations

$$\mathbf{z}_{t+1} = A\mathbf{z}_t$$

When $\mathbf{z_t} = \begin{pmatrix} x_t \\ y_t \end{pmatrix}$ and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$,

$$\begin{aligned} x_{t+1} &= \bar{a}x_t + \bar{b}y_t \\ y_{t+1} &= \bar{c}x_t + \bar{d}y_t \end{aligned}$$

Definition 4 (Coupled, Uncoupled)**.** *When $b = c = 0$, above system is* <u>*uncoupled*</u>*. Otherwise, above system is* <u>*coupled*</u>*. When $b = c = 0$,*

$$\mathbf{z}_n = A^n\mathbf{z}_0 = \begin{pmatrix} a^n & 0 \\ 0 & d^n \end{pmatrix} \mathbf{z}_0$$

The Leslie Population Model

Leslie Mode: Linear Population Dynamics

$$\begin{pmatrix} x_{t+1} \\ y_{t+1} \end{pmatrix} = \begin{pmatrix} b_1 & b_2 \\ 1-d_1 & 1-d_2 \end{pmatrix} \begin{pmatrix} x_t \\ y_t \end{pmatrix}$$

- b_i: birth rate of agents in the ith period
- d_i: death rate of agents in the ith period
- Agents live at most 2-periods. This means $d_2 = 1$
- x_t: the number of 0-period old population
- y_t: the number of 1-period old population

Ex23.7: $b_1 = 1, b_2 = 4, d_1 = 0.5$

M1 Transform to uncoupled system by ERO

M2 Find P s.t. $P^{-1}AP$ is a diagonal matrix (diagonalize)

General Two-Dimensional Systems

General Linear Difference Equation

$$\mathbf{z}_{t+1} = A\mathbf{z}_t$$

Let $\mathbf{z}_t = P\mathbf{Z}_t$ or $\mathbf{Z}_t = P^{-1}\mathbf{z}_t$. Then,

$$\mathbf{Z}_{t+1} = P^{-1}AP\mathbf{Z}_t$$

Let r_1, r_2 be eigenvalues of A and $\mathbf{v}_1, \mathbf{v}_2$ be corresponding eigenvectors (2×1 matrix). If $P = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{pmatrix}$,

$$A\begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{pmatrix} \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix} \quad \Longleftrightarrow \quad A\mathbf{v}_i = r_i\mathbf{v}_i \quad \forall i \tag{Th23.3}$$

This leads to:

$$P^{-1}AP = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix}$$

General k-Dimensional Systems

Theorem 4 (23.4)**.** *Let A be $k \times k$ matrix. Let r_i be k eigenvalues of A, and $\mathbf{v}_i$ be the corresponding eigenvectors. Form the matrix*

$$P = \begin{pmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_k \end{pmatrix}$$

If $\exists P^{-1}$,

$$P^{-1}AP = \begin{pmatrix} r_1 & 0 & \cdots & 0 \\ 0 & r_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & r_k \end{pmatrix}$$

Note: $\exists P^{-1}$ means that $\mathbf{v}_i$ s are linearly independent

The Powers of Diagonalized Matrix

Theorem 5 (23.7)**.** *Let A be a $k \times k$ matrix. Suppose that there is a nonsingular (invertible) matrix P s.t.*

$$P^{-1}AP = \begin{pmatrix} r_1 & 0 & \cdots & 0 \\ 0 & r_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & r_k \end{pmatrix} = D \qquad \text{(Jordan Canonical Form)}$$

Then,

$$A^n = PD^nP^{-1}$$

And the solution of the corresponding system of difference equations $\mathbf{z}_{t+1} = A\mathbf{z}_t$ with given initial vector $\mathbf{z}_0$ is:

$$\mathbf{z}_n = PD^nP^{-1}\mathbf{z}_0 = P \begin{pmatrix} r_1^n & 0 & \cdots & 0 \\ 0 & r_2^n & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & r_k^n \end{pmatrix} P^{-1}\mathbf{z}_0$$

Dynamic Stability

Definition 5 (Asymptotic Stability)**.** *$\mathbf{z}_t$ is asymptotically stable if:*

$$\lim_{n \to \infty} \mathbf{z}_n = \mathbf{0}$$

Theorem 6 (23.8)**.** *If $A \in M_k$ has k distinct real eigenvalues r_i, every solution of the general system of linear difference equation is asymptotically stable iff $|r_i| < 1 \ \forall i$*

$$\mathbf{z}_{t+1} = A\mathbf{z}_t \quad \wedge \quad \lim_{n \to \infty} \mathbf{z}_n = \mathbf{0} \quad \iff \quad |r_i| < 1 \quad \forall i$$

3 Properties of Eigenvalues

Properties of Eigenvalues

Definition 6 (Trace). *Let a_{ii} be i, ith element of $A \in M_k$.*

$$traceA := \sum_i^k a_{ii}$$

Theorem 7 (23.9). *Let $A \in M_k$ with eigenvalues $r_1, \cdots, r_k$. Then,*

1. $\sum_i^k r_i = traceA$
2. $\prod_i^k r_i = \det A$

4 Repeated Eigenvalues

Repeated Eigenvalues

Definition 7 (Defective, Nondiagonalizable). *$A \in M_k$ is <u>defective</u> (or nondiaglonalizable) if $\nexists P$ such that diagonalize A*

Definition 8 (Generalized Eigenvector). *Let r^* be an eigenvalue of the matrix A. A vector $\mathbf{v} \neq \mathbf{0}$ such that $(A - r^*I)\mathbf{v} \neq \mathbf{0}$ and $(A - *I)^m\mathbf{v} - \mathbf{0}$ for some integer $m > 1$ is <u>generalized eigenvector</u> for A corresponding to r^**

When $A \in M_2$

Theorem 8 (23.11). *Let $A \in M_2$ with repeated eigenvalues r^*. Then,*

1. *$A = r^*I$, or*
2. *A has only one independent eigenvector (say $\mathbf{v}_1$). In this case, there is a generalized eigenvector $\mathbf{v}_2$ such that $(A - r^*I)\mathbf{v}_2 = \mathbf{v}_1$. If $P = \begin{pmatrix}\mathbf{v}_1 & \mathbf{v}_2\end{pmatrix}$,*

$$P^{-1}AP = \begin{pmatrix} r^* & 1 \\ 0 & r^* \end{pmatrix}$$

Theorem 9 (23.12). *If A is the case 2 in theorem 23.11, general solution of the system of difference equations $\mathbf{z}_{t+1} = A\mathbf{z}_t$ is:*

$$\mathbf{z}_n = (z_{1,0}r^n + nr^{n-1}z_{2,0})\mathbf{v}_1 + r^n z_{2,0}\mathbf{v}_2$$

Generalized Eigenvector: Example

Example: Jordan Canonical Forms
When $A \in M_4$, there are four cases of repeated eigenvectors

1. r_1, r_2, r_3, r_3 (2 repeated eigenvectors)
2. r_1, r_2, r_2, r_2 (3 repeated eigenvectors)
3. r_1, r_1, r_1, r_1 (4 repeated eigenvectors)

4. r_1, r_1, r_2, r_2 (two 2 repeated eigenvectors)

$$(1)\begin{pmatrix} r_1 & 0 & 0 & 0 \\ 0 & r_2 & 0 & 0 \\ 0 & 0 & r_3 & 1 \\ 0 & 0 & 0 & r_3 \end{pmatrix}, \quad (2)\begin{pmatrix} r_1 & 0 & 0 & 0 \\ 0 & r_2 & 1 & 0 \\ 0 & 0 & r_2 & 1 \\ 0 & 0 & 0 & r_2 \end{pmatrix},$$

$$(3)\begin{pmatrix} r_1 & 1 & 0 & 0 \\ 0 & r_1 & 1 & 0 \\ 0 & 0 & r_1 & 1 \\ 0 & 0 & 0 & r_1 \end{pmatrix}, \quad (4)\begin{pmatrix} r_1 & 1 & 0 & 0 \\ 0 & r_1 & 0 & 0 \\ 0 & 0 & r_2 & 1 \\ 0 & 0 & 0 & r_2 \end{pmatrix}$$

5 Complex Eigenvalues and Eigenvectors

Complex Eigenvalues

Theorem 10 (23.13)**.** *Let $A \in M_k$ with real entries. Then,*

- *If $r = \alpha + i\beta$ is an eigenvalue of A, so is $\bar{r} = \alpha - i\beta$.*
- *If $\mathbf{u} + i\mathbf{v}$ is an eigenvector for r, then $\mathbf{u} - i\mathbf{v}$ is an eigenvector for $\bar{r}$.*
- *If k is odd, A must have at least one real eigenvalue.*

If there is no repeated eigenvalues, A is diagonalizable even if r is complex number.