Euclidean Spaces

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Objects in Euclidean Spaces

Objects in n-dimensional Euclidean Spaces

Dimension	Object	Representation
0	point	$\emptyset$
1	line	$x_1 \in \mathbb{R}^1$
2	plane	$(x_1, x_2) \in \mathbb{R}^2$
3	3d space	$(x_1, x_2, x_3) \in \mathbb{R}^3$
n	nd space	$(x_1, \cdots, x_n) \in \mathbb{R}^n$

Vector

Definition ((Euclidean) Vector, displacement)

n-tuples of real numbers $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ are (Euclidean) Vectors that represent displacement in $\mathbb{R}^n$ space (or Cartesian coordinate system)
Let coordination of $\mathbf{p} = (p_1, \cdots, p_n)$, $\mathbf{q} = (q_1, \cdots, q_n)$. Then the displacement from $\mathbf{p}$ to $\mathbf{q}$ is defined as $\overrightarrow{\mathbf{pq}} := (q_1 - p_1, \cdots, q_n - p_n)$. In this definition, $\mathbf{p}$ is an origin, and $\mathbf{q}$ is a destination.

Note: Any vector $\mathbf{p} = (p_1, \cdots, p_n)$ can be interpreted as a location ($(p_1, \cdots, p_n)$) or, displacement with origin $\mathbf{0} := (0, \cdots, 0)$ (more explicit notation: $\begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix}$)

Let $\mathbf{u}, \mathbf{v}$ be the vectors in $\mathbb{R}^n$ space and $u_i, v_i \in \mathbb{R}^1$ be their i-th element.

Definition ($\pm$ of vectors)

$$(\mathbf{u} \pm \mathbf{v})_i := u_i \pm v_i \quad \forall i$$

or,

$$\mathbf{u} \pm \mathbf{v} := (u_1 \pm v_1, \cdots, u_n \pm v_n)$$

Geometrically, addition of vectors means a sequence of displacements.

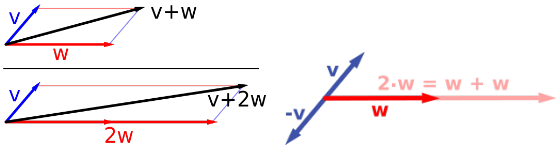

Figure: Geometrical Meaning: Addition, Subtraction, and Scalar Multiplication

Scalar Multiplication

Let r, s be scalars (or real numbers). $i.e.$, $r, s \in \mathbb{R}^1$.

Definition (Scalar Multiplication)

$$(r\mathbf{u})_i := ru_i \quad \forall i$$

Geometrically, scalar multiplcation means stretching or shrinking.

Algebraic Properties of Vector Operation

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u} \qquad \text{(Commutative Law)}$$
$$(r + s)\mathbf{u} = r\mathbf{u} + s\mathbf{u} \qquad \text{(Distributive Law 1)}$$
$$r(\mathbf{u} + \mathbf{v}) = r\mathbf{u} + r\mathbf{v} \qquad \text{(Distributive Law 2)}$$

In fact, any set of objects with a vector addition and scalar multiplication satisfying above laws is called vector space and vector is defined as the element of vector space. (Vector is defined by operations and their laws)

Length and Direction

Definition (Length of Vector $||\overrightarrow{\mathbf{pq}}||$)

$$||\overrightarrow{pq}|| := \sqrt{\sum_{i}^{n}(q_i - p_i)^2}$$

Theorem (10.1)

$$||r\mathbf{v}|| = |r| \cdot ||\mathbf{v}|| \quad \forall r \in \mathbb{R} \wedge \forall \mathbf{v} \in \mathbb{R}^n$$

Any vector has two kinds of information: (1) length, and (2) direction.

Definition (Unit Vector (Direction of a vector))

$$\textit{Unit vector of } \mathbf{v} := \frac{\mathbf{v}}{||\mathbf{v}||}$$

The Inner Product

Definition (Euclidean Inner Product (or dot product))

$$\mathbf{u} \bullet \mathbf{v} := \sum_{i}^{n} u_i v_i \in \mathbb{R}^1$$

Theorem (10.2: Properties of Inner Product)

1. $\mathbf{u} \bullet \mathbf{v} = \mathbf{v} \bullet \mathbf{u}$
2. $\mathbf{u} \bullet (\mathbf{v} + \mathbf{w}) = \mathbf{u} \bullet \mathbf{v} + \mathbf{u} \bullet \mathbf{w}$
3. $\mathbf{u} \bullet (r\mathbf{v}) = r(\mathbf{u} \bullet \mathbf{v}) = (r\mathbf{u}) \bullet \mathbf{v}$
4. $\mathbf{u} \bullet \mathbf{u} \geq 0$
5. $\mathbf{u} \bullet \mathbf{u} = 0 \quad \Longleftrightarrow \quad \mathbf{u} = \mathbf{0}$
6. $(\mathbf{u} + \mathbf{v}) \bullet (\mathbf{u} + \mathbf{v}) = \mathbf{u} \bullet \mathbf{u} + \mathbf{2u} \bullet \mathbf{v} + \mathbf{v} \bullet \mathbf{v}$

Inner Product and Angle between Two Vectors

Theorem (10.3)

Let θ be the angle between $\mathbf{u}, \mathbf{v}$. Then,

$$\mathbf{u} \bullet \mathbf{v} = ||\mathbf{u}|| \cdot ||\mathbf{v}|| \cos\theta$$

Theorem (10.4)

1. *θ is acute if $\mathbf{u} \bullet \mathbf{v} > 0$*
2. *θ is obtuse if $\mathbf{u} \bullet \mathbf{v} < 0$*
3. *θ is right if $\mathbf{u} \bullet \mathbf{v} = 0$*

Theorem (10.5: Triangle Inequality)

$$||\mathbf{u} + \mathbf{v}|| \leq ||\mathbf{u}|| + ||\mathbf{v}||$$

$$\Big|||\mathbf{u}|| - ||\mathbf{v}||\Big| \leq ||\mathbf{u} - \mathbf{v}||$$

Norms

Definition (Norms)

Any operation (X) *of a vector to a real number satisfying below three properties is norm. Length of vector is a norm.*

1. $X(\mathbf{u}) \geq 0 \wedge X(\mathbf{u}) = 0$ *only when* $\mathbf{u} = \mathbf{0}$
2. $X(r\mathbf{u}) = |r| X(\mathbf{u})$
3. $X(\mathbf{u} + \mathbf{v}) \leq X(\mathbf{u}) + X(\mathbf{v})$

Norm is set of distance measures between two vectors.

1D,2D Objects in $\mathbb{R}^n$ Spaces

Lines:One dimensional objects in $\mathbb{R}^n$ Spaces

$\mathbf{x}$ representing a line passing $\overline{\mathbf{x_0}}$ with direction $\overline{\mathbf{v}}$ is:

$$\mathbf{x} = \overline{\mathbf{x_0}} + t\overline{\mathbf{v}} \quad \forall t \in \mathbb{R} \qquad \text{(Parametric Representation)}$$

$\mathbf{x}$ representing a line passing $\overline{\mathbf{x_0}}, \overline{\mathbf{x_1}}$ is:

$$\mathbf{x} = (1-t)\overline{\mathbf{x_0}} + t\overline{\mathbf{x_1}} \quad \forall t \in \mathbb{R} \qquad \text{(Parametric Representation)}$$

Planes: Two dimensional objects in $\mathbb{R}^n$ Spaces

$\mathbf{x}$ representing a plane passing $\overline{\mathbf{x_0}}$ with direction $\overline{\mathbf{v_1}}$ and $\overline{\mathbf{v_2}}$ is:

$$\mathbf{x} = \overline{\mathbf{x_0}} + t_1\overline{\mathbf{v_1}} + t_2\overline{\mathbf{v_2}} \quad \forall t_i \in \mathbb{R} \qquad \text{(Parametric Representation)}$$

$\mathbf{x}$ representing a plane containing $\overline{\mathbf{x_0}}, \overline{\mathbf{x_1}}, \overline{\mathbf{x_2}}$ is:

$$\mathbf{x} = (1-t_1-t_2)\overline{\mathbf{x_0}} + t_1\overline{\mathbf{x_1}} + t_2\overline{\mathbf{x_2}} \quad \forall t_i \in \mathbb{R} \quad \text{(Parametric Representation)}$$

Nonparametric Equations

Definition (Normal Vector)

A normal vector $\bar{\mathbf{n}}$ *of a* n*-dimensional object* $\mathbf{X}$ *is a vector which is perpendicular to any vectors in the object. Suppose* $\mathbf{x}$, $\bar{\mathbf{p}}$ *are location vectors in the object. Then,*

$$\bar{\mathbf{n}} \bullet (\mathbf{x} - \bar{\mathbf{p}}) = 0 \quad \forall \mathbf{x}, \mathbf{p} \in \mathbf{X}$$

Finding Normal Vector

1. For linearly independent vectors $\mathbf{u_i}$, solve below systems of equations satisfying
$$\mathbf{n} \bullet \mathbf{u_i} = 0 \quad \forall i$$
2. $\mathbf{n} = \mathbf{u} \times \mathbf{v}$ (Only for $\mathbb{R}^3$ space)

$$\mathbf{u} \times \mathbf{v} := \left(\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right)$$

Hyperplanes

Definition (Hyperplane)

$n-1$ *dimensional object in* $\mathbb{R}^n$ *space is a hyperplane with normal vector* $\bar{\mathbf{a}}$ *if:*

$$\bar{a}_1 x_1 + \bar{a}_2 x_2 + \cdots \bar{a}_n x_n = \bar{d}$$

or,

$$\bar{\mathbf{a}} \bullet \mathbf{x} = \bar{d}$$

Budget Constraint

Definition (Commodity Bundle, Price Vector, Budget Set)

Let $x_i \geq 0$ be the quantity of ith commodity.

$$\mathbf{x} := (x_1, \ldots, x_n), \quad x_i \geq 0 \quad \forall i \qquad \text{(Commodity Bundle)}$$

Let $p_i \geq 0$ be the price of ith commodity.

$$\mathbf{p} := (p_1, \cdots, p_n), \quad p_i \geq 0 \quad \forall i \qquad \text{(Price Vector)}$$

Then <u>budget set</u> $\mathbf{x}$ can be defined with given budget $\bar{I}$:

$$\bar{\mathbf{p}} \bullet \mathbf{x} \leq \bar{I} \qquad \text{(Budget Constraint)}$$