Eigenvalues and Eigenvectors (2)

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6 Markov Processes

Terms

State
In each period t, the system is in one and only one of k states $S_1, \cdots, S_k$.

Definition 1 (Stochastic Process). *A stochastic process is a rule which gives the probability of the state i at the period $t = n + 1$ given the probabilities of all previous states ($t = 1, 2, \cdots, n$)*

Note: $\mathbf{x}_t = (x_{1,t}, \cdots, x_{k,t})$ is the probabilities of all k possible states at time t

Definition 2 (Markov Process). *A stochastic process that the probability of state i at $t = n+1$ depends only on what state the system was in at $t = n$ is a Markov process.*

Note: Markov processes are memoryless.

Markov Processes

Definition 3 (Transition Matrix). *M is a transition matrix for stochastic process $\mathbf{x}_t$ if:*

$$\mathbf{x}_{t+1} = M\mathbf{x}_t$$

If $\sum_i M_{ij} = 1 \quad \forall j$ (i.e., all column sums are 1), this process is a Markov process. Here, nonnegative scalar M_{ij} is transition probabilities that the process will be in state i at $t = n + 1$ if it is in state j at $t = n$

If M_{ij}, transition probabilities are fixed and independent of time indices t, this process is time-homogeneous or that M_{ij} are stationary.

Regular Markov Matrix

Definition 4 (Regular Markov Matrix). *M is a regular Markov matrix if:*

1. $\sum_i M_{ij} = 1 \quad \forall j$
2. $M_{ij} \geq 0 \quad \forall i, j$
3. $\exists r \in \mathbb{N}$ *s.t.* $M^r > 0 \quad \forall i, j$
4. *Condition 3 hold when $r = 1$*

Th23.15

Theorem 1 (23.15)**.** *Let M be a regular Markov matrix. Then,*

1. 1 *is an eigenvalue of M of multiplicity* 1 *(i.e.,* 1 *is not a repeated root)*
2. *For every other eigenvalue r of M, $|r| < 1$*
3. $\mathbf{w}_1$*, Eigenvector for eigenvalue* 1 *has strict positive components*
4. *If $\mathbf{v}_1 = \mathbf{w}_1/||\mathbf{w}_1||$, $\mathbf{v}_1$ is a probability vector and if $\mathbf{x}_{t+1} = M\mathbf{x}_t$,*

$$\lim_{n\to\infty} \mathbf{x}_n = \mathbf{v}_1$$

Note: example of non-regular Markov process. If $\exists i$ s.t. $M_{ii} = 1$, This state i is absorbing state. *I.e.*, once process reach state i, this state does not change forever. Therefore, this process will eventually reach one of these states i and then stay there forever.

7 Symmetric Matrices

Symmetric Matrices

Example of Symmetric Matrices in Economics

- (Bordered) Hessians in optimization problem
- Variance-covariance matrices in statistics

Fortunately, symmetric matrices do not have complex eigenvalues.

Definition 5 (Orthogonal Matrix)**.** *A matrix P satisfies the condition $P^{-1} = P^T$, (i.e., $P^TP = I$) is orthogonal matrix.*

We can find uncoupled system when A is symmetric.

Properties of Symmetric Matrices

Theorem 2 (23.16)**.** *Let $A \in M_k$ and $A^T = A$. Then,*

- *All k roots of* $\det A - rI = 0$ *are real numbers.*
- *All corresponding eigenvectors $\mathbf{w}_i$ are orthogonal*
- *$\exists P$ satisfying:*
  - $\mathbf{w}_i$*s are normalized eigenvectors for each eigenvalues r_i: $||\mathbf{w}_i|| = 1 \quad \forall i$*
  - *Matrix $\begin{bmatrix}\mathbf{w}_1 & \cdots & \mathbf{w}_k\end{bmatrix}$ is nonsingular*
  - *$\mathbf{w}_i\mathbf{w}_j = 0 \quad \forall i \neq j$ (orthogonal to each other)*
  - *$P^{-1} = P^T$*
  - $$P^{-1}AP = P^TAP = \begin{pmatrix} r_1 & 0 & \cdots & 0 \\ 0 & r_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & r_k \end{pmatrix}$$

8 Definiteness of Quadratic Forms

Quadratic Forms

Quadratic Forms
Every quadratic form $Q(\mathbf{x})$ can be represented by symmetric matrix A:

$$Q(\mathbf{x}) = \mathbf{x}^T A \mathbf{x} \quad \wedge \quad A^T = A$$

Always, we can find uncoupled system by taking $P^T\mathbf{x} = \begin{bmatrix}\mathbf{w}_1 & \cdots & \mathbf{w}_k\end{bmatrix}^T \mathbf{x}$ when $\mathbf{w}_i$ are correspoinding normalized eigenvalues $r_1, \cdots, r_k$. Let the transformed uncoupled system be $\mathbf{y} = P^T\mathbf{x}$. Then,

$$Q(\mathbf{x}) = Q(P\mathbf{y}) = (P\mathbf{y})^T A(P\mathbf{y}) = \mathbf{y}^T(P^T AP)\mathbf{y}$$

Note: $\mathbf{y}$ is a linear change of coordinates from $\mathbf{x}$.

Definiteness and Eigenvalues

Theorem 3 (23.17)**.** *Let $A^T = A \in M_k$ and $r_1, \cdots, r_k$ are eigenvalues of A. Then,*

1. *A is PD $\iff r_i > 0 \quad \forall i$*
2. *A is ND $\iff r_i < 0 \quad \forall i$*
3. *A is PSD $\iff r_i \geq 0 \quad \forall i$*
4. *A is NSD $\iff r_i \leq 0 \quad \forall i$*
5. *A is ID $\iff \exists i, j \quad s.t. \quad r_i < 0 \wedge r_j > 0$*

Theorem 4 (23.18)**.** *Let $A^T = A \in M_k$. Then the below statements are equivalent:*

1. *A is PD*
2. *$\exists B \quad s.t. \quad A = B^T B \wedge \exists B^{-1}$*
3. *$\exists Q \quad s.t. \quad Q^T AQ = I \wedge \exists Q^{-1}$*