Limits and Open Sets

Ch.12

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September 5, 2016

1 Sequences of Real Numbers

(Sub)Sequences of Real Number: Definition

Definition 1 (Sequence of Real Number). *$\{x_n\}_{n=1}^{\infty}$ is a sequence of real number if:*

$$x : \mathbb{N} \to \mathbb{R}, \quad x(i) = x_i$$

I.e., sequence of real number is just a real function whose domain is $\mathbb{N}$ (the set of (all) natural numbers, or the set of (all) positive integers)

Definition 2 (Subsequence). *Let $M = \{n_i\}_{i=1}^{\infty}$ be any infinite subset of $\mathbb{N}$ and $n_i > n_j \forall i > j$. (I.e., increasing sequence of natural numbers). A sequence $\{y_n\}_{n=1}^{\infty}$ is a subsequence of $\{x_n\}_{n=1}^{\infty}$ if:*

$$y_j = x_{n_j}, \quad j \in \mathbb{N}$$

Limit and Convergence: Definition

Definition 3 (Limit of a Sequence, Convergence). *$\bar{r} \in \mathbb{R}$ is the limit of a sequence of $\{x_n\}_{n=1}^{\infty}$ if:*

$$\forall \epsilon > 0, \quad \exists \bar{N} \in \mathbb{N} \quad s.t. \quad \forall n \geq \bar{N} \quad |x_n - \bar{r}| < \epsilon$$

$$\text{Then, } \lim x_n = \bar{r} \quad \text{or} \quad \lim_{n\to\infty} x_n = \bar{r} \quad \text{or} \quad x_n \to \bar{r} \qquad (x_n \text{ converges to } \bar{r})$$

Note 1: Sometimes, $\epsilon \in (0, \bar{\alpha})$ is used (for all small positive real numbers)

Note 2: $|x_n - \bar{r}| < \epsilon$ has alternative notation: ϵ-interval: $x_n \in I_\epsilon(\bar{r})$

Definition 4 (Limit of a Real Function ($\lim_{x\to\bar{x}_0} f(x) = \bar{r}$)).

$$\forall \epsilon > 0, \exists \delta > 0 \quad s.t. \quad x \in D \wedge 0 < |x - \bar{x}_0| < \delta \Rightarrow |f(x) - \bar{r}| < \epsilon$$

Algebraic Properties of Limits

Theorem 1 (12.1). *A sequence can have at most one limit.*

Theorem 2 (12.2).

$$\text{If} \quad x_n \to \bar{x} \quad \wedge \quad y_n \to \bar{y},$$

1. *$x_n \pm y_n \to \bar{x} \pm \bar{y}$ (Th 12.2)*
2. *$x_n y_n \to \bar{x}\bar{y}$ (Th 12.3)*
3. *$x_n / y_n \to \bar{x}/\bar{y}$*

Theorem 3 (12.4).

$$x_n \to \bar{x} \quad \wedge \quad x_n \leq [\geq] \bar{b} \quad \forall n \Rightarrow \bar{x} \leq [\geq] \bar{b}$$

2 Sequences in $\mathbb{R}^m$

Convergence in $\mathbb{R}^m$ Space

Definition 5 (Sequence of Vector)**.** $\{\mathbf{x_n}\}_{n=1}^{\infty}$ *is a* *sequence of vector* *if:*

$$\mathbf{x} : \mathbb{N} \to \mathbb{R}^m, \quad \mathbf{x}(i) = \mathbf{x_i}$$

Definition 6 (ϵ-ball about $\bar{r}$)**.** $B_\epsilon(\mathbf{r})$*,* *ϵ-ball about $\mathbf{r}$* *is defined as:*

$$B_\epsilon(\mathbf{r}) := \{\mathbf{x} \in \mathbb{R}^m : ||\mathbf{x} - \mathbf{r}|| < \epsilon\}$$

Note: Geometrically, ϵ-ball is hyperball in m dimensions, or bounded by an $m - 1$ sphere

Definition 7 (Limit of a Sequence of Vector)**.**

$$\mathbf{x_n} \to \mathbf{x} \quad \textit{if} \quad \forall \epsilon > 0, \quad \exists \bar{N} \quad s.t. \quad \forall n \geq \bar{N}, \quad \mathbf{x_n} \in B_\epsilon(\mathbf{x})$$

Convergence of Vectors

Theorem 4 (12.5)**.** *Let* $\mathbf{x_n} = (x_{1n}, \cdots, x_{mn})$*.* $\mathbf{x_n}$ *converges iff:*

$$x_{in} \to \bar{x}_{in} \quad \forall i$$

Theorem 5 (12.6)**.** *If* $\mathbf{x_n} \to \mathbf{x}^*$*,* $\mathbf{y_n} \to \mathbf{y}^*$*, and* $c_n \to c^*$*, then*

$$c_n\mathbf{x_n} + \mathbf{y_n} \to c^*\mathbf{x}^* + \mathbf{y}^*$$

3 Open Sets

Open: Definition

Definition 8 (Open)**.** *A set* $S \in \mathbb{R}^m$ *is* *open* *if*

$$\forall \mathbf{x} \in S \quad \Rightarrow \quad \exists \epsilon > 0 \quad s.t. \quad B_\epsilon(\mathbf{x}) \subset S$$

Geometrically, open set has no boundary.

Theorem 6 (12.7)**.** *Open balls are open sets*

Theorem 7 (12.8)**.** *1. Any union of open set is open*

2. The finite intersection of open sets is open

Interior

Definition 9 (Interior)**.** *intS, or* *Interior* *of S is union of all open sets contained in S*

Note: Interior is the largest open subset of S

Open and Closed

	Open	Not Open
Closed		
Not Closed		

4 Closed Sets

Closed: Definition

Definition 10 (Closed)**.** *A set $S \in \mathbb{R}^m$ is closed if, the limits of all convergent sequence $\{\mathbf{x}_n\}_{n=1}^{\infty} \in S$ are contained in S*

Note: Closed set must contain all its boundary points.

Theorem 8 (12.9)**.** *$S \in \mathbb{R}^m$ is closed iff $S^c = \mathbb{R}^m - S$ is open*

Theorem 9 (12.10)**.**

1. *Any intersection of closed sets is closed*
2. *The finite union of closed sets is closed*

Closure, Boundary

Definition 11 (Closure)**.** *clS or $\bar{S}$ is closure of S if It is the intersection of all closed sets containing S*

Intuitively, closure is the smallest closed set contains S

Definition 12 (Bounadry)**.** *$\mathbf{x}$ is in the boundary of a set S if*

$$\forall \epsilon > 0, \quad B_\epsilon(\mathbf{x}) \cap S \neq \varnothing \quad \wedge \quad B_\epsilon(\mathbf{x}) \cap S^c \neq \varnothing$$

Theorem 10 (12.12)**.** *Boundary of S $= clS \cap clS^c$*

5 Compact Sets

Bounded, Compact

Definition 13 (bounded)**.** *$S \in \mathbb{R}^n$ is bounded if:*

$$\exists b \in \mathbb{R} \quad s.t. \quad ||\mathbf{x}|| \leq b \quad \forall \mathbf{x} \in S$$

Definition 14 (Compact)**.** *$S \in \mathbb{R}^n$ is compact iff S is closed and bounded*

Theorem 11 (12.13-14)**.**

- *Any sequence contained in the compact set $[0, 1]$ has a convergent subsequence (Th 12.13)*
- *Any sequence contained in the compact set $C \in \mathbb{R}^n$ has a convergent subsequence whose limit lies in C (Bolzano-Weierstrass Theorem)*