Constrained Optimization (II)

Ch.19

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Main Topics

Three Aspects of the Lagrangian Approach

1. Sensitivity of the solution to changes in the parameters
2. SOCs
3. CQs

Note:

- To analyze the sensitivity, we should (1) get $\mathbf{x}^*$, the solution of the optimization problem when parameters $\mathbf{a}$ are given. And then, we should (2) see the solution $\mathbf{x}^*$ as the function of the parameters.
- You should be able to distinguish between variables (*e.g.*, $\mathbf{x}$, μ) and parameters (*e.g.*, $\mathbf{a}$). To distinguish explicitly, we denote a function f of variables $\mathbf{x}$ and parameters $\mathbf{a}$ by:

$$f(\mathbf{x}; \mathbf{a})$$

1 The Meaning of The Multiplier

Multiplier μ

Think about the solution of the optimization problem in terms of a parameter a, the level of the equality constraint.

Theorem 1 (19.1)**.** *Let f, h be C^1 functions of 2 vars. For any fixed a (parameter), let $(x_1^*(a), x_2^*(a))$ be the solution of max(min)imization problem* $\arg\max_{\mathbf{x}} f(\mathbf{x}) \quad s.t. \quad h(\mathbf{x}) = a$ *with corresponding multiplier $\mu^*(a)$ Suppose (1) $\mathbf{x}^*, \mu^*$ are C^1 functions of a and (2) NDCQ holds at $(\mathbf{x}^*(a), \mu^*(a))$. Then,*

$$\mu^*(a) = \frac{d}{da} f(\mathbf{x}^*(a); a) = \frac{d}{da} f^*(a)$$

In this view, $f^*(a) := f(\mathbf{x}^*(a); a)$, *i.e.*, the function of optimized value with regard to a, and $\mu^*(a)$ is the slope of the f^*: changes in maximum value ($f^*(a)$) in terms of a

Generalization

Theorem 2 (19.2)**.** *Let f, $\mathbf{H}$ be C^1 functions of n vars. For any fixed $\mathbf{a} \in \mathbb{R}^m$ (parameters), let $(\mathbf{x}^*(\mathbf{a}))$ be the solution of max(min)imization problem*

$$\arg\max_{\mathbf{x}} f(\mathbf{x}) \quad s.t. \quad \mathbf{H}(\mathbf{x}) = \mathbf{a}$$

with corresponding multiplier $\mu^*(\mathbf{a}) = (\mu_1^*(\mathbf{a}), \cdots, \mu_m^*(\mathbf{a}))$ Suppose (1) $\mathbf{x}^*, \mu^*$ are C^1 functions of $\mathbf{a}$ and (2) NDCQ holds at $(\mathbf{x}^*(\mathbf{a}), \mu^*(\mathbf{a}))$. Then,

$$\mu^*(\mathbf{a}) = Df_{\mathbf{a}}(\mathbf{x}^*(\mathbf{a}); \mathbf{a}) = Df_{\mathbf{a}}^*(\mathbf{a})$$

Geographical Explanation:

$$\arg\max_{\mathbf{x}} (-x_1^2 - x_2^2) \quad s.t. \quad x_1 + x_2 = a$$

Inequality Constraints

Theorem 3 (19.3)**.** *Let $\mathbf{a}^* \in \mathbb{R}^k$. Consider the max(min)imization problem*

$$\arg\max_{\mathbf{x}} f(\mathbf{x}) \quad s.t. \quad \mathbf{G}(\mathbf{x}) \leq \mathbf{a}$$

Let $\mathbf{x}^*(\mathbf{a}^*)$ be the solution of above problem, and let $\lambda^*(\mathbf{a}^*) = (\lambda_1^*(\mathbf{a}^*), \cdots, \lambda_k^*(\mathbf{a}^*))$ be the corresponding Lagrange multipliers. Suppose $n + k$ functions $\mathbf{x}^*(\mathbf{a})$ and $\lambda^*(\mathbf{a})$ are differentiable around $\mathbf{a}^*$ and NDCQ holds at $\mathbf{a}^*$. Then,

$$\lambda(\mathbf{a}^*)^* = Df_{\mathbf{a}}(\mathbf{x}(\mathbf{a}^*); \mathbf{a}^*) = Df_{\mathbf{a}}^*(\mathbf{a}^*)$$

When $\mathbf{x}^*$ is interior solution, $\lambda^*(\mathbf{a}^*) = Df_{\mathbf{a}}^*(\mathbf{a}^*) = 0$. When f is a profit function, $\lambda_j^*(\mathbf{a})$ can be interpreted as the <u>shadow price</u> of input j. Geographical Explanation:

$$\arg\max_{\mathbf{x}} (-x_1^2 - x_2^2) \quad s.t. \quad x_1 + x_2 \leq a$$

2 Envelope Theorems

Envelope Theorems: Unconstrained Problems

Theorem 4 (19.4)**.** *Let $f(\mathbf{x} : a)$ be a C^1 function of $\mathbf{x} \in \mathbb{R}^n$ and scalar a. For a given parameter a, consider the unconstrained max(min)imization problem*

$$\arg\max_{\mathbf{x}} f(\mathbf{x}; a)$$

And let $\mathbf{x}^*(a)$ be a solution of above problem. Suppose that $\mathbf{x}^*(a)$ is a C^1 function of a. Then,

$$\frac{d}{da} f(\mathbf{x}^*(a); a) = \frac{\partial}{\partial a} f(\mathbf{x}^*(a); a) = \frac{\partial}{\partial a} f^*(a)$$

Proof: From chain rule and FOC of unconstrained optimization problem

$$\frac{d}{da} f(\mathbf{x}^*(a); a) = \overset{0 \because FOC}{\cancel{Df_{\mathbf{x}}(\mathbf{x}^*(a); a)}} \frac{d\mathbf{x}^*}{da}(a) + \frac{\partial f}{\partial a}(\mathbf{x}^*(a); a) \overset{1}{\cancel{\frac{da}{da}}}$$

Envelope Theormes: Constrained Problems

Theorem 5 (19.5)**.** *Let $f, \mathbf{H} = (H_1, \cdots, H_k)$ be C^1 functions of $\mathbf{x} \in \mathbb{R}^n$. Let $\mathbf{x}^*(a)$ denote the solution of the max(min)imization problem for any fixed parameter a:*

$$\arg\max_{\mathbf{x}} f(\mathbf{x}; a) \quad s.t. \quad \mathbf{H}(\mathbf{x}; a) = 0$$

Suppose that $\mathbf{x}^*(a)$ and the Lagrange multipliers $\mu^*(a)$ are C^1 functions of a and that the NDCQ holds. Then,

$$\frac{d}{da} f(\mathbf{x}^*(a); a) = \frac{\partial}{\partial a} L(\mathbf{x}^*(a), \mu^*(a); a) = \frac{\partial}{\partial a} L^*(a)$$

3 Second Order Conditions

Strict Local Equality Constrained Max(Min)

Theorem 6 (19.6)**.** *Let $f, \mathbf{H}$ be C^2 functions on $\mathbf{x} \in \mathbb{R}^n$. Consider the equality constrained max(min)imization problem*

$$\arg\max_{\mathbf{x}} f(\mathbf{x}) \quad s.t. \quad \mathbf{H}(x) = \mathbf{c}$$

Let $L := f(\mathbf{x}) + \mu(\mathbf{c} - \mathbf{H})$ and suppose that

1. $\mathbf{H}(\mathbf{x}^*) = \mathbf{c}$ *(Satisfies constraint)*
2. $DL_{\mathbf{x},\mu}(\mathbf{x}^*, \mu^*) = \mathbf{0}$ *(Satisfies FOC)*
3. *Hession of $L = D^2_{\mathbf{x}} L(\mathbf{x}^*, \mu^*)$ is ND on the linear constraint set $\{\mathbf{v} : D\mathbf{H}(\mathbf{x}^*)\mathbf{v} = \mathbf{0}\}$ (Satisfies Sufficient SOC)*

Then, $\mathbf{x}^*$ is a <u>strict local constrained max(min)</u> of f on the constrainted set

$\mathbf{v}$ is the tangent vector on the constraint set around $\mathbf{x}^*$. Proof: See Ch.30

Calculation Procedure

Sufficient SOC: Calculation Procedure
Suppose $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{H} \in \mathbb{R}^m$, and NCDQ holds.

1. Form a Lagrangian function L

$$L := f(\mathbf{x}) + \mu(\mathbf{c} - \mathbf{H})$$

2. Get points $\mathbf{x}^*, \mu^*$ satisfy FOCs
3. Make a bordered Hessian

$$H := D^2 L_{\mu,\mathbf{x}}(\mathbf{x}^*, \mu^*) = \begin{pmatrix} \mathbf{0} & D\mathbf{H}_{\mathbf{x}}(\mathbf{x}^*) \\ D\mathbf{H}_{\mathbf{x}}(\mathbf{x}^*)^T & D^2 L_{\mathbf{x}}(\mathbf{x}^*, \mu^*) \end{pmatrix}$$

4. If H is PD, then $\mathbf{x}^*$ is strict local min. If ND, $\mathbf{x}^*$ is strict local max.
   (a) If $\text{sign}(\det H) = \text{sign}((-1)^m)$ and all $n - m$ LPMs have same sign, H is <u>PD</u> on the constraint set
   (b) If $\text{sign}(\det H) = \text{sign}((-1)^n)$ and following $n - m$ LPMs alternates in sign, H is <u>ND</u> on the constraint set

Sufficient SOC: Mixed Constraints

Theorem 7 (19.8)**.** *Let $f, \mathbf{H} \in \mathbb{R}^m, \mathbf{G} \in \mathbb{R}^k$ be C^2 functions on $\mathbf{x} \in \mathbb{R}^n$. Consider the mixed constrained max(min)imization problem* $\arg\max_{\mathbf{x}} f(\mathbf{x}) \quad s.t. \quad \mathbf{H}(\mathbf{x}) = \mathbf{c} \wedge \mathbf{G}(\mathbf{x}) \leq \mathbf{b}$.

1. *Form the Lagrangian function L*

$$L := f(\mathbf{x}) + \mu(\mathbf{c} - \mathbf{H}(\mathbf{x})) + \lambda(\mathbf{b} - \mathbf{G}(\mathbf{x}))$$

2. *Suppose $\exists \mathbf{x}^*, \mu^*, \lambda^*$ satisfying FOCs (Theorem 18.5)*

3. *For convenience, suppose $\mathbf{G}_E := (G_1, \cdots, G_e)$ are binding at $\mathbf{x}^*$ and the others $\mathbf{G}_{-E} := (G_{e+1}, \cdots, G_k)$ are not binding. Let λ_E be the corresponding multiplier of $\mathbf{G}_E$. Then if the bordered Hessian $D^2 L_{\lambda_{\mathbf{E}}, \mu, \mathbf{x}}(\lambda^*_{\mathbf{E}}, \mu^*, \mathbf{x}^*)$ is PD(ND), $\mathbf{x}^*$ is a strict local mixed constrained max(min) of f.*

Sufficient SOC

Bordered Hessian (Mixed Constraints)

$$H = D^2 L_{\lambda_{\mathbf{E}}, \mu, \mathbf{x}}(\lambda^*_{\mathbf{E}}, \mu^*, \mathbf{x}^*) = \begin{pmatrix} \mathbf{0} & \mathbf{0} & D\mathbf{G}_{\mathbf{E}\mathbf{x}} \\ \mathbf{0} & \mathbf{0} & D\mathbf{H}_{\mathbf{x}} \\ D\mathbf{G}_{\mathbf{E}\mathbf{x}}^T & D\mathbf{H}_{\mathbf{x}}^T & D^2 L_{\mathbf{x}} \end{pmatrix}\Bigg|_{(\lambda_{\mathbf{E}}, \mu, \mathbf{x}) = (\lambda^*_{\mathbf{E}}, \mu^*, \mathbf{x}^*)}$$

Let $F_{ab} := \frac{\partial}{\partial a}\frac{\partial}{\partial b} F$. Then

$$H = \begin{pmatrix} \cancelto{\mathbf{0}}{L_{\lambda_E \lambda_E}} & \cancelto{\mathbf{0}}{L_{\lambda_E \mu}} & \cancelto{D\mathbf{G}_{\mathbf{E}\mathbf{x}}}{L_{\lambda \mathbf{x}}} \\ \cancelto{\mathbf{0}}{L_{\mu \lambda_E}} & \cancelto{\mathbf{0}}{L_{\mu\mu}} & \cancelto{D\mathbf{H}_{\mathbf{x}}}{L_{\mu \mathbf{x}}} \\ \cancelto{D\mathbf{G}_{\mathbf{E}\mathbf{x}}^T}{L_{\mathbf{x}\lambda_E}} & \cancelto{D\mathbf{H}_{\mathbf{x}}^T}{L_{\mathbf{x}\mu}} & L_{\mathbf{x}\mathbf{x}} \end{pmatrix}\Bigg|_{(\lambda_{\mathbf{E}}, \mu, \mathbf{x}) = (\lambda^*_{\mathbf{E}}, \mu^*, \mathbf{x}^*)}$$

Deteriminig Definiteness of Bordered Hessian (Mixed Constraints)

Determining Definiteness

(a) If $\text{sign}(\det H) = \text{sign}((-1)^{(m+e)})$ and all $n - (m+e)$ LPMs have same sign, H is $\underline{\text{PD}}$ on the constraint set

(b) If $\text{sign}(\det H) = \text{sign}((-1)^n)$ and following $n - (m+e)$ LPMs alternates in sign, H is $\underline{\text{ND}}$ on the constraint set

4 Smooth Dependence of the Parameters

5 Constraint Qualifications

6 Proofs of the First Order Conditions