Constrained Optimization (II)

Ch.19

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Main Topics

Three Aspects of the Lagrangian Approach

1. Sensitivity of the solution to changes in the parameters
2. SOCs
3. CQs

Note:

- To analyze the sensitivity, we should (1) get $\mathbf{x}^*$, the solution of the optimization problem when parameters $\mathbf{a}$ are given. And then, we should (2) see the solution $\mathbf{x}^*$ as the function of the parameters.
- You should be able to distinguish between variables ($e.g.$, $\mathbf{x}$, μ) and parameters ($e.g.$, $\mathbf{a}$). To distinguish explicitly, we denote a function f of variables $\mathbf{x}$ and parameters $\mathbf{a}$ by:

$$f(\mathbf{x}; \mathbf{a})$$

Multiplier μ

Think about the solution of the optimization problem in terms of a parameter a, the level of the equality constraint.

Theorem (19.1)

Let f, h be C^1 functions of 2 vars. For any fixed a (parameter), let $(x_1^*(a), x_2^*(a))$ be the solution of max(min)imization problem $\arg\max_{\mathbf{x}} f(\mathbf{x}) \quad s.t. \quad h(\mathbf{x}) = a$ with corresponding multiplier $\mu^*(a)$ Suppose (1) $\mathbf{x}^*, \mu^*$ are C^1 functions of a and (2) NDCQ holds at $(\mathbf{x}^*(a), \mu^*(a))$. Then,

$$\mu^*(a) = \frac{d}{da} f(\mathbf{x}^*(a); a) = \frac{d}{da} f^*(a)$$

In this view, $f^*(a) := f(\mathbf{x}^*(a); a)$, $i.e.$, the function of optimized value with regard to a, and $\mu^*(a)$ is the slope of the f^*: changes in maximum value $(f^*(a))$ in terms of a

Generalization

Theorem (19.2)

Let f, $\mathbf{H}$ *be* C^1 *functions of* n *vars. For any fixed* $\mathbf{a} \in \mathbb{R}^m$ *(parameters), let* $(\mathbf{x}^*(\mathbf{a}))$ *be the solution of max(min)imization problem*

$$\arg\max_{\mathbf{x}} f(\mathbf{x}) \quad s.t. \quad \mathbf{H}(\mathbf{x}) = \mathbf{a}$$

with corresponding multiplier $\mu^*(\mathbf{a}) = (\mu_1^*(\mathbf{a}), \cdots, \mu_m^*(\mathbf{a}))$ *Suppose (1)* $\mathbf{x}^*, \mu^*$ *are* C^1 *functions of* $\mathbf{a}$ *and (2) NDCQ holds at* $(\mathbf{x}^*(\mathbf{a}), \mu^*(\mathbf{a}))$*. Then,*

$$\mu^*(\mathbf{a}) = Df_{\mathbf{a}}(\mathbf{x}^*(\mathbf{a}); \mathbf{a}) = Df_{\mathbf{a}}^*(\mathbf{a})$$

Geographical Explanation:

$$\arg\max_{\mathbf{x}} (-x_1^2 - x_2^2) \quad s.t. \quad x_1 + x_2 = a$$

Inequality Constraints

Theorem (19.3)

Let $\mathbf{a}^* \in \mathbb{R}^k$. *Consider the max(min)imization problem*

$$\arg\max_{\mathbf{x}} f(\mathbf{x}) \quad s.t. \quad \mathbf{G}(\mathbf{x}) \leq \mathbf{a}$$

Let $\mathbf{x}^*(\mathbf{a}^*)$ *be the solution of above problem, and let* $\lambda^*(\mathbf{a}^*) = (\lambda_1^*(\mathbf{a}^*), \cdots, \lambda_k^*(\mathbf{a}^*))$ *be the corresponding Lagrange multipliers. Suppose* $n+k$ *functions* $\mathbf{x}^*(\mathbf{a})$ *and* $\lambda^*(\mathbf{a})$ *are differentiable around* $\mathbf{a}^*$ *and NDCQ holds at* $\mathbf{a}^*$*. Then,*

$$\lambda(\mathbf{a}^*)^* = Df_{\mathbf{a}}(\mathbf{x}(\mathbf{a}^*); \mathbf{a}^*) = Df_{\mathbf{a}}^*(\mathbf{a}^*)$$

When $\mathbf{x}^*$ is interior solution, $\lambda^*(\mathbf{a}^*) = Df_{\mathbf{a}}^*(\mathbf{a}^*) = 0$. When f is a profit function, $\lambda_j^*(\mathbf{a})$ can be interpreted as the shadow price of input j.
Geographical Explanation:

$$\arg\max_{\mathbf{x}}(-x_1^2 - x_2^2) \quad s.t. \quad x_1 + x_2 \leq a$$

Theorem (19.4)

Let $f(\mathbf{x} : a)$ be a C^1 function of $\mathbf{x} \in \mathbb{R}^n$ and scalar a. For a given parameter a, consider the unconstrained max(min)imization problem

$$\arg\max_{\mathbf{x}} f(\mathbf{x}; a)$$

And let $\mathbf{x}^*(a)$ be a solution of above problem. Suppose that $\mathbf{x}^*(a)$ is a C^1 function of a. Then,

$$\frac{d}{da} f(\mathbf{x}^*(a); a) = \frac{\partial}{\partial a} f(\mathbf{x}^*(a); a) = \frac{\partial}{\partial a} f^*(a)$$

Proof: From chain rule and FOC of unconstrained optimization problem

$$\frac{d}{da} f(\mathbf{x}^*(a); a) = \cancelto{0 \because FOC}{Df_{\mathbf{x}}(\mathbf{x}^*(a); a)} \frac{d\mathbf{x}^*}{da}(a) + \frac{\partial f}{\partial a}(\mathbf{x}^*(a); a) \cancelto{1}{\frac{da}{da}}$$

Envelope Theormes: Constrained Problems

Theorem (19.5)

Let $f, \mathbf{H} = (H_1, \cdots, H_k)$ *be* C^1 *functions of* $\mathbf{x} \in \mathbb{R}^n$. *Let* $\mathbf{x}^*(a)$ *denote the solution of the max(min)imization problem for any fixed parameter* a*:*

$$\arg\max_{\mathbf{x}} f(\mathbf{x}; a) \quad s.t. \quad \mathbf{H}(\mathbf{x}; a) = 0$$

Suppose that $\mathbf{x}^*(a)$ *and the Lagrange multipliers* $\mu^*(a)$ *are* C^1 *functions of* a *and that the NDCQ holds. Then,*

$$\frac{d}{da} f(\mathbf{x}^*(a); a) = \frac{\partial}{\partial a} L(\mathbf{x}^*(a), \mu^*(a); a) = \frac{\partial}{\partial a} L^*(a)$$

Strict Local Equality Constrained Max(Min)

Theorem (19.6)

Let $f, \mathbf{H}$ be C^2 functions on $\mathbf{x} \in \mathbb{R}^n$. Consider the equality constrained max(min)imization problem

$$\arg\max_{\mathbf{x}} f(\mathbf{x}) \quad s.t. \quad \mathbf{H}(x) = \mathbf{c}$$

Let $L := f(\mathbf{x}) + \mu(\mathbf{c} - \mathbf{H})$ and suppose that

1. $\mathbf{H}(\mathbf{x}^*) = \mathbf{c}$ *(Satisfies constraint)*
2. $DL_{\mathbf{x},\mu}(\mathbf{x}^*, \mu^*) = \mathbf{0}$ *(Satisfies FOC)*
3. *Hession of $L = D^2_{\mathbf{x}} L(\mathbf{x}^*, \mu^*)$ is ND on the linear constraint set $\{\mathbf{v} : D\mathbf{H}(\mathbf{x}^*)\mathbf{v} = \mathbf{0}\}$ (Satisfies Sufficient SOC)*

Then, $\mathbf{x}^*$ is a strict local constrained max(min) of f on the constrainted set

$\mathbf{v}$ is the tangent vector on the constraint set around $\mathbf{x}^*$. Proof: See Ch.30

Calculation Procedure

Sufficient SOC: Calculation Procedure

Suppose $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{H} \in \mathbb{R}^m$, and NCDQ holds.

1. Form a Lagrangian function L

$$L := f(\mathbf{x}) + \mu(\mathbf{c} - \mathbf{H})$$

2. Get points $\mathbf{x}^*, \mu^*$ satisfy FOCs
3. Make a bordered Hessian

$$H := D^2 L_{\mu,\mathbf{x}}(\mathbf{x}^*, \mu^*) = \begin{pmatrix} \mathbf{0} & D\mathbf{H}_{\mathbf{x}}(\mathbf{x}^*) \\ D\mathbf{H}_{\mathbf{x}}(\mathbf{x}^*)^T & D^2 L_{\mathbf{x}}(\mathbf{x}^*, \mu^*) \end{pmatrix}$$

4. If H is PD, then $\mathbf{x}^*$ is strict local min. If ND, $\mathbf{x}^*$ is strict local max.
   1. If $\operatorname{sign}(\det H) = \operatorname{sign}((-1)^m)$ and all $n - m$ LPMs have same sign, H is PD on the constraint set
   2. If $\operatorname{sign}(\det H) = \operatorname{sign}((-1)^n)$ and following $n - m$ LPMs alternates in sign, H is ND on the constraint set

Sufficient SOC: Mixed Constraints

Theorem (19.8)

Let $f, \mathbf{H} \in \mathbb{R}^m, \mathbf{G} \in \mathbb{R}^k$ *be* C^2 *functions on* $\mathbf{x} \in \mathbb{R}^n$*. Consider the mixed constrained max(min)imization problem*
$\arg\max_{\mathbf{x}} f(\mathbf{x}) \quad s.t. \quad \mathbf{H}(\mathbf{x}) = \mathbf{c} \wedge \mathbf{G}(\mathbf{x}) \leq \mathbf{b}$.

1. *Form the Lagrangian function* L

$$L := f(\mathbf{x}) + \mu(\mathbf{c} - \mathbf{H}(\mathbf{x})) + \lambda(\mathbf{b} - \mathbf{G}(\mathbf{x}))$$

2. *Suppose* $\exists \mathbf{x}^*, \mu^*, \lambda^*$ *satisfying FOCs (Theorem 18.5)*
3. *For convenience, suppose* $\mathbf{G}_E := (G_1, \cdots, G_e)$ *are binding at* $\mathbf{x}^*$ *and the others* $\mathbf{G}_{-E} := (G_{e+1}, \cdots, G_k)$ *are not binding. Let* λ_E *be the corresponding multiplier of* $\mathbf{G}_E$*. Then if the bordered Hessian* $D^2 L_{\lambda_{\mathbf{E}}, \mu, \mathbf{x}}(\lambda^*_{\mathbf{E}}, \mu^*, \mathbf{x}^*)$ *is PD(ND),* $\mathbf{x}^*$ *is a strict local mixed constrained max(min) of* f.

Sufficient SOC

Bordered Hessian (Mixed Constraints)

$$H = D^2 L_{\lambda_{\mathbf{E}},\mu,\mathbf{x}}(\lambda^*_{\mathbf{E}}, \mu^*, \mathbf{x}^*) = \left.\begin{pmatrix} \mathbf{0} & \mathbf{0} & D\mathbf{G}_{\mathbf{E}_\mathbf{x}} \\ \mathbf{0} & \mathbf{0} & D\mathbf{H}_\mathbf{x} \\ D\mathbf{G}_{\mathbf{E}_\mathbf{x}}^T & D\mathbf{H}_\mathbf{x}^T & D^2 L_\mathbf{x} \end{pmatrix}\right|_{(\lambda_{\mathbf{E}},\mu,\mathbf{x})=(\lambda^*_{\mathbf{E}},\mu^*,\mathbf{x}^*)}$$

Let $F_{ab} := \frac{\partial}{\partial a}\frac{\partial}{\partial b}F$. Then

$$H = \left.\begin{pmatrix} \cancelto{\mathbf{0}}{L_{\lambda_E\lambda_E}} & \cancelto{\mathbf{0}}{L_{\lambda_E\mu}} & \cancelto{D\mathbf{G}_{\mathbf{E}_\mathbf{x}}}{L_{\lambda\mathbf{x}}} \\ \cancelto{\mathbf{0}}{L_{\mu\lambda_E}} & \cancelto{\mathbf{0}}{L_{\mu\mu}} & \cancelto{D\mathbf{H}_\mathbf{x}}{L_{\mu\mathbf{x}}} \\ \cancelto{D\mathbf{G}_{\mathbf{E}_\mathbf{x}}^T}{L_{\mathbf{x}\lambda_E}} & \cancelto{D\mathbf{H}_\mathbf{x}^T}{L_{\mathbf{x}\mu}} & L_{\mathbf{x}\mathbf{x}} \end{pmatrix}\right|_{(\lambda_{\mathbf{E}},\mu,\mathbf{x})=(\lambda^*_{\mathbf{E}},\mu^*,\mathbf{x}^*)}$$

Deterimig Definiteness of Bordered Hessian (Mixed Constraints)

Determining Definiteness

(a) If $\text{sign}(\det H) = \text{sign}((-1)^{(m+e)})$ and all $n-(m+e)$ LPMs have same sign, H is PD on the constraint set

(b) If $\text{sign}(\det H) = \text{sign}((-1)^{n})$ and following $n-(m+e)$ LPMs alternates in sign, H is ND on the constraint set