Matrix Algebra

Ch.8

econMath.namun@gmail.com

2017년 4월 6일

Table of Contents

Matrix

Definition (Matrix)

Matrix is a rectangular array of numbers (scalars)

Let $a_{ij} \in \mathbb{R}$ or $A_{ij} \in \mathbb{R}$ be the ith row and jth column element of matrix A

Definition (Equal)

$$A = B \quad \Longleftrightarrow \begin{cases} \textit{same size} \\ a_{ij} = b_{ij} \quad \forall i, j \end{cases}$$

Addition, Subtraction

Let A, B be $n \times k$ matrices and $r \in \mathbb{R}$

Definition (Addition)

$$(A + B)_{ij} := a_{ij} + b_{ij} \quad \forall i, j$$

Important note: the first $+$ and the second $+$ are not same operators

Definition (Subtraction)

$$(A - B)_{ij} := a_{ij} - b_{ij} \quad \forall i, j$$

Multiplications of Matrices

Definition (Scalar Multiplication)

$$(rA)_{ij} := rA_{ij} \quad \forall i, j$$

Let A be $n \times k$ matrix and B be $k \times m$ matrix. Then AB is $n \times m$ matrix.

Definition (Matrix Multiplication)

$$(AB)_{ij} := A_{i1}B_{1j} + A_{i2}B_{2j} + \cdots + A_{ik}B_{kj} = \sum_{r=1}^{k} A_{ir}B_{rj}$$

For $n \times n$ matrices, identity matrix I_n is a multiplicative identity.

$$AI = IA = A$$

Laws of Matrix Algebra

Laws of Matrix Algebra

$$(A+B)+C = A+(B+C) \qquad \text{(Associative Law for Addition)}$$

$$(AB)C = A(BC) \qquad \text{(Associative Law for Multiplication)}$$

$$A+B = B+A \qquad \text{(Commutative Law for Addition)}$$

$$A(B+C) = AB+AC \qquad \text{(Distributive Law)}$$

$$(A+B)C = AC+BC \qquad \text{(Distributive Law)}$$

Important Note: $AB \neq BA$

Transpose

Definition (Transpose)

$A^\intercal$ $(n \times m)$ is a transpose of A $(m \times n)$ if:

$$(A^\intercal)_{ij} := A_{ji} \quad \forall i, j$$

$$(A \pm B)^\intercal = A^\intercal \pm B^\intercal$$

$$(A^\intercal)^\intercal = A$$

$$(rA)^\intercal = rA^\intercal$$

$$(AB)^\intercal = B^\intercal A^\intercal \qquad \text{(Theorem 8.1)}$$

Special Kinds of Matrices (1)

Suppose A is $k \times n$ matrix. Then,

Definition (Special Kinds of Matrices (1))

- *A is a <u>square matrix</u> if $k = n$*
- *A is a <u>column matrix</u> if $n = 1$*
- *A is a <u>row matrix</u> if $k = 1$*
- *A is a <u>diagonal matrix</u> if $k = n$ and $a_{ij} = 0 \quad \forall i \neq j$*
- *A is a <u>scalar matrix</u> if $A = tI_n$*
- *A is an <u>upper-triangular matrix</u> if $a_{ij} = 0 \quad \forall i > j$*
- *A is a <u>lower-triangular matrix</u> if $a_{ij} = 0 \quad \forall i < j$*

Special Kinds of Matrices (2)

Definition (Special Kinds of Matrices (2))

- *A is a symmetric matrix if A is square matrix and $a_{ij} = a_{ji} \quad \forall i, j$. Or, $A^\intercal = A$*
- *A is an Idempotent matrix if $AA = A$*
- *A is a permutation matrix if A is the result of I_n with ERO_1 (row exchange)*
- *A is a nonsingular matrix if $rank A = \#row = \#column$*

If a coefficient matrix of a system of linear equations is nonsingular, this system has only one solution $\mathbf{x} = A^{-1}\mathbf{b}$

Elementary Matrix

Let E be an elementary matrix of some EROs. Then,

Theorem (8.3)

ERO with a matrix A is equivalent to EA

Theorem (8.2)

- *Let $E1_{ij}$ be the permutation matrix with interchanging R_i and R_j of I_n, then $E1_{ij}$ is equivalent to $ERO_1(i,j)$*
- *Let $E2_{k,j,i}$ be the result of $ERO_2(k,j,i)$ from I_n, then $E2_{k,j,i}$ is equivalent to $ERO_2(k,j,i)$*
- *Let $E3_{k,i}$ be the result of $ERO_3(k,i)$ from I_n, then $E3_{k,i}$ is equivalent to $ERO_3(k,i)$*

Elementary Matrix

Definition (Elementary Matrix)

$E1, E2, E3$ *are elementary matrices corresponding to their* $EROs$

Theorem (8.4)

Let $A \in M_n$ *(set of* $n \times n$ *matrices),* $E_i \in EM$ *(set of elementary matrices), and* $(R)REFM$ *be the set of (R)REF matrices. Then:*

$$\exists E_i \quad i = 1, 2, \cdots, m \quad s.t. \quad \prod_{i=m}^{1} E_i A \in (R)REFM$$

or

$$E_m E_{m-1} \cdots E_2 E_1 A \in (R)REFM$$

Inverse of Matrices

Suppose $A, B \in M_n$

Definition (Inverse, Invertible)

B is (left, or right) inverse for A if:

$$\underbrace{AB}_{B:\text{ Right inverse}} = \overbrace{BA}^{B:\text{ Left inverse}} = I$$

A is invertible if $\exists B$

Notation: $B = A^{-1}$

Theorem (8.5:Uniquenes of Inverse)

$A \in M_n$ can have at most one inverse. (left inverse $=$ right inverse)

Inverse Matrices and the Solution of Linear Systems

Theorem (8.6)

For $A \in M_n$,

$$\exists A^{-1} \quad \Rightarrow \quad \begin{cases} A \text{ is nonsingular} \\ \text{Unique solution of } A\mathbf{x} = \mathbf{b} \quad \Rightarrow \quad \mathbf{x} = A^{-1}\mathbf{b} \end{cases}$$

Proof: easy

Theorem (8.7: inverse of Th8.6)

$$A \in M_n \text{ is nonsingular} \quad \Rightarrow \quad \exists A^{-1}$$

Proof: difficult

Calculation of Inverse Matrix

Calculation of Inverse Matrix

$$[A|I] \xrightarrow{EROs} [I|A^{-1}]$$

If RREF is not I_n, $\nexists A^{-1}$

Theorem (8.8)

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2$. A *is nonsingular iff* $ad - bc \neq 0$

For general case ($A \in M_n$), see Ch.9

Equivalent statements

Theorem (8.9)

For $A \in M_n$, the following statements are equivalent

1. $\exists A^{-1}$
2. A *has right inverse*
3. A *has left inverse*
4. $A\mathbf{x} = \mathbf{b}$ *has at least one solution for every* $\mathbf{b}$
5. $A\mathbf{x} = \mathbf{b}$ *has at most one solution for every* $\mathbf{b}$
6. A *is nonsingular*
7. $rank A = n$

Properties of Inverse Matrices and Their Exponentials

Theorem (8.10)

If $A, B \in M_n$ *and* $\exists A^{-1}, B^{-1}$,

1. $(A^{-1})^{-1} = A$
2. $(A^{\intercal})^{-1} = (A^{-1})^{\intercal}$
3. $\exists (AB)^{-1} \wedge (AB)^{-1} = B^{-1}A^{-1}$

Definition (Matrix Exponential)

$$A^m := \prod_{i=1}^{m} A$$

$$A^{-m} := (A^{-1})^m$$

Expoenetial Properties of Invertible Matrices

Theorem (8.11)

$$\exists A^{-1} \quad \Rightarrow \quad \begin{cases} \exists A^{-m} \quad \forall m \in \mathbb{N} \\ A^r A^s = A^{r+s} \quad \forall r, s \in \mathbb{N} \\ \forall r \in \mathbb{R} - \{0\}, \quad \exists (rA)^{-1} \wedge (rA)^{-1} = \frac{1}{r} A^{-1} \end{cases}$$

Important Note: $(AB)^k \neq A^k B^k$

Partitioned Matrices

Somtimes, matrix of matrices can be more convenient.

Definition (Submatrix, Partitioned matrix)

- *A submatrix of matrix A is a matrix obtained by deleting some R_i or C_j*
- *A partitioned matrix is a matrix partitioned into submatrices by horizontal and/or vertical lines which extended along entire rows or columns of a matrix A*

Partitioned Matrices

Theorem (8.15)

Let A be a square matrix partitioned as

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

and $A_{11}, A_{22} \in M_n$. Then,

$$\exists A_{22}^{-1} \wedge \exists D^{-1} \wedge D = A_{11} - A_{12}A_{22}^{-1}A_{21}$$
$$\Rightarrow \quad A^{-1} = \begin{pmatrix} D^{-1} & -D^{-1}A_{12}A_{22}^{-1} \\ -A_{22}^{-1}A_{21}D^{-1} & A_{22}^{-1}(I + A_{21}D^{-1}A_{12}A_{22}^{-1}) \end{pmatrix}$$