

Matrix Algebra

Ch.8

`mailto:eyeofyou@korea.ac.kr`

2020년 6월 30일

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Matrix

Definition (Matrix)

Matrix is a rectangular array of numbers (scalars)

Let $a_{ij} \in \mathbb{R}$ or $A_{ij} \in \mathbb{R}$ be the i th row and j th column element of matrix A

Definition (Equal)

$$A = B \iff \begin{cases} \text{same size} \\ a_{ij} = b_{ij} \quad \forall i, j \end{cases}$$

Addition, Subtraction

Let A, B be $n \times k$ matrices and $r \in \mathbb{R}$

Definition (Addition)

$$(A + B)_{ij} := a_{ij} + b_{ij} \quad \forall i, j$$

Important note: the first $+$ and the second $+$ are not same operators

Definition (Subtraction)

$$(A - B)_{ij} := a_{ij} - b_{ij} \quad \forall i, j$$

Multiplications of Matrices

Definition (Scalar Multiplication)

$$(rA)_{ij} := rA_{ij} \quad \forall i, j$$

Let A be $n \times k$ matrix and B be $k \times m$ matrix. Then AB is $n \times m$ matrix.

Definition (Matrix Multiplication)

$$(AB)_{ij} := A_{i1}B_{1j} + A_{i2}B_{2j} + \cdots + A_{ik}B_{kj} = \sum_{r=1}^k A_{ir}B_{rj}$$

For $n \times n$ matrices, identity matrix I_n is a multiplicative identity.

$$AI = IA = A$$

Laws of Matrix Algebra

Laws of Matrix Algebra

$$(A + B) + C = A + (B + C) \quad (\text{Associative Law for Addition})$$

$$(AB)C = A(BC) \quad (\text{Associative Law for Multiplication})$$

$$A + B = B + A \quad (\text{Commutative Law for Addition})$$

$$A(B + C) = AB + AC \quad (\text{Distributive Law})$$

$$(A + B)C = AC + BC \quad (\text{Distributive Law})$$

Important Note: $AB \neq BA$

Transpose

Definition (Transpose)

A^T ($n \times m$) is a transpose of A ($m \times n$) if:

$$(A^T)_{ij} := A_{ji} \quad \forall i, j$$

$$(A \pm B)^T = A^T \pm B^T$$

$$(A^T)^T = A$$

$$(rA)^T = rA^T$$

$$(AB)^T = B^T A^T \quad (\text{Theorem 8.1})$$

Special Kinds of Matrices (1)

Suppose A is $k \times n$ matrix. Then,

Definition (Special Kinds of Matrices (1))

- A is a square matrix if $k = n$
- A is a column matrix if $n = 1$
- A is a row matrix if $k = 1$
- A is a diagonal matrix if $k = n$ and $a_{ij} = 0 \quad \forall i \neq j$
- A is a scalar matrix if $A = tI_n$
- A is an upper-triangular matrix if $a_{ij} = 0 \quad \forall i > j$
- A is a lower-triangular matrix if $a_{ij} = 0 \quad \forall i < j$

Special Kinds of Matrices (2)

Definition (Special Kinds of Matrices (2))

- A is a symmetric matrix if A is square matrix and $a_{ij} = a_{ji} \quad \forall i, j$.
Or, $A^T = A$
- A is an Idempotent matrix if $AA = A$
- A is a permutation matrix if A is the result of I_n with ERO_1 (row exchange)
- A is a nonsingular matrix if $\text{rank} A = \# \text{row} = \# \text{column}$

If a coefficient matrix of a system of linear equations is nonsingular, this system has only one solution $\mathbf{x} = A^{-1}\mathbf{b}$

Elementary Matrix

Let E be an elementary matrix of some ERO s. Then,

Theorem (8.3)

ERO with a matrix A is equivalent to EA

Theorem (8.2)

- Let $E1_{ij}$ be the permutation matrix with interchanging R_i and R_j of I_n , then $E1_{ij}$ is equivalent to $ERO_1(i, j)$
- Let $E2_{k,j,i}$ be the result of $ERO_2(k, j, i)$ from I_n , then $E2_{k,j,i}$ is equivalent to $ERO_2(k, j, i)$
- Let $E3_{k,i}$ be the result of $ERO_3(k, i)$ from I_n , then $E3_{k,i}$ is equivalent to $ERO_3(k, i)$

Elementary Matrix

Definition (Elementary Matrix)

E_1, E_2, E_3 are elementary matrices corresponding to their EROs

Theorem (8.4)

Let $A \in M_n$ (set of $n \times n$ matrices), $E_i \in EM$ (set of elementary matrices), and $(R)REFM$ be the set of $(R)REF$ matrices. Then:

$$\exists E_i \quad i = 1, 2, \dots, m \quad \text{s.t.} \quad \prod_{i=m}^1 E_i A \in (R)REFM$$

or

$$E_m E_{m-1} \cdots E_2 E_1 A \in (R)REFM$$

Inverse of Matrices

Suppose $A, B \in M_n$

Definition (Inverse, Invertible)

B is (left, or right) inverse for A if:

$$\underbrace{AB}_{B: \text{Right inverse}} = \underbrace{BA}_{B: \text{Left inverse}} = I$$

A is invertible if $\exists B$

Notation: $B = A^{-1}$

Theorem (8.5: Uniqueness of Inverse)

$A \in M_n$ can have at most one inverse. (left inverse = right inverse)

Inverse Matrices and the Solution of Linear Systems

Theorem (8.6)

For $A \in M_n$,

$$\exists A^{-1} \Rightarrow \begin{cases} A \text{ is nonsingular} \\ \text{Unique solution of } A\mathbf{x} = \mathbf{b} \Rightarrow \mathbf{x} = A^{-1}\mathbf{b} \end{cases}$$

Proof: easy

Theorem (8.7: inverse of Th8.6)

$$A \in M_n \text{ is nonsingular} \Rightarrow \exists A^{-1}$$

Proof: difficult

Calculation of Inverse Matrix

Calculation of Inverse Matrix

$$[A|I] \xrightarrow{EROs} [I|A^{-1}]$$

If RREF is not I_n , $\nexists A^{-1}$

Theorem (8.8)

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2$. A is nonsingular iff $ad - bc \neq 0$

For general case ($A \in M_n$), see Ch.9

Equivalent statements

Theorem (8.9)

For $A \in M_n$, the following statements are equivalent

- ① $\exists A^{-1}$
- ② A has right inverse
- ③ A has left inverse
- ④ $A\mathbf{x} = \mathbf{b}$ has at least one solution for every \mathbf{b}
- ⑤ $A\mathbf{x} = \mathbf{b}$ has at most one solution for every \mathbf{b}
- ⑥ A is nonsingular
- ⑦ $\text{rank}A = n$

Properties of Inverse Matrices and Their Exponentials

Theorem (8.10)

If $A, B \in M_n$ and $\exists A^{-1}, B^{-1}$,

- ① $(A^{-1})^{-1} = A$
- ② $(A^{\top})^{-1} = (A^{-1})^{\top}$
- ③ $\exists (AB)^{-1} \wedge (AB)^{-1} = B^{-1}A^{-1}$

Definition (Matrix Exponential)

$$A^m := \prod_{i=1}^m A$$

$$A^{-m} := (A^{-1})^m$$

Exponential Properties of Invertible Matrices

Theorem (8.11)

$$\exists A^{-1} \Rightarrow \begin{cases} \exists A^{-m} \quad \forall m \in \mathbb{N} \\ A^r A^s = A^{r+s} \quad \forall r, s \in \mathbb{N} \\ \forall r \in \mathbb{R} - \{0\}, \quad \exists (rA)^{-1} \wedge (rA)^{-1} = \frac{1}{r} A^{-1} \end{cases}$$

Important Note: $(AB)^k \neq A^k B^k$

Partitioned Matrices

Sometimes, matrix of matrices can be more convenient.

Definition (Submatrix, Partitioned matrix)

- A submatrix of matrix A is a matrix obtained by deleting some R_i or C_j
- A partitioned matrix is a matrix partitioned into submatrices by horizontal and/or vertical lines which extended along entire rows or columns of a matrix A

Partitioned Matrices

Theorem (8.15)

Let A be a square matrix partitioned as

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

and $A_{11}, A_{22} \in M_n$. Then,

$$\exists A_{22}^{-1} \wedge \exists D^{-1} \wedge D = A_{11} - A_{12}A_{22}^{-1}A_{21}$$

$$\Rightarrow A^{-1} = \begin{pmatrix} D^{-1} & -D^{-1}A_{12}A_{22}^{-1} \\ -A_{22}^{-1}A_{21}D^{-1} & A_{22}^{-1}(I + A_{21}D^{-1}A_{12}A_{22}^{-1}) \end{pmatrix}$$