Functions of Several Variables

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Definitions

Definition (Function, Domain, Target, Image: General Definitions)

Function $f : A \to B$ *is a rule that assigns each object of* A *(domain) to one object in* B *(target space). Image of* f *is* $\{f(\mathbf{x}) | \mathbf{x} \in A\} \subset B$

Examples: $f : \mathbb{R}^n \to \mathbb{R}$

$$f(\mathbf{x}) = \bar{\mathbf{a}} \bullet \mathbf{x} \qquad \text{(Linear)}$$

$$f(\mathbf{x}) = \bar{k} \prod_i x_i^{\bar{b}_i} \qquad \text{(Cobb-Douglas)}$$

$$f(\mathbf{x}) = \bar{k} \left(\sum_i \bar{c}_i x_i^{-\bar{a}} \right)^{-\bar{b}/\bar{a}} \qquad \text{(CES)}$$

$\mathbf{f}: \mathbb{R}^k \to \mathbb{R}^m$

Let $f_i : \mathbb{R}^k \to \mathbb{R}$. Then $\mathbf{f}: \mathbb{R}^k \to \mathbb{R}^m$ can be represented by f_i $(i = 1, 2, \cdots, m)$

$$\mathbf{f}(\mathbf{x}) := (f_1(\mathbf{x}), \cdots, f_m(\mathbf{x}))$$

Examples

- Production function with k input factors and m products
- Utility mapping $\mathbf{u}: \mathbb{R}^{km} \to \mathbb{R}^m$
  - $u_i : \mathbb{R}^k \to \mathbb{R}$: Individual utility function of customer i
  - k: # of goods
  - m: # of consumers
  - $\mathbf{x_i}$: consumption of customer i

$$\mathbf{u}(\mathbf{x_1}, \cdots, \mathbf{x_m}) = (u_1(\mathbf{x_1}), \cdots, u_m(\mathbf{x_m}))$$

$\mathbf{f}: \mathbb{R} \to \mathbb{R}^m$

$\mathbf{f}(t)$

$$\mathbf{f}(t) := (f_1(t), \cdots, f_m(t))$$

Geometrically, $\mathbf{f}(t)$ is a parametric curve on $\mathbb{R}^m$ space (cf. parametric line)

Level Curves

Level Curves

Let $f : \mathbb{R}^2 \to \mathbb{R}^1$. Then level curves of f are curves on domain space with same $f(\mathbf{x})$. *I.e.,*

$$\{\mathbf{x} | f(\mathbf{x}) = \bar{c}\}$$

- Isoquant: level curve of production function
- Indifference curve: level curve of utility function
- Generally, when $f : \mathbb{R}^k \to \mathbb{R}$ it is called level set and this is k dimensional nonlinear object

Linear Functions on $\mathbb{R}^k$

Definition (Linear Function from $\mathbf{R}^k$ to $\mathbf{R}^m$)

$\mathbf{f}$ *is a linear function when*

1. $\mathbf{f}(\mathbf{x_1} + \mathbf{x_2}) = \mathbf{f}(\mathbf{x_1}) + \mathbf{f}(\mathbf{x_2})$
2. $\mathbf{f}(r\mathbf{x}) = r\mathbf{f}(\mathbf{x})$

Theorem (13.1,2)

- $f : \mathbb{R}^k \to \mathbb{R}$ *is a linear function* $\Rightarrow f(\mathbf{x}) = \bar{\mathbf{a}} \bullet \mathbf{x}$, $\mathbf{a} \in \mathbb{R}^k$
- $\mathbf{f} : \mathbb{R}^k \to \mathbb{R}^m$ *is a linear function* $\Rightarrow \mathbf{f}(\mathbf{x}) = \bar{A} \bullet \mathbf{x}$, $A : m \times k$ *matrix*

Quadratic Forms

Definition (Quadratic Form on $\mathbb{R}^k$)

$f : \mathbb{R}^k \to \mathbb{R}$ *is of the quadratic form if:*

$$f(\mathbf{x}) = \sum_{i,j}^{k} \bar{a}_{ij} x_i x_j$$

more elegantly,

$$f(\mathbf{x}) = \mathbf{x}^T \bar{A} \mathbf{x}$$

In this case, A *can be always symmetric.*

Monomial

Definition (Monomials on $\mathbb{R}^k$)

$f : \mathbb{R}^k \to \mathbb{R}$ is a monomial if:

$$f(\mathbf{x}) = \bar{c} \prod_i^k x_i^{\bar{a}_i}, \quad a_i \in \mathbb{N} \cup \{0\}$$

The degree of above monomial is $\sum_i^k a_i$

Continuous

Definition (Continuous Function on $\mathbb{R}^k$)

$\mathbf{f} = (f_1, \cdots, f_m)$ *is continuous at* $\mathbf{x}$ *iff all* f_i *are continuous at* $\mathbf{x}$