

# Eigenvalues and Eigenvectors (1)

Ch.23

econMath.namun+2016sp@gmail.com

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## 1 Definitions and Examples

### Eigenvalues

**Definition 1** (Eigenvalue). Let  $A \in M_n$ . A scalar  $r$  is an eigenevalue of  $A$  iff:

$$\det(A - rI) = 0$$

**Theorem 1** (23.1). The diagonal entries  $a_{ii}$  of a diagonal matrix  $A$  are eigenvalues of  $A$ .

**Theorem 2** (23.2). A matrix  $A \in M_n$  is singular iff  $0$  is an eigenvalue of  $A$ .

### Characteristic Polynomial

**Definition 2** (Characteristic Polynomial). An  $P_A(r)$ , the  $n$ th order polynomial of variable  $r$  is an polynomial of  $A \in M_n$  when:

$$P_A(r) = \det(A - rI)$$

$r$  is eigenvalue of  $A$  if  $P_A(r) = 0$

For general  $2 \times 2$  matrix  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ ,

$$P_A(r) = \det \begin{pmatrix} a-r & b \\ c & d-r \end{pmatrix} = r^2 - (a+d)r + ad - bc$$

$n \times n$  matrices can have at most  $n$  eigenvalues

### Eigenvectors

**Definition 3** (Eigenvectors).  $\mathbf{v}$  is an eigenvector of  $A$  if

$$\det(A - rI) = 0 \quad \wedge \quad (A - rI)\mathbf{v} = \mathbf{0}$$

or,

$$\det(A - rI) = 0 \quad \wedge \quad A\mathbf{v} = r\mathbf{v}$$

Note1: Get the simplest nonzero vector from eigenspace of  $A$  with respect to each eigenvalue

Note2:  $A - rI \in M_n$  is singular iff  $\exists \mathbf{v} \neq \mathbf{0}$  s.t.  $(A - rI)\mathbf{v} = \mathbf{0}$  (See Ch.8)

### Th23.3

**Theorem 3** (23.3). Let  $A \in M_n$ , and  $r \in \mathbb{R}$ . Then, following statements are equivalent:

1.  $A - rI$  is singular
2.  $\det(A - rI) = 0$
3.  $\exists \mathbf{v} \neq \mathbf{0}$  s.t.  $(A - rI)\mathbf{v} = \mathbf{0}$
4.  $A\mathbf{v} = r\mathbf{v}$  and  $\mathbf{v} \neq \mathbf{0}$

### Examples

#### Ex 23.6

Find the eigenvalues and eigenvectors of

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 5 & 0 \\ 3 & 0 & 2 \end{pmatrix}$$

Step 1) Get eigenvalues from characteristic polynomials

$$\det(A - rI) = 0$$

Step 2) Get eigenvectors from corresponding eigenvalues  $r = 5, 4, -1$  by solving  $(A - rI)\mathbf{v} = \mathbf{0}$

- $r = 5$
- $r = 4$
- $r = -1$

## 2 Solving Linear Difference Equations

### One Dimensional Linear Difference Equations

#### One-Dimensional Equations

$$\begin{aligned} y_{t+1} &= \bar{a}y_t, \quad t \in \mathbb{N} + \{0\} \\ \Rightarrow y_n &= \bar{a}^n y_0 \end{aligned}$$

Note: The simplest dynamic – time dependent – model (cf. static model is time-invariant). In general, dynamic model is more difficult to solve.

Above system can extend to general  $n$ -dimensional linear difference equations

$$\mathbf{z}_{t+1} = A\mathbf{z}_t, \quad \mathbf{z}_t \in \mathbb{R}^n, \quad A \in M_n$$

However, solution is similar only if system is uncoupled. If the system is coupled, transform it to uncoupled system using eigenvalues and eigenvectors.

## Two Dimensional Linear Difference Equations

### Two-Dimensional Equations

$$\mathbf{z}_{t+1} = A\mathbf{z}_t$$

When  $\mathbf{z}_t = \begin{pmatrix} x_t \\ y_t \end{pmatrix}$  and  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ ,

$$x_{t+1} = \bar{a}x_t + \bar{b}y_t$$

$$y_{t+1} = \bar{c}x_t + \bar{d}y_t$$

**Definition 4** (Coupled, Uncoupled). When  $b = c = 0$ , above system is uncoupled. Otherwise, above system is coupled. When  $b = c = 0$ ,

$$\mathbf{z}_n = A^n \mathbf{z}_0 = \begin{pmatrix} a^n & 0 \\ 0 & d^n \end{pmatrix} \mathbf{z}_0$$

### The Leslie Population Model

#### Leslie Mode: Linear Population Dynamics

$$\begin{pmatrix} x_{t+1} \\ y_{t+1} \end{pmatrix} = \begin{pmatrix} b_1 & b_2 \\ 1 - d_1 & 1 - d_2 \end{pmatrix} \begin{pmatrix} x_t \\ y_t \end{pmatrix}$$

- $b_i$ : birth rate of agents in the  $i$ th period
- $d_i$ : death rate of agents in the  $i$ th period
- Agents live at most 2-periods. This means  $d_2 = 1$
- $x_t$ : the number of 0-period old population
- $y_t$ : the number of 1-period old population

Ex23.7:  $b_1 = 1, b_2 = 4, d_1 = 0.5$

M1 Transform to uncoupled system by ERO

M2 Find  $P$  s.t.  $P^{-1}AP$  is a diagonal matrix (diagonalize)

### General Two-Dimensional Systems

#### General Linear Difference Equation

$$\mathbf{z}_{t+1} = A\mathbf{z}_t$$

Let  $\mathbf{z}_t = P\mathbf{Z}_t$  or  $\mathbf{Z}_t = P^{-1}\mathbf{z}_t$ . Then,

$$\mathbf{Z}_{t+1} = P^{-1}AP\mathbf{Z}_t$$

Let  $r_1, r_2$  be eigenvalues of  $A$  and  $\mathbf{v}_1, \mathbf{v}_2$  be corresponding eigenvectors ( $2 \times 1$  matrix). If  $P = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{pmatrix}$ ,

$$A \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{pmatrix} \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix} \iff A\mathbf{v}_i = r_i\mathbf{v}_i \quad \forall i \quad (\text{Th23.3})$$

This leads to:

$$P^{-1}AP = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix}$$

## General $k$ -Dimensional Systems

**Theorem 4** (23.4). *Let  $A$  be  $k \times k$  matrix. Let  $r_i$  be  $k$  eigenvalues of  $A$ , and  $\mathbf{v}_i$  be the corresponding eigenvectors. Form the matrix*

$$P = (\mathbf{v}_1 \quad \cdots \quad \mathbf{v}_k)$$

If  $\exists P^{-1}$ ,

$$P^{-1}AP = \begin{pmatrix} r_1 & 0 & \cdots & 0 \\ 0 & r_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r_k \end{pmatrix}$$

Note:  $\exists P^{-1}$  means that  $\mathbf{v}_i$  s are linearly independent

## The Powers of Diagonalized Matrix

**Theorem 5** (23.7). *Let  $A$  be a  $k \times k$  matrix. Suppose that there is a nonsingular (invertible) matrix  $P$  s.t.*

$$P^{-1}AP = \begin{pmatrix} r_1 & 0 & \cdots & 0 \\ 0 & r_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r_k \end{pmatrix} = D \quad (\text{Jordan Canonical Form})$$

Then,

$$A^n = PD^nP^{-1}$$

And the solution of the corresponding system of difference equations  $\mathbf{z}_{t+1} = A\mathbf{z}_t$  with given initial vector  $\mathbf{z}_0$  is:

$$\mathbf{z}_n = PD^nP^{-1}\mathbf{z}_0 = P \begin{pmatrix} r_1^n & 0 & \cdots & 0 \\ 0 & r_2^n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r_k^n \end{pmatrix} P^{-1}\mathbf{z}_0$$

## Dynamic Stability

**Definition 5** (Asymptotic Stability).  $\mathbf{z}_t$  is asymptotically stable if:

$$\lim_{n \rightarrow \infty} \mathbf{z}_n = \mathbf{0}$$

**Theorem 6** (23.8). *If  $A \in M_k$  has  $k$  distinct real eigenvalues  $r_i$ , every solution of the general system of linear difference equation is asymptotically stable iff  $|r_i| < 1 \ \forall i$*

$$\mathbf{z}_{t+1} = A\mathbf{z}_t \quad \wedge \quad \lim_{n \rightarrow \infty} \mathbf{z}_n = \mathbf{0} \quad \Longleftrightarrow \quad |r_i| < 1 \quad \forall i$$

### 3 Properties of Eigenvalues

#### Properties of Eigenvalues

**Definition 6** (Trace). Let  $a_{ii}$  be  $i, i$ th element of  $A \in M_k$ .

$$\text{trace} A := \sum_i^k a_{ii}$$

**Theorem 7** (23.9). Let  $A \in M_k$  with eigenvalues  $r_1, \dots, r_k$ . Then,

1.  $\sum_i^k r_i = \text{trace} A$
2.  $\prod_i^k r_i = \det A$

### 4 Repeated Eigenvalues

#### Repeated Eigenvalues

**Definition 7** (Defective, Nondiagonalizable).  $A \in M_k$  is defective (or nondiagonalizable) if  $\nexists P$  such that diagonalize  $A$

**Definition 8** (Generalized Eigenvector). Let  $r^*$  be an eigenvalue of the matrix  $A$ . A vector  $\mathbf{v} \neq \mathbf{0}$  such that  $(A - r^*I)\mathbf{v} \neq \mathbf{0}$  and  $(A - r^*I)^m \mathbf{v} = \mathbf{0}$  for some integer  $m > 1$  is generalized eigenvector for  $A$  corresponding to  $r^*$

When  $A \in M_2$

**Theorem 8** (23.11). Let  $A \in M_2$  with repeated eigenvalues  $r^*$ . Then,

1.  $A = r^*I$ , or
2.  $A$  has only one independent eigenvector (say  $\mathbf{v}_1$ ). In this case, there is a generalized eigenvector  $\mathbf{v}_2$  such that  $(A - r^*I)\mathbf{v}_2 = \mathbf{v}_1$ . If  $P = (\mathbf{v}_1 \ \mathbf{v}_2)$ ,

$$P^{-1}AP = \begin{pmatrix} r^* & 1 \\ 0 & r^* \end{pmatrix}$$

**Theorem 9** (23.12). If  $A$  is the case 2 in theorem 23.11, general solution of the system of difference equations  $\mathbf{z}_{t+1} = A\mathbf{z}_t$  is:

$$\mathbf{z}_n = (z_{1,0}r^n + nr^{n-1}z_{2,0})\mathbf{v}_1 + r^n z_{2,0}\mathbf{v}_2$$

#### Generalized Eigenvector: Example

##### Example: Jordan Canonical Forms

When  $A \in M_4$ , there are four cases of repeated eigenvectors

1.  $r_1, r_2, r_3, r_3$  (2 repeated eigenvectors)
2.  $r_1, r_2, r_2, r_2$  (3 repeated eigenvectors)
3.  $r_1, r_1, r_1, r_1$  (4 repeated eigenvectors)

4.  $r_1, r_1, r_2, r_2$  (two 2 repeated eigenvectors)

$$\begin{aligned}
 (1) \begin{pmatrix} r_1 & 0 & 0 & 0 \\ 0 & r_2 & 0 & 0 \\ 0 & 0 & r_3 & 1 \\ 0 & 0 & 0 & r_3 \end{pmatrix}, \quad (2) \begin{pmatrix} r_1 & 0 & 0 & 0 \\ 0 & r_2 & 1 & 0 \\ 0 & 0 & r_2 & 1 \\ 0 & 0 & 0 & r_2 \end{pmatrix}, \\
 (3) \begin{pmatrix} r_1 & 1 & 0 & 0 \\ 0 & r_1 & 1 & 0 \\ 0 & 0 & r_1 & 1 \\ 0 & 0 & 0 & r_1 \end{pmatrix}, \quad (4) \begin{pmatrix} r_1 & 1 & 0 & 0 \\ 0 & r_1 & 0 & 0 \\ 0 & 0 & r_2 & 1 \\ 0 & 0 & 0 & r_2 \end{pmatrix}
 \end{aligned}$$

## 5 Complex Eigenvalues and Eigenvectors

### Complex Eigenvalues

**Theorem 10** (23.13). *Let  $A \in M_k$  with real entries. Then,*

- *If  $r = \alpha + i\beta$  is an eigenvalue of  $A$ , so is  $\bar{r} = \alpha - i\beta$ .*
- *If  $\mathbf{u} + i\mathbf{v}$  is an eigenvector for  $r$ , then  $\mathbf{u} - i\mathbf{v}$  is an eigenvector for  $\bar{r}$ .*
- *If  $k$  is odd,  $A$  must have at least one real eigenvalue.*

If there is no repeated eigenvalues,  $A$  is diagonalizable even if  $r$  is complex number.