

Functions of Several Variables

Ch.13

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1 Functions Between Euclidean Spaces

Definitions

Definition 1 (Function, Domain, Target, Image: General Definitions). *Function* $f : A \rightarrow B$ is a rule that assigns each object of A (domain) to one object in B (target space). Image of f is $\{f(\mathbf{x}) | \mathbf{x} \in A\} \subset B$

Examples: $f : \mathbb{R}^n \rightarrow \mathbb{R}$

$$f(\mathbf{x}) = \bar{\mathbf{a}} \bullet \mathbf{x} \quad (\text{Linear})$$

$$f(\mathbf{x}) = \bar{k} \prod_i x_i^{\bar{b}_i} \quad (\text{Cobb-Douglas})$$

$$f(\mathbf{x}) = \bar{k} \left(\sum_i \bar{c}_i x_i^{-\bar{a}} \right)^{-\bar{b}/\bar{a}} \quad (\text{CES})$$

$$\mathbf{f} : \mathbb{R}^k \rightarrow \mathbb{R}^m$$

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Let $f_i : \mathbb{R}^k \rightarrow \mathbb{R}$. Then $\mathbf{f} : \mathbb{R}^k \rightarrow \mathbb{R}^m$ can be represented by f_i ($i = 1, 2, \dots, m$)

$$\mathbf{f}(\mathbf{x}) := (f_1(\mathbf{x}), \dots, f_m(\mathbf{x}))$$

Examples

- Production function with k input factors and m products
- Utility mapping $\mathbf{u} : \mathbb{R}^{km} \rightarrow \mathbb{R}^m$
 - $u_i : \mathbb{R}^k \rightarrow \mathbb{R}$: Individual utility function of customer i
 - k : # of goods
 - m : # of consumers
 - \mathbf{x}_i : consumption of customer i

$$\mathbf{u}(\mathbf{x}_1, \dots, \mathbf{x}_m) = (u_1(\mathbf{x}_1), \dots, u_m(\mathbf{x}_m))$$

$$\mathbf{f} : \mathbb{R} \rightarrow \mathbb{R}^m$$

$$\mathbf{f}(t)$$

$$\mathbf{f}(t) := (f_1(t), \dots, f_m(t))$$

Geometrically, $\mathbf{f}(t)$ is a parametric curve on \mathbb{R}^m space (cf. parametric line)

2 Geometric Representation of Functions

Level Curves

Level Curves

Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}^1$. Then level curves of f are curves on domain space with same $f(\mathbf{x})$. *I.e.*,

$$\{\mathbf{x} | f(\mathbf{x}) = \bar{c}\}$$

- Isoquant: level curve of production function
- Indifference curve: level curve of utility function
- Generally, when $f : \mathbb{R}^k \rightarrow \mathbb{R}$ it is called level set and this is k dimensional nonlinear object

3 Special Kinds of Functions

Linear Functions on \mathbb{R}^k

Definition 2 (Linear Function from \mathbb{R}^k to \mathbb{R}^m). \mathbf{f} is a linear function when

1. $\mathbf{f}(\mathbf{x}_1 + \mathbf{x}_2) = \mathbf{f}(\mathbf{x}_1) + \mathbf{f}(\mathbf{x}_2)$
2. $\mathbf{f}(r\mathbf{x}) = r\mathbf{f}(\mathbf{x})$

Theorem 1 (13.1,2). • $f : \mathbb{R}^k \rightarrow \mathbb{R}$ is a linear function $\Rightarrow f(\mathbf{x}) = \bar{\mathbf{a}} \bullet \mathbf{x}$, $\mathbf{a} \in \mathbb{R}^k$

- $\mathbf{f} : \mathbb{R}^k \rightarrow \mathbb{R}^m$ is a linear function $\Rightarrow \mathbf{f}(\mathbf{x}) = \bar{A} \bullet \mathbf{x}$, $A : m \times k$ matrix

Quadratic Forms

Definition 3 (Quadratic Form on \mathbb{R}^k). $f : \mathbb{R}^k \rightarrow \mathbb{R}$ is of the quadratic form if:

$$f(\mathbf{x}) = \sum_{i,j}^k \bar{a}_{ij} x_i x_j$$

more elegantly,

$$f(\mathbf{x}) = \mathbf{x}^T \bar{A} \mathbf{x}$$

In this case, A can be always symmetric.

Monomial

Definition 4 (Monomials on \mathbb{R}^k). $f : \mathbb{R}^k \rightarrow \mathbb{R}$ is a monomial if:

$$f(\mathbf{x}) = \bar{c} \prod_i^k x_i^{\bar{a}_i}, \quad a_i \in \mathbb{N} \cup \{0\}$$

The degree of above monomial is $\sum_i^k a_i$

4 Continuous Functions

Continuous

Definition 5 (Continuous Function on \mathbb{R}^k). $\mathbf{f} = (f_1, \dots, f_m)$ is continuous at \mathbf{x} iff all f_i are continuous at \mathbf{x}