Eigenvalues and Eigenvectors (2)

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Terms

State

In each period t, the system is in one and only one of k states $S_1, \cdots, S_k$.

Definition (Stochastic Process)

A stochastic process is a rule which gives the probability of the state i at the period $t = n+1$ given the probabilities of all previous states ($t = 1, 2, \cdots, n$)

Note: $\mathbf{x}_t = (x_{1,t}, \cdots, x_{k,t})$ is the probabilities of all k possible states at time t

Definition (Markov Process)

A stochastic process that the probability of state i at $t = n+1$ depends only on what state the system was in at $t = n$ is a Markov process.

Note: Markov processes are memoryless.

Markov Processes

Definition (Transition Matrix)

M *is a* *transition matrix* *for stochastic process* $\mathbf{x}_t$ *if:*

$$\mathbf{x}_{t+1} = M\mathbf{x}_t$$

If $\sum_i M_{ij} = 1 \quad \forall j$ *(*$i.e.$*, all column sums are* 1*), this process is a Markov process. Here, nonnegative scalar* M_{ij} *is* *transition probabilities* *that the process will be in state* i *at* $t = n + 1$ *if it is in state* j *at* $t = n$

If M_{ij}*, transition probabilities are fixed and independent of time indices* t*, this process is* *time-homogeneous* *or that* M_{ij} *are* *stationary*.

Regular Markov Matrix

Definition (Regular Markov Matrix)

M is a <u>regular Markov matrix</u> if:

1. $\sum_i M_{ij} = 1 \quad \forall j$
2. $M_{ij} \geq 0 \quad \forall i, j$
3. $\exists r \in \mathbb{N} \ s.t. \ M^r > 0 \quad \forall i, j$
4. *Condition 3 hold when $r = 1$*

Th23.15

Theorem (23.15)

Let M be a regular Markov matrix. Then,

1. 1 *is an eigenvalue of M of multiplicity 1 ($i.e.$, 1 is not a repeated root)*
2. *For every other eigenvalue r of M, $|r| < 1$*
3. $\mathbf{w}_1$, *Eigenvector for eigenvalue 1 has strict positive components*
4. *If $\mathbf{v}_1 = \mathbf{w}_1/||\mathbf{w}_1||$, $\mathbf{v}_1$ is a probability vector and if $\mathbf{x}_{t+1} = M\mathbf{x}_t$,*

$$\lim_{n\to\infty} \mathbf{x}_n = \mathbf{v}_1$$

Note: example of non-regular Markov process. If $\exists i$ s.t. $M_{ii} = 1$, This state i is absorbing state. $I.e.$, once process reach state i, this state does not change forever. Therefore, this process will eventually reach one of these states i and then stay there forever.

Symmetric Matrices

Example of Symmetric Matrices in Economics

- (Bordered) Hessians in optimization problem
- Variance-covariance matrices in statistics

Fortunately, symmetric matrices do not have complex eigenvalues.

Definition (Orthogonal Matrix)

A matrix P satisfies the condition $P^{-1} = P^T$, ($i.e.$, $P^T P = I$) is orthogonal matrix.

We can find uncoupled system when A is symmetric.

Properties of Symmetric Matrices

Theorem (23.16)

Let $A \in M_k$ *and* $A^T = A$. *Then,*

- *All* k *roots of* $\det A - rI = 0$ *are real numbers.*
- *All corresponding eigenvectors* $\mathbf{w}_i$ *are orthogonal*
- $\exists P$ *satisfying:*
  - $\mathbf{w}_i$*s are normalized eigenvectors for each eigenvalues* r_i*:* $||\mathbf{w}_i|| = 1 \quad \forall i$
  - *Matrix* $\begin{bmatrix} \mathbf{w}_1 & \cdots & \mathbf{w}_k \end{bmatrix}$ *is nonsingular*
  - $\mathbf{w}_i \mathbf{w}_j = 0 \quad \forall i \neq j$ *(orthogonal to each other)*
  - $P^{-1} = P^T$
  - $$P^{-1}AP = P^TAP = \begin{pmatrix} r_1 & 0 & \cdots & 0 \\ 0 & r_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & r_k \end{pmatrix}$$

Quadratic Forms

Quadratic Forms

Every quadratic form $Q(\mathbf{x})$ can be represented by symmetric matrix A:

$$Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} \quad \wedge \quad A^T = A$$

Always, we can find uncoupled system by taking $P^T\mathbf{x} = \begin{bmatrix}\mathbf{w}_1 & \cdots & \mathbf{w}_k\end{bmatrix}^T \mathbf{x}$ when $\mathbf{w}_i$ are correspoinding normalized eigenvalues $r_1, \cdots, r_k$. Let the transformed uncoupled system be $\mathbf{y} = P^T\mathbf{x}$. Then,

$$Q(\mathbf{x}) = Q(P\mathbf{y}) = (P\mathbf{y})^T A(P\mathbf{y}) = \mathbf{y}^T (P^T AP)\mathbf{y}$$

Note: $\mathbf{y}$ is a linear change of coordinates from $\mathbf{x}$.

Definiteness and Eigenvalues

Theorem (23.17)

Let $A^T = A \in M_k$ *and* $r_1, \cdots, r_k$ *are eigenvalues of* A. *Then,*

1. A *is PD* $\iff r_i > 0 \quad \forall i$
2. A *is ND* $\iff r_i < 0 \quad \forall i$
3. A *is PSD* $\iff r_i \geq 0 \quad \forall i$
4. A *is NSD* $\iff r_i \leq 0 \quad \forall i$
5. A *is ID* $\iff \exists i, j \quad s.t. \quad r_i < 0 \wedge r_j > 0$

Theorem (23.18)

Let $A^T = A \in M_k$. *Then the below statements are equivalent:*

1. A *is PD*
2. $\exists B \quad s.t. \quad A = B^T B \wedge \exists B^{-1}$
3. $\exists Q \quad s.t. \quad Q^T A Q = I \wedge \exists Q^{-1}$