

# One-Variable Calculus: Chain Rule

CH4

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## 1 Composite Functions and the Chain Rule

### Derivative of Composite Functions

**Definition 1** (Composition of Functions).

$$(h \circ g)(x) := h(g(x))$$

$h$ : outside function,  $g$ : inside function

In general,  $h \circ g \neq g \circ h$

### Chain Rule

Suppose  $\hat{f}_i = (f_i \circ f_{i+1} \circ \cdots \circ f_n)(x) = f_i(f_{i+1}(\cdots(f_n(x))))$ . then,

$$\frac{d(f_1 \circ f_2 \circ \cdots \circ f_n)(x)}{dx} = \frac{df_1(f_2(\cdots(f_n(x))))}{dx} = \frac{d\hat{f}_1}{d\hat{f}_2} \frac{d\hat{f}_2}{d\hat{f}_3} \cdots \frac{d\hat{f}_n}{dx}$$

Exercise:  $[\cos((\sin(x^3 + 4x))^{500})]' = ?$

## 2 Inverse Functions and Their Derivatives

### Inverse of a Function

**Definition 2** (Inverse Function). Suppose  $f : E_1 \rightarrow E_2$ . Then  $g : E_2 \rightarrow E_1$  is an inverse of  $f$  if

$$g(f(x)) = x \quad \forall x \in E_1$$

$$f(g(x)) = x \quad \forall x \in E_2$$

- Notation:  $g(x) = f^{-1}(x)$

- $(f^{-1} \circ f)(x) = (f \circ f^{-1})(x) = x$
- Geometrical meaning: Graph of  $f^{-1}$  is reflection of the graph of  $f$  across 45 degree line
- $\exists f^{-1}$  (i.e.,  $f$  is invertible) iff  $f$  is monotonically increasing [decreasing]

### Derivative of the Inverse Function

**Theorem 1** (4.3: Inverse Function Theorem). *Suppose  $f : I \rightarrow \mathbb{R}$  is  $\mathbf{C}^1$  Function,  $f' \neq 0, x \in I$ . Then,*

1.  $\exists f^{-1}$  on  $I$
2.  $f^{-1} \in \mathbf{C}^1$  on interval  $f(I)$
3.  $(f^{-1})' = \frac{1}{f'(f^{-1})}$ . more intuitively,

$$\frac{df^{-1}}{dx} = \frac{1}{\frac{df(f^{-1})}{df^{-1}}} = \frac{1}{\frac{dx}{df^{-1}}}$$

### The Derivative of $x^{m/n}$

**Theorem 2** (4.4).

$$\forall n \in \mathbb{N}, \quad (x^{1/n})' = \frac{1}{n} x^{(1/n)-1}$$

**Theorem 3** (4.5).

$$\forall m, n \in \mathbb{N}, \quad (x^{m/n})' = \frac{m}{n} x^{(m/n)-1}$$

In general,

$$(x^r)' = rx^{(r-1)} \quad \forall r \in \mathbb{R}$$