Eigenvalues and Eigenvectors (2)
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Terms

State

In each period t, the system is in one and only one of k states $S_1, \cdots, S_k$.

Definition (Stochastic Process)

A stochastic process is a rule which gives the probability of the state i at the period $t = n + 1$ given the probabilities of all previous states $(t = 1, 2, \cdots, n)$

Note: $\mathbf{x}_t = (x_{1,t}, \cdots, x_{k,t})$ is the probabilities of all k possible states at time t

Definition (Markov Process)

A stochastic process that the probability of state i at $t = n + 1$ depends only on what state the system was in at $t = n$ is a <u>Markov process</u>.

Note: Markov processes are memoryless.

Markov Processes

Definition (Transition Matrix)

M is a <u>transition matrix</u> for stochastic process $\mathbf{x}_t$ if:

$$\mathbf{x}_{t+1} = M\mathbf{x}_t$$

If $\sum_i M_{ij} = 1 \quad \forall j$ (i.e., all column sums are 1), this process is a Markov process. Here, nonnegative scalar M_{ij} is <u>transition probabilities</u> that the process will be in state i at $t = n + 1$ if it is in state j at $t = n$
If M_{ij}, transition probabilities are fixed and independent of time indices t, this process is <u>time-homogeneous</u> or that M_{ij} are <u>stationary</u>.

Regular Markov Matrix

Definition (Regular Markov Matrix)

M is a _regular Markov matrix_ if:

1. $\sum_i M_{ij} = 1 \quad \forall j$
2. $M_{ij} \geq 0 \quad \forall i, j$
3. $\exists r \in \mathbb{N} \ s.t. \ M^r > 0 \quad \forall i, j$
4. Condition 3 hold when $r = 1$

Theorem (23.15)

Let M be a regular Markov matrix. Then,

1. 1 *is an eigenvalue of M of multiplicity 1 (i.e., 1 is not a repeated root)*
2. *For every other eigenvalue r of M, $|r| < 1$*
3. $\mathbf{w}_1$, *Eigenvector for eigenvalue 1 has strict positive components*
4. *If $\mathbf{v}_1 = \mathbf{w}_1/||\mathbf{w}_1||$, $\mathbf{v}_1$ is a probability vector and if $\mathbf{x}_{t+1} = M\mathbf{x}_t$,*

$$\lim_{n \to \infty} \mathbf{x}_n = \mathbf{v}_1$$

Note: example of non-regular Markov process. If $\exists i$ s.t. $M_{ii} = 1$, This state i is absorbing state. *I.e.*, once process reach state i, this state does not change forever. Therefore, this process will eventually reach one of these states i and then stay there forever.

Symmetric Matrices

Example of Symmetric Matrices in Economics

- (Bordered) Hessians in optimization problem
- Variance-covariance matrices in statistics

Fortunately, symmetric matrices do not have repeated or complex eigenvalues.

Definition (Orthogonal Matrix)

A matrix P satisfies the condition $P^{-1} = P^T$, (i.e., $P^T P = I$) is <u>*orghogonal matrix*</u>.

We can find uncoupled system when A is symmetric.

Properties of Symmetric Matrices

Theorem (23.16)

Let $A \in M_k$ and $A^T = A$. Then,

- All k roots of $\det A - rI = 0$ are real numbers.
- All corresponding eigenvectors $\mathbf{w}_i$ are orthogonal
- $\exists P$ staisfying:
  - $\mathbf{w}_i$s are normalized eigenvectors for each eigenvalues r_i: $\|\mathbf{w}_i\| = 1 \quad \forall i$
  - Matrix $\begin{bmatrix} \mathbf{w}_1 & \cdots & \mathbf{w}_k \end{bmatrix}$ is nonsingular
  - $\mathbf{w}_i \mathbf{w}_j = 0 \quad \forall i \neq j$ (orthogonal to each other)
  - $P^{-1} = P^T$
  -
$$P^{-1}AP = P^TAP = \begin{pmatrix} r_1 & 0 & \cdots & 0 \\ 0 & r_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & r_k \end{pmatrix}$$

Quadratic Forms

Quadratic Forms

Every quadratic form $Q(\mathbf{x})$ can be represented by symmetric matrix A:

$$Q(\mathbf{x}) = \mathbf{x}^T A \mathbf{x} \quad \wedge \quad A^T = A$$

Always, we can find uncoupled system by taking
$P^T \mathbf{x} = \begin{bmatrix} \mathbf{w}_1 & \cdots & \mathbf{w}_k \end{bmatrix}^T \mathbf{x}$ when $\mathbf{w}_i$ are correspoinding normalized eigenvalues $r_1, \cdots, r_k$. Let the transformed uncoupled system be $\mathbf{y} = P^T \mathbf{x}$. Then,

$$Q(\mathbf{x}) = Q(P\mathbf{y}) = (P\mathbf{y})^T A (P\mathbf{y}) = \mathbf{y}^T (P^T A P) \mathbf{y}$$

Note: $\mathbf{y}$ is a linear chage of coordinates from $\mathbf{x}$.

Definiteness and Eigenvalues

Theorem (23.17)

Let $A^T = A \in M_k$ and $r_1, \cdots, r_k$ are eigenvalues of A. Then,

1. A is PD $\iff r_i > 0 \quad \forall i$
2. A is ND $\iff r_i < 0 \quad \forall i$
3. A is PSD $\iff r_i \geq 0 \quad \forall i$
4. A is NSD $\iff r_i \leq 0 \quad \forall i$
5. A is ID $\iff \exists i, j \quad s.t. \quad r_i < 0 \wedge r_j > 0$

Theorem (23.18)

Let $A^T = A \in M_k$. Then the below statements are equivalent:

1. A is PD
2. $\exists B \quad s.t. \quad A = B^T B \wedge \exists B^{-1}$
3. $\exists Q \quad s.t. \quad Q^T A Q = I \wedge \exists Q^{-1}$