

Systems of Linear Equations

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1 Gaussian and Gauss-Jordan Elimination

General Linear System

General Linear System

$$\begin{aligned}\bar{a}_{11}x_1 + \bar{a}_{12}x_2 + \cdots + \bar{a}_{1n}x_n &= \bar{b}_1 \\ \bar{a}_{21}x_1 + \bar{a}_{22}x_2 + \cdots + \bar{a}_{2n}x_n &= \bar{b}_2 \\ &\vdots \\ \bar{a}_{m1}x_1 + \bar{a}_{m2}x_2 + \cdots + \bar{a}_{mn}x_n &= \bar{b}_m\end{aligned}$$

More elegantly,

$$\bar{A}\mathbf{x} = \bar{\mathbf{b}}$$

(Compare with one-var version: $\bar{a}x = \bar{b}$)

Solution of Linear System: $\mathbf{x} = (x_1, x_2, \cdots, x_n)$

- Solution of linear system is $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ satisfies all equations in equation system
- Main considerations:
 - Existence of Solution
 - # of Solutions
 - Efficient deriving methods
 - * Substitution
 - * Elimination of variables
 - * Matrix methods

2 Elementary Row Operations (EROs)

Elementary Row Operations (EROs)

Solving Procedure: Big Picture

$$(A|\mathbf{b}) \xrightarrow{EROs} (A_{REF}|\mathbf{b}^*) \xrightarrow{EROs} (A_{RREF}|\mathbf{b}^{**}) \xrightarrow{EROs} (I|\mathbf{b}^{***})$$

Solution: $\mathbf{x} = \mathbf{b}^{***}$

General linear system

We should solve

$$\begin{aligned} \bar{a}_{11}x_1 + \bar{a}_{12}x_2 + \cdots + \bar{a}_{1n}x_n &= \bar{b}_1 \\ \bar{a}_{21}x_1 + \bar{a}_{22}x_2 + \cdots + \bar{a}_{2n}x_n &= \bar{b}_2 \\ &\vdots \\ \bar{a}_{m1}x_1 + \bar{a}_{m2}x_2 + \cdots + \bar{a}_{mn}x_n &= \bar{b}_m \end{aligned}$$

Coefficient Matrix, Augment Matrix

Definition 1 (Coefficient Matrix). A is a coefficient matrix for general linear system

$$A := \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

Definition 2 (Augment Matrix). \hat{A} is a augment matrix for general linear system

$$\hat{A} := (A|\mathbf{b})$$

Elementary Row Operations (EROs)

Definition 3. Let R_i be the i th row matrix of A , then,

$$R_i \leftrightarrow R_j \quad (ERO_1(i, j))$$

$$R_i \leftarrow \bar{k}R_j + R_i \quad (ERO_2(k, j, i))$$

$$R_i \leftarrow \bar{k}R_i \quad (ERO_3(k, i))$$

Above operations do not change solution

- Notations

- R_i : i th row (of A)
- C_i : i th column (of A)
- a_{ij} : Element of i th row, j th column (of A)

Row Echelon Form

Definition 4 (k -leading zeros, pivot, Row Echelon Form (REF)). R_i has k -leading zeros if:

$$a_{ij} = \begin{cases} 0, & \forall j \leq k \\ \text{not } 0, & j = k + 1 \end{cases}$$

Above $a_{i,k+1}$ is a pivot of R_i

Let k_i be # of leading zeros of R_i . Then \hat{A}_{REF} is row echelon form (REF) of \hat{A} if:

- $k_i > k_j \quad \forall i > j$
- Zero rows are placed on the bottom of A

Reduced Row Echelon Form (RREF)

Definition 5 (Reduced Row Echelon Form (RREF)). A_{RREF} is a reduced row echelon form (RREF) of A if:

- A_{RREF} is REF
- If C_i has pivot, only pivot is non-zero element in C_i
- Pivot = 1

Solving Procedure: Gauss-Jordan Elimination

$$(A|\mathbf{b}) \xrightarrow{EROs} (A_{REF}|\mathbf{b}^*) \xrightarrow{EROs} (A_{RREF}|\mathbf{b}^{**}) \xrightarrow{EROs} (I|\mathbf{b}^{***})$$

Solution: If the linear system $A\mathbf{x} = \mathbf{b}$ has a unique solution (A is $m \times n$ matrix), $A_{RREF} = I$ ($m = n$) or, $R_i = I_i$ ($i \leq n$) and $R_i = \mathbf{0}$ ($i > n$)

3 Systems with Many or No Solutions

3.1 Systems with No Solution

Systems with No Solution

During EROs for $\hat{A} = A|\mathbf{b}$, if you encounter with $\mathbf{0}|k \neq 0$, i.e., $(000 \cdots 0|k \neq 0)$, this system has no solution

Meaning:

$$0x_1 + 0x_2 + \cdots + 0x_n = 0 = k \neq 0 \quad (\text{contradiction!})$$

No \mathbf{x} can satisfy this equation

Theorem 1 (Fact7.2). \hat{A} has a solution iff

$$\text{rank} \hat{A} = \text{rank} A$$

3.2 Systems with Many Solutions

Systems with Many Solutions

Here, w stands for independent zero or non-zero element (can have any values) and $*$ is nonzero pivot.

REF of systems with many solutions

Systems with many solutions have following form of RREF (blank means zero)

$$\left(\begin{array}{cccccccc|c} * & w & w & w & w & w & w & w & w \\ & & & * & w & w & w & w & w \\ & & & & * & w & w & & w \\ & & & & & * & w & & w \\ & & & & & & * & & w \end{array} \right) \quad (\text{REF})$$

RREF of Systems with many solutions

Important note: This is just an example of systems with many solutions.

RREF of systems with many solutions

Systems with many solutions have following form of RREF (blank means zero)

$$\left(\begin{array}{cccccccc|c} 1 & w & w & 0 & w & 0 & 0 & 0 & w \\ & & & 1 & w & 0 & 0 & 0 & w \\ & & & & & 1 & 0 & 0 & w \\ & & & & & & 1 & 0 & w \\ & & & & & & & 1 & w \end{array} \right) \quad (\text{RREF})$$

In systems with many solutions, there exist C_i with no pivot (in our example, C_2, C_3, C_5)

Solutions of linear systems with many solutions

- Two types of solutions
 1. Basic variables are dependent on free variables or, fixed value:
 - Dependent on free variables: x_1, x_4 (in our example)
 - Fixed value: x_6, x_7, x_8 (in our example)
 2. Free variables can have any value: x_2, x_3, x_5 (in our example)
 - If C_i has no pivot, x_i is free variable
- In solving systems with many solutions, above types should be addressed explicitly

4 Rank - The Fundamental Criterion

Rank

Definition 6 (Rank). The Rank of a matrix is # of the nonzero rows in its REF (or RREF)

Property of Rank

Rank = # of pivots = # of nonzero rows of REF (or RREF)

- Implication: Rank means the number of effective (meaningful) equations

Theorem 2 (Fact7.11). A system of m equations and n unknowns (its coefficient matrix is $m \times n$ matrix)

$A\mathbf{x} = \mathbf{b}$	
$m < n$	# of solutions
$\mathbf{b} = \mathbf{0}$	∞
$\forall \mathbf{b}$	$0 \vee \infty$
$\text{rank} A = m$	$\infty(\forall \mathbf{b})$
$m > n$	
$\mathbf{b} = \mathbf{0}$	$1 \vee \infty$
$\forall \mathbf{b}$	$0 \vee 1 \vee \infty$
$\text{rank} A = n$	$0 \vee 1(\forall \mathbf{b})$
$m = n$	
$\mathbf{b} = \mathbf{0}$	$1 \vee \infty$
$\forall \mathbf{b}$	$0 \vee 1 \vee \infty$
$\text{rank} A = n = m$	$1(\forall \mathbf{b})$

5 The Linear Implicit Function Theorem

Example: IS-LM model

IS-LM Model

$$sY + ar = I^0 + G \quad (\text{IS})$$

$$mY - hr = M_s - M^0 \quad (\text{LM})$$

- Endogenous variables: Y, r
- Exogenous parameters: $s, a, m, h, I^0, G, M_s, M^0$
 - Policy variables: G, M_s
 - Behavioral variables: s, a, m, h, I^0, M^0

- Coefficient matrix:

$$A = \begin{pmatrix} s & a \\ m & -h \end{pmatrix}$$

Linear Implicit Function Theorem

Theorem 3 (Linear Implicit Function Theorem). *A general linear model with m equations and n unknowns can have a unique solution (and suppose x_i are arranged by endogeneity) iff:*

- x_1, \dots, x_k are endogenous variables
- x_{k+1}, \dots, x_n are exogenous variables (i.e., constant in this system)
- $k = m$
- $\text{rank} A = k$

Linear IFT

Implication of Linear IFT: treat exogenous variables as constant.

$$\begin{aligned} \bar{a}_{11}x_1 + \bar{a}_{12}x_2 + \dots + \bar{a}_{1n}x_n &= \bar{b}_1 \\ &\vdots \\ \bar{a}_{m1}x_1 + \bar{a}_{m2}x_2 + \dots + \bar{a}_{mn}x_n &= \bar{b}_m \end{aligned}$$

Above system can be solvable iff $\text{rank} \tilde{A} = k = m$

$$\tilde{A}\mathbf{x} = \tilde{\mathbf{b}}$$

$$\tilde{A} := \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kk} \end{pmatrix}, \tilde{\mathbf{b}} = \begin{pmatrix} b_1 - a_{1,k+1}x_{k+1} - \cdots - a_{1n}x_n \\ b_2 - a_{2,k+1}x_{k+1} - \cdots - a_{2n}x_n \\ \vdots \\ b_k - a_{k,k+1}x_{k+1} - \cdots - a_{kn}x_n \end{pmatrix}$$