One-Variable Calculus: Applications

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Positive Derivative implies Increasing Function

Suppose f is continuously differentiable at x_0 ($i.e.$, $\exists f' \wedge f'$ is continuous)

Theorem (3.1)

1. $f'(x_0) > 0 \Rightarrow \exists \bar{\alpha}, \bar{\beta} \in \mathbb{R}$ *s.t.* $x_0 \in (\bar{\alpha}, \bar{\beta}) \wedge f$ *is increasing on* $(\bar{\alpha}, \bar{\beta})$
2. $f'(x_0) < 0 \Rightarrow \exists \bar{\alpha}, \bar{\beta} \in \mathbb{R}$ *s.t.* $x_0 \in (\bar{\alpha}, \bar{\beta}) \wedge f$ *is decreasing on* $(\bar{\alpha}, \bar{\beta})$

Theorem (3.2)

1. $f' > 0$ *on* $(\bar{a}, \bar{b}) \subset D \Rightarrow f$ *is increasing on* $(\bar{a}, \bar{b})$
2. $f' < 0$ *on* $(\bar{a}, \bar{b}) \subset D \Rightarrow f$ *is decreasing on* $(\bar{a}, \bar{b})$
3. f *is increasing on* $(\bar{a}, \bar{b}) \Rightarrow f' \geq 0$ *on* $(\bar{a}, \bar{b})$
4. f *is decreasing on* $(\bar{a}, \bar{b}) \Rightarrow f' \leq 0$ *on* $(\bar{a}, \bar{b})$

Graph Sketching using First Derivatives

Procedure

(STEP 1) Find all x_i^* s.t. $f'(x_i^*) = 0$ (critical points), boundary points, and around undefined points

(STEP 2) Calculate $f(x_i^*)$

(STEP 3) Make table for graph sketch

- $f' > 0 \Rightarrow \nearrow$
- $f' < 0 \Rightarrow \searrow$

$$f(x) = x^4 - 8x^3 + 18x^2 - 11 \tag{Ex3.1}$$

Convexity and Concavity

Definition (Convex (Concave up), Concave (Concave down))

f is convex on $(\bar{\alpha}, \bar{\beta})$ iff:

$$f\left((1-t)\bar{a} + t\bar{b}\right) \leq (1-t)f(\bar{a}) + tf(\bar{b}), \quad \forall t \in [0,1] \quad \forall \bar{a}, \bar{b} \in [\alpha, \beta]$$

f is concave on $(\bar{\alpha}, \bar{\beta})$ iff:

$$f\left((1-t)\bar{a} + t\bar{b}\right) \geq (1-t)f(\bar{a}) + tf(\bar{b}), \quad \forall t \in [0,1] \quad \forall \bar{a}, \bar{b} \in [\alpha, \beta]$$

	$f' > 0$	$f' < 0$
$f'' > 0$		
$f'' < 0$		

Using Second Derivative for Graph Sketch

Procedure

- (STEP 1) Find all x_i^* s.t. $f'(x_i^*) = 0$ (critical points), $\underline{f''(x_i^*) = 0}$, boundary points, and around undefined points
- (STEP 2) Calculate $f(x_i^*)$
- (STEP 3) Make table for graph sketch

$$f(x) = x^4 - 8x^3 + 18x^2 - 11 \tag{Ex3.1}$$

Graphing Rational Function

Procedure

- (STEP 1) Find all x_i^* s.t. $f'(x_i^*) = 0$ (critical points), $f''(x_i^*) = 0$, boundary points, convergence toward undefined points, and tail ($i.e.$, convergence toward $\pm\infty$)
- (STEP 2) Calculate $f(x_i^*)$
- (STEP 3) Make table for graph sketch

$$f(x) = \frac{16(x+1)}{(x-2)^2} \tag{Ex3.6}$$

Tail

Tails of Polynomial

Only two cases: diverge to $\pm\infty$

Tails of Rational Function

$$g(x) = \frac{\bar{a}_0 x^{\bar{k}} + \bar{a}_1 x^{\bar{k}-1} + \cdots + \bar{a}_{\bar{k}}}{\bar{b}_0 x^{\bar{m}} + \cdots + \bar{b}_{\bar{m}}}$$

Tails of $g(x)$ is determined by $\frac{\bar{a}_0}{\bar{b}_0} \frac{x^{\bar{k}}}{x^{\bar{m}}}$

- $k > m$: Same as polynomials with degree $k - m$
- $k = m$: Converges to $\frac{a_0}{b_0}$ (Horizontal asymptote)
- $k < m$: converges to 0 (Horizontal asymptote)

Boundary Max and Interior Max

Theorem (3.3: First Order Condition (FOC))

x_0 *is an interior max or min of* $f \Rightarrow x_0$ *is a critical point of* f. $i.e.,$

$f'(x_0) = 0$ *(Inverse is not always true)*

Theorem (3.4: Second Order Condition (SOC))

1. $f'(x_0) = 0 \wedge f''(x_0) < 0 \Rightarrow x_0$ *is local max of* f
2. $f'(x_0) = 0 \wedge f''(x_0) > 0 \Rightarrow x_0$ *is local min of* f
3. $f'(x_0) = 0 \wedge f''(x_0) = 0 \Rightarrow x_0$ *can be max, min, or neither*

Global Maxima and Minima

- Finding global max (or min) is not easy problem
- These cases guarantee the existence of global max (or min)
  - (1) Domain of f is an interval $\wedge$ (2) f has only one critical point $\wedge$ (3) $f'' > 0$ (g.min) $\vee$ $f'' < 0$ (g.max) in domain of f
  - Domain of f is compact (closed and bounded) ($\exists$ global max, global min)
- Below case guarantees the nonexistence of global max (or min)
  - Strictly increasing (or decreasing) functions with open domain

Producer's Problem in Perfect Competative Market

Producer's Problem in perfect competitive market

$$\arg\max_{x} \Pi(x)$$

$$x = f(L) \qquad \text{(Production Function)}$$

Exogenous (Given) variables

- $\bar{w}$: unit price of labor
- $\bar{p}$: unit price of end product

Assumptions

Assumptions

- $f : D \to \mathbb{R} \subset \mathbf{C}^2$
- f is increasing: $f'(L) > 0 \forall L \in D$
- $\exists \bar{a} \geq 0$ s.t. (1) $f''(L) > 0 \forall L \in [0, \bar{a})$ ($i.e.$, convex on $[0, \bar{a})$) and (2) $f''(L) < 0 \forall L \in (\bar{a}, \infty)$ ($i.e.$, concave on $(\bar{a}, \infty)$)
- Quantity of input (labor) L is the only factor for production

Cost Functions

Big Picture for problem solving

Production Function $\rightarrow$ Cost Function (in terms of x) $\rightarrow$ Profit Function $\Pi(x)$ $\rightarrow$ Finding x^* maximizing $\Pi(x)$

Definition (Total Cost, Marginal Cost, and Average Cost)

- $TC(x)$: *Total cost for producing* x
- $MC(x) := TC'(x)$
- $AC(x) := \frac{TC(x)}{x}$

Theorem (3.7c)

At interior minimum of AC ($i.e.$, $AC' = 0$), $AC = MC$

Revenue and Profit Functions

Definition (Total Revenue, Marginal Revenue)

$$TR(x) := \bar{p}x$$

$$MR(x) := TR'(x)$$

Definition (Profit Function)

$$\Pi(x) := TR - TC$$

Producer's problem in Monopoly Case

In monopoly, p is endogenous

Producer's Problem in Monopoly

$$\arg\max_{p,x} \Pi(x)$$

However, firm is facing demand directly in monopoly

$$x = D(p) \qquad \text{(Demand Function)}$$

Elasticity

Definition (A Elasticity of B)

$$\epsilon_{B,A} := \frac{\frac{dB}{B}}{\frac{dA}{A}} = \frac{A}{B}\frac{dB}{dA}$$

- $\frac{\Delta x}{x}$: rate of change
- Elasticity: ratio of rate of change
  - $|\epsilon| < 1$: inelastic
  - $|\epsilon| > 1$: elastic
  - $|\epsilon| = 1$: unit elastic

Functions with Constant Demand

- Elasticity of linear demand function is not constant (not realistic)

$$x = D(p) = \bar{a} - \bar{b}p, \quad \bar{a}, \bar{b} > 0$$

- Example of constant elasticity demand function (more realistic)

$$x = D(p) = \bar{k}p^{-\bar{r}}, \quad \bar{k}, \bar{r} > 0$$