

Limits and Open Sets

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1 Sequences of Real Numbers

(Sub)Sequences of Real Number: Definition

Definition 1 (Sequence of Real Number). $\{x_n\}_{n=1}^{\infty}$ is a sequence of real number if:

$$x : \mathbb{N} \rightarrow \mathbb{R}, \quad x(i) = x_i$$

I.e., sequence of real number is just a real function whose domain is \mathbb{N} (the set of (all) natural numbers, or the set of (all) positive integers)

Definition 2 (Subsequence). Let $M = \{n_i\}_{i=1}^{\infty}$ be any infinite subset of \mathbb{N} and $n_i > n_j \forall i > j$. (I.e., increasing sequence of natural numbers). A sequence $\{y_n\}_{n=1}^{\infty}$ is a subsequence of $\{x_n\}_{n=1}^{\infty}$ if:

$$y_j = x_{n_j}, \quad j \in \mathbb{N}$$

Limit and Convergence: Definition

Definition 3 (Limit of a Sequence, Convergence). $\bar{r} \in \mathbb{R}$ is the limit of a sequence of $\{x_n\}_{n=1}^{\infty}$ if:

$$\forall \epsilon > 0, \quad \exists \bar{N} \in \mathbb{N} \quad \text{s.t.} \quad \forall n \geq \bar{N} \quad |x_n - \bar{r}| < \epsilon$$

$$\text{Then, } \lim x_n = \bar{r} \quad \text{or} \quad \lim_{n \rightarrow \infty} x_n = \bar{r} \quad \text{or} \quad x_n \rightarrow \bar{r} \quad (x_n \text{ converges to } \bar{r})$$

Note 1: Sometimes, $\epsilon \in (0, \bar{\alpha})$ is used (for all small positive real numbers)

Note 2: $|x_n - \bar{r}| < \epsilon$ has alternative notation: ϵ -interval: $x_n \in I_{\epsilon}(\bar{r})$

Definition 4 (Limit of a Real Function ($\lim_{x \rightarrow \bar{x}_0} f(x) = \bar{r}$)).

$$\forall \epsilon > 0, \exists \delta > 0 \quad \text{s.t.} \quad x \in D \wedge 0 < |x - \bar{x}_0| < \delta \Rightarrow |f(x) - \bar{r}| < \epsilon$$

Algebraic Properties of Limits

Theorem 1 (12.1). A sequence can have at most one limit.

Theorem 2 (12.2).

$$\text{If } x_n \rightarrow \bar{x} \quad \wedge \quad y_n \rightarrow \bar{y},$$

$$1. \quad x_n \pm y_n \rightarrow \bar{x} \pm \bar{y} \quad (\text{Th 12.2})$$

$$2. \quad x_n y_n \rightarrow \bar{x} \bar{y} \quad (\text{Th 12.3})$$

$$3. \quad x_n / y_n \rightarrow \bar{x} / \bar{y}$$

Theorem 3 (12.4).

$$x_n \rightarrow \bar{x} \quad \wedge \quad x_n \leq [\geq] \bar{b} \quad \forall n \Rightarrow \bar{x} \leq [\geq] \bar{b}$$

2 Sequences in \mathbb{R}^m

Convergence in \mathbb{R}^m Space

Definition 5 (Sequence of Vector). $\{\mathbf{x}_n\}_{n=1}^{\infty}$ is a sequence of vector if:

$$\mathbf{x} : \mathbb{N} \rightarrow \mathbb{R}^m, \quad \mathbf{x}(i) = \mathbf{x}_i$$

Definition 6 (ϵ -ball about $\bar{\mathbf{r}}$). $B_\epsilon(\mathbf{r})$, ϵ -ball about \mathbf{r} is defined as:

$$B_\epsilon(\mathbf{r}) := \{\mathbf{x} \in \mathbb{R}^m : \|\mathbf{x} - \mathbf{r}\| < \epsilon\}$$

Note: Geometrically, ϵ -ball is hyperball in m dimensions, or bounded by an $m - 1$ sphere

Definition 7 (Limit of a Sequence of Vector).

$$\mathbf{x}_n \rightarrow \mathbf{x} \quad \text{if} \quad \forall \epsilon > 0, \quad \exists \bar{N} \quad \text{s.t.} \quad \forall n \geq \bar{N}, \quad \mathbf{x}_n \in B_\epsilon(\mathbf{x})$$

Convergence of Vectors

Theorem 4 (12.5). Let $\mathbf{x}_n = (x_{1n}, \dots, x_{mn})$. \mathbf{x}_n converges iff:

$$x_{in} \rightarrow \bar{x}_{in} \quad \forall i$$

Theorem 5 (12.6). If $\mathbf{x}_n \rightarrow \mathbf{x}^*$, $\mathbf{y}_n \rightarrow \mathbf{y}^*$, and $c_n \rightarrow c^*$, then

$$c_n \mathbf{x}_n + \mathbf{y}_n \rightarrow c^* \mathbf{x}^* + \mathbf{y}^*$$

3 Open Sets

Open: Definition

Definition 8 (Open). A set $S \in \mathbb{R}^m$ is open if

$$\forall \mathbf{x} \in S \quad \Rightarrow \quad \exists \epsilon > 0 \quad \text{s.t.} \quad B_\epsilon(\mathbf{x}) \subset S$$

Geometrically, open set has no boundary.

Theorem 6 (12.7). Open balls are open sets

Theorem 7 (12.8). 1. Any union of open set is open

2. The finite intersection of open sets is open

Interior

Definition 9 (Interior). $\text{int}S$, or Interior of S is union of all open sets contained in S

Note: Interior is the largest open subset of S

Open and Closed

	Open	Not Open
Closed		
Not Closed		

4 Closed Sets

Closed: Definition

Definition 10 (Closed). A set $S \in \mathbb{R}^m$ is closed if, the limits of all convergent sequence $\{\mathbf{x}_n\}_{n=1}^{\infty} \in S$ are contained in S

Note: Closed set must contain all its boundary points.

Theorem 8 (12.9). $S \in \mathbb{R}^m$ is closed iff $S^c = \mathbb{R}^m - S$ is open

Theorem 9 (12.10). 1. Any intersection of closed sets is closed

2. The finite union of closed sets is closed

Closure, Boundary

Definition 11 (Closure). clS or \bar{S} is closure of S if It is the intersection of all closed sets containing S

Intuitively, closure is the smallest closed set contains S

Definition 12 (Boundary). \mathbf{x} is in the boundary of a set S if

$$\forall \epsilon > 0, \quad B_{\epsilon}(\mathbf{x}) \cap S \neq \emptyset \quad \wedge \quad B_{\epsilon}(\mathbf{x}) \cap S^c \neq \emptyset$$

Theorem 10 (12.12). Boundary of $S = clS \cap clS^c$

5 Compact Sets

Bounded, Compact

Definition 13 (bounded). $S \in \mathbb{R}^n$ is bounded if:

$$\exists b \in \mathbb{R} \quad s.t. \quad ||\mathbf{x}|| \leq b \quad \forall \mathbf{x} \in S$$

Definition 14 (Compact). $S \in \mathbb{R}^n$ is compact iff S is closed and bounded

Theorem 11 (12.13-14). • Any sequence contained in the compact set $[0, 1]$ has a convergent subsequence (Th 12.13)

- Any sequence contained in the compact set $C \in \mathbb{R}^n$ has a convergent subsequence whose limit lies in C (Bolzano-Weierstrass Theorem)