

Implicit Functions and Their Derivatives

Ch.15

`econMath.namun+2016su@gmail.com`

May 23, 2017

Table of Contents

- 1 Implicit Functions
- 2 Level Curves and Their Tangents
- 3 Systems of Implicit Functions

Explicit Function, Implicit Function

Explicit Function

$$x_{n+1} = x_{n+1}(\mathbf{x})$$

In explicit functions, all input $\mathbf{x} = (x_1, \dots, x_n)$ are free (or exogenous) variables. In this form, exogenous variables (x_1, \dots, x_n) and endogenous variable (x_{n+1}) can be distinguished easily.

Implicit Function

Let $x_{n+1} = x_{n+1}(\mathbf{x})$. Then, we can find alternative representation

$$G = G(\mathbf{x}, x_{n+1}) = 0$$

G is not a function but an equation (implicit equation). In this representation, x_{n+1} is an implicit function of the exogeneous variables $\mathbf{x} = (x_1, \dots, x_n)$. In this form, we can not distinguish easily between exogenous and endogenous variables.

Implicit Functions: Example

Representing Implicit Function by Explicit Function(s)

$$G(x, y) = x^2 + y^2 - 1 = 0$$

y can be an implicit function of x . On the other hand, x also can be an implicit function of y .

$$y = \begin{cases} \sqrt{1-x^2}, & y \geq 0 \\ -\sqrt{1-x^2}, & y < 0 \end{cases}$$

$$x = \begin{cases} \sqrt{1-y^2}, & x \geq 0 \\ -\sqrt{1-y^2}, & x < 0 \end{cases}$$

We cannot find well-defined functional relationship on the boundary of these explicit functions.

The Implicit Function Theorem (IFT) for \mathbb{R}^2

Main Question

- 1 Does $G(x, y) = \bar{c}$ determine y as a well-defined continuous function of x for around \bar{x}_0 and \bar{y}_0 ?
- 2 If (1) is true, $y' = \frac{\partial y}{\partial x} = ?$

We can get IFT on \mathbb{R}^2 by differentiating $G(x, y(x)) = \bar{c}$ with regard to x at \bar{x}_0 (Use Chain Rule I: Th14.1)

Chain Rule I

Let $g(t) = f(\mathbf{x}(t))$, $g : \mathbb{R} \rightarrow \mathbb{R}$, $f : \mathbb{R}^n \rightarrow \mathbb{R}$, $\mathbf{x} : \mathbb{R} \rightarrow \mathbb{R}^n$. Then,

$$\frac{dg}{dt} = Df_{\mathbf{x}(\mathbf{x})} \frac{\mathbf{x}(t)}{dt} = \frac{\partial f}{\partial x_1} \frac{dx_1}{dt} + \cdots + \frac{\partial f}{\partial x_n} \frac{dx_n}{dt}$$

Theorem (15.1 (IFT))

Let $G(x, y)$ be a C^1 function on $B_\epsilon(\bar{x}_0, \bar{y}_0)$ in \mathbb{R}^2 . Suppose that $G(\bar{x}_0, \bar{y}_0) = \bar{c}$ and consider the implicit equation

$$G(x, y) = \bar{c}$$

If $\frac{\partial G}{\partial y}(\bar{x}_0, \bar{y}_0) \neq 0$, (i.e., tangent line is not vertical) then $\exists y = y(x) \in C^1$ on $I = I_\epsilon(\bar{x}_0)$ s.t.,

① $G(x, y(x)) \equiv \bar{c} \quad \forall x \in I$

② $y(\bar{x}_0) = \bar{y}_0$

③ and

$$y'(\bar{x}_0) = -\frac{\frac{\partial G}{\partial x}(\bar{x}_0, \bar{y}_0)}{\frac{\partial G}{\partial y}(\bar{x}_0, \bar{y}_0)}$$

We can extend IFT on \mathbb{R}^n

Theorem (15.2)

Let $G(\mathbf{x}, f)$ be a C^1 function on $B_\epsilon(\overline{\mathbf{x}}_0, \overline{f}_0)$ in \mathbb{R}^n . Suppose that $G(\overline{\mathbf{x}}_0, \overline{f}_0) = \overline{c}$ and consider the implicit equation

$$G(\mathbf{x}, f) = \overline{c}$$

If $\frac{\partial G}{\partial f}(\overline{\mathbf{x}}_0, \overline{f}_0) \neq 0$ (i.e., tangent hyperplane is not vertical), then $\exists f = f(\mathbf{x}) \in C^1$ on $B = B_\epsilon(\overline{\mathbf{x}}_0)$ s.t.,

- ① $G(\mathbf{x}, f(\mathbf{x})) \equiv \overline{c} \quad \forall \mathbf{x} \in B$
- ② $f(\overline{\mathbf{x}}_0) = \overline{f}_0$
- ③ and

$$\frac{\partial f}{\partial x_i}(\overline{\mathbf{x}}_0) = -\frac{\frac{\partial G}{\partial x_i}(\overline{\mathbf{x}}_0, \overline{f}_0)}{\frac{\partial G}{\partial f}(\overline{\mathbf{x}}_0, \overline{f}_0)} \quad \forall i$$

Theorem (15.3)

Let (x_0, y_0) is on the $G(x, y) = \bar{c}$ in the plane and $G \in C^1$.

Case 1 If $\frac{\partial G}{\partial y}(x_0, y_0) \neq 0$, $\exists y = y(x) \in C^1$ around $x = x_0$ with slope

$$-\frac{\frac{\partial G}{\partial x}(\bar{x}_0, \bar{y}_0)}{\frac{\partial G}{\partial y}(\bar{x}_0, \bar{y}_0)}$$

Case 2 If $\frac{\partial G}{\partial y}(x_0, y_0) = 0$,

Case 2-1 If $\frac{\partial G}{\partial x}(x_0, y_0) \neq 0$, $\exists x = x(y) \in C^1$ around $y = y_0$ with slope

$$-\frac{\frac{\partial G}{\partial y}(\bar{x}_0, \bar{y}_0)}{\frac{\partial G}{\partial x}(\bar{x}_0, \bar{y}_0)}$$

Case 2-2 If $\frac{\partial G}{\partial x}(x_0, y_0) = 0$, there is no well-defined function around (x_0, y_0) (irregular point)

Definition (Regular Point)

(x_0, y_0) is a regular point of the $G(x, y) \in C^1$ if:

$$DG_{(x,y)}(x_0, y_0) = \left(\frac{\partial G}{\partial x}(x_0, y_0), \frac{\partial G}{\partial y}(x_0, y_0) \right) \neq \mathbf{0} = (0, 0)$$

We can find well-defined explicit function form around regular point.
Geometrically, this implies smooth curve (or 1d manifold, 1d object) in \mathbb{R}^2

Theorem (15.4)

Let $G \in C^1$ around (x_0, y_0) and this point is regular. Then, $\nabla G(x_0, y_0)$ is perpendicular to the level set of G at (x_0, y_0)

$$\nabla G(x_0, y_0) \bullet \left(1, -\frac{\frac{\partial G}{\partial x}(\bar{x}_0, \bar{y}_0)}{\frac{\partial G}{\partial y}(\bar{x}_0, \bar{y}_0)} \right) = 0$$

Extention to \mathbb{R}^n Space

Definition (Regular Point on \mathbb{R}^n)

\mathbf{x}_0 is a regular point of the $G(\mathbf{x}) \in C^1$ if:

$$\nabla G(\mathbf{x}_0) = DG_{\mathbf{x}}(\mathbf{x}_0) \neq \mathbf{0}$$

We can find well-defined explicit function form around regular point. Geometrically, this implies smooth hypersurface (or $n - 1$ dimensional manifold, $n - 1$ dimensional object) in \mathbb{R}^n

Theorem (15.6)

If $f : \mathbb{R}^n \rightarrow \mathbb{R} \in C^1$, $\mathbf{x}^* \in \mathbb{R}^n$, and $\nabla f(\mathbf{x}^*) \neq \mathbf{0}$, Then:

- 1 The level set of f through \mathbf{x}^* ,

$$\mathcal{F}_{f(\mathbf{x}^*)} \equiv \{\mathbf{x} : f(\mathbf{x}) = f(\mathbf{x}^*)\}$$

can be viewed as the graph of real-valued C^1 function of $(n - 1)$ variables in a neighborhood of \mathbf{x}^*

- 2 $\nabla f(\mathbf{x}^*)$ is perpendicular to the tangent hyperplane of $\mathcal{F}_{f(\mathbf{x}^*)}$ at \mathbf{x}^*
- 3 \mathbf{v} is a tangent vector of $\mathcal{F}_{f(\mathbf{x}^*)}$ at \mathbf{x}^* iff $Df_{\mathbf{x}}(\mathbf{x}^*) \bullet \mathbf{v} = 0$

Systems of Implicit Functions

Definition (System of implicit functions)

A set of m equations in $m + n$ unknowns

$$\mathbf{f}(x_1, \dots, x_{m+n}) = \mathbf{c} \in \mathbb{R}^m$$

is called a system of implicit functions if there is a partition of the variables into n exogenous variables and m endogenous variables, so that if exogenous variables are given, the resulting system can be solved uniquely.

By linearization, we can solve df_1, \dots, df_m from given dx_1, \dots, dx_n around $(\mathbf{f}, \mathbf{x}) = (\mathbf{f}^*, \mathbf{x}^*)$

Linearized System

We can get a linearized system from nonlinear system

$$F_1(f_1, \dots, f_m, x_1, \dots, x_n) = \bar{c}_1$$

$$F_2(f_1, \dots, f_m, x_1, \dots, x_n) = \bar{c}_2$$

...

$$F_m(f_1, \dots, f_m, x_1, \dots, x_n) = \bar{c}_m$$

by taking derivative on a given point $(\mathbf{f}, \mathbf{x}) = (\mathbf{f}^*, \mathbf{x}^*)$,

$$\frac{\partial F_1}{\partial f_1} df_1 + \dots + \frac{\partial F_1}{\partial f_m} df_m + \frac{\partial F_1}{\partial x_1} dx_1 + \dots + \frac{\partial F_1}{\partial x_n} dx_n = 0$$

\vdots

\vdots

$$\frac{\partial F_m}{\partial f_1} df_1 + \dots + \frac{\partial F_m}{\partial f_m} df_m + \frac{\partial F_m}{\partial x_1} dx_1 + \dots + \frac{\partial F_m}{\partial x_n} dx_n = 0$$

Solving Linearized System

Solving Procedure

$$\begin{aligned}\frac{\partial F_1}{\partial f_1}df_1 + \cdots + \frac{\partial F_1}{\partial f_m}df_m &= - \left(\frac{\partial F_1}{\partial x_1}dx_1 + \cdots + \frac{\partial F_1}{\partial x_n}dx_n \right) \\ \vdots \\ \frac{\partial F_m}{\partial f_1}df_1 + \cdots + \frac{\partial F_m}{\partial f_m}df_m &= - \left(\frac{\partial F_m}{\partial x_1}dx_1 + \cdots + \frac{\partial F_m}{\partial x_n}dx_n \right)\end{aligned}$$

In this system, $d\mathbf{f}$ is unknown and others are given explicitly. Therefore,

$$d\mathbf{f} = -(D\mathbf{F}_{\mathbf{f}}(\mathbf{f}^*, \mathbf{x}^*))^{-1} \cdot D\mathbf{F}_{\mathbf{x}}(\mathbf{f}^*, \mathbf{x}^*)d\mathbf{x}$$

and when $d\mathbf{x} = d\mathbf{x}^*$, $\mathbf{f} = \mathbf{f}^* + d\mathbf{f}$