

# Constrained Optimization (I): FOC

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## 1 Constrained Optimization

### Terms

#### General Max(Min)imization Problem

$$\arg \max_{\mathbf{x}} f(\mathbf{x}) \quad s.t. \quad \begin{cases} \mathbf{G}(\mathbf{x}) \leq \bar{\mathbf{b}} \\ \mathbf{H}(\mathbf{x}) = \bar{\mathbf{a}} \end{cases}$$

- $f$ : Object function
- $\mathbf{G}(\mathbf{x}) \leq \bar{\mathbf{b}}$ : Inequality Constraints
- $\mathbf{H}(\mathbf{x}) = \bar{\mathbf{a}}$ : Equality Constraints

Note: From now on, we define inequality of vector as:

$$G_i(\mathbf{x}) \leq \bar{b}_i \quad \forall i$$

Note2: For minimization problem, use  $\arg \min$  instead.

### Examples

#### Utility Maximization Problem

$$\arg \max_{\mathbf{x}} U(\mathbf{x}) \quad s.t. \quad \mathbf{p} \bullet \mathbf{x} \leq Income \wedge \mathbf{x} \geq \mathbf{0}$$

- $U$ : Utility function  $\mathbf{x}$ : consumption bundle  $\mathbf{p}$ : price vector

#### Profit Maximization Problem of a Competitive Firm

$$\arg \max_{\mathbf{x}} \Pi(\mathbf{x})$$

$$\Pi(\mathbf{x}) := \bar{p}f(\mathbf{x}) - \mathbf{w} \bullet \mathbf{x}, \quad \Pi \geq 0 \wedge \mathbf{x} \geq \mathbf{0}$$

- $f$ : production function,  $p$ : price of final product
- $\mathbf{x}$ : quantity bundle of factors for production
- $\mathbf{w}$ : price vector of each factor

## 2 Equality Constraints

**FOC of Constrained Max(Min)imization of  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$**

**Theorem 1** (18.1). *Let  $f, h$  be  $C^1$  functions of  $\mathbf{x} \in \mathbb{R}^2$ . Suppose  $\mathbf{x}^*$  is a solution of the max(min)imization problem*

$$\arg \max_{\mathbf{x}} f(\mathbf{x}) \quad s.t. \quad h(\mathbf{x}) = \bar{a}$$

*If  $\mathbf{x}^*$  is not a critical point of  $h$ , Then  $\exists(\mathbf{x}^*, \mu^*)$  s.t.*

$$L(\mathbf{x}, \mu) := f(\mathbf{x}) + \mu(\bar{a} - h(\mathbf{x})) \quad \wedge \quad DL_{\mathbf{x}, \mu} = \mathbf{0}$$

Geometrically,  $L$  (Lagrangian function) comes from the fact that if  $\mathbf{x}^*$  is a solution of max(min)imization problem, then both gradient vectors of  $f, h$  on  $\mathbf{x}^*$  should be (1) perpendicular to the level sets of  $f, h$  respectively and (2) the level sets of  $f, h$  have the same slope at  $\mathbf{x}^*$

$$\exists \mu^* \quad s.t. \quad \nabla f(\mathbf{x}^*) = \mu^* \nabla h(\mathbf{x}^*) \quad \wedge \quad \nabla h(\mathbf{x}^*) \neq \mathbf{0}$$

### Constraint Qualification

#### Constraint Qualification (CQ)

If  $Dh_{\mathbf{x}}(\mathbf{x}^*) = \mathbf{0}$  (i.e., there is critical point of constraint function), we cannot use Th 18.1. This condition is constraint qualification (CQ). If we have points which can not pass the constraint qualification, we should include these points among our candidates for a solution to the original constrained maximization problem, along with the critical points of  $L$

#### Nondegenerate CQ (NDCQ)

If there are  $m$  equality constraints, NDCQ is  $DH_{i\mathbf{x}}(\mathbf{x}^*) \neq \mathbf{0}$  for all  $i = 1, \dots, m$  and this condition should be valid even in its row echelon form (REF). Generally, NDCQ implies

$$\text{rank} DH_{\mathbf{x}}(\mathbf{x}^*) = m$$

Note: NDCQ is a regularity condition: passing NDCQ implies that the constraint set has a well-defined  $(n - m)$  dimensional tangent hyperplane at  $\mathbf{x}^*$ .

### FOCs of General Equality Constraints

We can extend Th 18.1 to general FOCs of constrained max(min)imization problem

#### General Max(Min)imization problem with Equality Constraints

$$\arg \max_{\mathbf{x}} f(\mathbf{x}) \quad s.t. \quad \mathbf{H}(\mathbf{x}) = \bar{\mathbf{a}}$$

Here equality constraints  $\mathbf{H}(\mathbf{x}) = \bar{\mathbf{a}}$  means:

$$\begin{aligned} H_1(\mathbf{x}) &= a_1 \\ &\vdots \\ H_m(\mathbf{x}) &= a_m \end{aligned}$$

## FOCs of General Equality Constraints

**Theorem 2** (18.2). Let  $f, \mathbf{H}$  be  $C^1$  functions of  $n$  variables (i.e.,  $\mathbf{x} \in \mathbb{R}^n$ ,  $\mathbf{H} : \mathbb{R}^n \rightarrow \mathbb{R}^m$ ). Consider the max(min)imization problem with  $m$  equality constraints:

$$\arg \max_{\mathbf{x}} f(\mathbf{x}) \quad \text{s.t.} \quad \mathbf{H}(\mathbf{x}) = \bar{\mathbf{a}}$$

Suppose (1)  $\mathbf{H}(\mathbf{x}^*) = \bar{\mathbf{a}}$ , (2)  $\mathbf{x}^*$  is a local max (or min) of  $f$  on  $\mathbf{H}(\mathbf{x}) = \bar{\mathbf{a}}$  and (3)  $\text{rank} D\mathbf{H}_{\mathbf{x}}(\mathbf{x}^*) = m$  (NDCQ). Then,  $(\mathbf{x}^*, \mu^*) \in \mathbb{R}^{n+m}$  is a critical point of the Lagrangian

$$L(\mathbf{x}, \mu) = f(\mathbf{x}) + \mu(\bar{\mathbf{a}} - \mathbf{H}(\mathbf{x}))$$

I.e.,

$$DL_{\mathbf{x}, \mu}(\mathbf{x}^*, \mu^*) = \mathbf{0}$$

Note:  $\mu = (\mu_1, \dots, \mu_m) \in \mathbb{R}^m$

## 3 Inequality Constraints

### One Inequality Constraints

#### Inequality Constraints: Main Concept

Inequality constrained solution = [Equality Constrained solution] (corner solution, binding) or [Unconstrained solution] (internal solution, not binding)

**Theorem 3** (18.3).  $f, g$  are  $C^1$  function on  $\mathbb{R}^2$  and  $\mathbf{x}^*$  max(min)imizes  $f$  on the inequality constraint set  $g(\mathbf{x}) \leq b$ . If  $g(\mathbf{x}^*) = b$ , and  $Dg_{\mathbf{x}}(\mathbf{x}^*) \neq \mathbf{0}$ , There is a multiplier  $\lambda^*$  satisfying:

1.  $L(\mathbf{x}, \lambda) := f(\mathbf{x}) + \lambda(\bar{b} - g(\mathbf{x}))$
2.  $DL_{\mathbf{x}, \lambda}(\mathbf{x}^*, \lambda) = \mathbf{0}$
3.  $\lambda^*(\bar{b} - g(\mathbf{x}^*)) = 0$
4.  $\lambda^* \geq 0$
5.  $\bar{b} - g(\mathbf{x}^*) \geq 0$

### General Inequality Constraints

**Theorem 4** (18.4). Suppose  $f, \mathbf{G}$  are  $C^1$  functions of  $n$  variables ( $\mathbf{G} : \mathbb{R}^n \rightarrow \mathbb{R}^k$ ). Suppose  $\mathbf{x}^* \in \mathbb{R}^n$  is a local max(min)imizer of  $f$  on the constraint set defined by the  $k$  inequalities  $\mathbf{G} \leq \bar{\mathbf{b}}$ . If (1)  $k_0$  constraints are binding at  $\mathbf{x}^*$  and the other  $k - k_0$  constraints are not binding, and (2)  $\text{rank} D\mathbf{G}_{\mathbf{0}\mathbf{x}}(\mathbf{x}^*) = k_0$  ( $\mathbf{G}_0$ : binding inequality constraints) (NDCQ). Then,  $\exists \lambda^*$  satisfying:

1.  $L(\mathbf{x}, \lambda) := f(\mathbf{x}) + \lambda(\bar{\mathbf{b}} - \mathbf{G}(\mathbf{x}))$
2.  $DL_{\mathbf{x}}(\mathbf{x}^*, \lambda^*) = \mathbf{0}$
3.  $\lambda_i^*(\bar{b}_i - G_i(\mathbf{x}^*)) = 0 \quad \forall i = 1, 2, \dots, k$
4.  $\lambda \geq \mathbf{0}$
5.  $\bar{\mathbf{b}} - \mathbf{G}(\mathbf{x}^*) \geq \mathbf{0}$

Note:  $\lambda := (\lambda_1, \dots, \lambda_k) \in \mathbb{R}^k$ . When  $i$ th constraint is not binding,  $\lambda_i = 0$  (like unconstraint) and when  $j$ th constraint is binding,  $\lambda_j > 0$  (like equality constraint).

## 4 Mixed Constraints

### Mixed Constraints

**Theorem 5** (18.5-1). Suppose  $f, \mathbf{H}, \mathbf{G}$  are  $C^1$  functions of  $n$  variables. Suppose  $\mathbf{x} \in \mathbb{R}^n$  is a local max(min)imizer of  $f$  on the constraint set defined  $m$  equalities and  $k$  inequalities:

$$\arg \max_{\mathbf{x}} f(\mathbf{x}) \quad s.t. \quad \begin{cases} \mathbf{H}(\mathbf{x}) = \bar{\mathbf{a}} \\ \mathbf{G}(\mathbf{x}) \leq \bar{\mathbf{b}} \end{cases}$$

Assume  $k_0$  inequality constraints are binding at  $\mathbf{x}^*$  and the other  $k - k_0$  inequality constraints are not binding at  $\mathbf{x}^*$ . And suppose that NDCQ is satisfied ( $\mathbf{G}_0$ : binding inequality constraints):

$$\text{rank} D \begin{pmatrix} \mathbf{G}_0 \\ \mathbf{H} \end{pmatrix}_{\mathbf{x}} (\mathbf{x}^*) = k_0 + m$$

### Mixed Constraints (2)

**Theorem 6** (18.5-2). Then,  $\exists \mu^* \in \mathbb{R}^m, \lambda^* \in \mathbb{R}^k$  satisfying:

1.  $L(\mathbf{x}, \mu, \lambda) := f(\mathbf{x}) + \mu(\bar{\mathbf{a}} - \mathbf{H}(\mathbf{x})) + \lambda(\bar{\mathbf{b}} - \mathbf{G}(\mathbf{x}))$
2.  $DL_{\mathbf{x}, \mu}(\mathbf{x}^*, \mu^*, \lambda^*) = \mathbf{0}$
3.  $\lambda^*(\bar{\mathbf{b}} - \mathbf{G}(\mathbf{x}^*)) = \mathbf{0}$
4.  $\lambda^* \geq \mathbf{0}$
5.  $\bar{\mathbf{b}} - \mathbf{G}(\mathbf{x}^*) \geq \mathbf{0}$

Note: When minimizing, the only difference is making  $L := f(\mathbf{x}) + \mu(\bar{\mathbf{a}} - \mathbf{H}(\mathbf{x})) - \lambda(\bar{\mathbf{b}} - \mathbf{G}(\mathbf{x}))$  (Equivalent to Th 18.6)

## 5 Constrained Minimization Problems

## 6 Kuhn-Tucker Formulation

### Kuhn-Tucker Formulation

### Kuhn-Tucker Formulation

$$\arg \max_{\mathbf{x}} f(\mathbf{x}) \quad s.t. \quad \mathbf{G}(\mathbf{x}) \leq \bar{\mathbf{b}} \quad \wedge \quad \mathbf{x} \geq \mathbf{0}$$

When Considering Kuhn-Tucker Lagrangian

$$\tilde{L} := f(\mathbf{x}) + \lambda(\bar{\mathbf{b}} - \mathbf{G}(\mathbf{x}))$$

FOCs are

- $D\tilde{L}_{\mathbf{x}}(\mathbf{x}^*, \lambda^*) \leq \mathbf{0}$
- $D\tilde{L}_{\lambda}(\mathbf{x}^*, \lambda^*) \geq \mathbf{0}$
- $x_i \frac{\partial \tilde{L}}{\partial x_i}(\mathbf{x}^*, \lambda^*) = 0$  for all  $i$
- $\lambda_i \frac{\partial \tilde{L}}{\partial \lambda_i}(\mathbf{x}^*, \lambda^*) = 0$  for all  $i$

Note: Above lagrangian does not contain  $n$  inequality constraints ( $\mathbf{x} \geq \mathbf{0}$ )