

Eigenvalues and Eigenvectors

Ch.23

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Eigenvalues

Definition (Eigenvalue)

Let $A \in M_n$. A scalar r is an eignevalue of A iff:

$$\det(A - rI) = 0$$

Theorem (23.1)

The diagonal entries a_{ii} of a diagonal matrix A are eigenvalues of A .

Theorem (23.2)

A matrix $A \in M_n$ is singular iff 0 is an eigenvalue of A .

Characteristic Polynomial

Definition (Characteristic Polynomial)

An $P_A(r)$, the n th order polynomial of variable r is an polynomial of $A \in M_n$ when:

$$P_A(r) = \det(A - rI)$$

r is eigenvalue of A if $P_A(r) = 0$

For general 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$,

$$P_A(r) = \det \begin{pmatrix} a - r & b \\ c & d - r \end{pmatrix} = r^2 - (a + d)r + ad - bc$$

$n \times n$ matrices can have at most n eigenvalues

Eigenvectors

Definition (Eigenvectors)

\mathbf{v} is an eigenvector of A if

$$\det(A - rI) = 0 \quad \wedge \quad (A - rI)\mathbf{v} = \mathbf{0}$$

or,

$$\det(A - rI) = 0 \quad \wedge \quad A\mathbf{v} = r\mathbf{v}$$

Note1: Get the simplest nonzero vector from eigenspace of A with respect to each eigenvalue

Note2: $A - rI \in M_n$ is singular iff $\exists \mathbf{v} \neq \mathbf{0}$ s.t. $(A - rI)\mathbf{v} = \mathbf{0}$ (See Ch.8)

Theorem (23.3)

Let $A \in M_n$, and $r \in \mathbb{R}$. Then, following statements are equivalent:

- ① $A - rI$ is singular
- ② $\det(A - rI) = 0$
- ③ $\exists \mathbf{v} \neq \mathbf{0}$ s.t. $(A - rI)\mathbf{v} = \mathbf{0}$
- ④ $A\mathbf{v} = r\mathbf{v}$ and $\mathbf{v} \neq \mathbf{0}$

Ex 23.6

Find the eigenvalues and eigenvectors of

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 5 & 0 \\ 3 & 0 & 2 \end{pmatrix}$$

Step 1) Get eigenvalues from characteristic polynomials

$$\det(A - rI) = 0$$

Step 2) Get eigenvectors from corresponding eigenvalues $r = 5, 4, -1$ by solving $(A - rI)\mathbf{v} = 0$

- $r = 5$
- $r = 4$
- $r = -1$

One Dimensional Linear Difference Equations

One-Dimensional Equations

$$y_{t+1} = \bar{a}y_t, \quad t \in \mathbb{N} + \{0\}$$

$$\Rightarrow y_n = \bar{a}^n \overline{y_0}$$

Note: The simplest dynamic – time dependent – model (cf. static model is time-invariant). In general, dynamic model is more difficult to solve. Above system can extend to general n -dimensional linear difference equations

$$\mathbf{z}_{t+1} = A\mathbf{z}_t, \quad \mathbf{z}_t \in \mathbb{R}^n, \quad A \in M_n$$

However, solution is similar only if system is uncoupled. If the system is coupled, transform it to uncoupled system using eigenvalues and eigenvectors.

Two Dimensional Linear Difference Equations

Two-Dimensional Equations

$$\mathbf{z}_{t+1} = A\mathbf{z}_t$$

When $\mathbf{z}_t = \begin{pmatrix} x_t \\ y_t \end{pmatrix}$ and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$,

$$x_{t+1} = \bar{a}x_t + \bar{b}y_t$$

$$y_{t+1} = \bar{c}x_t + \bar{d}y_t$$

Definition (Coupled, Uncoupled)

When $b = c = 0$, above system is uncoupled. Otherwise, above system is coupled. When $b = c = 0$,

$$\mathbf{z}_n = A^n \mathbf{z}_0 = \begin{pmatrix} a^n & 0 \\ 0 & d^n \end{pmatrix} \mathbf{z}_0$$

The Leslie Population Model

Leslie Mode: Linear Population Dynamics

$$\begin{pmatrix} x_{t+1} \\ y_{t+1} \end{pmatrix} = \begin{pmatrix} b_1 & b_2 \\ 1 - d_1 & 1 - d_2 \end{pmatrix} \begin{pmatrix} x_t \\ y_t \end{pmatrix}$$

- b_i : birth rate of agents in the i th period
- d_i : death rate of agents in the i th period
- Agents live at most 2-periods. This means $d_2 = 1$
- x_t : the number of 0-period old population
- y_t : the number of 1-period old population

Ex23.7: $b_1 = 1, b_2 = 4, d_1 = 0.5$

M1 Transform to uncoupled system by ERO

M2 Find P s.t. $P^{-1}AP$ is a diagonal matrix (diagonalize)

General Two-Dimensional Systems

General Linear Difference Equation

$$\mathbf{z}_{t+1} = A\mathbf{z}_t$$

Let $\mathbf{z}_t = P\mathbf{Z}_t$ or $\mathbf{Z}_t = P^{-1}\mathbf{z}_t$. Then,

$$\mathbf{Z}_{t+1} = P^{-1}AP\mathbf{Z}_t$$

Let r_1, r_2 be eigenvalues of A and $\mathbf{v}_1, \mathbf{v}_2$ be corresponding eigenvectors (2×1 matrix). If $P = (\mathbf{v}_1 \ \mathbf{v}_2)$,

$$A(\mathbf{v}_1 \ \mathbf{v}_2) = (\mathbf{v}_1 \ \mathbf{v}_2) \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix} \iff A\mathbf{v}_i = r_i\mathbf{v}_i \quad \forall i \quad (\text{Th23.3})$$

This leads to:

$$P^{-1}AP = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix}$$

General k -Dimensional Systems

Theorem (23.4)

Let A be $k \times k$ matrix. Let r_i be k eigenvalues of A , and \mathbf{v}_i be the corresponding eigenvectors. Form the matrix

$$P = (\mathbf{v}_1 \quad \cdots \quad \mathbf{v}_k)$$

If $\exists P^{-1}$,

$$P^{-1}AP = \begin{pmatrix} r_1 & 0 & \cdots & 0 \\ 0 & r_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r_k \end{pmatrix}$$

Note: $\exists P^{-1}$ means that \mathbf{v}_i s are linearly independent

The Powers of Diagonalized Matrix

Theorem (23.7)

Let A be a $k \times k$ matrix. Suppose that there is a nonsingular (invertible) matrix P s.t.

$$P^{-1}AP = \begin{pmatrix} r_1 & 0 & \cdots & 0 \\ 0 & r_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r_k \end{pmatrix} = D \quad (\text{Jordan Canonical Form})$$

Then,

$$A^n = PD^nP^{-1}$$

And the solution of the corresponding system of difference equations $\mathbf{z}_{t+1} = A\mathbf{z}_t$ with given initial vector \mathbf{z}_0 is:

$$\begin{pmatrix} r_1^n & 0 & \cdots & 0 \\ 0 & r_2^n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r_k^n \end{pmatrix}$$

Dynamic Stability

Definition (Asymptotic Stability)

\mathbf{z}_t is asymptotically stable if:

$$\lim_{n \rightarrow \infty} \mathbf{z}_n = \mathbf{0}$$

Theorem (23.8)

If $A \in M_k$ has k distinct real eigenvalues r_i , every solution of the general system of linear difference equation is asymptotically stable iff $|r_i| < 1 \ \forall i$

$$\mathbf{z}_{t+1} = A\mathbf{z}_t \quad \wedge \quad \lim_{n \rightarrow \infty} \mathbf{z}_n = \mathbf{0} \quad \Longleftrightarrow \quad |r_i| < 1 \quad \forall i$$

Properties of Eigenvalues

Definition (Trace)

Let a_{ii} be i, i th element of $A \in M_k$.

$$\text{trace} A := \sum_i^k a_{ii}$$

Theorem (23.9)

Let $A \in M_k$ with eigenvalues r_1, \dots, r_k . Then,

- ① $\sum_i^k r_i = \text{trace} A$
- ② $\prod_i^k r_i = \det A$

Repeated Eigenvalues

Definition (Defective, Nondiagonalizable)

$A \in M_k$ is defective (or nondiagonalizable) if $\nexists P$ such that diagonalize A

Definition (Generalized Eigenvector)

Let r^* be an eigenvalue of the matrix A . A vector $\mathbf{v} \neq \mathbf{0}$ such that $(A - r^*I)\mathbf{v} \neq \mathbf{0}$ and $(A - r^*I)^m \mathbf{v} = \mathbf{0}$ for some integer $m > 1$ is generalized eigenvector for A corresponding to r^*

When $A \in M_2$

Theorem (23.11)

Let $A \in M_2$ with repeated eigenvalues r^* . Then,

- ① $A = r^*I$, or
- ② A has only one independent eigenvector (say \mathbf{v}_1). In this case, there is a generalized eigenvector \mathbf{v}_2 such that $(A - r^*I)\mathbf{v}_2 = \mathbf{v}_1$. If $P = (\mathbf{v}_1 \ \mathbf{v}_2)$,

$$P^{-1}AP = \begin{pmatrix} r^* & 1 \\ 0 & r^* \end{pmatrix}$$

Theorem (23.12)

If A is the case 2 in theorem 23.11, general solution of the system of difference equations $\mathbf{z}_{t+1} = A\mathbf{z}_t$ is:

$$\mathbf{z}_n = (z_{1,0}r^n + nr^{n-1}z_{2,0})\mathbf{v}_1 + r^n z_{2,0}\mathbf{v}_2$$

Generalized Eigenvector: Example

Example: Jordan Canonical Forms

When $A \in M_4$, there are four cases of repeated eigenvectors

- ① r_1, r_2, r_3, r_3 (2 repeated eigenvectors)
- ② r_1, r_2, r_2, r_2 (3 repeated eigenvectors)
- ③ r_1, r_1, r_1, r_1 (4 repeated eigenvectors)
- ④ r_1, r_1, r_2, r_2 (two 2 repeated eigenvectors)

$$(1) \begin{pmatrix} r_1 & 0 & 0 & 0 \\ 0 & r_2 & 0 & 0 \\ 0 & 0 & r_3 & 1 \\ 0 & 0 & 0 & r_3 \end{pmatrix}, \quad (2) \begin{pmatrix} r_1 & 0 & 0 & 0 \\ 0 & r_2 & 1 & 0 \\ 0 & 0 & r_2 & 1 \\ 0 & 0 & 0 & r_2 \end{pmatrix},$$

$$(3) \begin{pmatrix} r_1 & 1 & 0 & 0 \\ 0 & r_1 & 1 & 0 \\ 0 & 0 & r_1 & 1 \\ 0 & 0 & 0 & r_1 \end{pmatrix}, \quad (4) \begin{pmatrix} r_1 & 1 & 0 & 0 \\ 0 & r_1 & 0 & 0 \\ 0 & 0 & r_2 & 1 \\ 0 & 0 & 0 & r_2 \end{pmatrix}$$

Complex Eigenvalues

Theorem (23.13)

Let $A \in M_k$ with real entries. Then,

- If $r = \alpha + i\beta$ is an eigenvalue of A , so is $\bar{r} = \alpha - i\beta$.
- If $\mathbf{u} + i\mathbf{v}$ is an eigenvector for r , then $\mathbf{u} - i\mathbf{v}$ is an eigenvector for \bar{r} .
- If k is odd, A must have at least one real eigenvalue.

If there is no repeated eigenvalues, A is diagonalizable even if r is complex number.