Linear Independence

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1 Linear Independence

LI: Definition

Definition 1 (Linear Combinations, Span, Linear (in)dependence)**.** *$\mathcal{L}$ is spanned set generated by linear combination of k vectors $\mathbf{v_1}, \cdots, \mathbf{v_k}$*

$$\mathcal{L}[\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_k} \in \mathbb{R}^{\mathbf{n}}] := \left\{ \sum_i^k r_i \mathbf{v}_i : \forall r_i \in \mathbb{R} \right\}$$

$\mathbf{v_1}, \cdots, \mathbf{v_k}$ are linearly independent iff:

$$\mathcal{L}[\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_k} \in \mathbb{R}^{\mathbf{n}}] \subset \mathbb{R}^k$$

Otherwise, $\mathbf{v_1}, \cdots, \mathbf{v_k}$ are linearly dependent

Note: Different vectors can span the same space. Canonical basis $\mathbf{e_i}$ can be the representative vector (basis) for $\mathbb{R}^n$ space.

LI: Alternative Definition

Definition 2 (Linear (in)dependence: Alternative definition)**.** *$\mathbf{v_1}, \cdots, \mathbf{v_k} \in \mathbb{R}^{\mathbf{n}}$ are linearly dependent iff:*

$$\exists c_1, \cdots, c_k \neq 0 \quad s.t. \quad \sum_i^k c_i \mathbf{v_i} = \mathbf{0}$$

$\mathbf{v_1}, \cdots, \mathbf{v_k} \in \mathbb{R}^{\mathbf{n}}$ are linearly independent iff:

$$\sum_i^k c_i \mathbf{v_i} = \mathbf{0} \quad \Rightarrow \quad c_1 = \cdots = c_k = 0$$

Theorems

Theorem 1 (11.1)**.** *$\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_k} \in \mathbb{R}^{\mathbf{n}}$ are linearly dependent iff*

$$A \begin{pmatrix} c_1 \\ \vdots \\ c_k \end{pmatrix} = \mathbf{0}$$

has nonzero solution $\mathbf{c}$, where A is $n \times k$ matrix whose $C_i = \mathbf{v_i}$. i.e.,

$$A = \begin{pmatrix} \mathbf{v_1} & \mathbf{v_2} & \cdots & \mathbf{v_k} \end{pmatrix}$$

Theorem 2 (11.2)**.** $\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_n} \in \mathbb{R}^\mathbf{n}$ *are linearly independent iff*

$$\det \begin{pmatrix} \mathbf{v_1} & \mathbf{v_2} & \cdots & \mathbf{v_n} \end{pmatrix} \neq 0$$

Checking LI

Procedure: Checking Linear (in)dependence

1. Stack $\mathbf{v_i}$ to make $k \times n$ matrix, A.
2. Calculate $rank(A)$: this is the dimension of $\mathcal{L}[\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_k} \in \mathbb{R}^\mathbf{n}]$
3. $\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_k} \in \mathbb{R}^\mathbf{n}$ are linearly independent iff $rank(A) = k$. Otherwise, they are linearly dependent

Note: $rank(A^T) = rank(A)$

Theorem 3 (11.3)**.** *If $k > n$, any set of k vectors in $\mathbb{R}^n$ is linearly dependent.*

2 Spanning Sets

3 Basis and Dimension in $\mathbb{R}^n$

Basis

Definition 3 (Basis)**.** *Let $V = \mathcal{L}[\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_k} \in \mathbb{R}^\mathbf{n}]$. If $\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_k} \in \mathbb{R}^\mathbf{n}$ are linearly independent, $\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_k} \in \mathbb{R}^\mathbf{n}$ is called a basis of V. More generally, $\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_k} \in \mathbb{R}^\mathbf{n}$ forms a basis of V if:*

1. $\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_k} \in \mathbb{R}^\mathbf{n}$ *span* V
2. $\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_k} \in \mathbb{R}^\mathbf{n}$ *are linearly independent*

Theorem 4 (11.7)**.** *Every basis of $\mathbb{R}^n$ contains n vectors.*

Linear Independence and Basis

Theorem 5 (11.8)**.** *Let $\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_n} \in \mathbb{R}^\mathbf{n}$ be a collection of n vectors in $\mathbb{R}^n$. And let $n \times n$ matrix $A = \begin{pmatrix} \mathbf{v_1} & \cdots & \mathbf{v_n} \end{pmatrix}$. Then the following statements are equivalent:*

1. $\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_n} \in \mathbb{R}^\mathbf{n}$ *are linearly independent.*
2. $\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_n} \in \mathbb{R}^\mathbf{n}$ *span* $\mathbb{R}^n$
3. $\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_n} \in \mathbb{R}^\mathbf{n}$ *form a basis of* $\mathbb{R}^n$
4. $\det(A) \neq 0$