3.2 Encontre a transformada de Laplace das seguintes funções:

(a) $f(t) = 1 + 2t$

$$\mathcal{L}(1+2t) = \mathcal{L}(1) + \mathcal{L}(2t) = \frac{1}{s} + \frac{2}{s^2} = \frac{1}{s}\cdot\left(1+\frac{2}{s}\right)$$

(b) $f(t) = 3 + 7t + t^2 + \delta(t)$

$$\mathcal{L}(3+7t+t^2+\delta(t)) = \mathcal{L}(3) + \mathcal{L}(7t) + \mathcal{L}(t^2) + \mathcal{L}(\delta(t)) = \frac{3}{s} + \frac{7}{s^2} + \frac{2}{s^3} + 1$$

(c) $f(t) = e^{-t} + 2e^{-2t} + te^{-3t}$

$$\mathcal{L}(e^{-t} + 2e^{-2t} + te^{-3t}) = \mathcal{L}(e^{-t}) + \mathcal{L}(2e^{-2t}) + \mathcal{L}(te^{-3t}) = \frac{1}{(s+1)} + \frac{2}{(s+2)} + \frac{1}{(s+3)^2}$$

(d) $f(t) = (t+1)^2$

$$\mathcal{L}((t+1)^2) = \mathcal{L}(t^2+2t+1) = \mathcal{L}(t^2) + \mathcal{L}(2t) + \mathcal{L}(1) = \frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}$$

(e) $f(t) = \operatorname{senh} t$ $\quad \mathcal{L}(\operatorname{senh} t) = \frac{h}{s^2-1}$

3.3 Encontre a transformada de Laplace das seguintes funções:

(a) $f(t) = 3\cos 6t$

$$\mathcal{L}(3\cos 6t) = 3\cdot\frac{s}{s^2+6^2} = \frac{3s}{s^2+36}$$

(b) $f(t) = \operatorname{sen} 2t + 2\cos 2t + e^{-t}\operatorname{sen} 2t$

$$\mathcal{L}(\operatorname{sen} 2t + 2\cos 2t + e^{-t}\operatorname{sen} 2t) = \mathcal{L}(\operatorname{sen} 2t) + 2\mathcal{L}(\cos 2t) + \mathcal{L}(e^{-t}\operatorname{sen} 2t)$$

$$\frac{2}{s^2+2^2} + 2\cdot\frac{s}{s^2+2^2} + \frac{2}{(s+1)^2+2^2}$$

$$\frac{2}{s^2+4} + \frac{2s}{s^2+4} + \frac{2}{(s+1)^2+4}$$

(c) $f(t) = t^2 + e^{-2t}$ sen $3t$

$$\mathcal{L}(t^2 + e^{-2t}\,\text{sen}\,3t) = \mathcal{L}(t^2) + \mathcal{L}(e^{-2t}\,\text{sen}\,3t) = \frac{2}{s^3} + \frac{3}{(s+2)^2+9}$$

3.5 Encontre a transformada de Laplace das seguintes funções

(a) $f(t) = \text{sen}\, t\, \text{sen}\, 3t$

$$\mathcal{L}(\text{sen}\,t \cdot \text{sen}\,3t) = \frac{1}{2}\mathcal{L}(\cos(t-3t) - \cos(3t+t))$$

$$\frac{1}{2}\mathcal{L}(\cos(-2t) - \cos(4t)) = \frac{1}{2}\cdot\left(\frac{s}{s^2+4} - \frac{s}{s^2+16}\right)$$

(b) $f(t) = \text{sen}^2\, t + 3\cos^2 t$

$$\mathcal{L}(\text{sen}^2 t + 3\cos^2 t) = \frac{1}{2}\left(\mathcal{L}(\cancel{\cos 0t}^{\,1} - \cos 2t) + 3\mathcal{L}(\cancel{\cos 0t}^{\,1} + \cos 2t)\right)$$

$$\frac{1}{2}\left(\frac{1}{s} - \frac{s}{s^2+4} + 3\left(\frac{1}{s} + \frac{s}{s^2+4}\right)\right)$$

$$\frac{2}{s} + \frac{s}{s^2+4} = \frac{2(s^2+4)+s^2}{s\cdot(s^2+4)} = \frac{3s^2+8}{s\cdot(s^2+4)}$$

(c) $f(t) = (\text{sen}\, t)/t$

$$\mathscr{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st}\,dt$$

$$\mathcal{L}\{f(t)\} = F(s) \Rightarrow \mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_0^{\infty} F(u)\,du$$

$f(t) = \text{sen}\,t$

$$\mathcal{L}\{\text{sen}\,t\} = \frac{1}{s^2+1} \Rightarrow \mathcal{L}\left\{\frac{\text{sen}\,t}{t}\right\} = \int_0^{\infty}\frac{1}{u^2+1} = \frac{\pi}{2} - \tan^{-1}(s)$$

3.7 Encontre a função no domínio do tempo que corresponde a cada uma das seguintes transformadas de Laplace, utiliza expansão em frações parciais:

(a) $F(s) = \frac{2}{s(s+2)}$

$$F(s) = \frac{2}{s(s+2)} = \frac{A}{s} + \frac{B}{(s+2)} = \frac{1}{s} - \frac{1}{(s+2)} = u(t) - e^{-2t}u(t)$$

$$A(s+2) + Bs = 2 \qquad s=0 \quad A = \frac{2}{2} = 1$$

$$s=-2 \Rightarrow \quad B = \frac{-2}{2} = -1$$

(d) $F(s) = \frac{3s^2+9s+12}{(s+2)(s^2+5s+11)}$

$$F(s) = \frac{3s^2+9s+12}{(s+2)(s^2+5s+11)} = \frac{K_1}{s+2} + \frac{K_2 s + K_3}{s^2+5s+11}$$

$$K_1 = \left.\frac{3s^2+9s+12}{(s^2+5s+11)}\right|_{(s=-2)} = \frac{6}{5}$$

$$K_3 = \left.\left(\frac{\cancel{3s^2}+\cancel{9s}+12}{(\cancel{s}+2)(\cancel{s^2}+\cancel{5s}+11)} - \frac{K_1}{\cancel{s}+2}\right)\cdot 11\right|_{(s=0)} = -\frac{3}{5}$$

Para encontrar K_2:

$$\left.s\cdot\left(\frac{3s^2+9s+12}{(s+2)(s^2+5s+11)}\right)\right|_{(s\to\infty)} = \left(\frac{K_1}{s+2} + \frac{K_2 s + K_3}{s^2+5s+11}\right)\cdot s$$

$$\Rightarrow 3 = K_1 + K_2 \qquad K_2 = -K_1 + 3 = \frac{9}{5}$$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{6}{5\cdot(s+2)} + \frac{9s+3}{5\cdot(s^2+5s+11)}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{6}{5\cdot(s+2)}\right\} = \frac{6}{5}\mathcal{L}^{-1}\left\{\frac{1}{(s+2)}\right\} = \frac{6}{5}e^{-2t}$$

$$\mathcal{L}^{-1}\left\{\frac{9s+3}{5\cdot(s^2+5s+11)}\right\} = \underline{A}$$

$$s^2+5s+11 = \left(s+\frac{5}{2}\right)^2 - \frac{25}{4} + 11 = \underbrace{\left(s+\frac{5}{2}\right)^2}_{(s+a)^2} + \underbrace{\frac{19}{4}}_{b}$$

$$\mathcal{L}^{-1}\left\{\frac{9s+3}{5\cdot(s^2+5s+11)}\right\} = \frac{\frac{9}{5}\left(s+\frac{5}{2}\right) - \frac{9}{5}\cdot\frac{5}{2} + \frac{3}{5}}{\left(s+\frac{5}{2}\right)^2 + \frac{19}{4}}$$

$$\frac{\frac{9}{5}\left(s+\frac{5}{2}\right) - \frac{39}{10}}{\left(s+\frac{5}{2}\right)^2 + \frac{19}{4}} = \frac{9}{5}\cdot\frac{\left(s+\frac{5}{2}\right)}{\left(s+\frac{5}{2}\right)^2+\frac{19}{4}} - \frac{39}{10}\frac{1}{\left(s+\frac{5}{2}\right)^2+\frac{19}{4}}$$

$$\mathcal{L}^{-1}\left\{\frac{9}{5}\cdot\frac{\left(s+\frac{5}{2}\right)}{\left(s+\frac{5}{2}\right)^2+\frac{19}{4}}\right\} = \frac{9}{5}\mathcal{L}^{-1}\left\{\frac{\left(s+\frac{5}{2}\right)}{\left(s+\frac{5}{2}\right)^2+\frac{19}{4}}\right\} = \frac{9}{5}e^{-\frac{5t}{2}}\cos\left(\frac{\sqrt{19}}{2}t\right)$$

$$\mathcal{L}^{-1}\left\{-\frac{39}{10}\frac{1}{\left(s+\frac{5}{2}\right)^2+\frac{19}{4}}\right\} = \frac{-39}{10}\mathcal{L}^{-1}\left\{\frac{1}{\left(s+\frac{5}{2}\right)^2+\frac{19}{4}}\right\} = \frac{-39}{10}e^{-\frac{5t}{2}}\frac{2}{\sqrt{19}}\operatorname{sen}\left(\frac{\sqrt{19}}{2}t\right)$$

$$\mathcal{L}^{-1}\left\{\frac{3s^2+9s+12}{(s+2)(s^2+5s+11)}\right\}=\frac{6}{5}e^{-2t}+e^{-\frac{5t}{2}}\left(\frac{9}{5}\cdot\cos\left(\frac{\sqrt{19}}{2}t\right)-\frac{39}{10}\cdot\frac{2}{\sqrt{19}}\,\text{sen}\left(\frac{\sqrt{19}}{2}t\right)\right)$$

(e) $F(s)=\frac{1}{s^2+4}$ $\quad \frac{b}{s^2+b^2}=\frac{2}{s^2+2^2}\cdot\frac{1}{2}=\frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+2^2}\right\}=\frac{1}{2}e^{-2t}\,\text{sen}(2t)$

$$F(s)=\frac{2(s+2)}{(s+1)(s^2+4)}=\frac{A}{(s+1)}+\frac{Bs+C}{(s^2+4)}$$

$$\frac{2(s+2)(s+1)\cancel{(s^2+4)}}{\cancel{(s+1)(s^2+4)}}=((s+1)(s^2+4))\cdot\frac{A}{(s+1)}+\frac{Bs+C}{(s^2+4)}$$

$$2(s+2)=(s^2+4)A+(s+1)(Bs+C)$$

$$2s+4=As^2+4A+Bs^2+Bs+Cs+C$$
$$2s+4=(A+B)s^2+(B+C)s+4A+C$$

$A+B=0 \quad A=-B \qquad \begin{array}{r} 4A+C=4 \\ -\;-A+C=2 \end{array} \rightarrow 5A=2$

$B+C=2 \qquad\qquad A=\frac{2}{5},\ B=-\frac{2}{5},\ C=\frac{12}{5}$

$4A+C=4$

$$\frac{2s+2}{(s^2+1)(s^2+4)}=\frac{2}{5\cdot(s+1)}+\frac{-2s+12}{5\cdot(s^2+4)}=\frac{2}{5\cdot(s+1)}-\frac{2}{5}\frac{s}{5\cdot(s^2+4)}+\frac{12}{5}\frac{1}{(s^2+4)}$$

$$\mathcal{L}^{-1}\left\{\frac{2}{5\cdot(s+1)}-\frac{2}{5}\frac{s}{5\cdot(s^2+4)}+\frac{12}{5}\frac{1}{(s^2+4)}\right\}=\frac{2}{5}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}-\frac{2}{5}\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\}+\frac{12}{5}\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\}$$

$$\mathcal{L}^{-1}\{F(s)\}=\frac{2}{5}\cdot\left(e^{-t}-\cos 2t\right)+\frac{12}{5}\cdot\frac{1}{2}\,\text{sen}\,2t$$

(i) $F(s) = \frac{4}{s^4+4}$

$$\frac{4}{s^4+4} = \frac{As+B}{(s^2+2+2s)} + \frac{Cs+D}{(s^2+2-2s)}$$

$$\frac{\cancel{(s^2+2+2s)}\cancel{(s^2+2-2s)}4}{\cancel{(s^2+2+2s)}\cancel{(s^2+2-2s)}} = \frac{\cancel{(s^2+2+2s)}(s^2+2-2s)As+B}{\cancel{(s^2+2+2s)}}$$

$$+\frac{(s^2+2+2s)\cancel{(s^2+2-2s)}Cs+D}{\cancel{(s^2+2-2s)}}$$

$$4 = (s^2+2-2s)(As+B)+(s^2+2+2s)(Cs+D)$$

$$As^3+2As-2As^2+Bs^2+2B-2Bs+Cs^3+2Cs+2Cs^2+Ds^2+2D+2Ds$$

$$(A+C)s^3+(-2A+B+2C+D)s^2-(B-A-C-D)2s+2(B+D)$$

$$\begin{cases} A+C=0 \\ -2A+B+2C+D=0 \\ 2A-2B+2C+2D=0 \\ 2B+2D=4 \end{cases}$$

$A=\frac{1}{2}$, $B=1$, $C=-\frac{1}{2}$, $D=1$

$$\frac{4}{s^4+4} = \frac{\frac{1}{2}s+1}{(s^2+2+2s)} + \frac{-\frac{1}{2}s+1}{(s^2+2-2s)}$$

$$\mathcal{L}^{-1}\left\{\frac{4}{s^4+4}\right\} = \mathcal{L}^{-1}\left\{\frac{\frac{1}{2}s+1}{(s^2+2+2s)}\right\} + \mathcal{L}^{-1}\left\{\frac{-\frac{1}{2}s+1}{(s^2+2-2s)}\right\}$$

$$\mathcal{L}\{(K_1\cos\omega t + K_2\sin\omega t)e^{-at}\} = \frac{K_1(s+a)+K_2\omega}{(s+a)^2+\omega^2} = \frac{K_1 s+(K_1 a+K_2\omega)}{s^2+2as+a+\omega^2} = \frac{Bs+C}{s^2+c_1 s+c_2}$$

$$\mathcal{L}^{-1}\left\{\frac{\frac{1}{2}s+1}{(s^2+2+2s)}\right\} \Rightarrow K_1\cancel{s} = A\cancel{s} \Rightarrow K_1 = A = \frac{1}{2}$$

$$a^2+\omega^2 = c_2$$

$$a^2+\omega^2 = 2$$

$$\omega = \sqrt{2-1} = 1$$

$K_1 a + K_2 w = B \qquad 2as = C_1 s = 2s \Rightarrow a = 1$

$\frac{a}{2} + K_2 w = 1 \qquad\qquad K_2 = 1 - \frac{1}{2} = \frac{1}{2}$

$$\mathcal{L}^{-1}\left\{\frac{\frac{1}{2}s+1}{(s^2+2+2s)}\right\} = \frac{e^{-\frac{t}{2}}}{2}.(\cos t + \operatorname{sen} t)$$

$$\mathcal{L}^{-1}\left\{\frac{-\frac{1}{2}s+1}{(s^2+2-2s)}\right\} \Rightarrow K_1 = Cs \Rightarrow K_1 = -\frac{1}{2} \qquad 2as = -2s \Rightarrow a = -1$$

$a + w^2 = 2$

$K_1 a + K_2 w = D \qquad w^2 = 2+1 \Rightarrow w = \sqrt{3}$

$$\frac{1}{2} + \sqrt{3}K_2 = 1 \Rightarrow K_2 = \frac{1}{2\sqrt{3}}.\frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{6}$$

$$\mathcal{L}^{-1}\left\{\frac{-\frac{1}{2}s+1}{(s^2+2-2s)}\right\} = \left(-\frac{1}{2}\cos\sqrt{3}t + \frac{\sqrt{3}}{6}\operatorname{sen}\sqrt{3}t\right)e^{t}$$

$$\mathcal{L}^{-1}\left\{\frac{4}{s^4+4}\right\} = \frac{e^{-\frac{t}{2}}}{2}.(\cos t + \operatorname{sen} t) + \left(-\frac{1}{2}\cos\sqrt{3}t + \frac{\sqrt{3}}{6}\operatorname{sen}\sqrt{3}t\right)e^{t}$$

(j) $F(s) = \frac{e^{-s}}{s^2} \quad = \quad \mathscr{L}[f(t-T)] \quad = e^{-sT}F(s)$

$$e^{-s}\cdot\frac{1}{s^2} = \mathcal{L}[f(t-T)] \qquad \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t\,u(t)$$

$$\mathcal{L}^{-1}\left\{e^{-s}\cdot\frac{1}{s^2}\right\} = (t-1)\cdot u(t-1)$$

3.9 Resolva as seguintes EDOs usando a transformada de Laplace:

(a) $\ddot{y}(t) + \dot{y}(t) + 3y(t) = 0;\ y(0) = 1,\ \dot{y}(0) = 2$

$$s^2 Y(s) - sy(0) - y'(0) + sY(s) - y(0) + 3Y(s) = 0$$

$$Y(s)(s^2+s+3) - sy(0) - y'(0) - y(0) = 0$$

$$Y(s)(s^2+s+3) - 1s - 2 - 1 = 0$$

$$Y(s) = \frac{s+3}{(s^2+s+3)} \qquad K_1 \cancel{8} = A = \cancel{8} \quad K_1 = 1 \qquad a = \frac{1}{2}$$

$$a^2 + \omega^2 = c_2 \qquad \frac{1}{4} + \omega^2 = 3 \qquad \omega = \sqrt{3 - \frac{1}{4}} = \frac{\sqrt{11}}{2}$$

$$K_1 a + K_2 \omega = B \qquad \frac{1}{2} + \frac{\sqrt{11}}{2} K_2 = 3 \Rightarrow K_2 = \frac{5\sqrt{11}}{11}$$

$$\mathcal{L}^{-1}\left\{ Y(s) = \frac{s+3}{(s^2+s+3)} \right\} = e^{-\frac{t}{2}}\left(\frac{1}{2} \cdot \cos\left(\frac{\sqrt{11}}{2} t\right) + \frac{5\sqrt{11}}{11} \operatorname{sen}\left(\frac{\sqrt{11}}{2}\right)\right)$$

(b) $\ddot{y}(t) - 2\dot{y}(t) + 4y(t) = 0;\ y(0) = 1,\ \dot{y}(0) = 2$

$$s^2 Y(s) - sy(0) - y'(0) - 2(sY(s) - y(0)) + 4Y(s) = 0$$

$$s^2 Y(s) - 2sY(s) + 4Y(s) - s - 2 + 4 = 0$$

$$Y(s)(s^2-2s+4)=s-2 \qquad Y(s)=\frac{s-2}{(s^2-2s+4)}$$

$$K_1=1 \qquad a=-1$$

$$a^2+\omega^2=4 \Rightarrow \omega=\sqrt{4-1}=\sqrt{3}$$

$$K_1 a + K_2 \omega = -2 \qquad K_2=\frac{-2+1}{\sqrt{3}}=\frac{-1}{\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}}=-\frac{\sqrt{3}}{3}$$

$$\mathcal{L}^{-1}\left\{\frac{s-2}{(s^2-2s+4)}\right\}=e^{t}\left(\cos(\sqrt{3}t)-\frac{\sqrt{3}}{3}\operatorname{sen}(\sqrt{3}t)\right)$$

(e) $\ddot{y}(t)+2\dot{y}(t)=e^{t}$; $y(0)=1$, $\dot{y}(0)=2$

$$s^2Y(s)-sy(0)-y'(0)+2(sY(s)-y(0))=\frac{1}{s-1}$$

$$Y(s)(s^2+2s)-1s-2-1=\frac{1}{s-1}$$

$$Y(s)=\frac{1}{(s-1)(s^2+2s)}+\frac{s+3}{(s^2+2s)}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-1)(s+2)(s-0)}+\frac{s+3}{(s+2)(s-0)}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-1)(s+2)(s-0)}\right\}=\frac{A}{(s-1)}+\frac{B}{(s+2)}+\frac{C}{(s-0)}$$

$$s=1 \quad A=\frac{1}{3} \qquad s=-2 \quad B=\frac{1}{6} \qquad s=0 \quad C=-\frac{1}{2}$$

$$\frac{1}{3(s-1)} + \frac{1}{6(s+2)} + \frac{1}{2s} = \frac{e^{t}}{3} + \frac{e^{-2t}}{6} - \frac{u(t)}{2}$$

$$\mathcal{L}^{-1}\left\{\frac{s+3}{(s+2)(s-0)}\right\} = \frac{A}{s+2} + \frac{B}{s}$$

$s=-2 \quad A = -\frac{1}{2} \qquad s=0 \quad B = \frac{3}{2}$

$$\mathcal{L}^{-1}\left\{-\frac{1}{2}\cdot\frac{1}{s+2} + \frac{3}{2}\cdot\frac{1}{s}\right\} = \frac{-e^{-2t}+3u(t)}{2}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \frac{e^{t}}{3} + \frac{e^{-2t}}{6} - \frac{u(t)}{2} - \frac{e^{-2t}-3u(t)}{2}$$

$$\frac{2e^{t}+e^{-2t}-3u(t)-3e^{-2t}+9u(t)}{6} = \frac{-2(e^{-2t}-e^{t})+6u(t)}{6} = -\frac{(e^{-2t}-e^{t})}{3}+u(t)$$

(f) $\ddot{y}(t) + y(t) = t;\ y(0) = 1,\ \dot{y}(0) = -1$

$$s^2Y(s) - sy(0) - y'(0) + Y(s) = \frac{1}{s^2}$$

$$Y(s) = \frac{1}{s^2(s^2+1)} + \frac{s+1}{s^2+1}$$

$K_1 = 1,\ a = 0,\quad a^2+\omega^2 = 1 \Rightarrow \omega = 1 \qquad \cancel{K_1 a}^{0} + K_2\omega = 1$

$K_2 = 1$

$\cos t + \operatorname{sen} t$

$$\frac{1}{s^2(s^2+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+1}$$

$S=0$

$$\frac{\cancel{s^2(s^2+1)}\,1}{\cancel{s^2(s^2+1)}} = \cancel{s^2}(s^2+1)\frac{A}{\cancel{s}} + \cancel{s^2}(s^2+1)\frac{B}{\cancel{s^2}} + s^2\cancel{(s^2+1)}\frac{Cs+D}{\cancel{s^2+1}}$$

$$s(s^2+1)A+(s^2+1)B+s^2(Cs+D)=1$$
$$s^3A+sA+s^2B+B+Cs^3+Ds^2=1$$
$$s^3(A+C)+s^2(B+D)+As+B=1$$

$A=0,\ B=1,\ C=0,\ D=-1$

$$\frac{\cancel{A}^{\,0}}{\cancel{s}}+\frac{1}{s^2}+\frac{\cancel{C}^{\,0}s-1}{s^2+1} = \mathcal{L}^{-1}\left\{\frac{1}{s^2}-\frac{1}{s^2+1}\right\} = t - \operatorname{sen} t$$

$$t - \cancel{\operatorname{sen} t} + \cos t + \cancel{\operatorname{sen} t} = \cos(t)+t$$