Задача:

Да се изследва сходимостта на приближеното решение на МКМ на

$$\int\int_V\int z\sqrt{x^2+y^2+z^2}\ dxdydz,$$

където V се дефинира от неравенствата:

$$\sqrt{x^2+y^2} \leq z \leq \sqrt{1-x^2-y^2}$$

Упътване: Да се използват сферични координати.

Решение:

$$\left|\begin{array}{l} x = \rho\sin\psi\cos\varphi \\ y = \rho\sin\psi\sin\varphi \\ z = \rho\cos\psi \\ 0 \leq \psi \leq \pi, 0 \leq \varphi \leq 2\pi \end{array}\right.$$

Така, за първото ограничение имаме:

$$\sqrt{\rho^2\sin^2\psi\cos^2\varphi + \rho^2\sin^2\psi\sin^2\varphi} \leq \rho\cos\psi$$
$$\sqrt{\rho^2\sin^2\psi(\cos^2\varphi + \sin^2\varphi)} \leq \rho\cos\psi$$
$$\sqrt{\rho^2\sin^2\psi} \leq \rho\cos\psi$$
$$\rho\sin\psi \leq \rho\cos\varphi$$
$$\sin\psi \leq \cos\psi$$
$$\Rightarrow \psi \in [0; \frac{\pi}{4}]$$

За второто ограничение:

$$\rho\cos\psi \leq \sqrt{1-\rho^2\sin^2\psi\cos^2\varphi - \rho^2\sin^2\psi\sin^2\varphi}$$
$$\rho\cos\psi \leq \sqrt{1-\rho^2\sin^2\psi(\cos^2\varphi + \sin^2\varphi)}$$
$$\rho\cos\psi \leq \sqrt{1-\rho^2\sin^2\psi}$$
$$\rho^2\cos^2\psi \leq 1-\rho^2\sin^2\psi$$
$$1-\rho^2\sin^2\psi - \rho^2\cos^2\psi \geq 0$$
$$1-\rho^2(\sin^2\psi + \cos^2\psi) \geq 0$$
$$1-\rho^2 \geq 0 \Rightarrow \rho^2 \leq 1$$
$$\Rightarrow \rho \in [-1; 1], \text{но } \rho \geq 0 \Rightarrow \rho \in [0; 1]$$

За Якобиана J имаме $|J| = \rho^2 \sin\psi$.
Пресмятаме интеграла:

$$\int\limits_0^1 \int\limits_0^{\frac{\pi}{4}} \int\limits_0^{2\pi} \left(\rho^2 \sin\psi.\rho\cos\psi\sqrt{\rho^2\sin^2\psi\cos^2\varphi + \rho^2\sin^2\psi\sin^2\varphi + \rho^2\cos^2\psi}\right) d\varphi d\psi d\rho =$$

$$= \int\limits_0^1 \int\limits_0^{\frac{\pi}{4}} \int\limits_0^{2\pi} \left(\rho^2 \sin\psi.\rho\cos\psi\sqrt{\rho^2\sin^2\psi(\cos^2\varphi + \sin^2\varphi) + \rho^2\cos^2\psi}\right) d\varphi d\psi d\rho =$$

$$= \int\limits_0^1 \int\limits_0^{\frac{\pi}{4}} \int\limits_0^{2\pi} \left(\rho^3 \sin\psi\cos\psi\sqrt{\rho^2(\sin^2\psi + \cos^2\psi)}\right) d\varphi d\psi d\rho =$$

$$= \int\limits_0^1 \int\limits_0^{\frac{\pi}{4}} \int\limits_0^{2\pi} \left(\rho^4 \sin\psi\cos\psi\right) d\varphi d\psi d\rho = 2\pi \int\limits_0^1 \rho^4\, d\psi \int\limits_0^{\frac{\pi}{4}} (\sin\psi\cos\psi)\, d\psi =$$

$$= 2\pi \cdot \frac{\rho^5}{5}\bigg|_0^1 \cdot \frac{1}{4} = 2\pi \cdot \frac{1}{5} \cdot \frac{1}{4} = \frac{2\pi}{20} = \frac{\pi}{10} \approx 0,31415$$