

Задача:

Да се изследва сходимостта на приближеното решение на МКМ на

$$\int \int \int_V z \sqrt{x^2 + y^2 + z^2} \, dx dy dz,$$

където V се дефинира от неравенствата:

$$\sqrt{x^2 + y^2} \leq z \leq \sqrt{1 - x^2 - y^2}$$

Упътване: Да се използват сферични координати.

Решение:

$$\left| \begin{array}{l} x = \rho \sin \psi \cos \varphi \\ y = \rho \sin \psi \sin \varphi \\ z = \rho \cos \psi \\ 0 \leq \psi \leq \pi, 0 \leq \varphi \leq 2\pi \end{array} \right.$$

Така, за първото ограничение имаме:

$$\sqrt{\rho^2 \sin^2 \psi \cos^2 \varphi + \rho^2 \sin^2 \psi \sin^2 \varphi} \leq \rho \cos \psi$$

$$\sqrt{\rho^2 \sin^2 \psi (\cos^2 \varphi + \sin^2 \varphi)} \leq \rho \cos \psi$$

$$\sqrt{\rho^2 \sin^2 \psi} \leq \rho \cos \psi$$

$$\rho \sin \psi \leq \rho \cos \psi$$

$$\sin \psi \leq \cos \psi$$

$$\Rightarrow \psi \in [0; \frac{\pi}{4}]$$

За второто ограничение:

$$\rho \cos \psi \leq \sqrt{1 - \rho^2 \sin^2 \psi \cos^2 \varphi - \rho^2 \sin^2 \psi \sin^2 \varphi}$$

$$\rho \cos \psi \leq \sqrt{1 - \rho^2 \sin^2 \psi (\cos^2 \varphi + \sin^2 \varphi)}$$

$$\rho \cos \psi \leq \sqrt{1 - \rho^2 \sin^2 \psi}$$

$$\rho^2 \cos^2 \psi \leq 1 - \rho^2 \sin^2 \psi$$

$$1 - \rho^2 \sin^2 \psi - \rho^2 \cos^2 \psi \geq 0$$

$$1 - \rho^2 (\sin^2 \psi + \cos^2 \psi) \geq 0$$

$$1 - \rho^2 \geq 0 \Rightarrow \rho^2 \leq 1$$

$$\Rightarrow \rho \in [-1; 1], \text{ но } \rho \geq 0 \Rightarrow \rho \in [0; 1]$$

За Якобиана J имаме $|J| = \rho^2 \sin \psi$.

Пресмятаме интеграла:

$$\begin{aligned}
& \int_0^1 \int_0^{\frac{\pi}{4}} \int_0^{2\pi} \left(\rho^2 \sin \psi \cdot \rho \cos \psi \sqrt{\rho^2 \sin^2 \psi \cos^2 \varphi + \rho^2 \sin^2 \psi \sin^2 \varphi + \rho^2 \cos^2 \psi} \right) d\varphi d\psi d\rho = \\
& = \int_0^1 \int_0^{\frac{\pi}{4}} \int_0^{2\pi} \left(\rho^2 \sin \psi \cdot \rho \cos \psi \sqrt{\rho^2 \sin^2 \psi (\cos^2 \varphi + \sin^2 \varphi) + \rho^2 \cos^2 \psi} \right) d\varphi d\psi d\rho = \\
& = \int_0^1 \int_0^{\frac{\pi}{4}} \int_0^{2\pi} \left(\rho^3 \sin \psi \cos \psi \sqrt{\rho^2 (\sin^2 \psi + \cos^2 \psi)} \right) d\varphi d\psi d\rho = \\
& = \int_0^1 \int_0^{\frac{\pi}{4}} \int_0^{2\pi} \left(\rho^4 \sin \psi \cos \psi \right) d\varphi d\psi d\rho = 2\pi \int_0^1 \rho^4 d\rho \int_0^{\frac{\pi}{4}} (\sin \psi \cos \psi) d\psi = \\
& = 2\pi \cdot \frac{\rho^5}{5} \Big|_0^1 \cdot \frac{1}{4} = 2\pi \cdot \frac{1}{5} \cdot \frac{1}{4} = \frac{2\pi}{20} = \frac{\pi}{10} \approx 0,31415
\end{aligned}$$