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MULTIVARIATE INTERPOLATION ON UNISOLVENT NODES LIFTING THE CURSE OF DIMENSIONALITY

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ABSTRACT. We present some basic ideas being part of current developments in Multivarite Interpolation (MIP).

1. INTRODUCTION

This is just a short extract and briefly sketched outline of current ideas. I hope it can help.

1.1. **Notation.** Let $m, n \in \mathbb{N}$, $p \geq 1$. We denote by $e_1 = (1, 0, \dots, 0), \dots, e_m = (0, \dots, 0, 1) \in \mathbb{R}^m$ the standard basis, by $\|\cdot\|$ the euclidean norm on $\mathbb{R}^m$, and by $\|M\|_p$ the l_p-norm of a matrix $M \in \mathbb{R}^{m \times m}$. Further, $A_{m,n,p} \subseteq \mathbb{N}^m$ denotes all multi-indices $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{N}^m$ with $\|\alpha\|_p = (\alpha_1^p + \dots + \alpha_m^p)^{1/p} \leq n$.

We order a finite set $A \subseteq \mathbb{N}^m$, $m \in \mathbb{N}$ of multi-indices with respect to the lexicographical order $\leq_L$ on $\mathbb{N}^m$ starting from x_m to x_1, e.g. $(5,3,1) \leq_L (1,0,3) \leq_L (1,1,3)$. Thereby, $\alpha_{\min}, \alpha_{\max}$ shall denote the minimum and maximum of $A = \{\alpha_{\min}, \dots, \alpha_{\max}\}$ with respect to $\leq_L$. We call A *complete* iff there is no $\beta = (b_1, \dots, b_m) \in \mathbb{N}^m \setminus A$ with $b_i \leq a_i$, $\forall\, i = 1, \dots, m$ for some $\alpha = (a_1, \dots, a_m) \in A$. Note that $A_{m,n,p}$ is complete for all $m, n \in \mathbb{N}$, $p \geq 1$. Given $A \subseteq \mathbb{N}^m$ complete and a matrix $R_A \in \mathbb{R}^{|A| \times |A|}$ we slightly abuse notation by denoting

$$R_A = (r_{\alpha,\beta})_{\alpha,\beta \in A} = (r_{i,j})_{1 \leq i,j \leq |A|}\,, \tag{1.1}$$

with α, β being the i-th, j-th entry of A ordered by $\leq_L$, respectively.

We consider the *real polynomial ring* $\mathbb{R}[x_1, \dots, x_m]$ in m variables and denote by Π_m the $\mathbb{R}$-*vector space of all real polynomials* in m variables. Further, $\Pi_A \subseteq \Pi_m$ denotes the *polynomial subspace* induced by A and generated by the *canonical basis* given by the monomials $x^\alpha = x_1^{\alpha_1} \cdots x_m^{\alpha_m}$ with $\alpha \in A$. For $A = A_{m,n,p}$ we write $\Pi_{m,n,p}$ to mean $\Pi_{A_{m,n,p}}$. Given a polynomial $Q(x) = \sum_{\alpha \in A} c_\alpha x^\alpha$, $A \subseteq \mathbb{N}^m$, we call $\max_{\alpha \in A\,, c_\alpha \neq 0} \|\alpha\|_p$ the l_p-degree of Q. As it will turn out, the notion of l_p-degree plays a crucial role for the approximation quality of polynomial interpolation. Whatsoever, while $A_{1,n,p} = \{0, \dots, n\}$, for $m > 1$ considering $A \subseteq \mathbb{N}^m$ *generalizes the concept of polynomial degree to multi-dimensions.*

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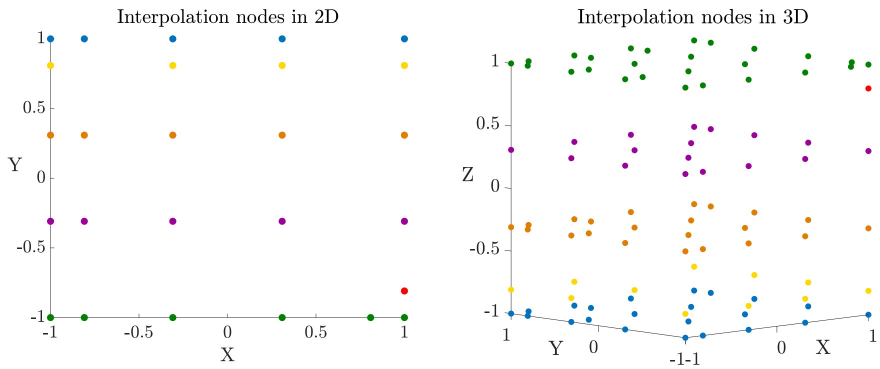

FIGURE 1. Unisolvent nodes P_A in 2D (left) and 3D (right) with respect to $A_{m,n,p}$ for dimensions $m = 2, 3$, $n = 5$, $p = 2$, and generating nodes $\mathrm{GP} = \oplus_{i=1}^m \mathrm{Cheb}_n^{\mathrm{2nd}}$. Nodes belonging to the same line/plane are colored equally.

Throughout this article $\Omega = [-1,1]^m$ denotes the m-dimensional *standard hypercube*. Finally, we use the standard *Landau symbols* $\mathcal{O}(\cdot), o(\cdot)$

$$f \in \mathcal{O}(g) \Longleftrightarrow \limsup_{x\to\infty} \frac{|f(x)|}{|g(x)|} \leq \infty\,, \qquad f \in o(g) \Longleftrightarrow \lim_{x\to\infty} \frac{|f(x)|}{|g(x)|} = 0\,.$$

2. UNISOLVENT NODES

In the following, we present a simplified and shortened version of unisolvent nodes and a specific notion of a *unisolvent grid* developed in our previous work [5]. We highly recommend to consider [5] for more detailed explanations and proofs.

Definition 2.1 (unisolvent nodes)**.** Let $m \in \mathbb{N}$ and $\Pi \subseteq \Pi_m$ be a polynomial subspace. We call a finite non–empty set $\emptyset \neq P \subseteq \mathbb{R}^m$ *unisolvent* with respect to Π iff there exists no non–zero polynomial

$$Q \in \Pi \setminus \{0\} \quad \text{with} \quad Q(p) = 0\,,\ \forall\, p \in P\,.$$

Definition 2.2 (Essential Assumptions)**.** We say that the *essential assumptions* hold with respect to $A \subseteq \mathbb{N}^m$ and $P_A \subseteq \mathbb{R}^m$, where $m \in \mathbb{N}$ and A is a complete set of multi-indices, if and only if there exist generating nodes

$$\mathrm{GP} = \oplus_{i=1}^m P_i\,, \quad P_i = \{p_{0,i}, \dots, p_{n_i,i}\} \subseteq \mathbb{R}\,,\ \ n_i = \max_{\alpha\in A}(\alpha_i)\,, \tag{2.1}$$

and the unisolvent nodes P_A are given by

$$P_A = \left\{ (p_{\alpha_1,1}\,, \dots, p_{\alpha_m,m}) \;\middle|\; \alpha \in A \right\}\,.$$

Unless further specified, the generating nodes GP are arbitrary.

In Figure 1, we illustrate examples of unisolvent nodes in two and three dimensions for the generating nodes $\mathrm{GP} = \oplus_{i=1}^m \mathrm{Cheb}_n^{\mathrm{2nd}}$, where the Chebyshev nodes $\mathrm{Cheb}_n^{\mathrm{1st}}$ of first and second kind $\mathrm{Cheb}_n^{\mathrm{2nd}}$ are defined in Eq. (2.3). For better visualization, all nodes belonging to the same line/plane are colored equally.

We will show that the unisolvent nodes P_A allow to deduce cubature formulas in any dimension. The following section provides the polynomial bases, which are needed in this regard.

Theorem 2.3 (Hecht et al.)**.** *Let $m \in \mathbb{N}$, $A \subseteq \mathbb{N}^m$ be a complete set of multi-indices, and $\Pi_A \subseteq \Pi_m$ by the polynomial sub-space induced by A. We consider the generating nodes given by the grid*

$$\mathrm{GP} = \oplus_{i=1}^m P_i\,, \quad P_i = \{p_{0,i}, \dots, p_{n_i,i}\} \subseteq \mathbb{R}\,, \;\; n_i = \max_{\alpha \in A}(\alpha_i)\,, \tag{2.2}$$

where the P_i are arbitrary finite sets. Then, the node set

$$P_A = \big\{(p_{\alpha_1,1}\,, \dots\,, p_{\alpha_m,m}) \;\big|\; \alpha \in A\big\}$$

is unisolvent with respect to Π_A.

The following nodes are of crucial importance *Chebyshev nodes of first and second kind*

$$\begin{aligned}
\mathrm{Cheb}_n^{\mathrm{1st}} &= \left\{\cos\left(\frac{2k-1}{2(n+1)}\pi\right) \;\Big|\; 1 \leq k \leq n\right\},\\
\mathrm{Cheb}_n^{\mathrm{2nd}} &= \left\{\cos\left(\frac{k\pi}{n+2}\right) \;\Big|\; 1 \leq k \leq n\right\},\\
\mathrm{Cheb}_n^{0} &= \left\{\cos\left(\frac{k\pi}{n}\right) \;\Big|\; 0 \leq k \leq n-1\right\}.
\end{aligned} \tag{2.3}$$

$\mathrm{Cheb}_n^{\mathrm{1st}}, \mathrm{Cheb}_n^{\mathrm{2nd}}$ are the roots of the associated *Chebyshev polynomials*

$$\begin{aligned}
t_{n+1}(x) &= \prod_{i=1}^{n+1}(x - p_i)\,, \;\; p_i \in \mathrm{Cheb}_n^{\mathrm{1st}}\\
u_{n+1}(x) &= \prod_{i=1}^{n+1}(x - q_i)\,, \;\; q_i \in \mathrm{Cheb}_n^{\mathrm{2nd}}\,.
\end{aligned} \tag{2.4}$$

of first and second kind, respectively. Indeed, $\mathrm{Cheb}_n^{\mathrm{1st}}$ are minimizers of the product $M_{P_n}(x) = \prod_{p \in P_n} |x - p|$, $|P_n| = n+1$, i.e.,

$$\min_{P_n \subseteq \Omega} \|M_{P_n}\|_{C^0(\Omega)} = \frac{1}{2^n} \quad \text{for } \; P_n = \mathrm{Cheb}_n^{\mathrm{1st}}\,. \tag{2.5}$$

The nodes Cheb_n^0 are the extrema of t_{n+1}, with values oscillating between $t_{n+1}(q) \in \{-1, 1\}$, $q \in \mathrm{Cheb}_n^{\mathrm{2nd}}$. Therefore, the term *Chebyshev extreme nodes* is also often used [3, 10]. **In practice, Cheb_n^0 is the best choice for yielding fast approximations, when aiming to approximate arbitrary functions.** Moreover, the following inclusions are crucial:

$$\mathrm{Cheb}_n^0 \subseteq \mathrm{Cheb}_{2n}^0\,, \quad \mathrm{Cheb}_n^{\mathrm{2nd}} \subseteq \mathrm{Cheb}_{2n+1}^{\mathrm{2nd}}\,, \quad \mathrm{Cheb}_n^{\mathrm{1st}} \subseteq \mathrm{Cheb}_{3n}^{\mathrm{1st}}$$

3. Multivariate Polynomial Bases

We recapture results of our prior work [5] to introduce the notion of *multivariate Newton and Lagrange bases* with respect to Π_A, $A \subseteq \mathbb{N}^m$. Indeed the generalized multi-dimensional notions coincide with the classic notion in dimension $m = 1$ or the *tensorial definitions* for $A_{m,n,\infty}$ [2, 3, 8, 9, 10].

Definition 3.1 (Multivariate Newton Polynomials)**.** Let the essential assumptions (Definition 2.2) be fulfilled with respect to $A \subseteq \mathbb{N}^m$ and $P_A \subseteq \mathbb{R}^m$. Then, we define the *multivariate Newton polynomials* by

$$N_\alpha(x) = \prod_{i=1}^{m} \prod_{j=0}^{\alpha_i - 1} (x_i - p_{j,i}) \,, \quad \alpha \in A \,. \tag{3.1}$$

Indeed, in dimension $m = 1$ this reduces to the classic definition of Newton polynomials [3, 9, 10].

Similar to the Newton case, our notion of unisolvent nodes also allows us to generalize the concept of Lagrange bases to multi–dimensions. For this, we define:

Definition 3.2 (Lagrange Polynomials)**.** Let the essential assumptions (Definition 2.2) be fulfilled with respect to a complete set $A \subseteq \mathbb{N}^m$ and $P_A \subseteq \mathbb{R}^m$. Then, we define the *multivariate Lagrange polynomials*

$$L_\alpha \in \Pi_A \quad \text{with} \quad L_\alpha(p_\beta) = \delta_{\alpha,\beta} \,, \;\; \alpha, \beta \in A \,, \tag{3.2}$$

where $\delta_{\cdot,\cdot}$ is the Kronecker delta.

Indeed, by considering $A = A_{m,n,\infty}$, $\alpha \in A$ and a regular grid P_A the definition recovers the notion of *tensorial mD Lagrange polynomials* [2, 8].

$$L_\alpha(x) = \prod_{i=1}^{m} l_{\alpha_i}(x) \,, \quad l_{\alpha_i}(x) = \prod_{j=0, j\neq\alpha_i}^{n} \frac{x_i - p_{j,i}}{p_{\alpha_i,i} - p_{j,i}} \,, \quad p_{i,j} \in P_i \,.$$

By summarizing results from [5] the following theorems generalize the classic facts known for 1D Newton and Lagrange polynomials.

Theorem 3.3 (Hecht et al.)**.** *Let the essential assumptions (Definition 2.2) be fulfilled with respect to any complete set of multi–indices $A \subseteq \mathbb{N}^m$ and $P_A \subseteq \mathbb{R}^m$. Then the multivariate Newton polynomials $\{N_\alpha\}_{\alpha\in A}$ and the multivariate Lagrange polynomials $\{L_\alpha\}_{\alpha\in A}$ are bases of Π_A.*

Corollary 3.4 (Lagrange interpolation)**.** *Let the essential assumptions (Definition 2.2) be fulfilled with respect to $A \subseteq \mathbb{N}^m$ and $P_A \subseteq \mathbb{R}^m$, and $f : \Omega \longrightarrow \mathbb{R}$ be a function. Then the uniquely determined interpolant $Q_{f,A} \in \Pi_A$ of f satisfying $Q_{f,A}(p_\alpha) = f(p_\alpha)$ for all $p_\alpha \in P_A$ is given by*

$$Q_{f,A} = \sum_{\alpha\in A} f(p_\alpha) L_\alpha(x) \,.$$

In other words : The Lagrange coefficients $C_{\mathrm{Lag}} \in \mathbb{R}^{|A|}$ of $Q_{f,A}$ are given by $C_{\mathrm{Lag}} = F$, $F = (f(p_0), \ldots, f(p_{\max})) \in \mathbb{R}^{|A|}$.

Proof. The proof follows directly from Theorem 3.3. □

In [5] we enabled to link the different kind of bases by proving the following statement.

Theorem 3.5 (LU-Decomposition)**.** *Let the essential assumptions (Definition 2.2) be fulfilled, $f : \mathbb{R}^m \longrightarrow \mathbb{R}$ be a function and $F = \big(f(p_{\alpha_{\min}}), \ldots, f(p_{\alpha_{\max}})\big) \in \mathbb{R}^{|A|}$. Then:*

i) *A lower triangular matrix* $\mathrm{NL}_A \in \mathbb{R}^{|A|\times|A|}$ *can be computed in* $\mathcal{O}(|A|^2)$ *operations, such that*

$$\mathrm{NL}_A \cdot C_{\mathrm{Newt}} = C_{\mathrm{Lag}} = F\,, \quad \textit{where} \quad C_{\mathrm{Newt}} = (c_{\alpha_{\min}}, \dots, c_{\alpha_{\max}}) \in \mathbb{R}^{|A|}$$

denote the uniquely determined Newton coefficients of $Q_{f,A} = \sum_{\alpha\in A} c_\alpha N_\alpha$.

ii) *A lower triangular matrix* $\mathrm{LN}_A \in \mathbb{R}^{|A|\times|A|}$ *can be computed in* $\mathcal{O}(|A|^2)$ *operations, such that*

$$\mathrm{LN}_A \cdot C_{\mathrm{Lag}} = C_{\mathrm{Newt}}\,, \quad \textit{where} \quad C_{\mathrm{Lag}} = (c_{\alpha_{\min}}, \dots, c_{\alpha_{\max}}) = F \in \mathbb{R}^{|A|}$$

denote the uniquely determined Lagrange coefficients of $Q_{f,A} = \sum_{\alpha\in A} c_\alpha L_\alpha$. *In particular,* $\mathrm{NL}_A^{-1} = \mathrm{LN}_A$.

iii) *An upper triangular matrix* $\mathrm{CN}_A \in \mathbb{R}^{|A|\times|A|}$ *can be computed in* $\mathcal{O}(|A|^3)$ *operations, such that*

$$\mathrm{CN}_A \cdot C_{\mathrm{can}} = C_{\mathrm{Newt}}\,, \quad \textit{where} \quad C_{\mathrm{can}} = (d_{\alpha_{\min}}, \dots, d_{\alpha_{\max}}) \in \mathbb{R}^{|A|}$$

denote the canonical coefficients of $Q_{f,A}(x) = \sum_{\alpha\in A} d_\alpha x^\alpha$.

iv) *An LU-decomposition of the multivariate Vandermonde matrix* $V(P_A) = (p_\alpha^\beta)_{\alpha,\beta\in A}$ *can be computed in* $\mathcal{O}(|A|^3)$ *operations and is given by*

$$V(P_A) = \mathrm{NL}_A \cdot \mathrm{CN}_A\,.$$

Remark 3.6. Note, that the LU–decomposition of $V(P_A)$ is not computed by a general decomposition scheme but by multivariate Newton evaluation [5]. In contrast to a general scheme, the structure of V_A is thereby incorporated in a feasible fashion yielding numerically stable computations of the entries of NL_A and CN_A. The sparsity of NL_A and CN_A and the efficient performance of the multivariate Newton–interpolation make the computations very accurate and fast in practice.

Remark 3.7. The LU–decomposition of $V(P_A)$ allows stable inversion, i.e.,

$$V(P_A)^{-1} = \mathrm{NC}_A \cdot \mathrm{LN}_A\,, \quad \mathrm{NC}_A = \mathrm{CN}_A^{-1}\,, \quad \mathrm{LN}_A = \mathrm{NL}_A^{-1}$$

Thereby, LN_A can be computed by multivariate Newton interpolation [5] instead of using a general inversion scheme, yielding more accurate results. Even though, the computation of CN_A requires general matrix inversion, the fact that CN_A is a sparse upper triangle matrix makes the computations reasonable accurate and efficient in practice.

Corollary 3.8. *Let the essential assumptions (Definition 2.2) be fulfilled with respect to a complete set* $A \subseteq \mathbb{N}^m$ *and* $P_A \subseteq \mathbb{R}^m$. *Then the Lagrange polynomials* L_α *are given in canonical form by*

$$L_\alpha(x) = \sum_{\beta\in A} d_\beta x^\beta\,, \quad D = (d_{\beta_{\min}}, \dots, d_{\beta_{\max}})\,, \tag{3.3}$$

where $D = V(P_A)^{-1}\cdot e_\alpha = \mathrm{NC}_A \cdot \mathrm{LN}_A \cdot e_\alpha$, *and* $e_\alpha \in \mathbb{R}^{|A|}$ *is the* α*–th standard basis vector, when ordering* $A = \{\alpha_{\min}, \dots, \alpha_{\max}\}$ *by lexicographical order as introduced.*

Proof. The proof follows directly from Theorem 3.5 and Remark 3.7. □

Corollary 3.9. *Let the essential assumptions (Definition 2.2) be fulfilled with respect to a complete set* $A \subseteq \mathbb{N}^m$ *and* $P_A \subseteq \mathbb{R}^m$. *Let* $Q \in \Pi_{m,n,p}$ *be given in Newton–form*

$$Q(x) = \sum_{\beta\in A} c_\alpha N_\alpha(x)\,. \tag{3.4}$$

Then there is an (iterative) algorithm that evaluates $Q(x_0) \in \mathbb{R}$ in $\mathcal{O}(m + n|A|)$ at any $x_0 \in \Omega$.

Proof. This is the new evaluation version coded by Jannik. □

Thus, we have to use the transformations in a smart way to get what we want...evaluation, integration differentiation, convolution etc ....

4. Polynomlets – recursive polynomial regression

Let $n \geq 2^k$, $m, k, p \in \mathbb{N}$ such that $m^{1/p} \cdot 2^k \leq n$ then $A_{m,2^k,\infty} \subseteq A_{m,n,p}$. We consider the nested nodes

$$\mathrm{Cheb}_n^0 = \mathrm{Cheb}_2^0 \cup (\mathrm{Cheb}_4^0 \setminus \mathrm{Cheb}_2^0) \cup \cdots \cup (\mathrm{Cheb}_n^0 \setminus \mathrm{Cheb}_{2^k}^0)$$

in the order as above. Set $\mathrm{GP} = \oplus_{i=1}^m \mathrm{Cheb}_n^0$ and let P_A, $A = A_{m,n,p}$ the corresponding unisolvent nodes. Consider the inclusion

$$\Pi_{m,2,\infty} \subseteq \Pi_{m,4,\infty} \subseteq \cdots \subseteq \Pi_{m,2^k,\infty} \subseteq \Pi_{m,n,p}$$

and the corresponding Lagrange bases

$$L_\alpha^1\,, \alpha \in A_{m,2,\infty}\,, \quad L_\alpha^2\,, \alpha \in A_{m,4,\infty} \setminus A_{m,2,\infty}\,, \ldots\,, \quad L_\alpha^n\,, \alpha \in A_{m,n,p} \setminus A_{m,2^k,\infty}$$

Then setting $A_1 = A_{m,2,\infty}$,$A_2 = A_{m,4,\infty} \setminus A_{m,2,\infty}$, $\ldots$, $A_n = A_{m,n,p} \setminus A_{m,2^k,\infty}$ yields

$$L_\alpha^j(p_\beta) = 0\,, \quad \forall\, \beta \in A_{m,j-1,\infty}\,, \quad \forall\, j \geq 2 \tag{4.1}$$

Given a function $f : \Omega \longrightarrow \mathbb{R}$ on some arbitrary data nodes $\{p_1, \ldots, p_M\} = P_0 \subseteq \Omega$, $F = (f(p_1), \ldots, f(p_M)) \in \mathbb{R}^M$. We consider the matrices

$$\begin{aligned} R_1 &= (r_{i,\beta})_{1 \leq i \leq M, \beta \in A_1}\,, \quad r_{i,\beta} = L_\beta^1(p_i) \\ R_2 &= (r_{i,\beta})_{1 \leq i \leq M, \beta \in A_2}\,, \quad r_{i,\beta} = L_\beta^2(p_i) \quad \ldots \\ R_n &= (r_{i,\beta})_{1 \leq i \leq M, \beta \in A_n}\,, \quad r_{i,\beta} = L_\beta^n(p_i) \end{aligned}$$

Due to Eq. (5.1) we observe that when computing

$$R_1 C_1 \approx F_1\,, \quad R_2 C_2 \approx F_2\,, \quad R_n C_n \approx F_n\,, \quad F_i = F - \sum_{j=1}^{i} R_j C_j$$

then we have defined a recursive regression of F.**The size of the matrices $R_j \in \mathbb{R}^{M \times J}$, $J = 2^j/2^m$ increases reasonable fast and allows polynomial regression even in high dimensions. Let's start with $m = 2, 3$.**

5. Nested orthogonal polynomials

Let $n > 2^k$, $m, k, p \in \mathbb{N}$ such that $m^{1/p} \cdot 2^k \leq n$ then $A_{m,2^k,\infty} \subseteq A_{m,n,p}$. We consider the nested nodes

$$\mathrm{Cheb}_{n+1}^0 = \mathrm{Cheb}_3^{2\mathrm{nd}} \cup (\mathrm{Cheb}_7^{2\mathrm{nd}} \setminus \mathrm{Cheb}_3^{2\mathrm{nd}}) \cup \cdots \cup (\mathrm{Cheb}_{n+1}^{2\mathrm{nd}} \setminus \mathrm{Cheb}_{n/2}^{2\mathrm{nd}})$$

in the order as above. Set $\mathrm{GP} = \oplus_{i=1}^m \mathrm{Cheb}_n^0$ and let P_A, $A = A_{m,n,p}$ the corresponding unisolvent nodes. Consider the inclusion

$$\Pi_{m,2,\infty} \subseteq \Pi_{m,5,\infty} \subseteq \cdots \subseteq \Pi_{m,(n-1)/2,\infty} \subseteq \Pi_{m,n,p}$$

and the corresponding Lagrange bases

$$L_\alpha^1\,, \alpha \in A_{m,2,\infty}\,, \quad L_\alpha^2\,, \alpha \in A_{m,4,\infty} \setminus A_{m,2,\infty}\,, \ldots\,, \quad L_\alpha^n\,, \alpha \in A_{m,n,p} \setminus A_{m,(n-1)/2,\infty}$$

Then setting $A_1 = A_{m,2,\infty}$,$A_2 = A_{m,5,\infty} \setminus A_{m,2,\infty}, \ldots, A_n = A_{m,n,p} \setminus A_{m,(n-1)/2,\infty}$ yields

$$L_\alpha^j(p_\beta) = L_\alpha^{j-1}(p_\beta)\,, \quad \forall\, \beta \in A_{m,j-1,\infty}\,, \quad \forall\, j \geq 2 \tag{5.1}$$

Conversely,

$$L_\alpha^{j-1}(x) = L_\alpha^j(x) + \sum_{\beta \in A_j \setminus A_{j-1}} L_\alpha^{j-1}(p_\beta) L_\beta^j(x) \tag{5.2}$$

Thus,

$$\begin{aligned}\int_\Omega \omega L_\alpha^n(x) d\Omega &= \int_\Omega \omega L_\alpha^{n-1}(x) d\Omega - \sum_{\beta \in A_n \setminus A_{n-1}} \int_\Omega \omega L_\alpha^{n-1}(p_\beta) L_\beta^n(x) d\Omega \\ &= \int_\Omega \omega L_\alpha^{n-1}(x) d\Omega - \sum_{\beta \in A_n \setminus A_{n-1}} \omega L_\alpha^{n-1}(p_\beta) \int_\Omega L_\beta^n(x) d\Omega\,.\end{aligned}$$

If both terms of the right hand side vanish then the integration weight $w_\alpha = 0$ vanishes as well ! This gives an recursive condition, which we can check, i.e.,

$$\int_\Omega \omega L_\alpha^{n-1}(x) d\Omega = 0\,? \quad L_\alpha^{n-1}(p_\beta) = 0 \quad \text{for all } \beta \text{ with } \int_\Omega \omega L_\beta^n(x) d\Omega \neq 0\,?$$

$\int_\Omega \omega L_\alpha^{j-1}(x) d\Omega = 0$ holds whenever $\alpha \in A_{m,2^j+1,\infty}$. The remaining term then becomes a recursive condition.

6. Sobolev theory

This brief summary of Sobolev theory is a rewritten and corrected version of aspects discussed in our previous work [5, 6]. We recommend [1] as an excellent overview of Sobolev theory and [4, 7] for specific aspects deduced here.

Let $C^0(\Omega, \mathbb{R})$ be the *Banach space* of continuous functions $f : \Omega \longrightarrow \mathbb{R}$ with norm $\|f\|_{C^0(\Omega)} = \sup_{x \in \Omega} |f(x)|$. We consider the Hilbert space $L^2(\Omega, R)$, $R = \mathbb{R}, \mathbb{C}$ of all real- or complex-valued Lebesgue-integrable functions [1] with associated scalar product and norm

$$\langle f, g \rangle_{L^2(\Omega)} = \frac{1}{|\Omega|} \int_\Omega f(x) g(x)\, \mathrm{d}x\,, \;\; \|f\|^2_{L^2(\Omega)} = \langle f, f \rangle_{L^2(\Omega)}\,.$$

For $k \in \mathbb{N}$, we consider the *Sobolev space* of all L^2 functions with existing and integrable weak derivatives

$$\begin{aligned} H^k(\Omega, R) &= \left\{ f \in L^2(\Omega, R) \;\middle|\; \partial^\alpha f \in L^2(\Omega, R) \text{ for all } \alpha \in A_{m,k,1} \right\}\,, \\ \langle f, g \rangle_{H^k(\Omega)} &= \sum_{\alpha \in A} \langle \partial^\alpha f, \partial^\alpha g \rangle_{L^2(\Omega)}\,, \quad \|f\|^2_{H^k(\Omega)} = \langle f, f \rangle_{H^k(\Omega)}\,,\; R = \mathbb{R}, \mathbb{C}\,. \end{aligned}$$

Indeed, $\left(H^k(\Omega, \mathbb{R}), \langle \cdot, \cdot \rangle_{H_A}\right)$ is a Hilbert space and by the Sobolev embedding Theorem for $k > m/2$, we have that $H^k(\Omega, R) \subseteq C^0(\Omega, R)$ [1]. Therefore, $H^k(\Omega, R)$ with $k > m/2$ is the largest Hilbert space contained in the space of continuous functions.

In our previous work [5] we answered the fundamental question of which functions can be approximated by polynomial interpolation at all. That is:

Theorem 6.1 (Approximation of Sobolev functions)**.** *Let the essential assumptions (Definition 2.2) be fulfilled with respect to $A_{m,n,p}$, $m,n \in \mathbb{N}$, $p \geq 1$, and the nodes $P_{A_{m,n,p}}$ generated by*

$$\mathrm{GP} = \oplus_{i=1}^m \mathrm{Cheb}_n^{2\mathrm{nd}}\,.$$

Assume that $f \in H^k(\Omega,\mathbb{R})$, $k > m/2$ and denote with $Q_{f,A_{m,n,p}}$ its unique interpolant with respect to $P_{A_{m,n,p}}$. Then

$$\|f - Q_{f,A_{m,n,p}}\|_{C^0(\Omega)} \xrightarrow[n\rightarrow\infty]{} 0\,.$$

6.1. **periodic functions.** We denote by $\mathbb{T}^m = \mathbb{R}^m/2\mathbb{Z}^m$ the torus with fundamental domain Ω and call a function $f : R^m \longrightarrow R$ periodic if and only if $f(x+2\mathbb{Z}^m) = f(x)$, i.e., $f : \mathbb{T}^m \longrightarrow R$ is well defined. Since $C^\infty(\Omega,R) \subseteq H^k(\Omega,R)$ is dense [1], $H^k(\mathbb{T}^m,R)$ can be defined as the completion of the space of all smooth periodic functions

$$H^k(\mathbb{T}^m,R) = \overline{C^\infty_{\mathrm{period}}(\Omega,R)}^{\|\cdot\|_{H^k(\Omega)}}\,.$$

Thus, $H^k(\mathbb{T}^m,R) \subseteq H^{k'}(\mathbb{T}^m,R)$ for all $k \geq k' \geq 0$ with $H^0(\mathbb{T}^m,R) = L^2(\mathbb{T}^m,R)$. Consequently, every $f \in H^k(\mathbb{T}^m,R)$ can be expanded in a Fourier series

$$f(x) = \sum_{\beta\in\mathbb{N}^m} c_\beta e^{\pi i\langle\beta,x\rangle}\,, \quad c_\beta \in \mathbb{C} \tag{6.1}$$

almost everywhere, i.e., Eq. (6.1) is violated only on a set $\Omega' \subseteq \Omega$ of Lebesgue measure zero. Since the $e^{\pi i\langle\beta,x\rangle}$ are an orthogonal L^2-basis, the norm of f is given by

$$||\,f\,||^2_{H^k(\Omega)} = \sum_{\alpha\in A_{m,k,1}} \langle\partial^\alpha f,\partial^\alpha f\rangle_{L^2(\Omega)} = \sum_{\alpha\in A_{m,k,1}}\sum_{\beta\in\mathbb{N}^m}\left(\pi^{\|\alpha\|_1}\beta^\alpha|c_\beta|\right)^2, \tag{6.2}$$

where $\beta^\alpha = \beta_1^{\alpha_1}\cdots\beta_m^{\alpha_m}$. Due to the *Sobolev and Rellich-Kondrachov Embedding Theorem* [1], we have that whenever $k > m/2$ then $H^k(\mathbb{T}^m,\mathbb{R}) \subseteq C^0(\mathbb{T}^m,\mathbb{R})$ and the embedding

$$i : H^k(\mathbb{T}^m,\mathbb{R}) \hookrightarrow C^0(\mathbb{T}^m,\mathbb{R})$$

is well defined, continuous, and compact. Thus, for $m \in \mathbb{N}$ there exists a constant $c = c(m,\Omega) \in \mathbb{R}^+$ such that

$$||\,f\,||_{C^0(\Omega)} \leq c\,||\,f\,||_{H^k(\Omega)}\,.$$

By the Trace Theorem [1], we observe furthermore that whenever $H \subseteq \mathbb{R}^m$ is a hyperplane of co-dimension 1, then the induced restriction

$$\varrho : H^k(\Omega,\mathbb{R}) \longrightarrow H^{k-1/2}(\Omega\cap H,\mathbb{R}) \tag{6.3}$$

is continuous, i.e., $||\,f_{|\Omega\cap H}\,||_{H^{k-1/2}(\Omega\cap H)} \leq d||\,f\,||_{H^k(\Omega)}$ for some $d = d(m,\Omega) \in \mathbb{R}^+$.

7. Laplacian and other differential operators

Consider $A_{m,n,p}$ and the Laplace operator on not necessarily periodic functions

$$\Delta : H^{k+2}(\Omega,\mathbb{R}) \longrightarrow H^k(\Omega,\mathbb{R})$$

Then we can approximate Δ by a finite dimensional approximation being representable by a matrix $\widetilde{\Delta} \in \mathbb{R}^{|A|\times|A|}$, i.e.,

$$\widetilde{\Delta} : \Pi_{m,n,p} \longrightarrow \Pi_{m,n,p}\,, \quad \widetilde{\Delta} = (d_{\alpha,\beta})_{\alpha,\beta \in A}\,, \quad \text{where} \sum_{\alpha \in A_{m,n,p}} d_{\alpha,\beta} L_\alpha = \Delta L_\beta\,.$$

The computation of $d_{\alpha,\beta}$ is thereby done by smartly combining the transformations and the known derivation w.r.t. canonical basis. In the case, where we consider periodic functions, i.e.,

$$\Delta : H^{k+2}(\mathbb{T}^m, \mathbb{R}) \longrightarrow H^k(\mathbb{T}^m, \mathbb{R})$$

the Eigenvalues and Eigenfunctions are given by the Fourier basis, i.e.,

$$\Delta e^{\pi i \langle \beta, x \rangle} = \pi^2 \beta^2 e^{\pi i \langle \beta, x \rangle}\,, \quad \beta^2 = \beta_1^2 + \cdots + \beta_m^2\,.$$

However, for non–periodic functions this does not longer hold and we have to study that further !

8. Polynomial Level Sets

To come ...sorry Sachin ;)

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