Independent Study

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Contents

1. Stochastic-differential Approach to Fin-Math

1.1 Brownian Motion

A collection of random variables indexed by “time” variable t: X(t), is called a Stochastic Process. Brownian Motion, is a very important stochastic process known for its stationary independent increments.

Definition 1.1.1 A continuous stochastic process B(t), t $\geq$ 0, to be a Brownian Motion if

- B(0) = 0;
- The increments of the process $B(t_1)-B(0), B(t_2)-B(t_1), \ldots, B(t_n)-B(t_{n-1})$, $\forall\ 0 \leq t_1 \leq t_2 \leq \ldots t_n$ are independent;
- Each of these increments is normally distributed with distribution $B(t_{i+1})-B(t_i) \sim N(0, t_{i+1}-t_i)$.

It is not obvious from the above definition that Brownian Motion exists. We show that it does exist by essentially constructing Brownian Motion.
We start with a simpler stochastic process that is called Symmetric Random Walk (SRW).

1.1.1 Symmetric Random Walk (SRW)

Construction:
To construct a SRW we repeatedly toss a fair coin: $P(H) = P(T) = 1/2$. We define the successive outcomes of the tosses by $\omega_1, \omega_2, \omega_3 \dots \in \{H, T\}$ and define random variable X_j by

$$X_j = \begin{cases} 1 & \text{if } \omega_j = H \\ -1 & \text{if } \omega_j = T \end{cases}$$

Define discrete-time stochastic process $M(n)$ in the following way:

$$M(0) = 0, M(n) = \sum_{j=1}^{n} X_j.$$

Increments of the SRW:

A random walk has independent increments. This means that if we choose nonnegative integers $0 = n_0 < n_1 < \cdots < n_k$, then random variables

$$M(n_1) = M(n_1) - M(n_0), M(n_2) - M(n_1), \ldots, M(n_k) - M(n_{k-1})$$

are indepedent. Each of these random variables

$$M(n_{i+1}) - M(n_i) = \sum_{j=n_i+1}^{n_{i+1}} X_j$$

is called an increment of the random walk. It is the change in the position of the random walk between times n_i and n_{i+1}. Increments over non-overlapping time intervals are independent because they depend on different coin tosses. Moreover,

$$\mathbb{E}(M(n_{i+1}) - M(n_i)) = \sum_{j=n_i+1}^{n_{i+1}} \mathbb{E}X_j = 0$$

and

$$Var(M(n_{i+1}) - M(n_i)) = \sum_{j=n_i+1}^{n_{i+1}} VarX_j = n_{i+1} - n_i \text{ since } VarX_j = 1$$

Martingale Property:

To check the martingale property of $M(n)$, we need to check whether $M(n)$ is $\mathscr{F}_n - measurable$ and whether $\mathbb{E}(M(t) \mid \mathscr{F}_s) = M(s), \forall t > s$

- $M(n) = \sum_{j=1}^{n} X_j$ is $\mathscr{F}_n - measurable$ since it depends only on $X_j, j \leq n$, i.e. on the information available at time n;
- $\mathbb{E}(M(t) \mid \mathscr{F}_s) = \mathbb{E}(M(t) - M(s) + M(s) \mid \mathscr{F}_s) = \mathbb{E}(M(t) - M(s) \mid \mathscr{F}_s) + \mathbb{E}(M(s) \mid \mathscr{F}_s) = \mathbb{E}(M(t) - M(s)) + \mathbb{E}(M(s)) = 0 + M(s) = M(s)$;

Quadratic Variation of the SRW:

Quadratic variation of a discrete stochastic process up to time t is defined as

$$< M, M >_t = \sum_{j=1}^{n} (M(j) - M(j-1))^2$$

Clearly, for the symmetric random walk increments can take values ±1 and thus

$$< M, M >_t = t$$

1.1.2 Scaled SRW

Construction

To approximate a Brownian motion we speed up time and scale down the step size of a SRW. More precisely, we fix a positive integer n and define the scaled SRW at rational points $\frac{k}{n}$ as

$$B^{(n)}(\frac{k}{n}) = \frac{1}{\sqrt{n}} M(k).$$

Expectation, Variance, Martingale, Quadratic Variation

Same Results can be proved.

Limiting Distribution

We are going to prove that $B^{(n)}(t)$ converges in distribution to the normal random variable $N(0,t)$ with the help of Characteristic Functions and Paul L´evy Theorem.

Definition 1.1.2 Let X be a random variable with distribution $\mathbb{P}$. Characteristic function of X is defined as

$$\varphi_X(u) = \mathbb{E}e^{iuX} = \int e^{iux}\,dP(x)$$

for every $u \in \mathbb{R}$.

Theorem 1.1.1 — Paul L´evy Theorem. Let Xn be a sequence of random variables on R. Then

1. If X_n converges to X in distribution then $\varphi_{X\,n}(u) \to \varphi_X(u)$ for every u and in fact this convergence is uniform.
2. If $\varphi_{X\,n}(u) \to \varphi_X(u)$ then X_n converges to X in distribution.
3. If $\varphi_{X\,n}(u)$ converges to some function $\varphi(u)$ and $\varphi(u)$ is continuous at 0 then there exists a unique random variable X such that φ is the characteristic function of X and X_n converges to X in distribution.

We use (2) of the theorem to prove that $B^{(n)}(t)$ converges to normal distribution with expectation 0 and variance t.

$$\varphi_{B^{(n)}}(u) = \mathbb{E}e^{iuB^{(n)}(t)} = \mathbb{E}e^{iu\frac{1}{\sqrt{n}}M_{nt}} = \mathbb{E}e^{iu\frac{1}{\sqrt{n}}\sum_{j=1}^{nt}X_j} = (\mathbb{E}e^{iu\frac{1}{\sqrt{n}}X_j})^{nt} = (\frac{1}{2}e^{iu\frac{1}{\sqrt{n}}} + \frac{1}{2}e^{-iu\frac{1}{\sqrt{n}}})^{nt}$$

Expanding in Taylor series,

$$\frac{1}{2}e^{iu\frac{1}{\sqrt{n}}} + \frac{1}{2}e^{-iu\frac{1}{\sqrt{n}}} = 1 - \frac{u^2}{2n} + O(\frac{1}{n^{\frac{3}{2}}}).$$

Using the fact that

$$(1+\frac{x}{n})^n = e^x$$

We have

$$\varphi_{B^{(n)}}(u) = (\frac{1}{2}e^{iu\frac{1}{\sqrt{n}}} + \frac{1}{2}e^{-iu\frac{1}{\sqrt{n}}})^{nt} \to e^{-\frac{y^2}{2n}\cdot nt} = e^{\frac{-u^2t}{2}},$$

which is exactly the characteristic function of the normal random variable with mean 0 and variance t.

1.1.3 Important Martingales Related to BM

These are two important martingales related to BM which are valuable tools in many applications.

- $Y(t) = B(t)^2 - t$;
- $Z(t) = e^{\lambda B(t) - \frac{1}{2}\lambda^2 t}$ (Exponential martingale);

We can check whether a stochastic process is a martingale by using either of the following two methods:

1. Use Ito's lemma and check whether the differential is null-drift.
2. Use the definition of a martingale and check whether $\mathbb{E}(M(t) \mid \mathscr{F}_s) = M(s), \forall t > s$

1.2 Itô Integral as a kind of Stochastic Integral

1.2.1 Motivation of Stochastic Integral

Consider an asset whose price per share is equal to $X_t, t \geq 0$ and a portfolio that initially consists of Δ_0 shares. Consider the following trading strategy: keep an initial position Δ_0 up to time $t_1 \geq t_0 = 0$ and then re-balance the portfolio by taking position Δ_1 in the asset. Keep it up to time $t_2 \geq t_1$ and then re-balance the portfolio again by taking position Δ_2 is the asset. In general, we re-balance the portfolio at trading date t_i by taking position Δ_i in the asset and keeping it till the next trading date t_{i+1}. What is the the profit $I_T(\Delta)$ of the above trading strategy at time T ? Clearly it should be

$$I_T(\Delta) = \Delta_0 (X_{t_1} - X_{t_0}) + \Delta_1 (X_{t_2} - X_{t_1}) + \cdots + \Delta_{n-1} \left(X_{t_n} - X_{t_{n-1}}\right)$$

and by analogy with the Riemann integral we write symbolically

$$I_T(\Delta) = \int_0^T \Delta(t) dX(t),$$

where $\Delta(t)$ is a piecewise constant function which is equal to Δ_i on $[t_i, t_{i+1}]$.

We fix an interval $[S, T]$ and try to make sense of

$$\int_S^T f(t,w) dX_t(w),$$

where $f(t,w)$ is a random function and $dX_t(w)$ refers to the increments of a stochastic process X_t.

If X_t is a differentiable function, then we can define

$$\int_S^T f(t,w) dX_t = \int_S^T f(t,w) X_t' dt$$

where the right-hand side is an ordinary Riemann integral with respect to time. This ordinary Riemann integral approach does NOT work for stochastic process X_t as we saw that the trajectories of X_t, which are driven by Brownian motions are not differentiable.

1.2.2 Construction of Stochastic integral for Simple Functions

Just like in the construction of the Riemann integral $\int_S^T f(t)dt$, where $f(t)$ is a deterministic function, we start with a construction of stochastic integral for a simple class of functions f and then extend by some approximation procedure.

For simplicity, we take $X_t = B_t$ in the following construction. Assume that $\Pi = \{t_0, t_1, \ldots, t_n\}$ is a partition of $[S, T]$, i.e.

$$S = t_0 \le t_1 \le \cdots \le t_n = T,$$

and that $f(t,w)$ is constant $e_{j+1}(\omega)$ in t on each subinterval $(t_j, t_{j+1}]$. Such a process $f(t,w)$ called a simple process. Consider interval $[t_0, t_1]$, on this interval $f(t,\omega) = e_1(\omega)$ is a random quantity, but independent of t and thus it is natural to define

$$\int_{t_0}^{t_1} f(t,\omega) dB_t(\omega) = \int_{t_0}^{t_1} e_1(\omega) dB_t(\omega) = e_1(\omega) \left(B(t_1) - B(t_0)\right).$$

Applying this procedure to intervals $[t_1, t_2], [t_2, t_3], \ldots, [t_{n-1}, t_n]$ we get

$$\begin{aligned}\int_S^T f(t,w) dB_t(w) &= e_1(w)(B_{t_1} - B_{t_0}) + e_2(w)(B_{t_2} - B_{t_1}) \\ &\quad + e_3(w)(B_{t_3} - B_{t_2}) + \ldots + e_n(w)\left(B_{t_n} - B_{t_{n-1}}\right)\end{aligned}$$

1.2.3 Problems in the Construction of Stochastic integral for General Functions

Naturally, to define stochastic integral for general functions $f(t,w)$ we approximate it with simple functions similarly to approximation of continuous functions by stepwise constant functions in the theory of Riemann integration. Recall that Riemann integral is a limit of Riemann sums:

$$\int_S^T f(t)dt \approx \sum_{i=0} f(t_i^*)(t_{i+1} - t_i)$$

where t_i^* is ANY point on the interval $[t_i, t_{i+1}]$. When the length of the longest interval in the partition tends to zero the limit is $\int_S^T f(t)dt$. We recall that it was not important what point t_i^* we took inside the interval $[t_i, t_{i+1}]$. For example, it could be t_i (left point approximation) or t_{i+1} (right point approximation).

But in stochastic integral, without any further assumption on approximating functions $e_i(w)$, namely whether we choose left endpoint, or midpoint or right endpoint to approximate $f(t,w)$ will give us different answers. Here is an example. Consider

$$\int_0^T B_t dB_t$$

Left point approximation:

$$I_1 \cong \sum_i B(t_i)\left(B(t_{i+1}) - B(t_i)\right)$$

Right point approximation:

$$I_2 \cong \sum_i B(t_{i+1})\left(B(t_{i+1}) - B(t_i)\right).$$

From the independence of increments of Brownian motion and the fact that $\mathbb{E}\left[\left(B(t_{i+1}) - B(t_i)\right)\right] = \mathbb{E}\left[B(t_1)\right] = 0$ we have

$$\begin{aligned}
\mathbb{E}(I_1) &= \sum_i \mathbb{E}\left[B(t_i)\left(B(t_{i+1}) - B(t_i)\right)\right] \\
&= \sum_i \mathbb{E}\left[B(t_i)\right]\mathbb{E}\left[\left(B(t_{i+1}) - B(t_i)\right)\right] = 0. \\
\mathbb{E}(I_2) &= \sum_i \mathbb{E}\left[B(t_{i+1})\left(B(t_{i+1}) - B(t_i)\right)\right] \\
&= \sum_i \mathbb{E}\left[B(t_{i+1})^2 - B(t_{i+1})B(t_i)\right] \\
&= \sum_i \left[t_{i+1} - t_i\right] = T
\end{aligned}$$

since $\mathbb{E}\left[B(t_{i+1})^2\right] = t_{i+1}$ and $\mathbb{E}\left[B(t_{i+1})B(t_i)\right] = t_i$ as follows from

$$\mathbb{E}(B(t)B(s)) = \min(s,t).$$

Thus we see that depending on the choice of the point t_i^* in the approximation we can get very different results. Function $f(t,w)$ is $\mathscr{F}_t$-measurable and thus it is reasonable to choose the approximating simple function to be $\mathscr{F}_t$-measurable as well. We therefore have to choose the left end point approximation. In what follows we choose

$$t_i^* = t_i \text{ (left end point approximation)}$$

which leads to the Itô integral.

Remark: If we choose midpoint, this will lead to Stratonovich Integral.

1.2.4 Properties of Itô integral for Simple Processes

Theorem 1.2.1 Ito integral is a martingale.

Intuition: The Itô integral is defined as the gain from treading in the martingale B_t. A martingale has no tendency to rise or fall and hence it is to be expected that

$$I_t(f) = \int_0^t f(s,w)dB_s$$

also has no tendency to rise or fall. Proof: see [Steven Shreve, Stochastic Calculus for Finance II: Continuous-Time Models], pages 128-129.

Theorem 1.2.2 — Itô's Isometry. The Itô integral satisfies

$$\mathbb{E}I_t^2(f) = \mathbb{E}\int_0^t f(s,w)^2 ds$$

Proof. For the simplicity of notation we introduce $\Delta B_i = B(t_{i+1}) - B(t_i), e_i = e_i(w)$. Then by definition

$$I_t(f) = \int_0^t f(s,\omega)dB_s = \sum_i e_i \Delta B_i$$

and

$$\left(\int_0^t f(s,\omega)dB_s\right)^2 = \left(\sum_i e_i \Delta B_i\right)^2 = \sum_{i,j} e_i e_j \Delta B_i \Delta B_j.$$

Taking expectation

$$\mathbb{E}\left(\int_0^t f(s,\omega)dB_s\right)^2 = \sum_{i,j} \mathbb{E}\left(e_i e_j \Delta B_i \Delta B_j\right).$$

$$\mathbb{E}\left(e_i e_j \Delta B_i \Delta B_j\right) = \begin{cases} 0 = \mathbb{E}\left(\Delta B_j\right), \text{ if } i < j \\ \mathbb{E}\left(e_i^2 \Delta B_i^2\right), \text{ if } i = j \end{cases}$$

For $i = j$ we use independence of increments property to conclude

$$\mathbb{E}\left(e_i^2 (\Delta B_i)^2\right) = \mathbb{E}\left(e_i^2\right)\mathbb{E}(\Delta B_i)^2 = \mathbb{E}\left(e_i^2\right)(t_{i+1} - t_i) = \mathbb{E}\left(e_i^2\right)\Delta t_i.$$

$$\sum_i \mathbb{E}\left(e_i^2\right)\Delta t_i = \mathbb{E}\sum_i \left(e_i^2\right)\Delta t_i = \mathbb{E}\int_0^t f(s,\omega)^2 dt$$

Theorem 1.2.3 Quadratic variation of the Ito integral is equal to

$$\int_0^t f^2(s,w)ds = \sum_i e_i^2 \Delta t_i.$$

Proof: Consider the quantity

$$\mathbb{E}\left(\sum_i (e_i \Delta B_i)^2 - \sum_i e_i^2 \Delta t_i\right)^2$$

and prove that it approaches 0 as $||\Pi|| \to 0$. We first rewrite it as

$$\mathbb{E}\left(\sum_i e_i^2 (\Delta B_i)^2 - \sum_i e_i^2 \Delta t_i\right)^2$$

$$= \mathbb{E}\left(\sum_i e_i^2 \left[(\Delta B_i)^2 - \Delta t_i\right]\right)^2$$

$$= \mathbb{E}\sum_{i,j} e_i^2 e_j^2 \left[(\Delta B_i)^2 - \Delta t_i\right]\left[(\Delta B_j)^2 - \Delta t_j\right]$$

Just as in the calculation of the quadratic variation of the Brownian motion let us split the above sum in two sums: in the first one we keep the terms with $i \neq j$ and in the second one we keep terms with $i = j$. Let us first look at terms with $i \neq j$, for instance $i < j$. Then $\left[(\Delta B_j)^2 - \Delta t_j\right]$ is independent of $e_i^2 e_j^2 \left[(\Delta B_i)^2 - \Delta t_i\right]$ an thus

$$\begin{aligned}\mathbb{E} e_i^2 e_j^2 \left[(\Delta B_i)^2 - \Delta t_i\right]\left[(\Delta B_j)^2 - \Delta t_j\right] &= \mathbb{E} e_i^2 e_j^2 \left[(\Delta B_i)^2 - \Delta t_i\right] \mathbb{E}\left[(\Delta B_j)^2 - \Delta t_j\right] \\ &= \mathbb{E} e_i^2 e_j^2 \left[(\Delta B_i)^2 - \Delta t_i\right] \cdot 0 = 0\end{aligned}$$

Let us now consider the case of $i = j$. Then

$$\mathbb{E} e_i^4 \left[(\Delta B_i)^2 - \Delta t_i\right]^2 = \mathbb{E} e_i^4 \mathbb{E}\left[(\Delta B_i)^2 - \Delta t_i\right]^2,$$

since random variables e_i^4 and $\left[(\Delta B_i)^2 - \Delta t_i\right]^2$ are independent. It follows from the fact that e_i is $\mathscr{F}_t$-measurable and thus ΔB_i is independent of e_i. But $\mathbb{E}\left[(\Delta B_i)^2 - \ \Delta t_i\right]^2 = 2\left(\Delta t_i\right)^2$ and thus

$$\begin{aligned}\mathbb{E}\left(\sum_i (e_i \Delta B_i)^2 - \sum_i e_i^2 \Delta t_i\right)^2 &= \sum_i 2\mathbb{E} e_i^4 \left(\Delta t_i\right)^2 \\ &\leq \|\Pi\| \sum_i 2\mathbb{E} e_i^4 \Delta t_i.\end{aligned}$$

Since $\sum_i 2\mathbb{E} e_i^4 \Delta t_i$ converges to $\int_0^T \mathbb{E} f^4 dt < \infty$. Thus

$$\mathbb{E}\left(\sum_i (e_i \Delta B_i)^2 - \sum_i e_i^2 \Delta t_i\right)^2 \to 0$$

and quadratic variation of $I(f)$ is proved to be

$$\int_0^t f^2(s,w)ds$$

1.2.5 Itô integral for general functions

We now describe the class of functions $f(s,w)$ for which the Itô integral will be defined.

Definition 1.2.1 Let V be the class of functions $f(s,\omega) : [0,\infty) \times \Omega \to \mathbb{R}$, such that:

- $f(s,\omega)$ is $\mathscr{F}_s$ - adapted;
- $\int_0^t f(s,w)^2 ds < \infty$.;

We claim that each function $f \in V$ can be approximated by a sequence $\{\varphi_n\}_{n=1,2,\ldots}$ of simple functions (or equivalently, by a sequence of simple processes) in the sense that as $n \to \infty$

$$\mathbb{E}\int_0^t (f - \varphi_n)^2 \to 0.$$

The approximation is done is three steps:

Step 1 (Approximate bounded continuous functions with simple functions) Let $g \in V$ be bounded, i. e., every trajectory $g(.,\omega)$ (ω is fixed and t changes) is continuous. Then, there exists a sequence of simple functions $\varphi_n \in V$, such that as $n \to \infty$

$$\mathbb{E}\int_0^t (g - \varphi_n)^2 ds \to 0.$$

Step 2 (Approximate bounded functions with bounded continuous functions) Let $h \in V$ be bounded, then there exists a sequence of bounded continuous functions g_n, such that

$$\mathbb{E}\int_0^t (h-g_n)^2\,ds \to 0$$

Step 3 (Approximate general functions with bounded functions)

Let $f \in V$, then there exists a sequence of bounded functions h_n, such that

$$\mathbb{E}\int_0^t (f-h_n)^2\,ds \to 0$$

Putting together steps 1,2 and 3 we get that for any function $f(s,\omega) \in V$ there exists a sequence of simple functions $\varphi_n(s,w)$ such that 30 is true. We define then the Itô integral of function $f(s,\omega)$ as

$$I_t(f) = \int_0^t f(s,\omega)dB_s = \lim_{n\to\infty} I(\varphi_n).$$

Question: Why does the limit exist and in what sense?

Answer: By Ito's Isometry we have that

$$\begin{aligned}\mathbb{E}\left(I(\varphi_n)-I(\varphi_m)\right)^2 &= \mathbb{E}\int_0^t (\varphi_n-\varphi_m)^2\,ds \\ &\le \mathbb{E}\int_0^t (f-\varphi_m)^2\,ds + \mathbb{E}\int_0^t (\varphi_n-f)^2\,ds \to 0\end{aligned}$$

Thus the sequence of random variables $\left\{\int_0^t \varphi_n(s,\omega)dB_t\right\}$ forms a Cauchy sequence in $L_2(\Omega,\mathscr{F},\mathbb{P})$. Since $L_2(\Omega,\mathscr{F},\mathbb{P})$ is a complete space then there exists a limit of $I(\varphi_n)$ as an element of $L_2(\Omega,\mathscr{F},\mathbb{P})$. This limit is by definition the Itô integral $I(f)$.

Example: Compute $\int_0^t B_s dB_s$

By definition

$$\int_0^t B_s dB_s = \lim_{n\to\infty}\int_0^t \varphi_n(s,\omega)dB_s$$

where φ_n is such that $\mathbb{E}\int_0^t (\phi_n - B_s)^2\,ds \to 0$ and φ_n is $\mathscr{F}_s$-adapted.

As we already saw in the beginning that we can approximate $f(s,w)=B_s$ by partitioning $[0,t]$ into $[0,t_1],[t_1,t_2],\ldots,[t_n-1,t_n=t]$ and defining $\varphi_n(s,w)=B(t_i)$ for $s\in[t_i,t_{i+1}]$. Let us first check that φ_n indeed approximated f:

$$\begin{aligned}\mathbb{E}\int_0^t (\phi_n-B_s)^2\,ds &= \mathbb{E}\sum_i\int_{t_i}^{t_{i+1}} (\phi_n-B_s)^2\,ds = \sum_i\int_{t_i}^{t_{i+1}} \mathbb{E}\left(B(t_i)-B(s)\right)^2 ds \\ &= \sum_i\int_{t_i}^{t_{i+1}} (s-t_i)\,ds = \sum_i\frac{(t_{i+1}-t_i)^2}{2}\end{aligned}$$

If we define $\max(t_{i+1}-t_i)=M_n$ then

$$\sum_i\frac{(t_{i+1}-t_i)^2}{2} \le \sum_i\frac{t_{i+1}-t_i}{2}M_n = \frac{M_n}{2}\sum_i(t_{i+1}-t_i) = \frac{M_n}{2}t \to 0.$$

Thus we have to compute $\int_0^t \phi_n dB_s = \sum_i B(t_i)\,\Delta B_i$, where $\Delta B_i = B(t_{i+1})-B(t_i)$.

We use the following identity

$$\begin{aligned}\Delta B_i^2 &= B(t_{i+1})^2 - B(t_i)^2 = \left(B(t_{i+1})-B(t_i)\right)^2 + 2B(t_{i+1})B(t_i) - 2B(t_i)^2 \\ &= \left(B(t_{i+1})-B(t_i)\right)^2 + 2B(t_i)\left(B(t_{i+1})-B(t_i)\right).\end{aligned}$$

Summing both parts over i we get

$$B_t^2 = \sum_i \left(B\left(t_{i+1}\right) - B\left(t_i\right)\right)^2 + 2I\left(\phi_n\right).$$

Therefore

$$I\left(\varphi_n\right) = \frac{B_t^2}{2} - \frac{1}{2}\sum_i \left(B_{i+1} - B_i\right)^2, \text{ but } \sum_i \left(B_{i+1} - B_i\right)^2 \xrightarrow[L_2]{} t.$$

Finally,

$$\int_0^t B_s dB_s = \frac{B_t^2}{2} - \frac{t}{2}$$

1.3 Itô Formula

We just defined Itô integral

$$\int_0^t f(s,\omega)dB_s = I(f)$$

for stochastic processes $f(t,w)$ that are $\mathscr{F}_t$-adapted and square-integrable. Since the procedure of calculating Itô integral from the definition is rather work consuming, we try to come up with a different approach.

Theorem 1.3.1 — Itô Formula. Let $f(t,x)$ be a function for which the partial derivatives $f_t(t,x), f_x(t,x)$ and $f_{xx}(t,x)$ are defined and continuous, and let B_t be a Brownian motion. Then for every $T \geq 0$

$$f\left(T,B_T\right) = f\left(0,B_0\right) + \int_0^T f_t\left(t,B_t\right)dt + \int_0^T f_x\left(t,B_t\right)dB_t + \frac{1}{2}\int_0^T f_{xx}\left(t,B_t\right)dt$$

Sketch of Proof: from the Taylor series expansion we obtain

$$\begin{aligned} f\left(t_{j+1},x_{j+1}\right) - f\left(t_j,x_j\right) &= f_t\left(t_j,x_j\right)\left(t_{j+1}-t_j\right) + f_x\left(t_j,x_j\right)\left(x_{j+1}-x_j\right) \\ &+ \frac{1}{2}f_{xx}\left(t_j,x_j\right)\left(x_{j+1}-x_j\right)^2 + f_{tx}\left(t_j,x_j\right)\left(x_{j+1}-x_j\right)\left(t_{j+1}-t_j\right) \\ &+ \frac{1}{2}f_{tt}\left(t_j,x_j\right)\left(t_{j+1}-t_j\right)^2 + \text{ higher order terms.} \end{aligned}$$

Apply this formula with $x_{j+1} = B\left(t_{j+1}\right), x_j = B\left(t_j\right)$ and sum over j :

$$\begin{aligned} f\left(T,B_T\right) - f\left(0,B_0\right) &= \sum_j \left(f\left(t_{j+1},B\left(t_{j+1}\right)\right) - f\left(t_j,B\left(t_j\right)\right)\right) \\ &= \sum_j f_t\left(t_j,B\left(t_j\right)\right)\left(t_{j+1}-t_j\right) \\ &+ \sum_j f_x\left(t_j,B\left(t_j\right)\right)\left(B\left(t_{j+1}\right)-B\left(t_j\right)\right) \\ &+ \sum_j \frac{1}{2}f_{xx}\left(t_j,B\left(t_j\right)\right)\left(B\left(t_{j+1}\right)-B\left(t_j\right)\right)^2 \\ &+ \sum_j f_{tx}\left(t_j,B\left(t_j\right)\right)\left(B\left(t_{j+1}\right)-B\left(t_j\right)\right)\left(t_{j+1}-t_j\right) \\ &+ \sum_j \frac{1}{2}f_{tt}\left(t_j,B\left(t_j\right)\right)\left(t_{j+1}-t_j\right)^2 + \text{ higher order terms.} \end{aligned}$$

As we take the limit $\|\Pi\| \to 0$ then the first term on the right-hand side converges to an ordinary Riemann integral

$$\sum_j f_t(t_j, B(t_j))(t_{j+1} - t_j) \to \int_0^T f_t(t, B_t)\, dt.$$

As $\|\Pi\| \to 0$ the second term converges to an Itô integral

$$\sum_j f_x(t_j, B(t_j))(B(t_{j+1}) - B(t_j)) \to \int_0^T f_x(t, B_t)\, dB_t$$

Let us study the third sum. To simplify notation put $a_j = f_{xx}(t_j, B_j), \Delta B_j = B(t_{j+1}) - B(t_j)$. Then

$$\sum_j \frac{1}{2} f_{xx}(t_j, B(t_j))(B(t_{j+1}) - B(t_j))^2 = \sum_j a_j (\Delta B_j)^2$$

Consider

$$\mathbb{E}\left[\left(\sum_j a_j (\Delta B_j)^2 - \sum_j a_j \Delta t_j\right)^2\right] = \sum_{i,j} \mathbb{E}\left[a_i a_j \left((\Delta B_j)^2 - \Delta t_j\right)\left((\Delta B_i)^2 - \Delta t_i\right)\right].$$

If $i < j$ then $a_i a_j \left((\Delta B_i)^2 - \Delta t_i\right)$ and $(\Delta B_j)^2 - \Delta t_j$ are independent and so the terms vanish in this case. Similarly for $i > j$. So we are left with

$$\begin{aligned}
\sum_j \mathbb{E}\left[a_j^2\left((\Delta B_j)^2 - \Delta t_j\right)^2\right] &= \sum_j \mathbb{E}\left[a_j^2\right]\mathbb{E}\left[(\Delta B_j)^4 - 2(\Delta B_j)^2 \Delta t_j + (\Delta t_j)^2\right] \\
&= \sum_j \mathbb{E}\left[a_j^2\right]\left[3(\Delta t_j)^2 - 2\Delta t_j \Delta t_j + (\Delta t_j)^2\right] \\
&= \sum_j \mathbb{E}\left[a_j^2\right](\Delta t_j)^2 \to 0 \text{ as } \Delta t_j \to 0.
\end{aligned}$$

Thus the third term converges to

$$\frac{1}{2}\int_0^T f_{xx}(t, B_t)\, dt$$

Theorem 1.3.2 — Differential Form of Itô Formula. One often writes Itô formula in the differential form:

$$df(t, B_t) = f_t(t, B_t)\, dt + f_x(t, B_t)\, dB_t + \frac{1}{2} f_{xx}(t, B_t)\, dt.$$

■ **Example 1.1** Previously, we applied the definition of the Itô integral to find

$$\int_0^T B_t dB_t = \frac{B_T^2}{2} - \frac{T}{2}.$$

Let us show how Itô formula simplifies the computation of Itô integrals. For example, with $f(t,x) = \frac{1}{2}x^2$ this formula says that

$$df(t, B_t) = f_t dt + f_x dB_t + \frac{1}{2} f_{xx} dt = 0 \cdot dt + B_t dB_t + \frac{1}{2} \cdot 1 \cdot dt = B_t dt + \frac{1}{2} dt.$$

Integrating we further obtain

$$\int_0^t d\left(\frac{1}{2}B_s^2\right) = \int_0^t \left(B_s dB_s + \frac{1}{2}ds\right),$$

$$\frac{1}{2}B_t^2 - \frac{1}{2}B_0^2 = \int_0^t B_s dB_s + \frac{t}{2},$$

$$\frac{1}{2}B_t^2 - \frac{t}{2} = \int_0^t B_s dB_s.$$

■

■ **Example 1.2** Let $\beta_k(t) = E\left[B_t^k\right]$. We will use Itô formula to prove recursive relation

$$\beta_k(t) = \frac{1}{2}k(k-1)\int_0^t \beta_{k-2}(s)ds \text{ for } k \geq 2.$$

Note that we can choose: $f(t, B_t) = B_t^k$. Then

$$df(t, B_t) = 0 \cdot dt + kB_t^{k-1}dB_t + \frac{1}{2}k(k-1)B_t^{k-2}dt.$$

Integrating

$$\int_0^t df_(s, B_s\Big) = f(t, B_t) - f(0, B_0) = B_t^k = \int_0^t kB_t^{k-1}dB_t + \frac{1}{2}k(k-1)\int_0^t B_t^{k-2}dt$$

$$\mathbb{E}\int_0^t f(s, B_s)\, dB_s = 0,$$

hence,

$$\mathbb{E}\int_0^t kB_t^{k-1}dB_t = 0.$$

We are left with $\mathbb{E}\left[\frac{1}{2}k(k-1)\int_0^t B_s^{k-2}ds\right] = \frac{1}{2}k(k-1)\int_0^t \beta_{k-2}(s)dt$.

We can simply obtain that

$$\mathbb{E}\left(B_t^4\right) = 3t^2.$$

Indeed

$$\mathbb{E}\left(B_t^4\right) = \beta_t^4 = \frac{1}{2}\cdot 4\cdot 3\int_0^t sds = 3t^2,$$

because $\int_0^t sds = \frac{t^2}{2}$

$$\mathbb{E}\left(B_t^{2k+1}\right) = 0, \forall k, \mathbb{E}\left(B_t^{2k}\right) = \frac{(2k)!t^k}{2^k k!}, \mathbb{E}\left(B_t^6\right) = 15t^3.$$

■

Previously, we only play with Brownian Motions, but now we're going to introduce a more complicated stochastic process driven by Brownian Motions.

Definition 1.3.1 — Itô process. An Itô process is a stochastic process X_t on $(\Omega, \mathscr{F}, \mathbb{P})$ of the form:

$$X_t = X_0 + \int_0^t \mu(s,\omega)ds + \int_0^t \nu(s,\omega)dB_s,$$

where $\nu \in V, \mathbb{P}\left(\int_0^t \nu^2(s,\omega)ds < \infty, \forall t \geq 0\right) = 1$. We also assume that μ is $\mathscr{F}_t$- adapted and $\mathbb{P}\left(\int_0^t |\mu(s,\omega)|ds < \infty, \forall t \geq 0\right) = 1$

Definition 1.3.2 — Differential Form of Itô process.

$$dX_t = \mu dt + \nu dB_t,$$

where μdt is called drift, νdB_t is called a volatile part.

We now extend Itô's Formula for Brownian motion to Itô's Lemma for Itô Process.

Theorem 1.3.3 — Itô's Lemma. Let X_t be an Itô process . Let $g(t,X) \in C^2([0,\infty) \times \mathbb{R})$, i. e., g is twice continuously differentiable on $[0,\infty) \times \mathbb{R}$. Then $Y_t = g(t,X_t)$ is again an Itô process and

$$dY_t = g_t'(t,X_t)\,dt + g_x'(t,X_t)\,dX_t + \frac{1}{2}g_{xx}''(t,X_t)\,(dX_t)^2,$$

where $(dX_t)^2$ is calculated according to the following rules

	dt	dB_t
dt	0	0
dB_t	0	dt

Thus $(dX_t)^2 = (\mu dt + \nu dB_t)^2 = \mu^2(dt)^2 + \nu^2(dB_t)^2 + 2\mu\nu dt dB_t = \nu^2 dt$.
Substituting dX_t,

$$\begin{aligned} dY_t &= g_t' dt + g_x'(\mu dt + \nu dB_t) + \frac{1}{2}g_{xx}''\nu^2 dt \\ &= \left(g_t' + g_x'\mu + \frac{1}{2}g_{xx}''\nu^2\right)dt + g_x'\nu dB_t \end{aligned}$$

with the first term representing new drift and the second representing new volatility term.

1.4 Stochastic Differential Equations

1.4.1 General SDE: Drifted Brownian Motion

Definition 1.4.1 — SDE. General Stochastic Differential Equation is an equation in the form of:

$$X_t = X_0 + \int_0^t b(s,w)ds + \int_0^t \sigma(s,w)dB_s$$

Why stochastic? Because $\int_0^t \sigma(s,w)dB_s$ is a random component, also $b = b(s,w)$ can be a random function. Thus The solution to an SDE is not a function, but a family of functions.

Definition 1.4.2 — Differential Form of SDE.

$$dX_t = bdt + \sigma dB_t, X(0) = X_0.$$

We can view SDE as a Brownian Motion with drift or an Ito process.

1.4.2 Geometric Brownian Motion

Could we use drifted brownian motion to model stock price? No! Because we observe that the right hand side of the differential form of general SDE can take negative values. Thus, we use drifted brownian motion to model rate of return.

Definition 1.4.3 — GBM. GBM is a process in the form of

$$\frac{dS_t}{S_t} = rdt + \sigma dB_t,$$

or equivalentlt,

$$dS_t = rS_t dt + \sigma S_t dB_t,$$

initial condition $S(0) = S_0$, where the term $\sigma S_t dB_t$ represents uncertainty.

- r - risk-free interest rate (assumed constant)
- σ-volatility (also assumed constant)
- B_t - standard Brownian motion
- S_t - stock price

1.4.3 Solve GBM

Consider the ordinary differential equation without uncertainty:

$$dS_t = rS_t dt, \text{ initial condition } S(0) = S_0.$$

Dividing both sides by S_t yields:

$$\frac{dS_t}{S_t} = rdt \Longrightarrow d(\log S_t) = \frac{dS_t}{S_t} \Longrightarrow d(\log S_t) = rdt.$$

Integrating both sides from 0 to t, we have

$$\log S_t - \log S_0 = rt, \Longrightarrow S_t = S_0 \exp(rt).$$

In other words, investing S_0 at time 0 results in $S_0 \exp(rt)$ at time t.

On the contrary, GBM $dS_t = rS_t dt + \sigma S_t dB_t$ is a formula for return on risk-free investment plus uncertainty.

By Ito formula: $dg(t, S_t) = g_t' dt + g_s' ds + \frac{1}{2} g_{ss}''(ds)^2$. We take $g(t, S_t) = \log S_t$

$$d\log S_t = \frac{dS_t}{S_t} - \frac{1}{2}\frac{(dS_t)^2}{S_t^2}$$

$$(dS_t)^2 = r^2 S_t^2 (dt)^2 + 2r\sigma S_t^2 dt dB_t + \sigma^2 S_t^2 (dB_t)^2 = \sigma^2 S_t^2 (dB_t)^2$$

$$d\log S_t = rdt + \sigma dB_t - \frac{1}{2}\sigma^2 dt = \left(r - \frac{\sigma^2}{2}\right) dt + \sigma dB_t$$

$$\log S_t - \log S_0 = \left(r - \frac{\sigma^2}{2}\right) t + \sigma B_t$$

$$S_t = S_0 \exp\left(\left(r - \frac{\sigma^2}{2}\right) t + \sigma B_t\right)$$

The solution to GBM implies that $S_t \geq 0$, as long as $S_0 \geq 0$.

Remark: This result allows us to calculate the prices for vanilla European options.

The call price $= e^{-rt}\mathbb{E}(S_t - K)^+ = S_0 N(d_1) - K\exp(-rt) N(d_2)$

$$d_1 = \frac{\left(\log \frac{S_0}{K} + rt\right) + \frac{\sigma^2 t}{2}}{\sigma\sqrt{t}}, d_2 = d_1 - \sigma\sqrt{t}, N(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) dy$$

1.4.4 Ornstein-Uhlenbeck SDE

Definition 1.4.4 OU SDE is equations in the form of

$$dX_t = (\beta - X_t) dt + \sigma dB_t, X(0) = X_0,$$

which is equivalent to

$$dX_t = \beta dt - X_t dt + \sigma dB_t$$

and

$$dX_t + X_t dt = \beta dt + \sigma dB_t$$

1.4.5 Properties of OU SDE

Take $g(t,x) = e^t X_t$, then $d(e^t X_t) = e^t X_t dt + e^t dX_t + 0 = e^t (X_t dt + dX_t)$
Thus, $d(e^t X_t) = \beta e^t dt + \sigma e^t dB_t$

It implies that

$$e^t X_t - e^0 X_0 = \beta \int_0^t e^s ds + \sigma \int_0^t e^s dB_s$$
$$e^t X_t - X_0 = \beta \left(e^t - 1\right) + \sigma \int_0^t e^s dB_s$$
$$X_t = X_0 e^{-t} + \beta \left(1 - e^{-t}\right) + \sigma e^{-t} \int_0^t e^s dB_s$$
$$\mathbb{E}[X_t] = X_0 e^{-t} + \beta \left(1 - e^{-t}\right)$$

because $\mathbb{E} \int_0^t e^s dB_s = 0$. Clearly, X_t is normally distributed, as long as the function in the integrand is deterministic, because that makes X_t the sum of normally distributed random variables

$$\int_0^t f(s) dB_s \approx \sum_{i=0}^{n-1} f(S_i)\left(B(S_{i+1}) - B(S_i)\right)$$

Note that, as $t \to \infty$, $\mathbb{E}[X_t] \to \beta$. Therefore, OU process is called mean reverting.

1.5 Derivations of Black-Scholes Equation

1.5.1 Through Δ-Hedging

We write Geometric Brownian Motion in the form of $dS_t = \mu S_t dt + \sigma S_t dB_t$. And Let $V(t, S_t)$ be the price of the option as a function of stock price S_t and time t. By definition 1.3.3: Ito's Lemma, we plug in $\mu = \mu S_t$ and $\nu = \sigma S_t$,we can deduce that

$$dV = \left(\frac{\partial V}{\partial t} + \frac{\partial V}{\partial S}\mu S + \frac{1}{2} \times \frac{\partial^2 V}{\partial S^2}(\sigma S)^2\right)dt + \frac{\partial V}{\partial S}\sigma S dB_t$$

From here we will derive the Black-Scholes equation. To do this let us construct a portfolio:

$$\Pi = V - \Delta S$$

which stands for the value of selling 1 portion of option and buying Δ portion of stock. The choice of Δ will be clear after computing $\frac{d\Pi}{dt}$, the jump in the value of this portfolio in one time step:

$$\begin{aligned}
d\Pi &= dV - \Delta dS \\
&= \left(\frac{\partial V}{\partial t} + \frac{\partial V}{\partial S}\mu S + \frac{1}{2} \times \frac{\partial^2 V}{\partial S^2}(\sigma S)^2\right)dt + \frac{\partial V}{\partial S}\sigma S dB_t - \Delta(\mu S_t dt + \sigma S_t dB_t) \\
&= \underbrace{\sigma S\left(\frac{\partial V}{\partial S} - \Delta\right)dB_t}_{\text{random part}} + \left(\mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} - \Delta S \mu\right)dt
\end{aligned}$$

We want to choose Δ so that instant rate of change in the value of portfolio Π with respect to t is NOT affected by stochastic volatility. Hence, it's natural to choose

$$\Delta = \frac{\partial V}{\partial S}$$

$$\implies d\Pi = (\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2})dt$$

At the same time, in order to prevent the appearance of arbitrage, we require

$$d\Pi = r\Pi dt$$

where the RHS represents the risk-free return over an infinitesimal time interval dt.
Plugging in $\Pi = V - \Delta S = V - \frac{\partial V}{\partial S}S$, we have

$$\begin{aligned} d\Pi &= r\Pi dt \\ (\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2})dt &= r(V - \Delta S)dt \\ &= r(V - \frac{\partial V}{\partial S}S)dt \end{aligned}$$

We deduce the famous BS-Equation:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} = rV$$

Note that, we can always make use of table to better formulate our idea: deduce BS-Equation by considering how cash flow evolves in a time interval dt.

time	risk-free	stock(portion)	option(portion)
t	V(t,S_t) $- \Delta S_t$	Δ	-1
t+dt	(V(t,S_t) $- \Delta S_t)e^{rdt}$	Δ	-1

We require that the second line minus the first line gives us 0.

$$-(V(t+dt, S_{t+dt}) - V(t,S_t)) + \Delta(S_{t+dt} - S_t) + (V(t,S_t) - \Delta S_t)(e^{rdt} - 1) = 0$$

Denoting $S_{t+dt} - S_t := dS_t$ and since $(dS_t)^2 = \sigma^2 S_t^2 dt$ by Ito's Lemma, we have that

$$\begin{aligned} &-(\frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS_t + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(dS_t)^2) + \Delta dS_t + (V - \Delta S_t)rdt \\ &= -(\frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS_t + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S_t^2 dt) + \Delta dS_t + (V - \Delta S_t)rdt \end{aligned}$$

Given that most people are risk-aversive, we take $\Delta = \frac{\partial V}{\partial S}$ to eliminate dS_t term.

$$\implies \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} = rV$$

1.5.2 Through Feymann-Kac Theorem

Theorem 1.5.1 — Feymann-Kac. Consider the SDE

$$dX(u) = \beta(u, X(u))du + \gamma(u, X(u))dW(u).$$

Let $h(y)$ be a Borel-measurable function. Fix $T > 0$, and let $t \in [0,T]$ be given. Define the function

$$g(t,x) = \mathbb{E}^{t,x}h(X(T)).$$

Then $g(t,x)$ satisfies the partial differential equation

$$g_t(t,x) + \beta(t,x)g_x(t,x) + \frac{1}{2}\gamma^2(t,x)g_{xx}(t,x) = 0$$

and the terminal condition

$$g(T,x) = h(x) \text{ for all } x.$$

Lemma:
Let $X(u)$ be a solution to the above SDE with initial condition given at time 0 . Let h(y) be a Borel-measurable function, fix T>0, and let $g(t,x) = \mathbb{E}^{t,x}h(X(T))$. Then the stochastic process

$$g(t,X(t)), \quad 0 \le t \le T,$$

is a martingale.
Proof: Let $0 \le s \le t \le T$ be given. Since

$$\begin{aligned}\mathbb{E}[h(X(T)) \mid \mathscr{F}(s)] &= g(s,X(s)),\\ \mathbb{E}[h(X(T)) \mid \mathscr{F}(t)] &= g(t,X(t)).\end{aligned}$$

Then

$$\begin{aligned}\mathbb{E}[g(t,X(t)) \mid \mathscr{F}(s)] &= \mathbb{E}[\mathbb{E}[h(X(T)) \mid \mathscr{F}(t)] \mid \mathscr{F}(s)]\\ &= \mathbb{E}[h(X(T)) \mid \mathscr{F}(s)]\\ &= g(s,X(s))\end{aligned}$$

Proof of Feymann Kac:

$$\begin{aligned}dg(t,X(t)) &= g_t dt + g_x dX + \frac{1}{2}g_{xx}dXdX\\ &= g_t dt + \beta g_x dt + \gamma g_x dW + \frac{1}{2}\gamma^2 g_{xx}dt\\ &= \left[g_t + \beta g_x + \frac{1}{2}\gamma^2 g_{xx}\right]dt + \gamma g_x dW.\end{aligned}$$

Since $g(t,X(t))$ is a martingale, it has null-drift, thus

$$g_t + \beta g_x + \frac{1}{2}\gamma^2 g_{xx} = 0$$

Theorem 1.5.2 — Discounted Feymann Kac. Consider the SDE

$$dX(u) = \beta(u,X(u))du + \gamma(u,X(u))dW(u).$$

Let $h(y)$ be a Borel-measurable function and let r be constant. Define the function

$$f(t,x) = \mathbb{E}^{t,x}\left[e^{-r(T-t)}h(X(T))\right].$$

Then $f(t,x)$ satisfies the partial differential equation

$$f_t(t,x) + \beta(t,x)f_x(t,x) + \frac{1}{2}\gamma^2(t,x)f_{xx}(t,x) = rf(t,x)$$

and the terminal condition
$$f(T,x) = h(x) \text{ for all } x.$$

Proof:
$$f(t,X(t)) = \mathbb{E}\left[e^{-r(T-t)}h(X(T)) \mid \mathscr{F}(t)\right].$$
However, it is not the case that $f(t,X(t))$ is a martingale. Indeed, if $0 \le s \le t \le T$, then

$$\begin{aligned}\mathbb{E}[f(t,X(t)) \mid \mathscr{F}(s)] &= \mathbb{E}\left[\mathbb{E}\left[e^{-r(T-t)}h(X(T)) \mid \mathscr{F}(t)\right] \mid \mathscr{F}(s)\right] \\ &= \mathbb{E}\left[e^{-r(T-t)}h(X(T)) \mid \mathscr{F}(s)\right],\end{aligned}$$

which is not the same as

$$f(s,X(s)) = \mathbb{E}\left[e^{-r(T-s)}h(X(T)) \mid \mathscr{F}(s)\right]$$

The difficulty here is that we need the random variable being estimated not to depend on t. To get a martingale, we "complete the discounting,"

$$e^{-rt}f(t,X(t)) = \mathbb{E}\left[e^{-rT}h(X(T)) \mid \mathscr{F}(t)\right]$$

The differential of this martingale is

$$\begin{aligned}d\left(e^{-rt}f(t,X(t))\right) &= e^{-rt}\left[-rf\,dt + f_t dt + f_x dX + \frac{1}{2}f_{xx}dXdX\right] \\ &= e^{-rt}\left[-rf + f_t + \beta f_x + \frac{1}{2}\gamma^2 f_{xx}\right]dt + e^{-rt}\gamma f_x dW\end{aligned}$$

In the case of GBM, we plug in $f = V, X_t = S_t, \beta = rS, \gamma = \sigma S$,then we can obtain BS-Equation:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} = rV$$

1.6 Derivations of Black-Scholes Formula

1.6.1 Derive BS formula by Solving BS Equation as a Heat Equation

First, start with the Black-Scholes equation:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

Then set $t = T - \frac{\tau}{\frac{1}{2}\sigma^2}$ and solve for τ :

$$\frac{\tau}{\frac{1}{2}\sigma^2} = T - t$$

$$\tau = (T-t)\frac{1}{2}\sigma^2$$

Next set $S = Ke^x$ and solve for x :

$$e^x = \frac{S}{K}$$
$$x = \ln\left(\frac{S}{K}\right)$$

With both of these equations, set:

$$V(S,t) = Kv(x,\tau)$$

The next step is to take the first and second derivatives of V with respect to stock price and the first derivative with respect to time:

$$\frac{\partial V}{\partial t} = K\frac{\partial v}{\partial \tau} * \frac{\partial \tau}{\partial t} = K\frac{\partial v}{\partial \tau}\left[(T-t)\frac{1}{2}\sigma^2\frac{\partial}{\partial t}\right] = K\frac{\partial v}{\partial \tau} * \frac{-\sigma^2}{2}$$

$$\frac{\partial V}{\partial S} = K\frac{\partial v}{\partial x} * \frac{\partial x}{\partial S} = K\frac{\partial v}{\partial x}\left[\ln\left(\frac{S}{K}\right)\frac{\partial}{\partial S}\right] = K\frac{\partial v}{\partial x} * \frac{1}{S}$$

Using $\frac{\partial x}{\partial S} = \frac{1}{\frac{S}{K}} * \frac{1}{K} = \frac{1}{S}$:

$$\begin{aligned}
\frac{\partial^2 V}{\partial S^2} &= \frac{\partial}{\partial S}\left(K\frac{\partial v}{\partial x} * \frac{1}{S}\right) \\
&= K\frac{\partial v}{\partial x} * \frac{-1}{S^2} + K\frac{\partial}{\partial S}\left(\frac{\partial v}{\partial x}\right)\frac{1}{S} \\
&= K\frac{\partial v}{\partial x} * \frac{-1}{S^2} + K\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right)\frac{\partial x}{\partial S} * \frac{1}{S} \\
&= K\frac{\partial v}{\partial x} * \frac{-1}{S^2} + K\frac{\partial^2 v}{\partial x^2} * \frac{1}{S^2}
\end{aligned}$$

With these equations, the terminal equation is set to:

$$V(S,T) = \max(S-K,0) = \max\left(Ke^x - K, 0\right)$$

$$V(S,T) = Kv(x,0) \text{ and } v(x,0) = \max\left(e^x - 1, 0\right)$$

Take the derivatives and plug them back into the BS-equation:

$$\left(K\frac{\partial v}{\partial \tau} * \frac{-\sigma^2}{2}\right) + \frac{\sigma^2}{2}S^2\left(K\frac{\partial v}{\partial x} * \frac{-1}{S^2} + K\frac{\partial^2 v}{\partial x^2} * \frac{1}{S^2}\right) + rS\left(K\frac{\partial v}{\partial x} * \frac{1}{S}\right) - rKv = 0$$

Simplify the equation by factoring out the K values, canceling out S and S^2 :

$$\left(\frac{\partial v}{\partial \tau} * \frac{-\sigma^2}{2}\right) + \frac{\sigma^2}{2}S^2\left(\frac{\partial v}{\partial x} * \frac{-1}{S^2} + \frac{\partial^2 v}{\partial x^2} * \frac{1}{S^2}\right) + rS\left(\frac{\partial v}{\partial x} * \frac{1}{S}\right) - rv = 0$$

$$\left(\frac{\partial v}{\partial \tau} * \frac{-\sigma^2}{2}\right) + \frac{\sigma^2}{2}\left(\frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial x}\right) + r\left(\frac{\partial v}{\partial x}\right) - rv = 0$$

Solve for $\frac{\partial v}{\partial \tau}$:

$$\frac{\partial v}{\partial \tau} * \frac{\sigma^2}{2} = \frac{\sigma^2}{2}\left(\frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial x}\right) + r\frac{\partial v}{\partial x} - rv$$

Factor out $\frac{\sigma^2}{2}$, let $k = \frac{r}{\frac{\sigma^2}{2}}$ to substitute, and combine like terms:

$$\frac{\partial v}{\partial \tau} = \left(\frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial x}\right) + \frac{r}{\frac{\sigma^2}{2}} * \frac{\partial v}{\partial x} - \frac{r}{\frac{\sigma^2}{2}}v$$

$$\frac{\partial v}{\partial \tau} = \left(\frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial x}\right) + k\frac{\partial v}{\partial x} - kv$$

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + (k-1)\frac{\partial v}{\partial x} - kv$$

This leaves one parameter, k, that has no dimension. From this, rescale the v equation so that:

$$v = e^{\alpha x + \beta\tau} u(x,\tau)$$

Derive according to x and τ :

$$\begin{aligned}
\frac{\partial v}{\partial \tau} &= \beta e^{\alpha x+\beta\tau} u + e^{\alpha x+\beta\tau}\frac{\partial u}{\partial \tau} \\
\frac{\partial v}{\partial x} &= \alpha e^{\alpha x+\beta\tau} u + e^{\alpha x+\beta\tau}\frac{\partial u}{\partial x} \\
\frac{\partial^2 v}{\partial x^2} &= \alpha^2 e^{\alpha x+\beta\tau} u + 2\alpha e^{\alpha x+\beta\tau}\frac{\partial u}{\partial x} + e^{\alpha x+\beta\tau}\frac{\partial^2 u}{\partial x^2}
\end{aligned}$$

Plug these derivatives back:

$$\beta e^{\alpha x+\beta\tau} u + e^{\alpha x+\beta\tau}\frac{\partial u}{\partial \tau} = \alpha^2 e^{\alpha x+\beta\tau} u + 2\alpha e^{\alpha x+\beta\tau}\frac{\partial u}{\partial x} + e^{\alpha x+\beta\tau}\frac{\partial^2 u}{\partial x^2} + (k-1)\left(\alpha e^{\alpha x+\beta\tau} u + e^{\alpha x+\beta\tau}\frac{\partial u}{\partial x}\right) - k e^{\alpha x+\beta\tau} u$$

Divide by $e^{\alpha x+\beta\tau}$ and combine like terms:

$$\begin{aligned}
\beta u + \frac{\partial u}{\partial \tau} &= \alpha^2 u + 2\alpha\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} + (k-1)\left(\alpha u + \frac{\partial u}{\partial x}\right) - ku \\
\beta u + \frac{\partial u}{\partial \tau} &= \alpha^2 u + 2\alpha\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} + k\alpha u + k\frac{\partial u}{\partial x} - \alpha u - \frac{\partial u}{\partial x} - ku \\
\frac{\partial u}{\partial \tau} &= \alpha^2 u + 2\alpha\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} + k\alpha u + k\frac{\partial u}{\partial x} - \alpha u - \frac{\partial u}{\partial x} - ku - \beta u \\
\frac{\partial u}{\partial \tau} &= \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}(k-1+2\alpha) + u\left(\alpha^2 + k\alpha - \alpha - k - \beta\right)
\end{aligned}$$

Set $\alpha = \frac{-(k-1)}{2}$ and $\beta = \alpha^2 + (k-1)\alpha - k = \frac{-(k+1)^2}{4}$. This will lead to the basis of the Heat Equation:

$$u_\tau = u_{xx}$$

The initial condition is then transformed into:

$$u(x,0) = \max\left(e^{\left(\frac{(k+1)}{2}x\right)} - e^{\left(\frac{(k-1)}{2}x\right)}, 0\right)$$

This leads to the Heat equation solution, which we will transform to use for the Black-Scholes equation:

$$u(x,\tau) = \frac{1}{2\sqrt{\pi\tau}}\int_{-\infty}^{\infty} u_0(s) e^{\frac{-(x-s)^2}{4\tau}} ds$$

Make a change of variable so that $s = z\sqrt{2\tau} + x$. The goal is to get the exponent into the form of $\frac{-y^2}{2}$, which is why $z = \frac{x-s}{\sqrt{2\tau}}$, to get the equation of the standard normal deviation. This will then be used later in this derivation to find the final solution. The derivatives of these equations will then be $ds = dx$ and $dx = \sqrt{2\tau}dz$:

$$u(x,\tau) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u_0(z\sqrt{2\tau}+x) e^{\frac{-z^2}{2}} dz$$

From this transformation,

$$u_0 = e^{\frac{k+1}{2}(x+z\sqrt{2\tau})} - e^{\frac{k-1}{2}(x+z\sqrt{2\tau})}$$

It must happen that $u_0 > 0$ because the time value cannot be less than 0 . So, $x > -\frac{x}{\sqrt{2\tau}}$ which transforms the base of the domain of the integral :

$$
\begin{aligned}
u(x,\tau) &= \frac{1}{\sqrt{2\pi}}\int_{-\frac{x}{\sqrt{2\tau}}}^{\infty} e^{\frac{k+1}{2}(x+z\sqrt{2\tau})}e^{-\frac{z^2}{2}} - e^{\frac{k-1}{2}(x+z\sqrt{2\tau})}e^{-\frac{z^2}{2}}dz \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\frac{x}{\sqrt{2\tau}}}^{\infty} e^{\frac{k+1}{2}(x+z\sqrt{2\tau})}e^{-\frac{z^2}{2}}dz - \frac{1}{\sqrt{2\pi}}\int_{-\frac{x}{\sqrt{2\tau}}}^{\infty} e^{\frac{k-1}{2}(x+z\sqrt{2\tau})}e^{-\frac{z^2}{2}}dz \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\frac{x}{\sqrt{2\tau}}}^{\infty} e^{\frac{k+1}{2}(x+z\sqrt{2\tau})-\frac{z^2}{2}}dz - \frac{1}{\sqrt{2\pi}}\int_{-\frac{x}{\sqrt{2\tau}}}^{\infty} e^{\frac{k-1}{2}(x+z\sqrt{2\tau})-\frac{z^2}{2}}dz
\end{aligned}
$$

After the split of the integral, take the first integral and complete the square of the exponent:

$$
\begin{aligned}
\frac{k+1}{2}(x+z\sqrt{2\tau}) - \frac{z^2}{2} &= -\frac{1}{2}\left[z^2 - z\sqrt{2\tau}(k+1)\right] + \frac{x(k+1)}{2} \\
&= -\frac{1}{2}\left[z^2 - z\sqrt{2\tau}(k+1) + \frac{\tau}{2}(k+1)^2\right] + \frac{x(k+1)}{2} - \left[-\frac{\tau(k+1)^2}{4}\right] \\
&= -\frac{1}{2}\left[z - \sqrt{\frac{\tau}{2}}(k+1)\right]^2 + \frac{x(k+1)}{2} + \frac{\tau(k+1)^2}{4}
\end{aligned}
$$

The first integral becomes:

$$
\frac{e^{\frac{x(k+1)}{2}+\frac{\tau(k+1)^2}{4}}}{\sqrt{2\pi}}\int_{\frac{-x}{\sqrt{2\tau}}}^{\infty} e^{-\frac{1}{2}\left[z-\sqrt{\frac{\tau}{2}}(k+1)\right]^2}dz
$$

Set $y = z - \sqrt{\frac{\tau}{2}}(k+1), dy = dz$, and $z = \frac{-x}{\sqrt{2\tau}}$ which in turn changes the domain once again:

$$
\frac{e^{\frac{x(k+1)}{2}+\frac{\tau(k+1)^2}{4}}}{\sqrt{2\pi}}\int_{\frac{-x}{\sqrt{2\tau}}-\sqrt{\frac{\tau}{2}(k+1)}}^{\infty} e^{-\frac{y^2}{2}}dy
$$

Then:

$$
u(x,\tau) = \frac{e^{\frac{x(k+1)}{2}+\frac{\tau(k+1)^2}{4}}}{\sqrt{2\pi}}\int_{-\infty}^{\frac{x}{\sqrt{2\tau}}+\sqrt{\frac{\tau}{2}}(k+1)} e^{-\frac{y^2}{2}}dy - \frac{e^{\frac{x(k-1)}{2}+\frac{\tau(k-1)^2}{4}}}{\sqrt{2\pi}}\int_{-\infty}^{\frac{-x}{\sqrt{2\tau}}-\sqrt{\frac{\tau}{2}(k+1)}} e^{-\frac{y^2}{2}}dy
$$

Thus,

$$
u(x,\tau) = e^{\frac{x(k+1)}{2}+\frac{\tau(k+1)^2}{4}}N(d_1) - e^{\frac{x(k-1)}{2}+\frac{\tau(k-1)^2}{4}}N(d_2)
$$

where

$$
\begin{aligned}
d_1 &= \frac{x}{\sqrt{2\tau}} + \sqrt{\frac{\tau}{2}}(k+1) \\
d_2 &= \frac{x}{\sqrt{2\tau}} + \sqrt{\frac{\tau}{2}}(k-1)
\end{aligned}
$$

Thus,

$$
\begin{aligned}
v(x,\tau) &= e^{\frac{-x(k-1)}{2}-\frac{\tau(k+1)^2}{4}}u(x,\tau) \\
&= e^{\frac{-x(k-1)}{2}-\frac{\tau(k+1)^2}{4}} * \left[e^{\frac{(k+1)x}{2}+\frac{\tau(k+1)^2}{4}}N(d_1) - e^{\frac{x(k-1)}{2}+\frac{\tau(k-1)^2}{4}}N(d_2)\right] \\
&= e^{x}N(d_1) - e^{-k\tau}N(d_2) \\
&= e^{ln(S/K)}N(d_1) - e^{-\frac{k}{2}\sigma^2(T-t)}N(d_2) \\
&= \frac{S}{K}N(d_1) - e^{-\frac{k}{2}\sigma^2(T-t)}N(d_2) \\
&= \frac{S}{K}N(d_1) - e^{-r(T-t)}N(d_2)
\end{aligned}
$$

For d-values,

$$
\begin{aligned}
d_1 &= \frac{\ln\left(\frac{S}{K}\right)}{\sqrt{2\left(\frac{1}{2}\sigma^2(T-t)\right)}} + \sqrt{\frac{\frac{1}{2}\sigma^2(T-t)}{2}}(k+1) \\
&= \frac{\ln\left(\frac{S}{K}\right)}{\sigma\sqrt{T-t}} + \frac{\sigma}{2}\sqrt{T-t}(k+1) \\
&= \frac{\ln\left(\frac{S}{K}\right)}{\sigma\sqrt{T-t}} + \frac{\frac{\sigma^2}{2}(T-t)(k+1)}{\sigma\sqrt{T-t}} \\
&= \frac{\ln\left(\frac{S}{K}\right)\left(\frac{\sigma^2}{2}k+\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \\
&= \frac{\ln\left(\frac{S}{K}\right)+\left(r+\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}
\end{aligned}
$$

Then

$$
\begin{aligned}
V(S,t) &= Kv(x,\tau) \\
&= K\frac{S}{K}N(d_1) - Ke^{-r(T-t)}N(d_2) \\
&= SN(d_1) - Ke^{-r(T-t)}N(d_2)
\end{aligned}
$$

We then have the solution to the Black-Scholes Equation:

$$V(S,t) = SN(d_1) - Ke^{-r(T-t)}N(d_2)$$

where,

$$
\begin{aligned}
d_1 &= \frac{\ln\left(\frac{S}{K}\right)+\left(r+\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \\
d_2 &= \frac{\ln\left(\frac{S}{K}\right)+\left(r-\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}
\end{aligned}
$$

1.6.2 Derive BS formula by Risk-Neutral Pricing

Is it possible to derive BS-formula without deriving BS equation? The answer is YES. We'll show that the discounted stock price is a martingale with respect to risk-neutral measure. And then we shall derive the formula for $V(S,t)$, which is. exactly BS formula.

Definition 1.6.1 Two probability measures P and $\tilde{P}$ are said to be equivalent if for every event $A, P(A) = 0$ if and only if $\tilde{P}(A) = 0$.

■ **Example 1.3** Let Z be a random variable such that $EZ = 1$ and $Z > 0$. Define a new measure $\tilde{P}$ by

$$\tilde{P}(A) = EZ\mathbf{1}_A = \int_A ZdP.$$

for every event A. Then P and $\tilde{P}$ are equivalent. ■

REMARK 1.3. The assumption $EZ = 1$ above is required to guarantee $\tilde{P}(\Omega) = 1$.

Definition 1.6.2 When $\tilde{P}$ is defined by Example 1.3, we say

$$d\tilde{P} = ZdP \quad \text{or} \quad Z = \frac{d\tilde{P}}{dP},$$

and Z is called the density of $\tilde{P}$ with respect to P.

Theorem 1.6.1 — Radon-Nikodym. Two measures P and $\tilde{P}$ are equivalent if and only if there exists a random variable Z such that $EZ = 1, Z > 0$ and $\tilde{P}$ is given by Example 1.3.

Suppose now $T > 0$ is fixed, and Z is a martingale. Define a new measure $\tilde{P} = \tilde{P}_T$ by

$$d\tilde{P} = d\tilde{P}_T = Z(T)dP.$$

We will denote expectations and conditional expectations with respect to the new measure by $\tilde{E}$. That is, given a random variable X,

$$\tilde{E}X = EZ(T)X = \int Z(T)XdP.$$

Also, given a σ-algebra $\mathscr{F}, \tilde{E}(X \mid \mathscr{F})$ is the unique $\mathscr{F}$-measurable random variable such that

$$\int_F \tilde{E}(X \mid \mathscr{F})d\tilde{P} = \int_F Xd\tilde{P},$$

holds for all $\mathscr{F}$ measurable events F. Of course, equation (1.2) is equivalent to requiring

$$\int_F Z(T)\tilde{E}(X \mid \mathscr{F})dP = \int_F Z(T)\tilde{X}dP,$$

for all $\mathscr{F}$ measurable events F.

The Girsanov Theorem

Now we introduce and prove the Girsanov theorem.

Theorem 1.6.2 — Girsanov Theorem. Let $b(t) = (b_1(t), b_2(t), \ldots, b_d(t))$ be a d-dimensional adapted process, W be a d-dimensional Brownian motion, and define Let Z be the process defined by

$$\tilde{W}(t) = W(t) + \int_0^t b(s)ds.$$

$$Z(t) = \exp\left(-\int_0^t b(s) \cdot dW(s) - \frac{1}{2}\int_0^t |b(s)|^2 ds\right),$$

and define a new measure $\tilde{P} = \tilde{P}_T$ by $d\tilde{P} = Z(T)dP$. If Z is a martingale then $\tilde{W}$ is a Brownian motion under the measure $\tilde{P}$ up to time T.

REMARK Above

$$b(s) \cdot dW(s) \stackrel{\text{def}}{=} \sum_{i=1}^d b_i(s)dW_i(s) \quad \text{and} \quad |b(s)|^2 = \sum_{i=1}^d b_i(s)^2.$$

REMARK Note $Z(0) = 1$, and if Z is a martingale then $EZ(T) = 1$ ensuring $\tilde{P}$ is a probability measure. You might, however, be puzzled at need for the assumption that Z is a martingale. Indeed, let $M(t) = \int_0^t b(s) \cdot dW(s)$, and $f(t,x) = \exp\left(-x - \frac{1}{2}\int_0^t b(s)^2 ds\right)$. Then, by Itô's formula,

$$\begin{aligned} dZ(t) &= d(f(t, M(t))) = \partial_t f dt + \partial_x f dM(t) + \frac{1}{2}\partial_x^2 f d[M,M](t) \\ &= -\frac{1}{2}Z(t)|b(t)|^2 dt - Z(t)b(t) \cdot dW(t) + \frac{1}{2}Z(t)|b(t)|^2 dt \end{aligned}$$

and hence

$$dZ(t) = -Z(t)b(t) \cdot dW(t).$$

Thus you might be tempted to say that Z is always a martingale, assuming it explicitly is unnecessary. However, we recall that Itô integrals are only guaranteed to be martingales under the square integrability condition

$$E\int_0^T |Z(s)b(s)|^2 ds < \infty.$$

Without this finiteness condition, Itô integrals are only local martingales, whose expectation need not be constant, and so $EZ(T) = 1$ is not guaranteed. Indeed, there are many examples of processes b where the finiteness condition (1.4) does not hold and we have $EZ(T) < 1$ for some $T > 0$.

The main idea behind the proof of the Girsanov theorem is the following: Clearly $\left[\tilde{W}_i, \tilde{W}_j\right] = [W_i, W_j] = \mathbf{1}_{i=j}$ t. Thus if we can show that $\tilde{W}$ is a martingale with respect to the new measure $\tilde{P}$, then Lévy's criterion will guarantee $\tilde{W}$ is a Brownian motion. We now develop the tools required to check when processes are martingales under the new measure.

Proposition 1.6.3 Let $0 \leqslant s \leqslant t \leqslant T$. If X is a $\mathscr{F}_t$-measurable random variable then

$$\tilde{E}\left(X \mid \mathscr{F}_s\right) = \frac{1}{Z(s)} E\left(Z(t)X \mid \mathscr{F}_s\right)$$

Proof: Let $A \in \mathscr{F}_s$ and observe that

$$\begin{aligned}\int_A \tilde{E}\left(X \mid \mathscr{F}_s\right) d\tilde{P} &= \int_A Z(T)\tilde{E}\left(X \mid \mathscr{F}_s\right) dP \\ &= \int_A E\left(Z(T)\tilde{E}\left(X \mid \mathscr{F}_s\right) \mid \mathscr{F}_s\right) dP = \int_A Z(s)\tilde{E}\left(X \mid \mathscr{F}_s\right) dP.\end{aligned}$$

Also,

$$\begin{aligned}\textstyle\int_A \tilde{E}\left(X \mid \mathscr{F}_s\right) d\tilde{P} = \int_A X d\tilde{P} = \int_A XZ(T)dP = \int_A E\left(XZ(T) \mid \mathscr{F}_t\right) dP \\ \textstyle= \int_A Z(t)XdP = \int_A E\left(Z(t)X \mid \mathscr{F}_s\right) dP\end{aligned}$$

Thus

$$\int_A Z(s)\tilde{E}\left(X \mid \mathscr{F}_s\right) dP = \int_A E\left(Z(t)X \mid \mathscr{F}_s\right) dP$$

for every $\mathscr{F}_s$ measurable event A. Since the integrands are both $\mathscr{F}_s$ measurable this forces them to be equal, the proposition is proved.

Proposition 1.6.4 An adapted process M is a martingale under $\tilde{P}$ if and only if MZ is a martingale under P.

Proof: Suppose first MZ is a martingale with respect to P. Then

$$\tilde{E}\left(M(t) \mid \mathscr{F}_s\right) = \frac{1}{Z(s)} E\left(Z(t)M(t) \mid \mathscr{F}_s\right) = \frac{1}{Z(s)} Z(s)M(s) = M(s),$$

showing M is a martingale with respect to P.

Conversely, suppose M is a martingale with respect to $\tilde{P}$. Then

$$E\left(M(t)Z(t) \mid \mathscr{F}_s\right) = Z(s)\tilde{E}\left(M(t) \mid \mathscr{F}_s\right) = Z(s)M(s),$$

and hence ZM is a martingale with respect to P.

Proof of Girsanov Theorem: Clearly $\tilde{W}$ is continuous and

$$d\left[\tilde{W}_i, \tilde{W}_j\right](t) = d\left[W_i, W_j\right](t) = \mathbf{1}_{i=j} dt.$$

Thus if we show that each $\tilde{W}_i$ is a martingale (under $\tilde{P}$), then by Lévy's criterion, $\tilde{W}$ will be a Brownian motion under $\tilde{P}$.

We now show that each $\tilde{W}_i$ is a martingale under $\tilde{P}$. By Lemma , $\tilde{W}_i$ is a martingale under $\tilde{P}$ if and only if $Z\tilde{W}_i$ is a martingale under P. To show $Z\tilde{W}_i$ is a martingale under P, we use the product rule and get

$$\begin{aligned} d\left(Z\tilde{W}_i\right) &= Zd\tilde{W}_i + \tilde{W}_i dZ + d\left[Z, \tilde{W}_i\right] \\ &= ZdW_i + Zb_i dt - \tilde{W}_i Zb \cdot dW - b_i Zdt = ZdW_i - \tilde{W}_i Zb \cdot dW. \end{aligned}$$

Thus $Z\tilde{W}_i$ is a martingale under P, and by Lemma , $\tilde{W}_i$ is a martingale under $\tilde{P}$. This finishes the proof.

Discounted stock price process

Consider a stock whose price is modelled by a generalized geometric Brownian

$$dS(t) = \alpha(t)S(t)dt + \sigma(t)S(t)dW(t),$$

where $\alpha(t), \sigma(t)$ are the (time dependent) mean return rate and volatility respectively. Here α and σ are no longer constant, but allowed to be adapted processes. We will, however, assume $\sigma(t) > 0$.

Suppose an investor has access to a money market account with variable interest rate $R(t)$. Again, the interest rate R need not be constant, and is allowed to be any adapted process. Define the discount process D by

$$D(t) = \exp\left(-\int_0^t R(s)ds\right),$$

and observe

$$dD(t) = -D(t)R(t)dt$$

Technically, we have to check the square integrability condition to ensure that $Z\tilde{W}_i$ is a martingale, and not a local martingale. This, however, follows quickly from the Cauchy-Schwartz inequality and our assumption.

Definition 1.6.3 A risk neutral measure is a measure $\tilde{P}$ that is equivalent to P under which the discounted stock price process $D(t)S(t)$ is a martingale.

Using the Girsanov theorem, we can compute the risk neutral measure explicitly. Observe

where

$$\begin{aligned} d(D(t)S(t)) &= -RDSdt + DdS = (\alpha - R)DSdt + DS\sigma dW(t) \\ &= \sigma(t)D(t)S(t)(\theta(t)dt + dW(t)) \end{aligned}$$

is known as the market price of risk.

$$\theta(t) \stackrel{\text{def}}{=} \frac{\alpha(t) - R(t)}{\sigma(t)}$$

Define a new process $\tilde{W}$ by

$$d\tilde{W}(t) = \theta(t)dt + dW(t),$$

and observe

$$d(D(t)S(t))dt = \sigma(t)D(t)S(t)d\tilde{W}(t).$$

Proposition 1.6.5 If Z is the process defined by

$$Z(t) = \exp\left(-\int_0^t \theta(s)dW(s) - \frac{1}{2}\int_0^t \theta(s)^2 ds\right),$$

then the measure $\tilde{P} = \tilde{P}_T$ defined by $d\tilde{P} = Z(T)dP$ is a risk neutral measure.

Proof. By the Girsanov theorem, we know $\tilde{W}$ is a Brownian motion under $\tilde{P}$. Thus we can see that the discounted stock price is a martingale.

Theorem 1.6.6 — Risk Neutral Pricing formula. Let $V(T)$ be a $\mathscr{F}_T$-measurable random variable that represents the payoff of a derivative security, and let $\tilde{P} = \tilde{P}_T$ be the risk neutral measure above. The arbitrage free price at time t of a derivative security with payoff $V(T)$ and maturity T is given by

$$V(t) = \tilde{E}\left(\exp\left(-\int_t^T R(s)ds\right)V(T) \mid \mathscr{F}_t\right).$$

To understand why this happens we note

$$dS(t) = \alpha(t)S(t)dt + \sigma(t)S(t)dW(t) = R(t)S(t)dt + \sigma(t)S(t)d\tilde{W}.$$

The fact that S and the money market account have exactly the same mean return rate (under $\tilde{P}$) is precisely what makes the replicating portfolio (or any self-financing portfolio for that matter) a martingale (under $\tilde{P}$).

Proposition 1.6.7 Let Δ be any adapted process, and $X(t)$ be the wealth of an investor with that holds $\Delta(t)$ shares of the stock and the rest of his wealth in the money market account. If there is no external cash flow (i.e. the portfolio is self financing), then the discounted portfolio $D(t)X(t)$ is a martingale under $\tilde{P}$.

Proof of the proposition: We know

$$dX(t) = \Delta(t)dS(t) + R(t)(X(t) - \Delta(t)S(t))dt.$$

Using (2.3) this becomes

$$\begin{aligned} dX(t) &= \Delta RSdt + \Delta\sigma Sd\tilde{W} + RXdt - R\Delta Sdt \\ &= RXdt + \Delta\sigma Sd\tilde{W}. \end{aligned}$$

Thus, by the product rule,

$$\begin{aligned} d(DX) &= DdX + XdD + d[D,X] = -RDXdt + DRXdt + D\Delta\sigma Sd\tilde{W} \\ &= D\Delta\sigma Sd\tilde{W}. \end{aligned}$$

Since $\tilde{W}$ is a martingale under $\tilde{P}, DX$ must be a martingale under $\tilde{P}$.

Proof of Risk Neutral Pricing formula Theorem:
Suppose $X(t)$ is the wealth of a replicating portfolio at time t. Then by definition we know $V(t) = X(t)$, and by the previous lemma we know DX is a martingale under $\tilde{P}$. Thus

$$V(t) = X(t) = \frac{1}{D(t)}D(t)X(t) = \frac{1}{D(t)}\tilde{E}(D(T)X(T) \mid \mathscr{F}_t) = \tilde{E}\left(\frac{D(T)V(T)}{D(t)} \mid \mathscr{F}_t\right),$$

which is precisely what is desired.

solve for stock price under risk-neutral measure

Suppose

$$dS(t) = \alpha S(t)dt + \sigma S(t)dW(t) = rS(t)dt + \sigma S(t)d\tilde{W}_t.$$

Use Ito's formula to get that $dln(S_t) = (lnS_t)'dS_t + \frac{1}{2}(lnS_t)''dS_tdS_t = \frac{dS_t}{S_t} - \frac{1}{2S_t^2}dS_tdS_t$
plugging $dS_t rS(t)dt + \sigma S(t)d\tilde{W}$, we get $ln(S_t) = (r - \frac{\sigma^2}{2})t + \sigma\tilde{W}_t$
Thus, $S_t = S_0e^{(r-\frac{\sigma^2}{2})t+\sigma\tilde{W}_t}$, where $\tilde{W}_t \sim N(0,t)$
Therefore, $V(S,t) = e^{-r(T-t)}\int_{-\infty}^{\infty}[S_0e^{(r-\frac{\sigma^2}{2})t+\sigma\sqrt{T}x} - K]^+e^{-\frac{x^2}{2}}\frac{dx}{\sqrt{2\pi}}$ Then compute the integral, we can get the same BS formula as written in the last subsection.

1.6.3 derive BS formula through the limit of binary tree

See Notes of Mathematics of Finance, section 8.

1.7 Pricing of Zero-Coupon Bonds

From this chapter, we'll focus on pricing derivatives based on interests rates. The pricing essence is still no-arbitrage condition.

1.7.1 Classification of zero-coupon bonds

First, we give the definition of zero-coupon bonds.

Definition 1.7.1 A zero-coupon bond is a contract priced $P_0(t,T)$ at time $t < T$ to deliver $P_0(T,T) = \$1$ at time T.

We define 3 different types of zero-coupon bonds based on the underlying short term interest rates process.

- The short term interest rate is a deterministic constant $r > 0$.
  In this case,

$$e^{r(T-t)}P_0(t,T) = P_0(T,T) = 1,$$

  Thus we have

$$P_0(t,T) = e^{-r(T-t)}, \quad 0 \le t \le T.$$

- The short term interest rate is time-dependent and deterministic.
  In this case,

$$P_0(t,T) = e^{-\int_t^T r_s ds}, \quad 0 \le t \le T.$$

- The short term interest rate is a Markovian stochastic process.
  In this case,

$$\begin{aligned} P_0(t,T) &= \mathbb{E}_{\mathbb{Q}}[e^{-\int_t^T r_s ds} \mid \mathscr{F}(t)], \\ &= \mathbb{E}_{\mathbb{Q}}[e^{-\int_t^T r_s ds} \mid r(t)] \text{ given Markovianess of r(t).} \end{aligned}$$

1.7.2 Bond Pricing PDE under Vasicek interest rate

Assume from now on that the underlying short rate process is given by SDE

$$dr_t = \mu(t,r_t)dt + \sigma(t,r_t)dB_t$$

In Vasicek model we have

$$\mu(t,x) = a - bx \quad \text{and} \quad \sigma(t,x) = \sigma.$$

Consider a change between equivalent probability measure $\mathbb{Q}$ and $\mathbb{P}$:

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = e^{-\int_0^\infty K_s dB_s - \frac{1}{2}\int_0^\infty |K_s|^2 ds}$$

such that

$$\hat{B}_t := B_t + \int_0^t K_s ds$$

then $dr(t)$ can be rewritten under risk-neutral measure Q

$$dr_t = \tilde{\mu}(t, r_t)\, dt + \sigma(t, r_t)\, d\hat{B}_t$$

where

$$\tilde{\mu}(t, r_t) := \mu(t, r_t) - \sigma(t, r_t) K_t.$$

Note that ,we call the process K_t, which is called the "market price of risk".

Since under Q measure, the discounted price process should be a martingale, we apply Itô's formula to $F(t, r_t) = P(t,T)$ in order to derive a PDE satisfied by $F(t, r_t)$.

$$\begin{aligned}
& d\left(e^{-\int_0^t r_s ds} P(t,T)\right) = -r_t e^{-\int_0^t r_s ds} P(t,T) dt + e^{-\int_0^t r_s ds} dP(t,T) \\
& = -r_t e^{-\int_0^t r_s ds} F(t, r_t)\, dt + e^{-\int_0^t r_s ds} dF(t, r_t) \\
& = -r_t e^{-\int_0^t r_s ds} F(t, r_t)\, dt + e^{-\int_0^t r_s ds} \frac{\partial F}{\partial x}(t, r_t)\left(\tilde{\mu}(t, r_t)\, dt + \sigma(t, r_t)\, d\hat{B}_t\right) \\
& + e^{-\int_0^t r_s ds}\left(\frac{1}{2}\sigma^2(t, r_t)\frac{\partial^2 F}{\partial x^2}(t, r_t)\, dt + \frac{\partial F}{\partial t}(t, r_t)\, dt\right) \\
& = e^{-\int_0^t r_s ds}\sigma(t, r_t)\frac{\partial F}{\partial x}(t, r_t)\, d\hat{B}_t \\
& + e^{-\int_0^t r_s ds}\left(-r_t F(t, r_t) + \tilde{\mu}(t, r_t)\frac{\partial F}{\partial x}(t, r_t)\right. \\
& \left. + \frac{1}{2}\sigma^2(t, r_t)\frac{\partial^2 F}{\partial x^2}(t, r_t) + \frac{\partial F}{\partial t}(t, r_t)\right) dt.
\end{aligned}$$

Setting term before dt to 0, we can get the following proposition.

Proposition 1.7.1 The bond pricing PDE for $P(t,T) = F(t, r_t)$ is written as

$$xF(t,x) = \tilde{\mu}(t,x)\frac{\partial F}{\partial x}(t,x) + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2 F}{\partial x^2}(t,x) + \frac{\partial F}{\partial t}(t,x),$$

subject to the terminal condition $F(T,x) = 1$

1.7.3 Solve Pricing PDE with probabilistic method

Our goal is now to solve the PDE by direct computation of the conditional expectation

$$P(t,T) = \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T r_s ds} \mid \mathscr{F}_t\right].$$

We will assume that the short term interest rate $r(t)$ has the expression

$$r_t = g(t) + \int_0^t h(t,s)dB_s,$$

where $g(t)$ and $h(t,s)$ are deterministic functions. Letting $u \vee t = \max(u,t)$, using the property of conditional expectations, we have

$$\begin{aligned}
P(t,T) &= \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T r_s ds} \mid \mathscr{F}_t\right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T (g(s)+\int_0^s h(s,u)dB_u)ds} \mid \mathscr{F}_t\right] \\
&= e^{-\int_t^T g(s)ds}\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T \int_0^s h(s,u)dB_u ds} \mid \mathscr{F}_t\right] \\
&= e^{-\int_t^T g(s)ds}\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_0^T \int_{u\vee t}^T h(s,u)dsdB_u} \mid \mathscr{F}_t\right] \\
&= e^{-\int_t^T g(s)ds}e^{-\int_0^t \int_{u\vee t}^T h(s,u)dsdB_u}\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T \int_{u\vee t}^T h(s,u)dsdB_u} \mid \mathscr{F}_t\right] \\
&= e^{-\int_t^T g(s)ds}e^{-\int_0^t \int_t^T h(s,u)dsdB_u}\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T \int_u^T h(s,u)dsdB_u} \mid \mathscr{F}_t\right] \\
&= e^{-\int_t^T g(s)ds}e^{-\int_0^t \int_t^T h(s,u)dsdB_u}\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T \int_u^T h(s,u)dsdB_u}\right] \\
&= e^{-\int_t^T g(s)ds}e^{-\int_0^t \int_t^T h(s,u)dsdB_u}e^{\frac{1}{2}\int_t^T \left(\int_u^T h(s,u)ds\right)^2 du}.
\end{aligned}$$

In the Vašiček model, we have

$$dr_t = (a - br_t)\,dt + \sigma dB_t,$$

We claim without proof that:

$$r_t = r_0 e^{-bt} + \frac{a}{b}\left(1 - e^{-bt}\right) + \sigma \int_0^t e^{-b(t-s)}dB_s,$$

Plug in to the above calculation yields

$$\begin{aligned}
P(t,T) &= \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T r_s ds} \mid \mathscr{F}_t\right] \\
&= e^{-\int_t^T \left(r_0 e^{-bs}+\frac{a}{b}\left(1-e^{-bs}\right)\right)ds}e^{-\sigma\int_0^t \int_t^T e^{-b(s-u)}dsdB_u} \\
&= e^{-\int_t^T \left(r_0 e^{-bs}+\frac{a}{b}\left(1-e^{-bs}\right)\right)ds}e^{-\frac{\sigma}{b}\left(1-e^{-b(T-t)}\right)\int_0^t e^{-b(t-u)}}dB_u \\
&= e^{-\frac{r_t}{b}\left(1-e^{-b(T-t)}\right)+\frac{1}{b}\left(1-e^{-b(T-t)}\right)\left(r_0 e^{-bt}+\frac{a}{b}\left(1-e^{-bt}\right)\right)} \\
&= e^{C(T-t)r_t+A(T-t)},
\end{aligned}$$

where

$$C(T-t) = -\frac{1}{b}\left(1 - e^{-b(T-t)}\right)$$

and

$$
\begin{aligned}
A(T-t) &= \frac{1}{b}\left(1-e^{-b(T-t)}\right)\left(r_0e^{-bt}+\frac{a}{b}\left(1-e^{-bt}\right)\right) \\
&- \int_t^T \left(r_0e^{-bs}+\frac{a}{b}\left(1-e^{-bs}\right)\right)ds \\
&+ \frac{\sigma^2}{2}\int_t^T e^{2bu}\left(\frac{e^{-bu}-e^{-bT}}{b}\right)^2 du \\
&= \frac{1}{b}\left(1-e^{-b(T-t)}\right)\left(r_0e^{-bt}+\frac{a}{b}\left(1-e^{-bt}\right)\right) \\
&- \frac{r_0}{b}\left(e^{-bt}-e^{-bT}\right)-\frac{a}{b}(T-t)+\frac{a}{b^2}\left(e^{-bt}-e^{-bT}\right) \\
&+ \frac{\sigma^2}{2b^2}\int_t^T\left(1+e^{-2b(T-u)}-2e^{-b(T-u)}\right)du \\
&= \frac{a}{b^2}\left(1-e^{-b(T-t)}\right)\left(1-e^{-bt}\right)-\frac{a}{b}(T-t)+\frac{a}{b^2}\left(e^{-bt}-e^{-bT}\right) \\
&+ \frac{\sigma^2}{2b^2}(T-t)+\frac{\sigma^2}{2b^2}e^{-2bT}\int_t^T e^{2bu}du-\frac{\sigma^2}{b^2}e^{-bT}\int_t^T e^{bu}du \\
&= \frac{a}{b^2}\left(1-e^{-b(T-t)}\right)+\frac{\sigma^2-2ab}{2b^2}(T-t) \\
&+ \frac{\sigma^2}{4b^3}\left(1-e^{-2b(T-t)}\right)-\frac{\sigma^2}{b^3}\left(1-e^{-b(T-t)}\right) \\
&= \frac{4ab-3\sigma^2}{4b^3}+\frac{\sigma^2-2ab}{2b^2}(T-t) \\
&+ \frac{\sigma^2-ab}{b^3}e^{-b(T-t)}-\frac{\sigma^2}{4b^3}e^{-2b(T-t)}.
\end{aligned}
$$

1.8 Forward Rate Modeling

Besides the needs of investment, companies may also have the needs of borrowing. In this section, we shall study how to give fair price of loans, which is based on the fair price of zero-coupon bonds.

1.8.1 Forward Contracts and Forward Rates

First, we give the definition of forward contract.

Definition 1.8.1 A forward contract is an agreement at a present time t for a loan to be delivered over a future period of time $[T,S]$ at a rate $f(t,T,S), t\le T\le S$, which repays a unit amount at time S. We call the rate f forward interest rate.

To give the fair price of forward interest r, we consider the non-arbitrage condition.

1. at time t, borrow \$1 at the price $P(t,S)$, to be repaid at time S. In other words, take short position of 1 S-maturity zero-coupon-bond.
2. since one only needs the money at time T, it makes sense to invest the amount $P(t,S)$ over the period $[t,T]$ in a bond with maturity T, that will yield $P(t,S)/P(t,T)$ at time T. As a consequence the investor will receive $P(t,S)/P(t,T)$ at time T and repay a unit amount \$1 at time S.

Using exponential compounding,

$$\frac{P(t,S)}{P(t,T)}\exp((S-T)f(t,T,S))=1,$$

thus,

$$f(t,T,S) = -\frac{\log P(t,S) - \log P(t,T)}{S-T}.$$

1.8.2 Spot Forward Rates

Taking T equals t, we get the so-called spot forward rate $F(t,T)$,

$$F(t,T) := f(t,t,T) = -\frac{\log P(t,T)}{T-t}.$$

Recall that in the Vasicek model,

$$dr_t = (a - br_t)\,dt + \sigma dB_t$$

we have

$$P(t,T) = e^{C(T-t)r_t + A(T-t)}$$

where

$$C(T-t) = -\frac{1}{b}\left(1 - e^{-b(T-t)}\right)$$

and

$A(T-t) = \frac{4ab-3\sigma^2}{4b^3} + \frac{\sigma^2-2ab}{2b^2}(T-t) + \frac{\sigma^2-ab}{b^3}e^{-b(T-t)} - \frac{\sigma^2}{4b^3}e^{-2b(T-t)}$,

hence

$$\log P(t,T) = A(T-t) + r_t C(T-t)$$

and

$$\begin{aligned}
f(t,T,S) &= -\frac{\log P(t,S) - \log P(t,T)}{S-T} \\
&= -\frac{r_t(C(S-t) - C(T-t)) + A(S-t) - A(T-t))}{S-T} \\
&= -\frac{\sigma^2 - 2ab}{2b^2} \\
&\quad -\frac{1}{S-T}\left(\left(\frac{r_t}{b} + \frac{\sigma^2 - ab}{b^3}\right)\left(e^{-b(S-t)} - e^{-b(T-t)}\right)\right. \\
&\quad \left. -\frac{\sigma^2}{4b^3}\left(e^{-2b(S-t)} - e^{-2b(T-t)}\right)\right)
\end{aligned}$$

1.8.3 Instantaneous Forward Rates

The instantaneous forward rate $f(t,T)$ is defined by taking the limit of $f(t,T,S)$ as $S \searrow T$, i.e.

$$
\begin{aligned}
f(t,T) := & -\lim_{S \searrow T} \frac{\log P(t,S) - \log P(t,T)}{S-T} \\
= & -\lim_{\varepsilon \searrow 0} \frac{\log P(t,T+\varepsilon) - \log P(t,T)}{\varepsilon} \\
= & -\frac{\partial \log P(t,T)}{\partial T} \\
= & -\frac{1}{P(t,T)} \frac{\partial P(t,T)}{\partial T} \\
= & -\frac{\partial \log P(t,T)}{\partial T}.
\end{aligned}
$$

For example, in the Vasicek stochastic interest rate model,

$$
\begin{aligned}
f(t,T) := & -\frac{\partial \log P(t,T)}{\partial T} \\
= & \, r_t e^{-b(T-t)} + \frac{a}{b}\left(1 - e^{-b(T-t)}\right) - \frac{\sigma^2}{2b^2}\left(1 - e^{-b(T-t)}\right)^2,
\end{aligned}
$$

1.9 Forward Rates on LIBOR Market

In previous section, we used exponential compounding in getting the fair forward rates. In London InterBank Offered Rates (LIBOR) market, people use linear compounding in getting the fair forward rates.

Using linear compounding,

$$\frac{P(t,S)}{P(t,T)}(1 + (S-T)L(t,T,S)) = 1,$$

thus,

$$L(t,T,S) = \frac{1}{S-T}\left(\frac{P(t,T)}{P(t,S)} - 1\right).$$

1.10 What can be done in the Future?

- HJM model, which gives the no-arbitrage condition for the instantaneous forward rates
- 2-factor Vasicek model, which plays with correlations between rates
- Jump Process, which introduces non-continuity in the modeling of process

1.11 Summary

In this semester, starting from Brownian motion, I spent most of my time collecting all the possible methods of deducing and solving Black-Scholes equation. In the end, I temporarily said goodbye to the option pricing based on stocks, and started to explore interest rates modeling. From these two parts, I experienced the essence of derivative pricing again and again, which is no-arbitrage condition and dynamic replication, which can be translated into math language: the preservation of martingale. Due to the limited time, I still have a lot that I am curious about but don't have time to explore. I believe this is only a start of my study in Fin-Math.

1.12 Bibliography

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