

Assignment 3

March 18, 2024

1 Exercises

1.1

If $P_1 = [\mathbf{I} \ 0]$ and $P_2 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, compute the fundamental matrix \mathbf{F} .

The fundamental matrix is given by

$$F = [e_2]_x A = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix} * \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ -2 & -2 & 0 \end{bmatrix}$$

Suppose that the point $x = (1, 1)$ is the projection of a 3D-point X into P_1 . Compute the epipolar line in the second image generated from x .

The epipolar line is simply

$$l = Fx = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -4 \end{bmatrix}$$

Which of the points $x_1 = (2, 0)$, $x_2 = (2, 1)$ and $x_3 = (4, 2)$ could be a projection of the same point X into P_2 ?

The epipolar constraint:

$$\tilde{x}^T F x = \tilde{x}^T l = 0$$

$$\tilde{x}_1^T l = [2 \ 0 \ 1] \begin{bmatrix} 2 \\ 0 \\ -4 \end{bmatrix} = 0$$

$$\tilde{x}_2^T l = [2 \ 1 \ 1] \begin{bmatrix} 2 \\ 0 \\ -4 \end{bmatrix} = 0$$

$$\tilde{x}_3^T l = [4 \ 2 \ 1] \begin{bmatrix} 2 \\ 0 \\ -4 \end{bmatrix} = 4$$

x_1 and x_2 could be.

1.2

If $P_1 = [\mathbf{I} \ 0]$ and $P_2 = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, compute epipoles by projecting camera centers C_1 and

C_2 . C_1 is given by nullspace of P_1 , which can easily be shown is $C_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$. For C_2 we compute

nullspace of P_2 :

$$P_2 \begin{bmatrix} X \\ Y \\ Z \\ W \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ W \end{bmatrix} = \begin{bmatrix} X + Y + Z + W \\ 2Y + 2W \\ Z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} X \\ Y \\ Z \\ W \end{bmatrix} = \begin{bmatrix} -s \\ -s \\ 0 \\ s \end{bmatrix}$$

Choosing $s=1$ gives $C_2 = \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$. Projecting camera centers:

$$e_1 = P_1 C_2 = [I \ 0] \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}$$

$$e_2 = P_2 C_1 = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$$

Compute the fundamental matrix, its determinant and verify that $e_2^T F = 0$ and $F e_1 = 0$.
Fundamental matrix computing:

$$F = [e_2]_x A = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & -2 \\ -2 & 2 & 0 \end{bmatrix} * \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & -2 \\ -2 & 2 & -2 \end{bmatrix}$$

The determinant:

$$\det \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & -2 \\ -2 & 2 & -2 \end{bmatrix} = 0 + 0 + 2 * 0 = 0$$

And

$$e_2^T F = [2 \ 2 \ 0] \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & -2 \\ -2 & 2 & -2 \end{bmatrix} = [0 + 0 + 0 \ 0 + 0 + 0 \ 4 - 4 + 0] = [0 \ 0 \ 0]$$

And

$$F e_1 = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & -2 \\ -2 & 2 & -2 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 + 0 + 0 \\ 0 + 0 + 0 \\ 2 - 2 + 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

1.3

When computing the fundamental matrix F using the 8-point algorithm it is recommended to use normalization. Suppose the image points have been normalized using $\tilde{x}_1 = N_1 x_1$ and $\tilde{x}_2 = N_2 x_2$. If \tilde{F} fulfills $\tilde{x}_2^T \tilde{F} \tilde{x}_1 = 0$ what is the fundamental matrix F that fulfills $x_2^T F x_1 = 0$ for the original un-normalized points? From the normalization we get

$$\tilde{x}_2^T \tilde{F} \tilde{x}_1 = x_2^T N_2^T \tilde{F} N_1 x_1 = 0$$

from which we can see that for the original un-normalized points:

$$F = N_2^T \tilde{F} N_1$$

1.4

Consider the fundamental matrix $F = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$. Verify that the projection of scene points $\mathbf{X}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{X}_2 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ and $\mathbf{X}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, into cameras $P_1 = [\mathbf{I} \ 0]$ and $P_2 = [[\mathbf{e}_2]_x F \ \mathbf{e}_2]$ fulfills the epipolar constraint ($x_2^T F x_1 = 0$).

Using that $F^T \mathbf{e}_2 = 0$ we can find \mathbf{e}_2

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} e_2 \\ e_1 + e_3 \\ e_1 + e_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow$$

$$\mathbf{e}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

This gives us

$$A = [\mathbf{e}_2]_x F = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 2 \\ -1 & 0 & 0 \end{bmatrix} \rightarrow$$

$$P_2 = [A \ t] = \begin{bmatrix} -1 & 0 & 0 & -1 \\ 0 & 2 & 2 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$

Projecting the 3D points into camera P_1 :

$$\lambda_1 x_1 = P_1 \begin{bmatrix} \mathbf{X}_1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\lambda_2 x_2 = P_1 \begin{bmatrix} \mathbf{X}_2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

$$\lambda_3 x_3 = P_1 \begin{bmatrix} \mathbf{X}_3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

And camera P_2 :

$$\tilde{\lambda}_1 \tilde{x}_1 = P_2 \begin{bmatrix} \mathbf{X}_1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 10 \\ 0 \end{bmatrix}$$

$$\tilde{\lambda}_2 \tilde{x}_2 = P_2 \begin{bmatrix} \mathbf{X}_2 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ 6 \\ -2 \end{bmatrix}$$

$$\tilde{\lambda}_3 \tilde{x}_3 = P_2 \begin{bmatrix} \mathbf{X}_3 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix}$$

Verifying epipolar constraint is fulfilled ($k_i = 1/(\lambda_i \tilde{\lambda}_i)$):

$$\tilde{x}_1^T F x_1 = k_1 \begin{bmatrix} -2 & 10 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = k_1 \begin{bmatrix} 10 & -2 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = k_1(10 - 4 - 6) = 0$$

$$\tilde{x}_2^T F x_2 = k_2 \begin{bmatrix} -4 & 6 & -2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = k_2 \begin{bmatrix} 6 & -6 & -6 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = k_2(6 * 3 - 2 * 6 - 6) = 0$$

$$\tilde{x}_3^T F x_3 = k_3 \begin{bmatrix} -2 & 2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = k_3 \begin{bmatrix} 2 & -2 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = k_3(2 + 0 - 2) = 0$$

We see that the epipolar constraint is fulfilled for all the points.

What is the camera center of P_2 There are formulas for this, but easiest way is still to calculate nullspace of P_2 :

$$P_2 \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & -1 \\ 0 & 2 & 2 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} -x - w \\ 2y + 2z \\ -x + w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ -t \\ t \\ 0 \end{bmatrix}$$

It seems the camera center C_2 is a point at infinity in the direction $\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

1.5

An essential matrix has the singular value decomposition $E = U \text{diag}([1, 1, 0]) V^T$ where

$$U = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } V = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}. \text{ Verify that } \det(UV^T) = 1.$$

$$\begin{aligned} \det(UV^T) &= \det\left(\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}\right) = \det\left(\begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & -1 & 0 \end{bmatrix}\right) \\ &= (1/\sqrt{2}) * (0 + 1/\sqrt{2}) + 0 - (1/\sqrt{2}) * (-1/\sqrt{2}) = 1/2 + 1/2 = 1 \end{aligned}$$

Compute the essential matrix and verify that $x_1 = (0, 0)$ in camera 1 and $x_2 = (1, 1)$ in camera 2 is a plausible correspondence

We have from decomp. that

$$\begin{aligned} E &= U \text{diag}([1, 1, 0]) V^T = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

Our possible corresponding image points should fulfill $x_2^T E x_1 = 0$:

$$x_2^T E x_1 = [1 \ 1 \ 1] \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = [\sqrt{2} \ 0 \ 0] \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 0$$

Possible correspondence!

If x_1 is the projection of X in $P_1 = [I \ 0]$ show that X must be one of the points

$$X(s) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ s \end{bmatrix}$$

We project $\mathbf{X} = \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$ into camera 1:

$$P_1 \mathbf{X} = P_1 \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} = \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} = \lambda x_1 = \lambda \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1/\lambda \end{bmatrix}$$

We get $X=0, Y=0, Z=\lambda$. The point(s) are of the form

$$\mathbf{X} = \begin{bmatrix} 0 \\ 0 \\ \lambda \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1/\lambda \end{bmatrix}$$

Setting $\lambda = 1/s$ we get the result in question.

For each of the solutions $P_2 = [UWV^T u_3]$ or $P_2 = [UWV^T - u_3]$ or $P_2 = [UW^T V^T u_3]$ or $P_2 = [UW^T V^T - u_3]$, compute s such that $\mathbf{X}(s)$ projects to x_2 . For which of the camera pairs is the point $\mathbf{X}(s)$ in front of both cameras?

We calculate for the cameras:

Option 1)

$$[UWV^T \quad u_3] \begin{bmatrix} 0 \\ 0 \\ 1 \\ s \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} & 0 & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ s \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} \\ -1/\sqrt{2} \\ s \end{bmatrix} \rightarrow$$

$$x = \begin{bmatrix} -1/s\sqrt{2} \\ -1/s\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \rightarrow s = -1/\sqrt{2}$$

Option 2)

$$[UWV^T \quad -u_3] \begin{bmatrix} 0 \\ 0 \\ 1 \\ s \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} & 0 & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} & 0 \\ 0 & -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ s \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} \\ -1/\sqrt{2} \\ -s \end{bmatrix} \rightarrow$$

$$x = \begin{bmatrix} 1/s\sqrt{2} \\ 1/s\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \rightarrow s = 1/\sqrt{2}$$

Option 3)

$$[UW^T V^T \quad u_3] \begin{bmatrix} 0 \\ 0 \\ 1 \\ s \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ s \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ s \end{bmatrix} \rightarrow$$

$$x = \begin{bmatrix} 1/s\sqrt{2} \\ 1/s\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \rightarrow s = 1/\sqrt{2}$$

Option 4)

$$[UW^T V^T \quad -u_3] \begin{bmatrix} 0 \\ 0 \\ 1 \\ s \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\ 0 & -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ s \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ -s \end{bmatrix} \rightarrow$$

$$x = \begin{bmatrix} 1/s\sqrt{2} \\ 1/s\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \rightarrow s = -1/\sqrt{2}$$

The depth of a point $\mathbf{X} = \begin{bmatrix} X \\ s \end{bmatrix}$ in camera $P = [A_{[3x3]} \quad a_{[3x1]}]$ is given by

$$\text{depth}(P, \mathbf{X}) = \frac{\text{sign}(\det(A))}{\|A_3\| \cdot s} \cdot [A_3^T \quad a_3] \cdot X.$$

We see it is in front of camera P_2 for option 3 and option 4. For camera P1 it's easy to see $\text{depth}(P_1, \mathbf{X}(\mathbf{s}))=1/s$, which means points are in front of camera P1 for $s > 0$. This means option 3 will give points in front of both cameras.

2 Computer exercises

2.1

Computer exercise 1 The fundamental matrix for the original un-normalized points in Computer Exercise 1:

$$F = \begin{bmatrix} 0 & 0 & 0.0058 \\ 0 & 0 & -0.00267 \\ -0.0072 & 0.263 & 1 \end{bmatrix}$$

The mean distance between epipolar lines was calculated. When calculated from normalized points:

$$\text{meandist} = 0.3612$$

When F was calculated from un-normalized points ($N_1 = N_2 = \mathbf{I}$):

$$\text{meandist} = 0.4878$$

The plot of the epipolar lines and the histogram of distance between epipolar lines and corresponding points are seen in figures 1 and 2

2.2

Computer exercise 2

2.3

Computer exercise 3

The essential matrix computed:

$$E = 1000 \cdot \begin{bmatrix} -0.0089 & -1.0058 & 0.3771 \\ 1.2525 & 0.0784 & -2.4482 \\ -0.4728 & 2.5502 & 0.0010 \end{bmatrix}$$

In figure 5 is seen a histogram of computed distances from image points in image 2 to their corresponding epipolar lines. The mean distance was calculated to 2.0838.

2.4

Computer exercise 4

The best choice for camera P2 was $P_2 = [UW(V^T) \quad -u_3]$ (camera two). This led to all 2008 points being in front of both cameras. The max distance between projected points and original image points is 3.1321 and the mean distance 1.0421, so the error is quite small.

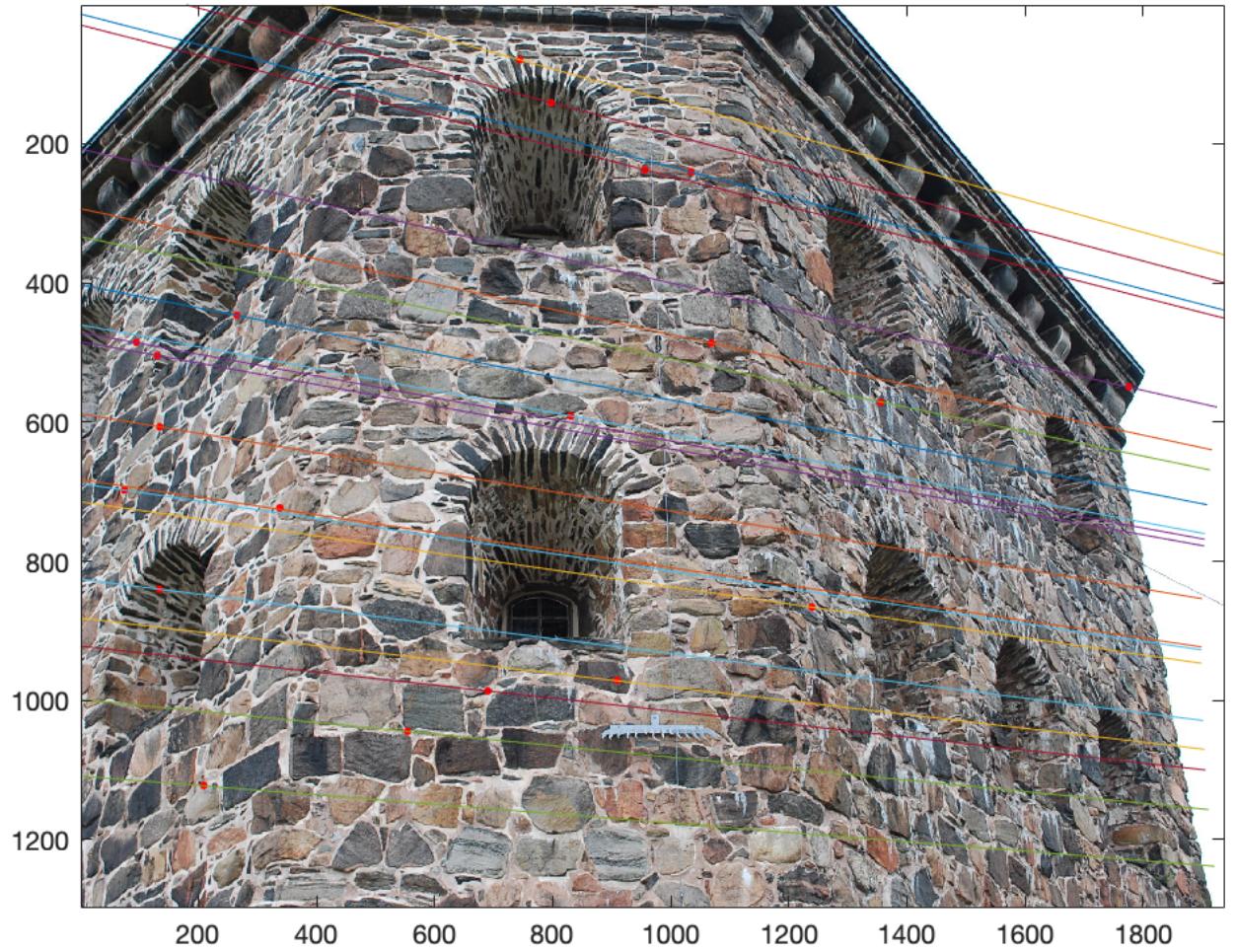


Figure 1: Plot of 20 randomly chosen image points and corresponding epipolar lines

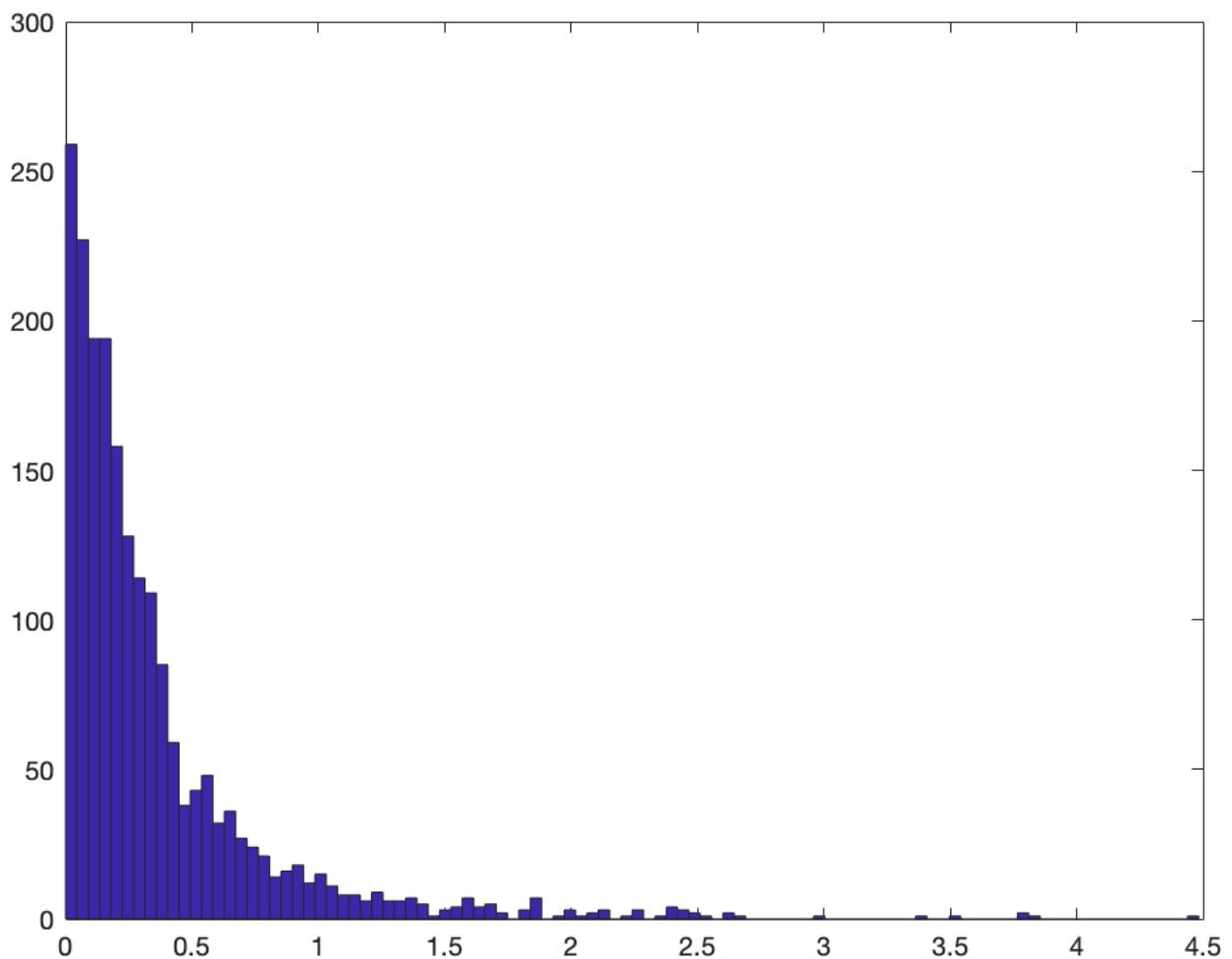


Figure 2: Histogram of distance between epipolar lines

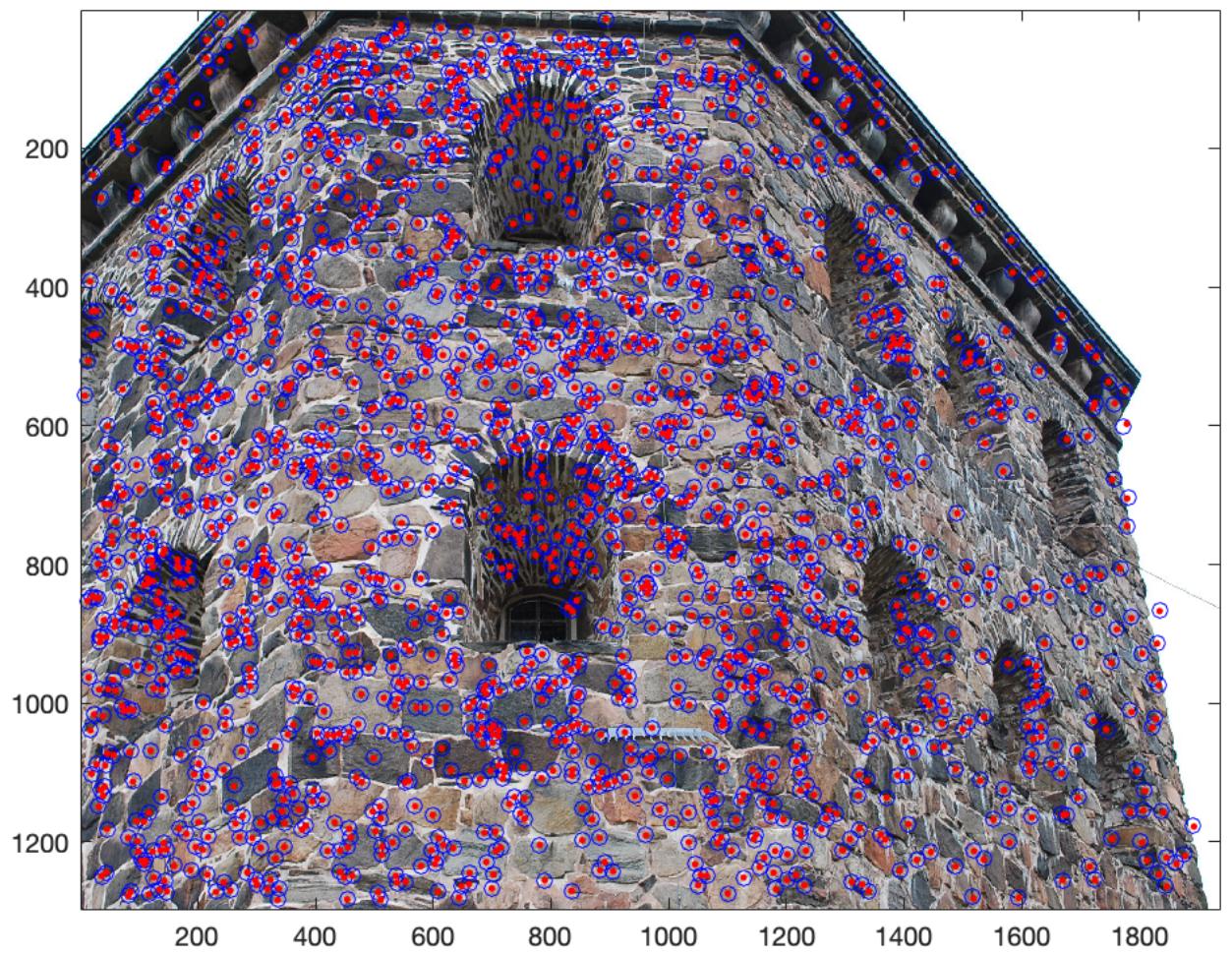


Figure 3: Projected 3D-points from computer exercise 2.

Reconstruction

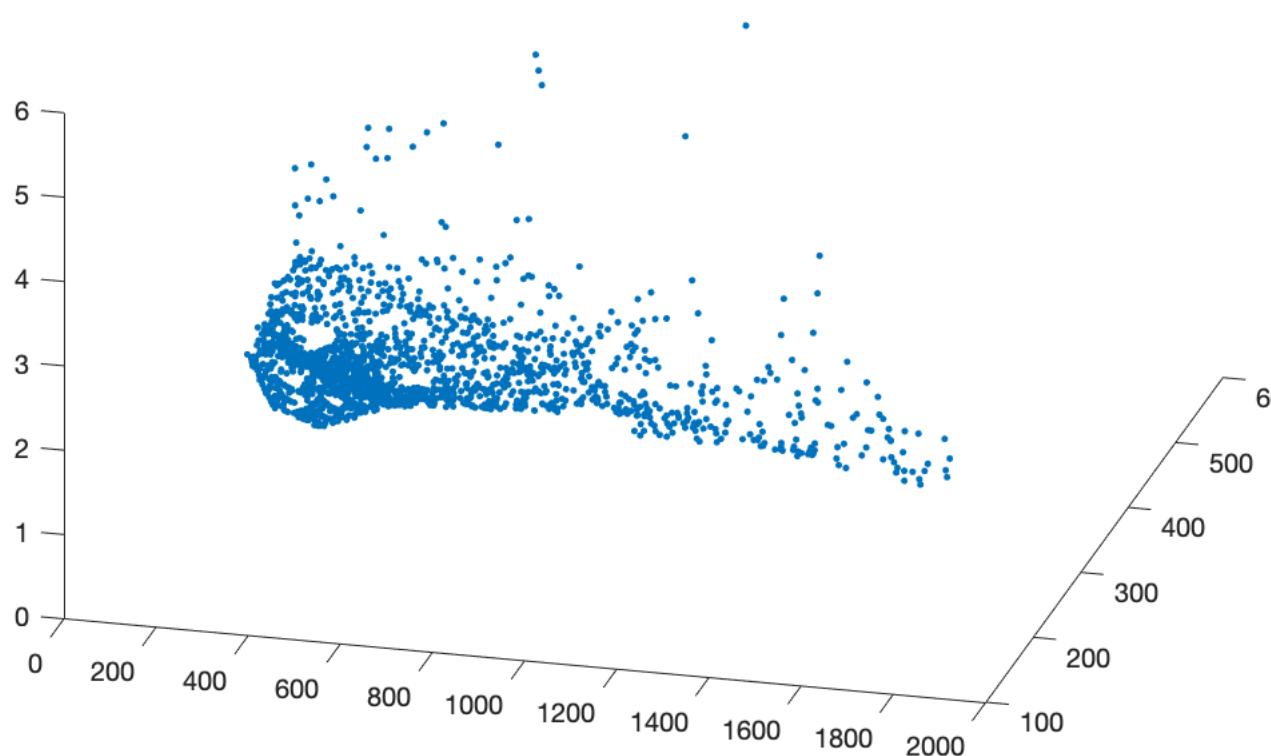


Figure 4: Plot of 3D-points from computer exercise 2

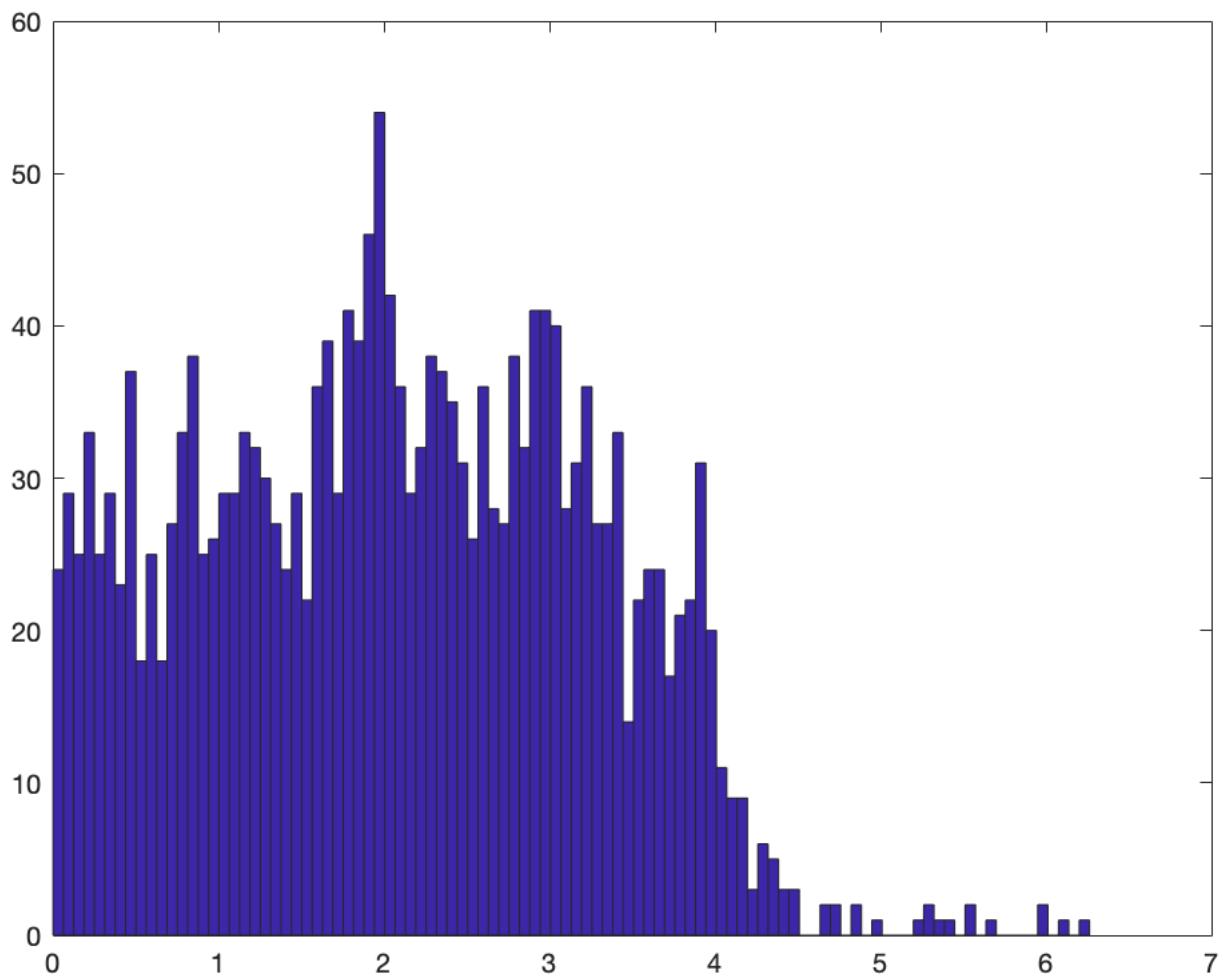


Figure 5: Plot shows a histogram with 100 bins of the distances between points in image 2 to their corresponding epipolar lines, from comp. exercise 3.

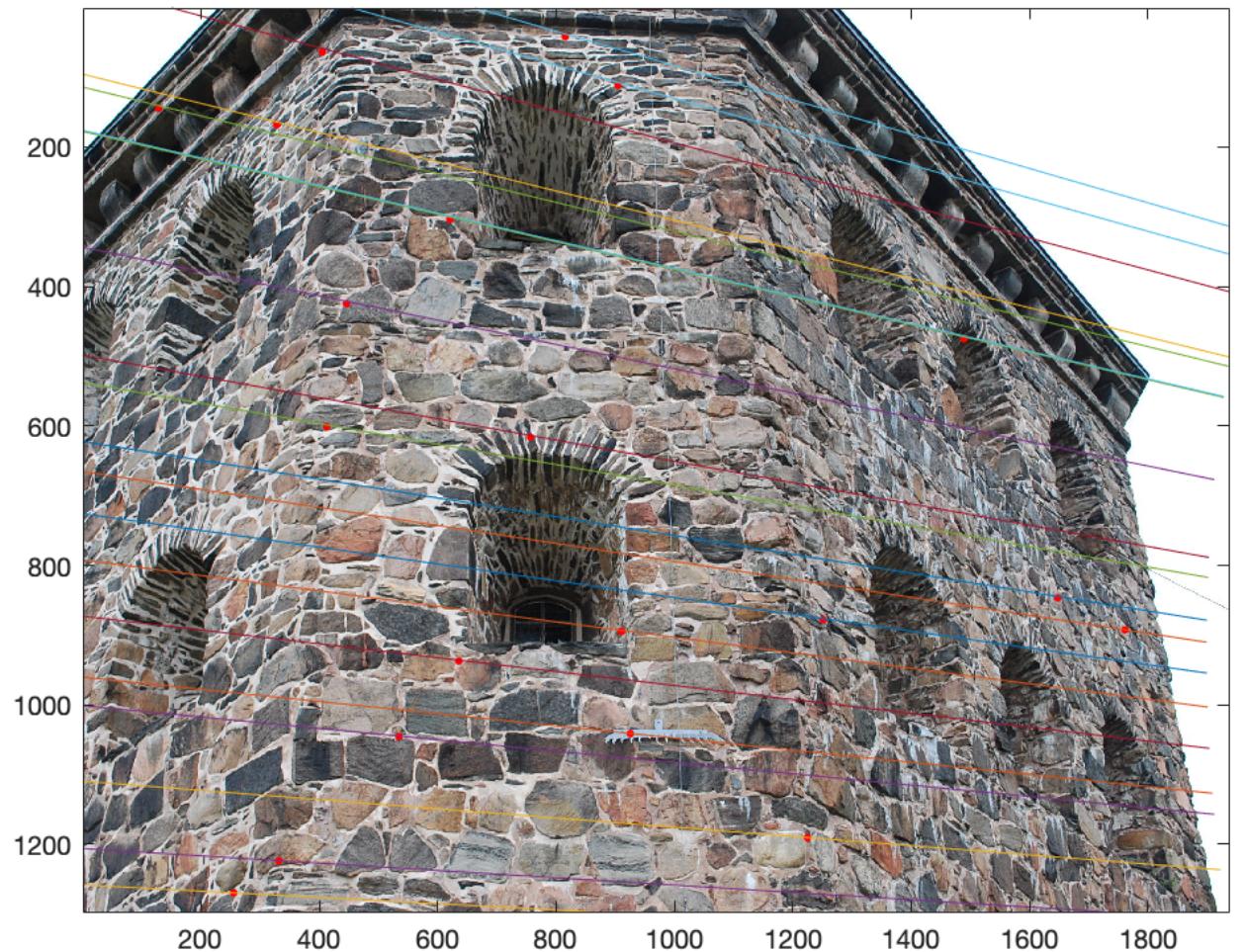


Figure 6: Plot of 20 randomly chosen points in image 2 and their corresponding epipolar lines, from computer exercise 3.

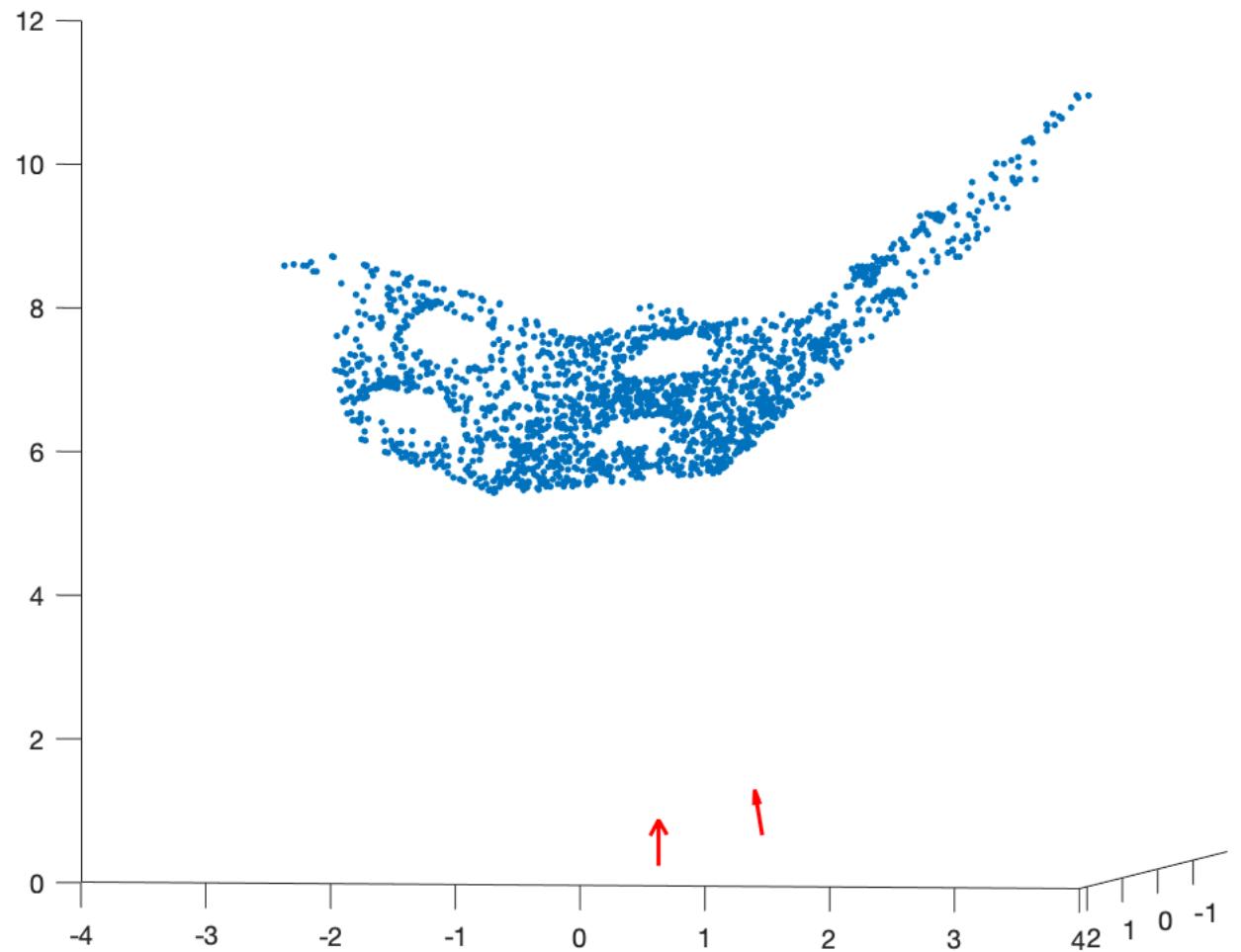


Figure 7: Plot of best 3D-points, from computer exercise 4



Figure 8: Projected image points from computer exercise 4