Lab 1 in Financial Statistics FMSN60

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I. Exercises

1: Autocorrelation function AR(1) and MA(1)

The AR(1) and the MA(1) processes are defined as:

$$\begin{aligned} \text{AR(1):} \quad & X_t = \phi X_{t-1} + \epsilon_t \\ \text{MA(1):} \quad & X_t = \epsilon_t + \theta \epsilon_{t-1} \end{aligned}$$

Autocovariance function of AR(1)

Let $\gamma(h)$ denote the autocovariance function.

$$\begin{aligned} \gamma(0) &= \text{Cov}(X_t, X_t) \\ &= \text{Cov}(\phi X_{t-1} + \epsilon_t, \phi X_{t-1} + \epsilon_t) \\ &= \phi^2 \text{Cov}(X_{t-1}, X_{t-1}) + 2\phi \text{Cov}(X_{t-1}, \epsilon_t) + \text{Cov}(\epsilon_t, \epsilon_t) \\ &= \phi^2 \gamma(0) + \sigma_\epsilon^2 \end{aligned}$$

Solving for $\gamma(0)$, we get ()

$$\gamma(0) = \frac{\sigma_\epsilon^2}{1 - \phi^2}$$

Next,

$$\gamma(1) = \text{Cov}(X_{t+1}, X_t) = \text{Cov}(\phi X_t + \epsilon_{t+1}, X_t) = \phi \gamma(0)$$

By induction, we can generalize that

$$\gamma(h) = \phi \gamma(h-1)$$

Thus, we find

$$\gamma(h) = \phi^h \gamma(0) = \phi^h \frac{\sigma_\epsilon^2}{1 - \phi^2}$$

Compute the Autocorrelation Function

The autocorrelation function $\rho(h)$ is defined as the autocovariance function normalized by the variance:

$$\rho(h) = \frac{\gamma(h)}{\gamma(0)} = \frac{\phi^h \frac{\sigma_\epsilon^2}{1-\phi^2}}{\frac{\sigma_\epsilon^2}{1-\phi^2}} = \phi^h$$

Therefore, the autocorrelation function of the AR(1) process is given by

$$\rho(h) = \phi^h$$

(b) Autocorrelation Function of an MA(1) Process

Autocovariance Function of MA(1)

Let $\gamma(h)$ denote the autocovariance function for the MA(1) process defined by:

$$X_t = \epsilon_t + \theta \epsilon_{t-1}$$

where ϵ_t is white noise with variance σ_ϵ^2.

Calculating $\gamma(0)$: Variance of X_t

$$\begin{aligned} \gamma(0) &= \text{Cov}(X_t, X_t) \\ &= \text{Cov}(\epsilon_t + \theta \epsilon_{t-1}, \epsilon_t + \theta \epsilon_{t-1}) \\ &= \text{Cov}(\epsilon_t, \epsilon_t) + \theta \text{Cov}(\epsilon_t, \epsilon_{t-1}) + \\ & \theta \text{Cov}(\epsilon_{t-1}, \epsilon_t) + \theta^2 \text{Cov}(\epsilon_{t-1}, \epsilon_{t-1}) \\ &= \sigma_\epsilon^2 + 0 + 0 + \theta^2 \sigma_\epsilon^2 \\ &= \sigma_\epsilon^2 (1 + \theta^2) \end{aligned}$$

Thus,

$$\gamma(0) = \sigma_\epsilon^2 (1 + \theta^2)$$

Calculating $\gamma(1)$: Covariance at Lag 1

$$\begin{aligned} \gamma(1) &= \text{Cov}(X_t, X_{t-1}) \\ &= \text{Cov}(\epsilon_t + \theta \epsilon_{t-1}, \epsilon_{t-1} + \theta \epsilon_{t-2}) \\ &= \text{Cov}(\epsilon_t, \epsilon_{t-1}) + \theta \text{Cov}(\epsilon_t, \epsilon_{t-2}) + \\ & \theta \text{Cov}(\epsilon_{t-1}, \epsilon_{t-1}) + \theta^2 \text{Cov}(\epsilon_{t-1}, \epsilon_{t-2}) \\ &= 0 + 0 + \theta \sigma_\epsilon^2 + 0 \\ &= \theta \sigma_\epsilon^2 \end{aligned}$$

For lags $h \geq 2$,

$$\gamma(h) = 0$$

Autocorrelation Function of MA(1)

The autocorrelation function $\rho(h)$ is defined as the autocovariance function normalized by the variance $\gamma(0)$:

$$\rho(h) = \frac{\gamma(h)}{\gamma(0)}$$

Substituting the values of $\gamma(h)$:

$$\rho(0) = \frac{\gamma(0)}{\gamma(0)} = 1$$

$$\rho(1) = \frac{\gamma(1)}{\gamma(0)} = \frac{\theta \sigma_\epsilon^2}{\sigma_\epsilon^2 (1 + \theta^2)} = \frac{\theta}{1 + \theta^2}$$

For $h \geq 2$,

$$\rho(h) = \frac{\gamma(h)}{\gamma(0)} = \frac{0}{\sigma_\epsilon^2 (1 + \theta^2)} = 0$$

Summary of Autocorrelation Function for MA(1)

$$\rho(h) = \begin{cases} 1 & \text{if } h = 0, \\ \dfrac{\theta}{1 + \theta^2} & \text{if } h = 1, \\ 0 & \text{if } h \geq 2. \end{cases}$$

2. Describe how one can estimate the parameters in an AR(1) process using Maximum Likelihood. How are the parameters estimated in an MA(1) process?

The maximum likelihood (ML) method estimates the parameters of a probability distribution by maximizing the likelihood function given observed data. The likelihood function is defined as $L(\theta|x) = f(x|\theta)$, where $f(x|\theta)$ represents the probability density or mass function of the data. The ML estimate $\hat{\theta}$ is the value of θ that maximizes $L(\theta|x)$, i.e.,

$$\hat{\theta} = \arg\max_{\theta} L(\theta|x).$$

Alternatively, if $\ell(\theta|x) = \ln L(\theta|x)$ is the log-likelihood function, the ML estimate can also be obtained by maximizing $\ell(\theta|x)$:

$$\hat{\theta} = \arg\max_{\theta} \ell(\theta|x).$$

The AR(1) process can be estimated by closed form ML techniques, but the MA(1) process cannot. The MA(1) process is estimated numerically or with Monte Carlo based methods.

AR(1) Parameter Estimation via Maximum Likelihood

Given the AR(1) model:

$$X_t = \phi X_{t-1} + \epsilon_t, \quad \epsilon_t \sim \mathcal{N}(0, \sigma_\epsilon^2)$$

1) **Likelihood Function:**

$$L(\phi, \sigma_\epsilon^2) = \prod_{t=2}^{T} \frac{1}{\sqrt{2\pi\sigma_\epsilon^2}} \exp\left(-\frac{(X_t - \phi X_{t-1})^2}{2\sigma_\epsilon^2}\right)$$

2) **Log-Likelihood:**

$$\ell(\phi, \sigma_\epsilon^2) = -\frac{T-1}{2}\ln(2\pi) - \frac{T-1}{2}\ln(\sigma_\epsilon^2) - \frac{1}{2\sigma_\epsilon^2}\sum_{t=2}^{T}(X_t - \phi X_{t-1})^2$$

3) **Estimators: (solving for ϕ and σ)**

$$\hat{\phi} = \frac{\sum_{t=2}^{T} X_{t-1}X_t}{\sum_{t=2}^{T} X_{t-1}^2}, \quad \hat{\sigma}_\epsilon^2 = \frac{1}{T-1}\sum_{t=2}^{T}(X_t - \hat{\phi}X_{t-1})^2$$

MA(1) Parameter Estimation

For the MA(1) model:

$$X_t = \epsilon_t + \theta\epsilon_{t-1}, \quad \epsilon_t \sim \mathcal{N}(0, \sigma_\epsilon^2)$$

Challenges: The likelihood involves past error terms, making it non-linear.

Summary

- AR(1): Parameters ϕ and σ_ϵ^2 are estimated via closed-form MLEs. - MA(1): Parameters θ and σ_ϵ^2 require numerical optimization or alternative estimation techniques due to model complexity.

3.

Question

Write down the likelihood function for the following model:

$$y_i = \beta_0 + \beta_1 x_i + \sigma\varepsilon_i,$$

where ε_i is iid $N(0,1)$ or $t(\nu)$.

Given the model:

$$y_i = \beta_0 + \beta_1 x_i + \sigma\varepsilon_i,$$

where ε_i are independent and identically distributed (iid) with $\varepsilon_i \sim N(0,1)$ or $\varepsilon_i \sim t(\nu)$, the likelihood function can be written as follows.

For $\varepsilon_i \sim N(0,1)$:

The probability density function (pdf) of y_i conditional on x_i is:

$$f(y_i \mid x_i) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(y_i - \beta_0 - \beta_1 x_i)^2}{2\sigma^2}\right)$$

Thus, the likelihood function for all observations is:

$$L(\beta_0, \beta_1, \sigma \mid \mathbf{y}, \mathbf{x}) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(y_i - \beta_0 - \beta_1 x_i)^2}{2\sigma^2}\right)$$

For $\varepsilon_i \sim t(\nu)$:

The pdf of y_i conditional on x_i is:

$$f(y_i \mid x_i) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\sigma\,\Gamma\left(\frac{\nu}{2}\right)}\left(1 + \frac{(y_i - \beta_0 - \beta_1 x_i)^2}{\nu\sigma^2}\right)^{-\frac{\nu+1}{2}}$$

Therefore, the likelihood function for all observations is:

$$L(\beta_0, \beta_1, \sigma \mid \mathbf{y}, \mathbf{x}) = \prod_{i=1}^{n} \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\sigma\,\Gamma\left(\frac{\nu}{2}\right)}\left(1 + \frac{(y_i - \beta_0 - \beta_1 x_i)^2}{\nu\sigma^2}\right)^{-\frac{\nu+1}{2}}$$

Summary:

The likelihood function for the given model depends on the distribution of the error terms ε_i:

4. Compute the Estimates and Covariance of the Estimates for the Linear Model with Gaussian Noise Using OLS and ML Methods

Linear Model with Gaussian Noise

Consider the simple linear regression model:

$$y_i = \beta_0 + \beta_1 x_i + \varepsilon_i, \quad i = 1, 2, \ldots, n,$$

where ε_i are independent and identically distributed random errors with $\varepsilon_i \sim N(0, \sigma^2)$.

(a) Ordinary Least Squares (OLS) Estimates

The OLS method estimates the parameters β_0 and β_1 by minimizing the sum of squared residuals:

$$RSS(\beta_0, \beta_1) = \sum_{i=1}^{n}(y_i - \beta_0 - \beta_1 x_i)^2.$$

OLS Estimators

Let $\bar{x}$ and $\bar{y}$ denote the sample means:

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i, \quad \bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i.$$

The OLS estimators are given by:

$$\hat{\beta}_1^{\text{OLS}} = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n}(x_i - \bar{x})^2},$$

$$\hat{\beta}_0^{\text{OLS}} = \bar{y} - \hat{\beta}_1^{\text{OLS}}\bar{x}.$$

Estimator of σ^2
The unbiased estimator of the error variance σ^2 is:

$$\hat{\sigma}^2_{\text{OLS}} = \frac{1}{n-2}\sum_{i=1}^{n}\left(y_i - \hat{\beta}_0^{\text{OLS}} - \hat{\beta}_1^{\text{OLS}}x_i\right)^2.$$

Covariance Matrix of OLS Estimates
The covariance matrix of the OLS estimators is:

$$\widehat{\text{Cov}}(\hat{\boldsymbol{\beta}}^{\text{OLS}}) = \hat{\sigma}^2_{\text{OLS}}(X^\top X)^{-1},$$

where $\hat{\boldsymbol{\beta}}^{\text{OLS}} = \begin{bmatrix}\hat{\beta}_0^{\text{OLS}} \\ \hat{\beta}_1^{\text{OLS}}\end{bmatrix}$ and X is the design matrix:

$$X = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix}.$$

The variances and covariance are:

$$\text{Var}(\hat{\beta}_0^{\text{OLS}}) = \hat{\sigma}^2_{\text{OLS}}\left(\frac{1}{n} + \frac{\bar{x}^2}{RSS_{xx}}\right),$$

$$\text{Var}(\hat{\beta}_1^{\text{OLS}}) = \frac{\hat{\sigma}^2_{\text{OLS}}}{RSS_{xx}},$$

$$\text{Cov}(\hat{\beta}_0^{\text{OLS}}, \hat{\beta}_1^{\text{OLS}}) = -\hat{\sigma}^2_{\text{OLS}}\frac{\bar{x}}{RSS_{xx}},$$

where $RSS_{xx} = \sum_{i=1}^{n}(x_i - \bar{x})^2$.
(b) Maximum Likelihood (ML) Estimates
Assuming normality of errors, the likelihood function is:

$$L(\beta_0, \beta_1, \sigma^2) = \prod_{i=1}^{n}\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(y_i - \beta_0 - \beta_1 x_i)^2}{2\sigma^2}\right).$$

The log-likelihood function is:

$$\ell(\beta_0, \beta_1, \sigma^2) = -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln(\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i - \beta_0 - \beta_1 x_i)^2.$$

ML Estimators
Maximizing the log-likelihood with respect to β_0 and β_1 yields the same estimators as OLS:

$$\hat{\beta}_0^{\text{ML}} = \hat{\beta}_0^{\text{OLS}}, \quad \hat{\beta}_1^{\text{ML}} = \hat{\beta}_1^{\text{OLS}}.$$

The ML estimator of σ^2 is:

$$\hat{\sigma}^2_{\text{ML}} = \frac{1}{n}\sum_{i=1}^{n}\left(y_i - \hat{\beta}_0^{\text{ML}} - \hat{\beta}_1^{\text{ML}}x_i\right)^2.$$

Covariance Matrix of ML Estimates
The covariance matrix of the ML estimators is:

$$\widehat{\text{Cov}}(\hat{\boldsymbol{\beta}}^{\text{ML}}) = \hat{\sigma}^2_{\text{ML}}(X^\top X)^{-1}.$$

Differences Between OLS and ML Methods
- The OLS and ML estimators of β_0, β_1 and σ are **identical**.

II. References

III. APPENDIX