Lab 2 in Financial Statistics FMSN60

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I. Exercises

Question 1: Difference between ARCH, GARCH, and EGARCH Processes

The ARCH, GARCH, and EGARCH models are modeling time-varying volatility in financial time series, with different mechanisms to capture volatility dynamics. ARCH is essentially the "original model" out of the three presented and the subsequent models GARCH and EGARCH are further building upon the ARCH model introducing new dynamics and adding complexity.

ARCH (Autoregressive Conditional Heteroskedasticity):

Engle (1982). The ARCH model specifies that the current conditional variance depends on past squared residuals. It states that the error has a purely random part for every innovation and a volatility term dependent on past squared residuals. Assume that μ is a suitable conditional mean for the time series y. μ is often assumed to be constant, but it could also represent a function of explanatory variables in more general models. It separates mean dynamics from the models volatility.

$$y_t = \mu + \epsilon_t, \quad \epsilon_t = \sigma_t z_t, \quad z_t \sim \mathcal{N}(0,1)$$

$$\sigma_t^2 = \alpha_0 + \alpha_1\epsilon_{t-1}^2 + \alpha_2\epsilon_{t-2}^2 + \cdots + \alpha_q\epsilon_{t-q}^2 = \alpha_0 + \sum_{i=1}^{q} \alpha_i \epsilon_{t-i}^2$$

where $\alpha_0 > 0$ and $\alpha_i \geq 0, \forall i \in Z_+$

GARCH (Generalized ARCH):

Bollerslev (1986), the GARCH model extends ARCH by including past conditional variances, allowing for representation of volatility persistence.

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{q} \alpha_i \epsilon_{t-i}^2 + \sum_{j=1}^{p} \beta_j \sigma_{t-j}^2$$

EGARCH (Exponential GARCH):

Nelson (1991), the EGARCH model captures asymmetries in volatility (leverage effects) and ensures positivity of the conditional variance by modeling its logarithm.

$$\log \sigma_t^2 = \omega + \sum_{i=1}^{q} \beta_i \log \sigma_{t-i}^2 + \sum_{j=1}^{p} \alpha_j \left(\frac{|\epsilon_{t-j}|}{\sigma_{t-j}} - \sqrt{\frac{2}{\pi}} \right) + \gamma_j \frac{\epsilon_{t-j}}{\sigma_{t-j}}$$

Differences

- **ARCH** models capture volatility clustering by relying solely on past squared residuals.
- **GARCH** builds further upon ARCH by having past conditional variances, and volatility persists stronger.
- **EGARCH** introduces asymmetry and ensures positive variances without parameter constraints.

To Show:

From a GARCH(1,1) process, X_t^2 follows the ARMA(1,1) equation:

$$X_t^2 - (\alpha + \beta)X_{t-1}^2 = \omega + \nu_t - \beta\nu_{t-1},$$

A GARCH(1,1) process is defined by:

$$X_t = \sigma_t \epsilon_t,$$

where ϵ_t is a sequence of i.i.d. standard normal

$$\sigma_t^2 = \omega + \alpha X_{t-1}^2 + \beta\sigma_{t-1}^2,$$

where $\omega > 0, \alpha \geq 0,$ and $\beta \geq 0$.

Since $X_t = \sigma_t \epsilon_t$, squaring both sides gives:

$$X_t^2 = \sigma_t^2 \epsilon_t^2.$$

Define the innovation ν_t as

$$\nu_t = X_t^2 - \sigma_t^2 = \sigma_t^2(\epsilon_t^2 - 1).$$

Note that $\epsilon_t^2 - 1$ has mean zero, ν_t is a mean-zero process.

the variance equation:

$$\sigma_t^2 = \omega + \alpha X_{t-1}^2 + \beta\sigma_{t-1}^2.$$

See that $X_{t-1}^2 = \sigma_{t-1}^2 + \nu_{t-1}$ from the definition of ν_{t-1}:

$$X_{t-1}^2 = \sigma_{t-1}^2 + \nu_{t-1}.$$

Plugging X_{t-1}^2 into the variance equation:

$$\sigma_t^2 = \omega + \alpha(\sigma_{t-1}^2 + \nu_{t-1}) + \beta\sigma_{t-1}^2.$$

Simplify:

$$\sigma_t^2 = \omega + (\alpha + \beta)\sigma_{t-1}^2 + \alpha\nu_{t-1}.$$

Since $\sigma_t^2 = X_t^2 - \nu_t$, we have:

$$X_t^2 - \nu_t = \omega + (\alpha + \beta)(X_{t-1}^2 - \nu_{t-1}) + \alpha\nu_{t-1}.$$

Rewriting and simplifying:

$$X_t^2 - \nu_t = \omega + (\alpha + \beta)X_{t-1}^2 - (\alpha + \beta)\nu_{t-1} + \alpha\nu_{t-1}.$$

Simplify the ν_{t-1} terms:

$$X_t^2 - \nu_t = \omega + (\alpha + \beta)X_{t-1}^2 - \beta\nu_{t-1}.$$

Move ν_t to the right-hand side:

$$X_t^2 - (\alpha + \beta)X_{t-1}^2 = \omega + \nu_t - \beta\nu_{t-1}.$$

And we are done.

II. References

III. APPENDIX