# Hand-in in 2 Spectral Analysis

Eliot Montesino Petrén, el6183mo-s, 990618-9130, LU Faculty of Engineering F19 , eliot.mp99@gmail.com

## I. Introduction

This report explores the fundamentals of time-frequency analysis, a key aspect of signal processing that examines signals through simultaneous time and frequency perspectives. The spectrogram is investigated, the Wigner distribution and the ambiguity function for comprehensive time-frequency insights. Additionally, a quadratic class definition for an alternative approach to time-frequency distributions is introduced. Some MATLAB exercises supplement the theoretical discussions highlighting signal characteristics and their applications. In summary, this is an examination of time-frequency analysis methods, their theoretical foundations, and practical implementations meant to equip readers with a better comprehension of this field.

## II. Tasks

### A. Spectrogram of a Gaussian-Windowed Signal

Show that the spectrogram of a signal $x(t) = (\frac{\beta}{\pi})^{1/4}e^{-\beta t^2/2}$, windowed with a Gaussian function $h(t) = (\frac{\alpha}{\pi})^{1/4}e^{-\alpha t^2/2}$, is given by a specific formula, see eq 1

$$S_x(t,f) = \frac{2\sqrt{\alpha\beta}}{\alpha+\beta}e^{-\frac{\alpha\beta t^2+4\pi^2 f^2}{\alpha+\beta}} \tag{1}$$

Given the signal

$$x(t) = \left(\frac{\beta}{\pi}\right)^{\frac{1}{4}} e^{-\frac{\beta t^2}{2}}, \tag{2}$$

and the windowing function

$$h(t) = \left(\frac{\alpha}{\pi}\right)^{\frac{1}{4}} e^{-\frac{\alpha t^2}{2}}, \tag{3}$$

we define the windowed signal as the product of $x(t)$ and $h(\tau - t)$:

$$x(\tau)h^*(\tau-t) = \left(\frac{\beta}{\pi}\right)^{\frac{1}{4}} e^{-\frac{\beta\tau^2}{2}} \left(\frac{\alpha}{\pi}\right)^{\frac{1}{4}} e^{-\frac{\alpha(\tau-t)^2}{2}}. \tag{4}$$

The Fourier Transform of the windowed signal is given by:

$$\mathcal{F}\{x(\tau)h^*(\tau-t)\} = \int_{-\infty}^{\infty} x(\tau)h^*(\tau-t)e^{-2\pi i f\tau}d\tau, \tag{5}$$

which, after substituting and simplifying yields:

$$\frac{(\alpha\beta)^{1/4}}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{\left(-\frac{\beta^2}{2}\tau^2-\frac{\alpha}{2}(\tau-t)^2\right)}e^{-i2\pi f\tau}\,d\tau \tag{6}$$

The (combined) exponents of the expression is quadratic in the variable that is being integrated and we should be able to solve it with some effort. This integral is generally known as the Gaussian integral and it apparently has the general solution

$$\int_{-\infty}^{\infty} e^{-(ax^2+bx+c)}\,dx = \sqrt{\frac{\pi}{a}}e^{-\frac{b^2}{4a}-c} \tag{7}$$

Now it remains to adapt the exponent to the desired form, thus we build

$$-\left(\frac{(\alpha+\beta)^2}{2}\tau^2 + (i2\pi f - \alpha t)\tau + \frac{\alpha}{2}t^2\right) \tag{8}$$

Using the known solution the integral it has been rendered trivial to find the Fourier Transform

$$\begin{aligned}\mathcal{F}\{x(\tau)h^*(\tau-t)\} &= X(t,f) = \\ &= \sqrt{\frac{2}{\alpha+\beta}}(\alpha\beta)^{1/4}e^{\left(\frac{(i2\pi f-\alpha t)^2}{2(\alpha+\beta)}-\frac{\alpha}{2}t^2\right)}\end{aligned}$$

To find the spectrogram $S_x(t,f)$, we take the magnitude squared of $X(f)$:

$$S_x(t,f) = |X(t,f)|^2 = X(t,f)X^*(t,f) \tag{9}$$

which simplifies to something that is obviously equivalent to the desired equation seen in 1, thus marking the end of this exercise.

$$S_x(t,f) = \frac{2\sqrt{\alpha\beta}}{\alpha+\beta}e^{-\frac{\alpha\beta t^2}{\alpha+\beta}-\frac{4\pi^2 f^2}{\alpha+\beta}}. \tag{10}$$

### B. Wigner distribution and Ambiguity function of a Complex-Valued Signal

*1) Calculate the Wigner distribution of $z(t) = e^{i2\pi f_1 t} + e^{i2\pi f_2 t}$ :*

Following the definition of the Wigner distribution (Lecture 3 Stationary and non-Stationary Spectral Analysis)

$$W_x(t,f) = \int_{-\infty}^{\infty} x\left(t+\frac{\tau}{2}\right)x^*\left(t-\frac{\tau}{2}\right)e^{-i2\pi f\tau}d\tau. \tag{11}$$

and substituting the signal with the one provided in the task the following expression is generated.

$$\begin{aligned}W_z(t,f) = \int_{-\infty}^{\infty} &\left(e^{i2\pi f_1\left(t+\frac{\tau}{2}\right)} + e^{i2\pi f_2\left(t-\frac{\tau}{2}\right)}\right) * \\ &* \left(e^{-i2\pi f_1\left(t+\frac{\tau}{2}\right)} + e^{-i2\pi f_2\left(t-\frac{\tau}{2}\right)}\right)e^{-2\pi i f\tau}d\tau\end{aligned}$$

The expression above kind of reminds of a squaring binomial of the left side $(a+b)^2 = a^2 + 2ab + b^2$ (but obviously every term is unique). It happens that when the two parentheses are multiplied in together and the "Fourier

Transform factor" to the right is multiplied with all terms two resulting terms result are quite nice to calculate. If we simplify with respect to the varible $\tau$ one last third term is obtained resulting in the expression below

$$\int_{-\infty}^{\infty} e^{i2\pi(f_1-f)}\, df + \int_{-\infty}^{\infty} e^{i2\pi(f_2-f)}\, df +$$
$$+ \int_{-\infty}^{\infty} \left(e^{i2\pi t(f_1-f_2)} + e^{-i2\pi t(f_1-f_2)}\right) e^{i2\pi\left(\frac{f_1+f_2}{2}-f\right)}\, df$$

All expressions that are $f$-variant are a common Fourier Transform equal to a dirac spike $\delta(f_x - f)$, both the symmetrical expressions in the first two integrals but also in the third integral and we identify the $f$-invariant expression in the third integral as a cosine with trigonometrical identities, $2\cos(2\pi t(f_1 - f_2))$. We finally we reach the Wigner distribution of $z(t)$.

$$W_z(t,f) = \delta(f_1 - f) + \delta(f_2 - f) +$$
$$+2\cos(2\pi t(f_1 - f_2))\delta\left(\frac{f_1+f_2}{2} - f\right)$$

*2) Calculate the ambiguity function of $z(t)$:*

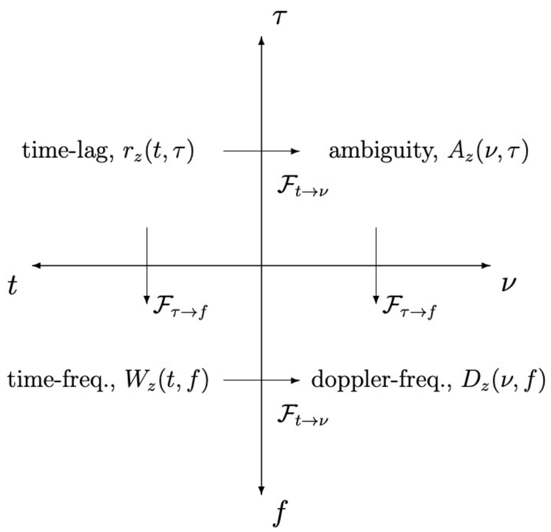


Fig. 1. The four possible domains of time-frequency analysis. [1]

Figure 1 taken from *Time-Frequency Analysis of Time-Varying Signals and Non-Stationary Processes - An Introduction* by Maria Sandsten shows how one can change between domains. Since we have acquired $W_z(t,f)$, we can obtain the ambiguity function $A_z(\nu,\tau)$ after two transformations.

$$r_z(t,\tau) = \mathcal{F}^{-1}_{f\to\tau}\{W_z(t,f)\} = \int_{-\infty}^{\infty} W_z(t,f)e^{-i2\pi f\tau}\, df =$$
$$= e^{-i2\pi f_1 t} + e^{-i2\pi f_2 t} + 2\cos(2\pi t(f_1 - f_2))e^{-i2\pi\left(\frac{f_1+f_2}{2}\right)t}$$

Now I perform the next and last transform finally obtaining the ambiguity function of $z(t)$.

$$A_z(\nu,\tau) = \mathcal{F}_{t\to\nu}\{r_z(t,\tau)\} = \int_{-\infty}^{\infty} r_z(t,\tau)e^{-i2\pi t\nu}\, dt =$$
$$e^{-i2\pi f_1\delta(\nu)} + e^{-i2\pi f_2\delta(\nu)} + e^{-i\pi\frac{f_1+f_2}{2}} *$$
$$* \left(\delta\left(\nu - (f_1 - f_2)\right) + \delta\left(\nu - (f_2 - f_1)\right)\right)$$

*3) Express the Wigner distribution and the ambiguity function of $x(t) = \cos(2\pi f_0 t)$ using the above results:*

The task indicate that we can identify the solution by some rewrites of the given signal. The Euler formula gives a new expression of the signal.

$$x(t) = \cos(2\pi f_0 t) = \frac{e^{i2\pi f_0 t} + e^{-i2\pi f_0 t}}{2} \tag{12}$$

It seems that $x(t) = \frac{1}{2}z^*(t)$, therefore must $W_x(t,f) = \frac{1}{2}W_z(t,f)$ as well (integral is a linear function for constants) where $f_1 = -f_2 = f_0$. The following Wigner distribution is obtained

$$W_x(t,f) = \frac{1}{2}\left(\delta(f_0 - f) + \delta(f_0 + f) + \cos(4\pi t f_0)\delta(f)\right) \tag{13}$$

The ambiguity function of $x$ is derived in a similar manner where $A_x(\nu,\tau) = \frac{1}{2}A_z(\nu,\tau)$ and $f_1 = -f_2 = f_0$ and we obtain the ambiguity function

$$A_x(\nu,\tau) = \delta(\nu)\cos(2\pi f_0\tau) + \frac{1}{2}\left(\delta(\nu + 2f_0) + \delta(\nu - 2f_0)\right) \tag{14}$$

*C. Quadratic Class Definition Analysis*

Use the quadratic class definition to show that it can be written in an alternative form involving $r_z(t,\tau)$ and $\rho(t,\tau)$.

Probably the easiest way to show how these expressions are connected is by "working backwards" and starting with the final result.

$$W_z^Q(t,f) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} r_z(u,\tau)\rho(t-u,\tau)e^{-i2\pi f\tau}\, du\, d\tau \tag{15}$$

Let us substitute $r_z(t,\tau)$ and $\rho(t,\tau)$ and obtain

$$W_x^Q(t,f) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} z\left(u + \frac{\tau}{2}\right) z^*\left(u - \frac{\tau}{2}\right) *$$
$$* \left(\int_{-\infty}^{\infty} \phi(\nu,\tau)e^{(i2\pi\nu(t-u))}d\nu\right) e^{(-i2\pi f\tau)} du d\tau$$

We are allowed to move the integral and the differential to the left-and rightmost ends as the rest of the expression is invariant with respect to $\nu$ outside. The simplified expression apparently is in the same format as the original expression and we are done.

$$W_x^Q(t,f) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} z\left(u+\frac{\tau}{2}\right) z^*\left(u-\frac{\tau}{2}\right) * \\ * \, \phi(\nu,\tau)\exp(i2\pi(\nu t - f\tau))\, du\, d\tau\, d\nu$$

All steps taken can be done "forwards" instead, i.e the steps are reversible. It almost feels like cheating but "cheating" when it is allowed is the best kind of math that there is, thus it has been shown how the quadratic equation can be rewritten in this way as the task asked for.

### *D. Signal Analysis with Matlab Functions*

There was not enough time to execute this task completely and in the Matlab files you will find just the start of the task at least successfully:

- Create a two-component signal with Gaussian windowed complex-valued sinusoids and compute and compare the spectrograms.

The rest was not done properly:

- Calculate the Wigner distributions of the constructed signals, analyzing the effects of window length choice, the appearance of cross-terms, and potential aliasing.

*1) Create complex signal and compute spectrogram:* A two-component signal with Gaussian windowed complex-valued sinusoids with time-freq centers $(80, 0.1)$ and $(180, 0.3)$ were generated according to instruction, please see the

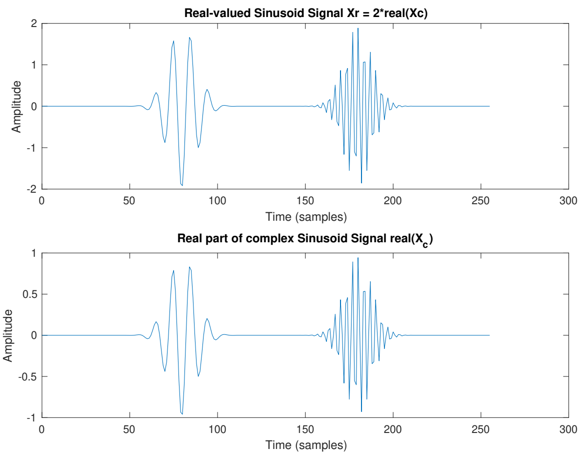


Fig. 2. Generated signals. Note that they have similar shape since they are based in the same signal but differing amplitudes.

Spectrograms of these signals were created in plots and a decision was made to find the optimal one. Several Hanning Window lenghts were tested as fractions of the total sample length divided by two $(128/n)$ were $n = 4, L = 128/4 = 32$ were found to be the best one generating circular shaped centers in the 2d plot. The circular shape corresponds to a balanced estimations of frequency and timing. Note that the spectrogram of the complex and the real valued signal are the same for a given length. When the Hanning window length were above or below the found optimal one the shape of the centers were rather oval shaped, streched out in one dimension. A long length stretched them out in time dimension and a low length in the frequency dimension.

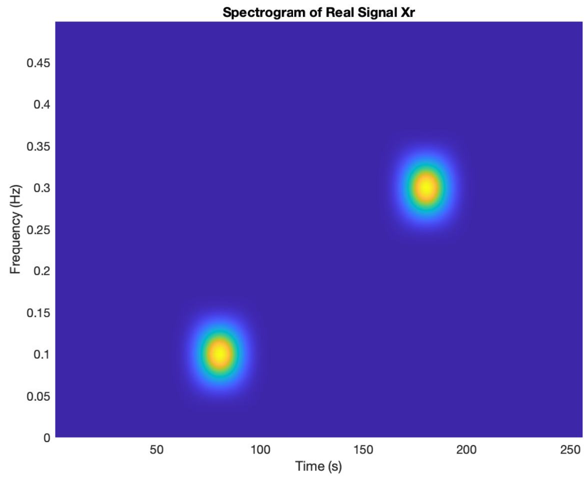


Fig. 3. Best found spectrogram of $X_r$ using Hanning window L = 32.

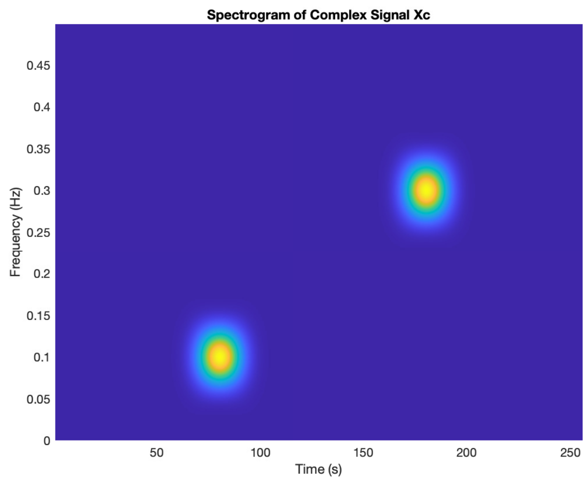


Fig. 4. Best found spectrogram of $X_c$ using Hanning window L = 32.

## III. References

[1] Maria Sandsten. *Time-Frequency Analysis of Time-Varying Signals and Non-Stationary Processes*. Lund University - Centre of Mathematical Statistics, 2024.