Sparse array signal processing

Cheng-Hsuan, Chien

June 12, 2024

Abstract

Antenna array is able to detect physical quantity of some objects such as DOA(Direction of arrival). The simplest array is uniform linear array (ULA), which can distinguish $N-1$ uncorrelated sources with N detectors. However, the mutual coupling of the detector causes more error. Sparse array has larger spacing between detector pairs, which reduced the mutual coupling effect [3]. At the same time, the number of uncorrelated sources which can be identified is of order $O(N^2)$ with N detectors. Sparse array saves resources and result in better detection accuracy. In this report, I simply review various kinds of sparse array and the MUSIC algorithm. Some MATLAB code are implemented to better understand the knowledge.

1 ULA (Uniform Linear Array)

The most straightforward way to set up a sensor array in 1-D is placing them with the same separation d uniformly. This configuration is called Uniform Linear Array. For monochromatic signal source with wavelength λ, the spacing of the sensors should be larger than $\frac{\lambda}{2}$ to prevent aliasing. In this report, we set the separation as $d = \frac{\lambda}{2}$. Suppose the total number of sources is D and their DOA(Direction of arrival) of the i^{th} source is θ_i. The configuration is shown in 1. The signal received by the sensor at position nd is:

$$x_n = \sum_{i=1}^{D} A_i \exp(j\frac{2\pi d \sin\theta}{\lambda} n) + noise(n) = \sum_{i=1}^{D} A_i \exp(j2\pi\bar{\theta}n) + noise(n), \tag{1}$$

where A_i is the complex amplitude of the i^{th} source. Normalized DOA is $\bar{\theta}_i = \frac{d\sin\theta}{\lambda} = \frac{\sin\theta}{2}$, which ranges from $\frac{-1}{2}$ to $\frac{1}{2}$. Summarizing the N received signals, we get a vector equation:

$$\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \cdots \\ x_{N-1} \end{bmatrix} = \sum_{i=1} A_i \begin{bmatrix} 1 \\ \exp(j2\pi\bar{\theta}) \\ \exp(j2\pi\bar{\theta}\cdot 2) \\ \cdots \\ \exp(j2\pi\bar{\theta}\cdot(N-1)) \end{bmatrix} + \begin{bmatrix} noise(0) \\ noise(1) \\ noise(2) \\ \cdots \\ noise(N-1). \end{bmatrix} \tag{2}$$

In brief, we write

$$\mathbf{x}_{\mathbb{S}} = \sum_{i=1}^{D} A_i \mathbf{v}_{\mathbb{S}}(\bar{\theta}) + \mathbf{n}_{\mathbb{S}}, \tag{3}$$

where $\mathbf{v}_{\mathbb{S}}$ is called the steering vector.

1.1 DOA estimation by MUSIC [1][4]

Early in 1986, a DOA estimation algorithm **multiple signal classification** (MUSIC) is proposed by Schmidt [11]. The description is as follows: We defined $\mathbb{R}_{\mathbb{S}} = \mathbb{E}[\mathbf{x}\mathbf{x}^H]$ (The S matrix in the original proposal).

$$\begin{aligned} \mathbb{R} &= \mathbb{E}[\mathbf{x}\mathbf{x}^H] && (4) \\ &= \mathbb{E}[\sum_{i,j}(A_iA_j^*)\mathbf{v}(\theta_i)\mathbf{v}^H(\theta_j)] + \mathbb{E}[\sum_i A_i\mathbf{v}(\theta_i)\mathbf{n}^H] + \mathbb{E}[\sum_j A_j^*\mathbf{n}\mathbf{v}^H(\theta_j)] + \mathbb{E}[\mathbf{n}\mathbf{n}^H] && (5) \\ &= \sum_{i,j}\mathbb{E}(A_iA_j^*)\mathbf{v}(\theta_i)\mathbf{v}^H(\theta_j) + \sum_i \mathbf{v}(\theta_i)\mathbb{E}(A_i\mathbf{n}^H) + \sum_j \mathbb{E}(A_j^*\mathbf{n})\mathbf{v}^H(\theta_j) + \mathbb{E}[\mathbf{n}\mathbf{n}^H]. && (6) \end{aligned}$$

The noise and the signal amplitude are uncorrelated:

$$\mathbb{E}(A_i\mathbf{n}^H) = \mathbf{0}^H, \tag{7}$$

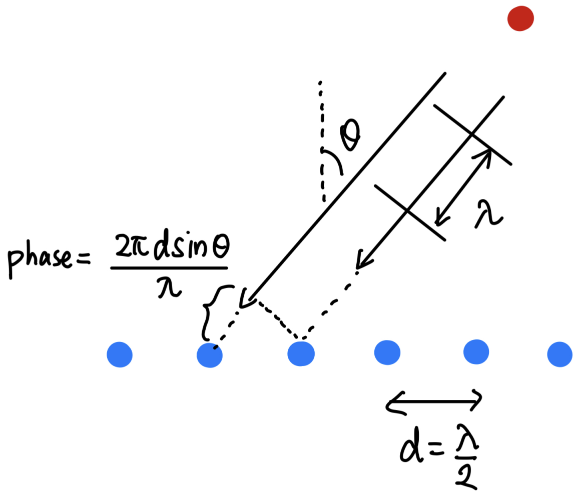

Figure 1: DOA estimation by ULA: The angle of the source (red dot) is θ Suppose the wavelength of the signal is a constant λ. In far-field regime, the phase difference between the two received signals are proportional to $\sin\theta$. This provides the information of DOA. The spacing between two receivers is set to be $\frac{\lambda}{2}$ to prevent aliasing effect.

$$\mathbb{E}(A_j^*\mathbf{n}) = \mathbf{0}. \tag{8}$$

The expectation value of $A_i A_j^*$ can be represented as $\delta_{i,j} p_i$, where $\delta_{i,j}$ is the Kronecker-delta symbol and $p_i = |A_i|^2$ is the power of the signal. The Kronecker-delta symbol implies signals with $i \neq j$ are uncorrelated. The noise term $\mathbb{E}(\mathbf{n}\mathbf{n}^H)$ is the expectation value of the power of the noise. Similarly, noise as non-zero covariance with itself, so it can be written as $\mathbb{E}(\mathbf{n}\mathbf{n}^H) = p_n \mathbf{I}$, where p_n is the expected power of the noise. Finally,

$$\mathbb{R} = \sum_i^D p_i \mathbf{v}(\theta_i)\mathbf{v}^H(\theta_i) + p_n\mathbf{I}. \tag{9}$$

The first term correspond to the signals (rank D) and the second term is the noise term. $\mathbb{R}$ is an N by N matrix. Assume the number of signal sources D is smaller than the number of detectors $N = |\mathbb{S}|$. We can decompose the first term (signal) as

$$\sum_i^D p_i \mathbf{v}(\theta_i)\mathbf{v}^H(\theta_i) = \sum_{i=1}^D \mu_i \mathbf{u}_i \mathbf{u}_i^H, \tag{10}$$

where μ_i are the eigenvalues and the set of N-vectors $\{\mathbf{u}_i\}_{i=1,2,3,\cdots,N}$ are orthogonal. The rest part (noise) can be written as

$$p_n\mathbf{I} = \sum_{i=1}^N p_n \mathbf{u}_i \mathbf{u}_i^H. \tag{11}$$

Overall, the correlation matrix $\mathbb{R}$ is decomposed as

$$\mathbb{R} = \sum_{i=1}^D (\mu_i + p_n)\mathbf{u}_i\mathbf{u}_i^H + \sum_{i=D+1}^N p_n \mathbf{u}_i\mathbf{u}_i^H. \tag{12}$$

The eigenvalue of $\mathbb{R}$ is λ_i with

$$\lambda_i = \mu_i + p_n, i = 1, 2, 3, \cdots, D. \tag{13}$$

$$\lambda_i = p_n, i = D+1, D+2, \cdots, N. \tag{14}$$

Since $\sum_i^D p_i \mathbf{v}(\theta_i)\mathbf{v}^H(\theta_i) = \sum_{i=1}^D \mu_i \mathbf{u}_i \mathbf{u}_i^H$ is positive semi-definite, the eigenvalues μ_i must be non-negative. So the eigenvector of the signal space must have large eigenvalues. Let

$$\mathbf{U}_s = \begin{bmatrix}\mathbf{u}_1 & \mathbf{u}_2 & \cdots \mathbf{u}_D.\end{bmatrix} \tag{15}$$

$$\mathbf{U}_n = \begin{bmatrix}\mathbf{u}_{D+1} & \mathbf{u}_{D+2} & \cdots \mathbf{u}_N.\end{bmatrix} \tag{16}$$

$\mathbf{U}_s$ is a $N \times D$ matrix corresponding to the signal space and $\mathbf{U}_s$ is a $N \times (N-D)$ matrix of the noise space. Due to the orthogonality of $\{\mathbf{u}_i\}_{i=1,2,3,\cdots,N}$, we have the following relation.

$$\mathbf{U}_n^H \mathbf{v}(\theta_i) = \mathbf{0}_{(N-D)}. \tag{17}$$

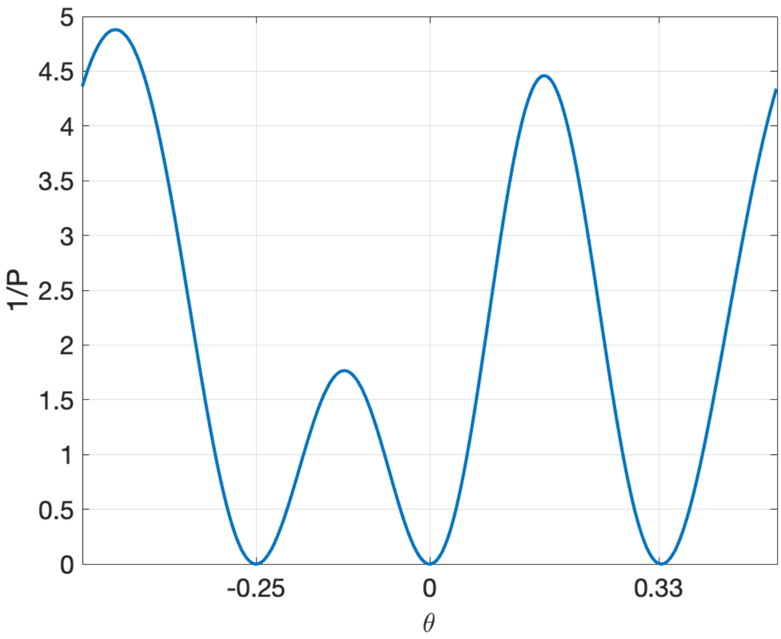

Figure 2: The reciprocal of the MUSIC spectrum $\frac{1}{P(\theta)}$. The zeros indicated the DOA of the sources.

In other words, when the angle $\theta = \theta_i$, we have

$$P(\theta) = \frac{1}{||\mathbf{U}_n^H \mathbf{v}(\theta)||^2} = \infty. \tag{18}$$

$P(\theta)$ is called the MUSIC spectrum. Here, a simple example is implemented. Let $D = 3$ and $N = 5$. The three DOA $\bar{\theta}$ are respectively $-\frac{1}{4}, 0, \frac{1}{3}$. So

$$\mathbf{v}(\bar{\theta_1}) = \begin{bmatrix} 1 \\ \exp(-j\frac{\pi}{2}) \\ \exp(-2j\frac{\pi}{2}) \\ \exp(-3j\frac{\pi}{2}) \\ \exp(-4j\frac{\pi}{2}). \end{bmatrix} \tag{19}$$

$\mathbf{v}(\bar{\theta}_{2,3})$ are similar. Let $p_i = 10$ and $p_n = 1$. Then the covariance matrix is

$$10(\mathbf{v}_1\mathbf{v}_1^H + \mathbf{v}_2\mathbf{v}_2^H + \mathbf{v}_3\mathbf{v}_3^H) + 1\mathbf{I}. \tag{20}$$

We eigen-decompose the matrix and calculate the reciprocal of the MUSIC spectrum $1/P(\bar{\theta})$. The result is shown in 2. We can see the reciprocal of the spectrum drops to zero when the normalized DOA is exactly the same as the source DOA.

1.2 MUSIC in practice

In real world, we cannot precisely know the value of the covariance matrix. We can get closed to the expectation value by taking K snapshot and calculate the average:

$$\tilde{\mathbb{R}} = \frac{1}{K}\sum_{j=1}^{K} \mathbf{x}_j \mathbf{x}_j^H. \tag{21}$$

$\mathbf{x}_j$ is the measurement of the j^{th} snapshot.

1.3 Drawbacks of physical array MUSIC [3]

For ULA with N sensors with uniform distance d. There are $N-1 = O(N)$ different distances between two sensors. Having more different distances improves the performance of DOA estimation. When N is already large, adding one more sensor can hardly improve the performance. This is because ULA has larger **spatial redundancy**. For example, there are $N-1$ sensor pairs having distance d, which is unhelpful in DOA estimation.

To remove such redundancy, using different spacing (sparse array) is one solution. Another drawback is the **mutual coupling** [12] between sensors. Physically, sensors are antenna collecting electromagnetic wave. The current in one antenna can contribute to the received signal in other closed antennas. The effect of mutual coupling can be characterized by a mutual coupling matrix $\mathbf{C}$. The steering vector $\mathbf{v}_{\mathbb{S}}$ should be modified as a version multiplied by the mutual coupling matrix: $\mathbf{C}\mathbf{v}_{\mathbb{S}}$.

2 Basics of sparse array [4]

In this chapter, the basic definitions about sparse array are shown. Most of the content is summarized from [4].

Definition 2.1 (Physical array) *The physical array correspond to the sensor in real world. The symbol $\mathbb{S}$ is used to indicate the positions of the sensors. The first sensor is indexed with $n = 0$.*

Definition 2.2 (Difference coarray) *For each possible signal array $\mathbb{S}$, we can defined its difference coarray $\mathbb{D}$ to characterise the property of sparse array. The formal definition of $\mathbb{D}$ is*

$$\mathbb{D} = \{x = n_1 - n_2 | n_1, n_2 \in \mathbb{S}\}. \tag{22}$$

*The cardinility of $\mathbb{D}$ is call the **degree of freedom** (DOF) of the sparse array.*

Definition 2.3 (central ULA segmment) *If $\mathbb{D}$ has a **largest ULA subset** in the middle, then this part is called central ULA segment and denoted by $\mathbb{U}$. The formal definition is*

$$\mathbb{U} = \{0, \pm 1, \pm 2, \cdots, \pm m\}, \tag{23}$$

*where m is the largest number such that $\mathbb{U} \subseteq \mathbb{D}$. The cardinanity of the set $\mathbb{U}$ is called the **uniform degree of freedom** (uniform DOF)of the sparse array*

Definition 2.4 (Smallest ULA containing $\mathbb{D}$) *The **smallset ULA set** containing the difference coarray is denoted by $\mathbb{V}$:*

$$\mathbb{V} = \{0, \pm 1, \pm 2, \cdots, \pm m\}, \tag{24}$$

where m is the smallest number such that $\mathbb{D} \subseteq \mathbb{V}$.

Definition 2.5 (Holes in $\mathbb{D}$) *The holes is defined as the set composed of those elements in $\mathbb{V}$ but not in $\mathbb{D}$:*

$$\mathbb{H} = \mathbb{V} \backslash \mathbb{D}. \tag{25}$$

*where m is the smallest number such that $\mathbb{D} \subseteq \mathbb{V}$. A difference coarray $\mathbb{D}$ is called **hole-free** if $\mathbb{H}$ is an empty set. The corresponding sparse array $\mathbb{S}$ is called **restricted**. If $\mathbb{D}$ is not hole-free, then the sparse array is called **general**.*

Definition 2.6 (Weight function) *The weight function of a sparse array is defined as the number of a specific spatial separation of sensors pair.*

$$w(m) = |\{(n_1, n_2) | n_1, n_2 \in \mathbb{S}, n_1 - n_2 = m\}|. \tag{26}$$

The weight function characterize the redundancy of the sensor separation. For instance, ULA has spatial separation d between k^{th} and $k + 1^{th}$ sensor (totally $n - 1$ pair, the weight function at $m = 1$ has value $(n - 1)$. Since the sources are considered to be in far field region, the correlations between the signal pairs are all the same. To acquire higher detection efficiency, we want the value of weight function to be as small as possible and spreads as large as possible. The weight function is called spatial sensitivity sometimes [8]. An example of these sets are shown in 3

3 Types of sparse array

3.1 Nested array [2][9]

For a pair of positive integers N_1 and N_2, the nested array is defined as

$$\mathbb{S}_{nested} = \{0, 1, 2, \cdots, N_1, N_1 + 1, 2(N_1 + 1), \cdots, N_2(N_1 + 1)\}. \tag{27}$$

Geometrically, it is a union of two ULA. The large one is with spacing $N_1 + 1$ and the small one is with spacing 1. An example is shown in 4. The small ULA "nested" in one "hole" of the large ULA. The intuition of using this array is that all the cites which is in the hole of the large ULA is and element in $\mathbb{D}$ because the small ULA is hole-free. The generated difference coarray is hole-free as well. The nested array can also be regarded as a "straightened" version of a simple 2D array (5).

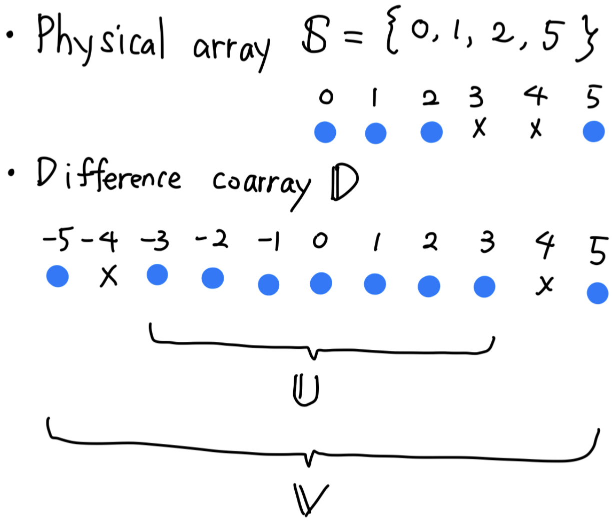

Figure 3: Example of $\mathbb{S}, \mathbb{D}, \mathbb{U}$ and $\mathbb{V}$. In this example, there are two holes $\{-4, 4\}$ in the difference coarray since no n_1, n_2 exists in $\mathbb{S}$ such that $n_1 - n_2 = \pm 4$

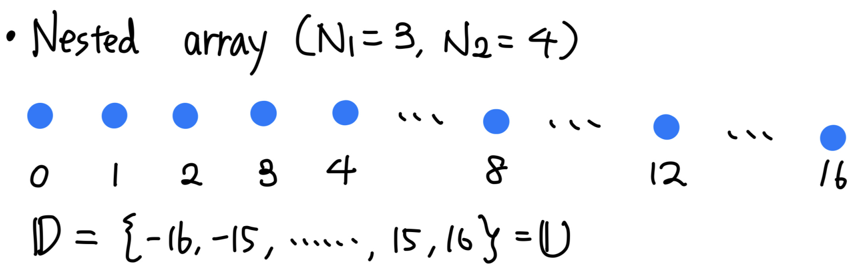

Figure 4: An example of nested array. The central ULA is ranging from -16 to 16, which is exactly the same as the difference coarray.

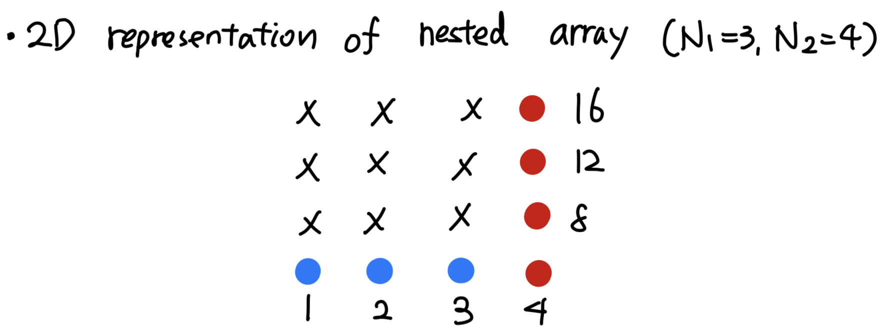

Figure 5: 2D representation of a nested array with $N_1 = 3, N_2 = 4$.

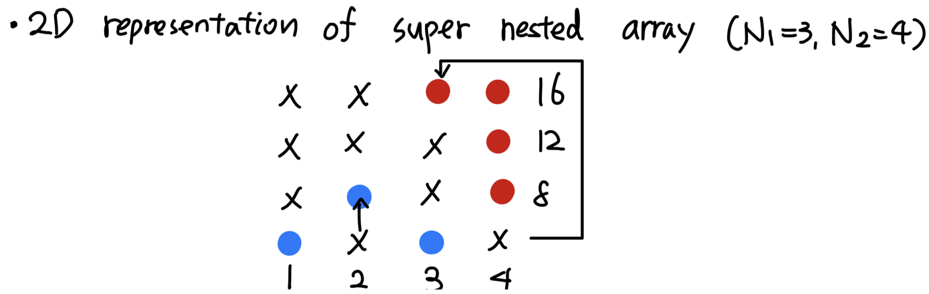

Figure 6: An example of super nested array with $N_1 = 3$ and $N_2 = 4$. The graph is modified by moving several node of the 2D representation of the nested array.

3.2 Super nested array [6]

Super nested array's 2D representation is modified by the one of nested array. Similar as nested array, the difference coarray is hole-free. However, the physical array $\mathbb{S}$ of super nested array is sparser than the nested array, which results in smaller mutual coupling. A super nested array's 2D representation is shown in 6. A super nested array is a union of 6 ULA:

$$\mathbb{S} = \mathbb{X}_1 \cup \mathbb{X}_2 \cup \mathbb{Y}_1 \cup \mathbb{Y}_2 \cup \mathbb{Z}_1 \cup \mathbb{Z}_2. \tag{28}$$

$$\mathbb{X}_1 = \{1 + 2l | 0 \le l \le A_1\}, \tag{29}$$

$$\mathbb{X}_2 = \{(N_1 + 1) + (2 + 2l) | 0 \le l \le A_2\}, \tag{30}$$

$$\mathbb{Y}_1 = \{(N_1 + 1) - (1 + 2l) | 0 \le l \le B_1\}, \tag{31}$$

$$\mathbb{Y}_2 = \{2(N_1 + 1) - (2 + 2l) | 0 \le l \le B_2\}, \tag{32}$$

$$\mathbb{Z}_1 = \{l(N_1 + 1) | 2 \le l \le N_2\}, \tag{33}$$

$$\mathbb{Z}_2 = \{N_2(N_1 + 1) - 1\}. \tag{34}$$

A_1, A_2, B_1, B_2 are constants related to N_1.

3.3 Coprime array [10][13]

The configuration of a coprime array is characterized by two coprime integers M, N, with $M + N$ sensors, the spatial sensitivity is $O(MN)$ [14]. The version with $2M + N$ sensor[13] was proposed as well. The sensor array can be mathematically defined as:

$$\mathbb{S}_{coprime} = \{kM | k = 0, 1, 2, \cdots, N - 1\} \cup \{kN | k = 0, 1, 2, \cdots, 2M - 1\}. \tag{35}$$

The consecutive difference coarray $\mathbb{D}$ is at least ranging from $-2MN$ to $2MN$. The intuitive reason for choosing coprime number for me is that when $gcd(M, N) = 1$ (7, 9), there must be a, b such that $aM - bN = 1$. The spacing between aM and bN is 1. Similarly, the spacing between $2aM$ and $2bN$ is 2. We can use this to construct a series of consecutive integers. The following graphs are showing the weight function and the uniform degree of freedom. An example with $gcd(M, N) \neq 1$ is shown in 8.

4 Coarray-based DOA estimators

4.1 Spatial smoothing[5]

The n_1, n_2 entry of the covariance matrix is

$$\mathbb{R}_{n_1,n_2} = \sum_{i=1}^{D} p_i v_{n_1}(\theta_i) v^*_{n_2}(\theta_i) + p_n \delta_{n_1,n_2} \tag{36}$$

The produce of the two element of the steering vector is

$$v_{n_1}(\theta_i) v^*_{n_2}(\theta_i) = exp(j2\pi(n_1 - n_2)\bar{\theta}_i). \tag{37}$$

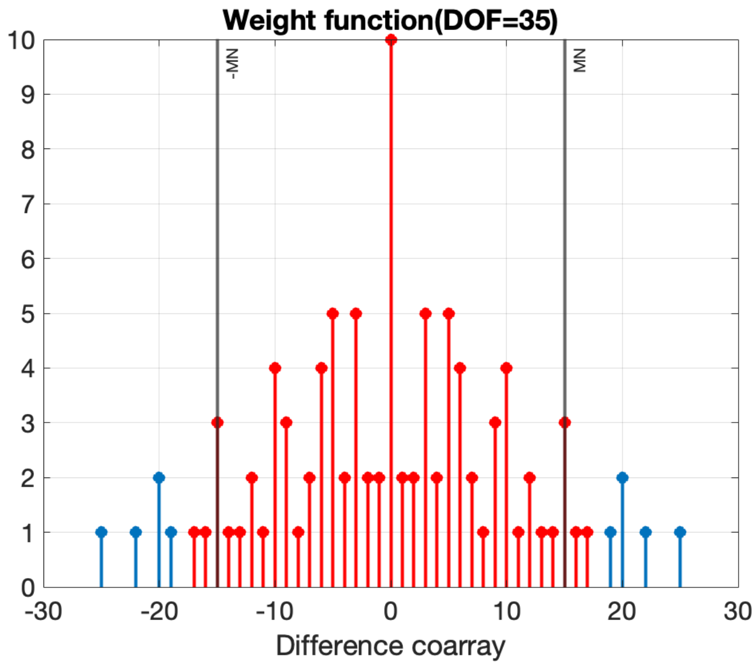

Figure 7: $M = 3, N = 5$ are coprime with each other. The parameters generates the central ULA(shown by the red stems) with length 35.

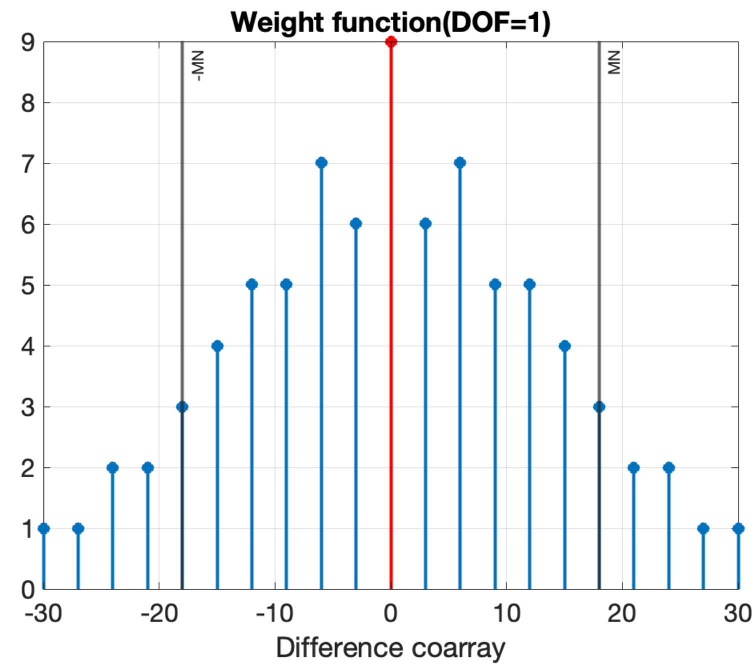

Figure 8: $M = 3, N = 6$ are not coprime with each other. The weight function of the difference coarray has non-zero value only when the index is the multiple of $gcd(3, 6) = 3$. The total uniform degree of freedom is 1.

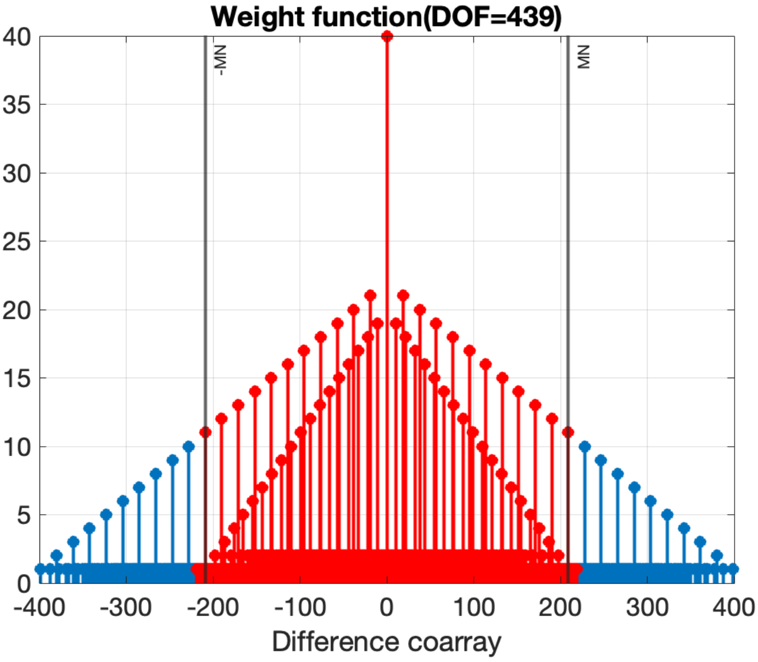

Figure 9: $M = 11, N = 19$ are coprime with each other. The uniform ULA still covers from -MN to MN.

The element of the covariance matrix is only related to the **difference** of the indexes. Hence we can include the concept of **difference coarray** $\mathbb{D}$ and rewrite the autocorrelation vector defined by the difference coarray

$$\mathbf{x}_{\mathbb{D}} = \sum_{i=1}^{D} p_i \mathbf{v}_D(\bar{\theta}_i) + p_n \mathbf{e}_0, \tag{38}$$

where $(\mathbf{e}_0)_m = \delta_{m,0}$. $m \in \mathbb{D}$. Similarly, we can define the autocorrelation vector related to the central ULA $\mathbb{U}$:

$$\mathbf{x}_{\mathbb{U}} = \sum_{i=1}^{D} p_i \mathbf{v}_D(\bar{\theta}_i) + p_n \mathbf{e}'_0, \tag{39}$$

where $(\mathbf{e}'_0)_m = \delta_{m,0}$. $m \in \mathbb{U}$. In practice, these autocorrelation vectors are calculated by averaging the corresponding element of the measured covariance $\tilde{\mathbb{R}}$:

$$(\mathbf{x}_{\mathbb{D}})_m = \frac{1}{w(m)} \sum_{n_1 - n_2 = m} \tilde{\mathbb{R}}_{n_1,n_2}. \tag{40}$$

5 Cramér-Rao Bounds (CRB)[7]

Let $\mathbf{x}$ to be a real-valued random vector and $\mathbf{a}$ is a real-valued parameter vector. The probability density function is $p(\mathbf{x}, \mathbf{a})$. The Fisher information matrix is

$$[\mathcal{I}(\mathbf{a})]_{i,j} = -\mathbb{E}_{\mathbf{x}}[\frac{\partial^2}{\partial \mathbf{a}_i \partial \mathbf{a}_j} \log(p(\mathbf{x}, \mathbf{a}))], \tag{41}$$

which is positive semi-definite. The CRB is defined as the inverse of the Fisher information matrix:

$$CRB(\mathbf{a}) = \mathcal{I}^{-1}(\mathbf{a}). \tag{42}$$

Let $\hat{\mathbf{a}}(\mathbf{x})$ to be the estimator of $\mathbf{a}$ based on the observation $\mathbf{x}$. The covariance matrix must be lower bounded by CRB:

$$\mathbb{E}_{\mathbf{x}}[\hat{\mathbf{a}}\hat{\mathbf{a}}^T] \succcurlyeq CRB(\mathbf{a}). \tag{43}$$

In ULA system with K snapshots, we set the vector $\mathbf{a}$ as

$$\mathbf{a} = \begin{bmatrix} \bar{\theta}_i & \Re(A_i(k)) & \Im(A_i(k)) & p_n \end{bmatrix}^T, \tag{44}$$

where $k = 1, 2, 3, \cdots, K$, $i = 1, 2, 3, \cdots, D$.

6 MATLAB code

6.1 MUSIC algorithm with $N = 5, D = 3$

```
N = 1:5;
N = N';

%steering factor
v1 = exp(-1i*2*pi/4).^(N-1);
v2 = exp(-1i*2*pi*0).^(N-1);
v3 = exp(1i*2*pi/3).^(N-1);

%covariance matrix and noise space
R = 10*(v1*v1'+v2*v2'+v3*v3')+eye(5);
[V,D] = eig(R);
Un = V(:,1:2);

% find the reciprocal of P and plot
theta = -1/2:0.003:1/2;
Pinv = [];
for k = 1:length(theta)
    v = exp(1i*2*pi*theta(k)).^(N-1);
    Pinv = [Pinv norm(Un'*v).^2];
end
plot(theta,P,"LineWidth",2)
xlabel("\theta")
xticks([-0.25,0,0.33])
ylabel("1/P")
set(gca,"FontSize",16)
grid
```

6.2 Coprime array

```
M = 11;
N = 19;

D1 = coarray(coprime(M,N));
table = tabulate(D1);
A = (DOF(D1)-1)/2;
a = find(table(:,1) == -A);
b = find(table(:,1) == A);
stem(table(:,1),table(:,2),"filled","LineWidth",2)
hold on
stem(table(a:b,1),table(a:b,2),"filled","-r","LineWidth",2)
xline(-M*N,"-","-MN","LineWidth",2);
xline(M*N,"-","MN","LineWidth",2);
grid
title("Weight function(DOF="+DOF(D1)+")")
xlabel("Difference coarray")
set(gca,"FontSize",16)

%calculate the physical array by M and N
function [S] = coprime(M,N)
    S = [];
    for i = 1:N
        if ismember((i-1)*M,S) == 0
            S = [S (i-1)*M];
        end
    end
    for i = 1:2*M
```

```
        if ismember((i-1)*N,S) == 0
            S = [S (i-1)*N];
        end
    end
    sort(S);
end

%calculate the difference coarray by the physical array
function [D] = coarray(S)
    D = [];
    for i = 1:length(S)
        for j = 1:length(S)
            D = [D S(i)-S(j)];
        end
    end
    D = sort(D);
end

%calculate the degree of freedom
function [output] = DOF(D)
    output = 0;
    for i = 1:max(D)
        if ismember(i,D)
            output = output + 1;
        else
            break
        end
    end
    output = 2*output + 1;
end
```

References

[1] Pooja Gupta and S.P. Kar. "MUSIC and improved MUSIC algorithm to estimate direction of arrival". In: *2015 International Conference on Communications and Signal Processing (ICCSP)*. 2015, pp. 0757–0761. DOI: 10.1109/ICCSP.2015.7322593.

[2] Keyong Han. *Statistical Nested Sensor Array Signal Processing*. Washington University in St. Louis, 2015.

[3] C. Larmour et al. "Sparse Array Mutual Coupling Reduction". In: *IEEE Open Journal of Antennas and Propagation* 5.1 (2024), pp. 201–216. DOI: 10.1109/OJAP.2023.3339368.

[4] Chun-Lin Liu. *Sparse array signal processing: New array geometries, parameter estimation, and theoretical analysis*. California Institute of Technology, 2018.

[5] Chun-Lin Liu and P. P. Vaidyanathan. "Remarks on the Spatial Smoothing Step in Coarray MUSIC". In: *IEEE Signal Processing Letters* 22.9 (2015), pp. 1438–1442. DOI: 10.1109/LSP.2015.2409153.

[6] Chun-Lin Liu and P. P. Vaidyanathan. "Super nested arrays: Sparse arrays with less mutual coupling than nested arrays". In: *2016 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP)*. 2016, pp. 2976–2980. DOI: 10.1109/ICASSP.2016.7472223.

[7] Chun-Lin Liu and P.P. Vaidyanathan. "Cramér–Rao bounds for coprime and other sparse arrays, which find more sources than sensors". In: *Digital Signal Processing* 61 (2017). Special Issue on Coprime Sampling and Arrays, pp. 43–61. ISSN: 1051-2004. DOI: https://doi.org/10.1016/j.dsp.2016.04.011. URL: https://www.sciencedirect.com/science/article/pii/S1051200416300264.

[8] A. Moffet. "Minimum-redundancy linear arrays". In: *IEEE Transactions on Antennas and Propagation* 16.2 (1968), pp. 172–175. DOI: 10.1109/TAP.1968.1139138.

[9] Piya Pal and P. P. Vaidyanathan. "Nested Arrays: A Novel Approach to Array Processing With Enhanced Degrees of Freedom". In: *IEEE Transactions on Signal Processing* 58.8 (2010), pp. 4167–4181. DOI: 10.1109/TSP.2010.2049264.

[10] Si Qin, Yimin D. Zhang, and Moeness G. Amin. "Generalized Coprime Array Configurations for Direction-of-Arrival Estimation". In: *IEEE Transactions on Signal Processing* 63.6 (2015), pp. 1377–1390. DOI: 10.1109/TSP.2015.2393838.

[11] R. Schmidt. “Multiple emitter location and signal parameter estimation”. In: *IEEE Transactions on Antennas and Propagation* 34.3 (1986), pp. 276–280. DOI: 10.1109/TAP.1986.1143830.

[12] Hema Singh, HL Sneha, RM Jha, et al. “Mutual coupling in phased arrays: A review”. In: *International Journal of Antennas and Propagation* 2013 (2013).

[13] Zhao Tan, Yonina C. Eldar, and Arye Nehorai. “Direction of Arrival Estimation Using Co-Prime Arrays: A Super Resolution Viewpoint”. In: *IEEE Transactions on Signal Processing* 62.21 (2014), pp. 5565–5576. DOI: 10.1109/TSP.2014.2354316.

[14] Palghat P. Vaidyanathan and Piya Pal. “Sparse Sensing With Co-Prime Samplers and Arrays”. In: *IEEE Transactions on Signal Processing* 59.2 (2011), pp. 573–586. DOI: 10.1109/TSP.2010.2089682.