CHAPTER 7

The Discrete Fourier Transform

Tutorial Problems

1. (a) Solution:
   The CTFT of $x_{\mathrm{c}}(t)$ is:

$$X_{\mathrm{c}}(\mathrm{j}2\pi F) = \int_{-\infty}^{\infty} x_{\mathrm{c}}(t)\mathrm{e}^{-\mathrm{j}\Omega t}\,\mathrm{d}t = \int_{0}^{\infty} 5\mathrm{e}^{-10t}\sin(20\pi t)\mathrm{e}^{-\mathrm{j}\Omega t}\,\mathrm{d}t$$
$$= \frac{100\pi}{(20\pi)^2 + (\mathrm{j}\Omega + 10)^2} = \frac{100\pi}{(20\pi)^2 + (\mathrm{j}20\pi F + 10)^2}$$

   (b) See plot below.

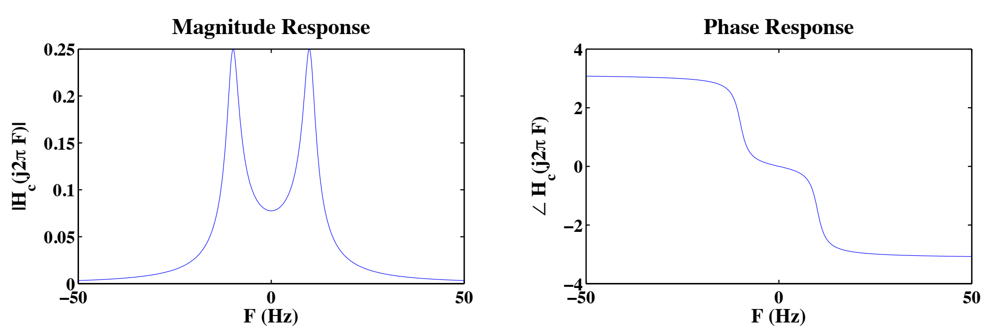


FIGURE 7.1: Magnitude and phase responses of $X_{\mathrm{c}}(\mathrm{j}2\pi F)$ over $-50 \leq F \leq 50$ Hz.

   (c) See plot below.

   MATLAB script:

```
% P0701: Numerical DFT approximation of CTFT
```

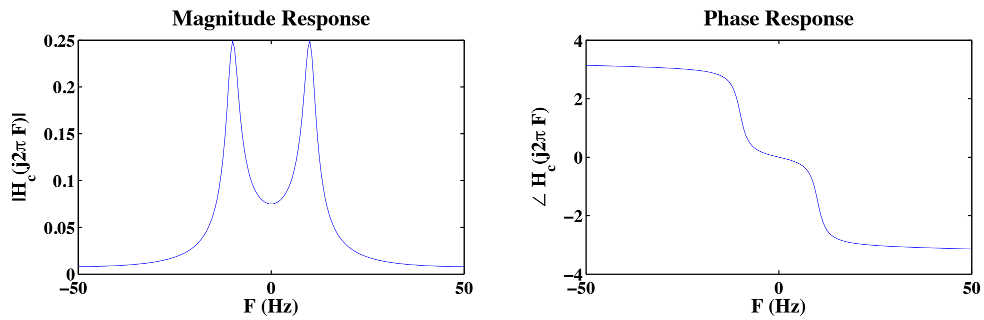


FIGURE 7.2: Approximated magnitude and phase responses of $X_c(j2\pi F)$ over $-50 \le F \le 50$ Hz using fft function.

```
close all; clc
%% Part (b):
t1 = 0; t2 = 2;
dF = 0.1;
F = -50:dF:50;
Xc = 100*pi./((20*pi)^2+(j*2*pi*F+10).^2);
%% Part (c):
Fs = 100;
T = Fs\1;
nT = t1:T:t2;
N = length(nT);
xn = 5*exp(-10*nT).*sin(20*pi*nT);
X = fftshift(fft(xn));
w = linspace(-pi,pi,N);
Xc_approx = T*X;
%% Plot:
hfa = figconfg('P0701a','long');
subplot(121)
plot(F,abs(Xc))
xlabel('F (Hz)','fontsize',LFS)
ylabel('|H_c(j2\pi F)|','fontsize',LFS)
title('Magnitude Response','fontsize',TFS)
subplot(122)
plot(F,angle(Xc))
xlabel('F (Hz)','fontsize',LFS)
ylabel('\angle H_c(j2\pi F)','fontsize',LFS)
```

```
title('Phase Response','fontsize',TFS)

hfb = figconfg('P0701b','long');
subplot(121)
plot(w/T/2/pi,abs(Xc_approx))
xlabel('F (Hz)','fontsize',LFS)
ylabel('|H_c(j2\pi F)|','fontsize',LFS)
title('Magnitude Response','fontsize',TFS)
subplot(122)
plot(w/T/2/pi,angle(Xc_approx))
xlabel('F (Hz)','fontsize',LFS)
ylabel('\angle H_c(j2\pi F)','fontsize',LFS)
title('Phase Response','fontsize',TFS)
```

2. (a) Solution:
   $T_0 = 5$, the fundamental period $\Omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{5}$.
   The CTFS of $\tilde{x}_c(t)$ is:

$$c_k = \frac{1}{T_0}\int_0^{T_0} \tilde{x}_c(t)\mathrm{e}^{-\mathrm{j}k\Omega_0 t}\mathrm{d}t = \frac{1}{5}\int_0^5 \mathrm{e}^{-t}\cdot \mathrm{e}^{-\mathrm{j}k\Omega_0 t}dt$$
$$= \frac{1-\mathrm{e}^{-5}}{5+\mathrm{j}2\pi k}$$

   (b) See plot below.

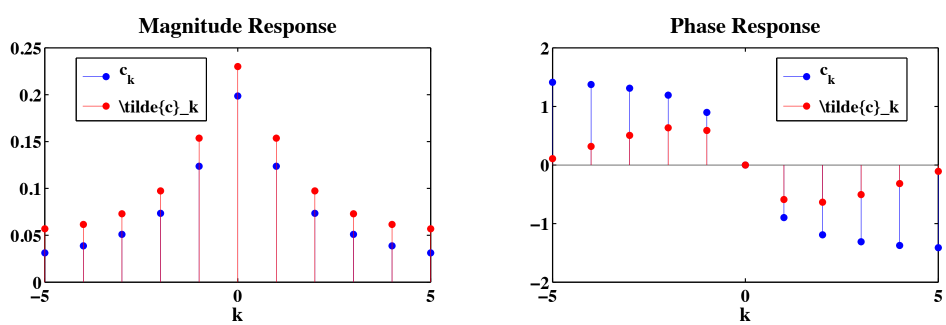


FIGURE 7.3: Magnitude and phase responses of c_k and $\hat{c}_k$ when sampling interval $T = 0.5$s.

   (c) See plot below.
   (d) See plot below.

MATLAB script:

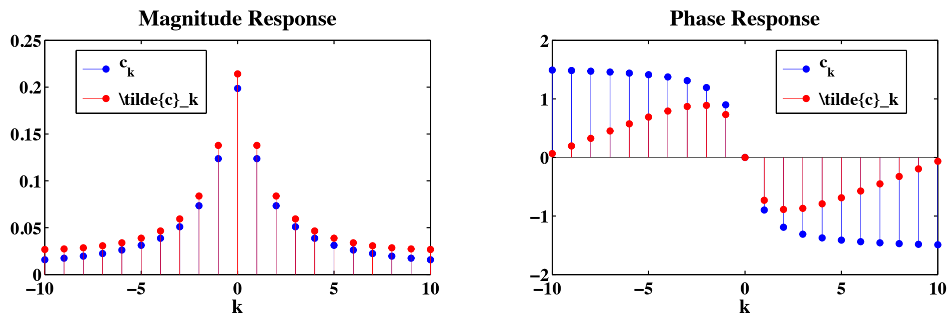


FIGURE 7.4: Magnitude and phase responses of c_k and $\hat{c}_k$ when sampling interval $T = 0.25$s.

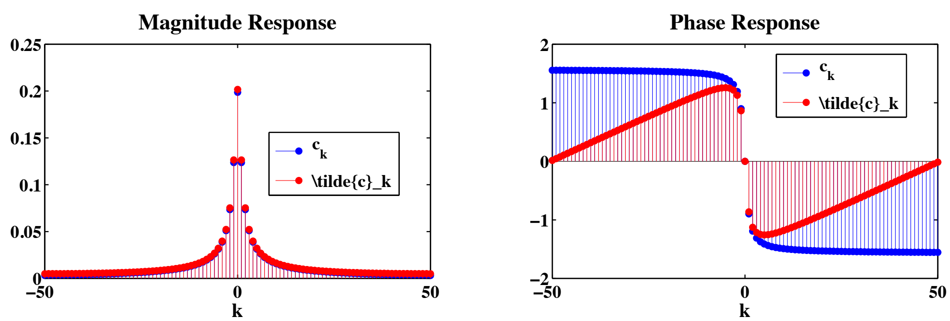


FIGURE 7.5: Magnitude and phase responses of c_k and $\hat{c}_k$ when sampling interval $T = 0.05$s.

```
% P0702: Numerical DFT approximation of CTFS
close all; clc
t1 = 0; t2 = 5;
T = 0.5; % Part (b)
% T = 0.25; % Part (c)
% T = 0.05; % Part (d)
N = t2/T;
k = -N/2:N/2;
ck = (1-exp(-5))./(5+j*2*pi*k);
nT = t1:T:t2;
xn = exp(-nT);
ck_approx = length(xn)\fftshift(fft(xn));
%% Plot:
```

```
hfa = figconfg('P0702a','long');
subplot(121)
stem(k,abs(ck),'filled');hold on
stem(k,abs(ck_approx),'filled','color','red');
xlabel('k','fontsize',LFS)
title('Magnitude Response','fontsize',TFS)
legend('c_k','\tilde{c}_k','location','best')
subplot(122)
stem(k,angle(ck),'filled');hold on
stem(k,angle(ck_approx),'filled','color','red');
xlabel('k','fontsize',LFS)
title('Phase Response','fontsize',TFS)
legend('c_k','\tilde{c}_k','location','best')
```

3. (a) Solution:
   The DTFT of $(0.9)^n u[n]$ is:

$$\frac{1}{1-0.9\mathrm{e}^{-\mathrm{j}\omega}}$$

   The DTFT of $x[n]$ is:

$$\begin{aligned}\tilde{X}(\mathrm{e}^{\mathrm{j}\omega}) &= (-\mathrm{j})\frac{\mathrm{d}}{\mathrm{d}\omega}\left(\frac{1}{1-0.9\mathrm{e}^{-\mathrm{j}\omega}}\right) \\ &= \frac{0.9\mathrm{e}^{-\mathrm{j}\omega}}{(1-0.9\mathrm{e}^{-\mathrm{j}\omega})^2}\end{aligned}$$

   (b) See plot below.
   (c) See plot below.
   (d) See plot below.

MATLAB script:

```
% P0703: Numerical DFT approximation of DTFT
close all; clc
w = linspace(0,2,1000)*pi;
X = 0.9*exp(-j*w)./(1-0.9*exp(-j*w)).^2;
N = 20; % Part (b)
% N = 50; % Part (c)
% N = 100; % Part (d)
n = 0:N-1;
```

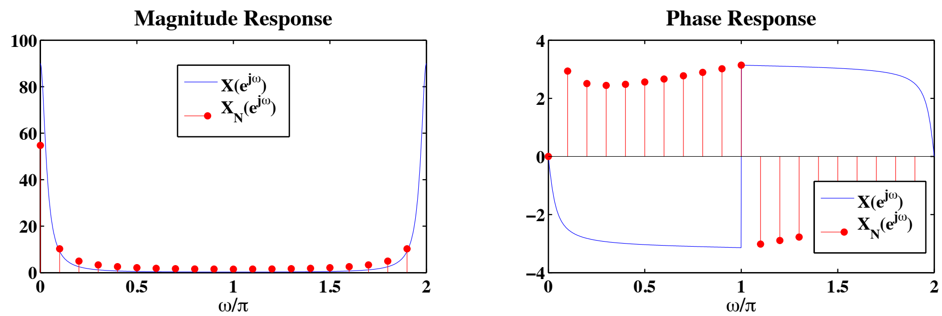


FIGURE 7.6: Magnitude and phase responses of $\tilde{X}(e^{j\omega})$ and $\tilde{X}_N(e^{j\omega})$ when $N = 20$.

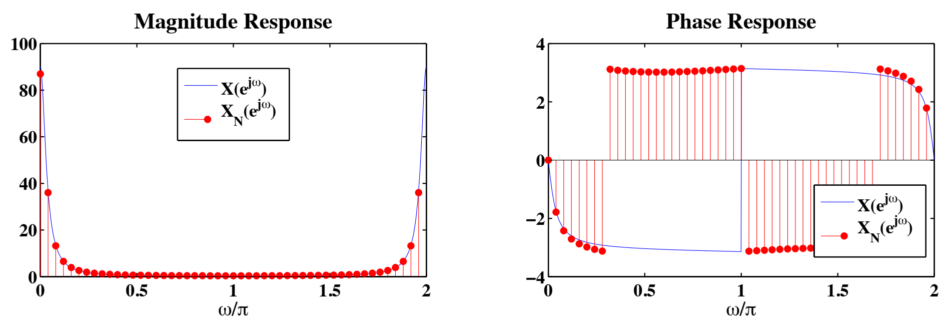


FIGURE 7.7: Magnitude and phase responses of $\tilde{X}(e^{j\omega})$ and $\tilde{X}_N(e^{j\omega})$ when $N = 50$.

```
xn = n.*0.9.^n;
XN = fft(xn);
wk = 2/N*(0:N-1);
%% Plot:
hfa = figconfg('P0703a','long');
subplot(121)
plot(w/pi,abs(X));hold on
stem(wk,abs(XN),'filled','color','red');
xlabel('\omega/\pi','fontsize',LFS)
title('Magnitude Response','fontsize',TFS)
legend('X(e^{j\omega})','X_N(e^{j\omega})','location','best')
subplot(122)
plot(w/pi,angle(X));hold on
```

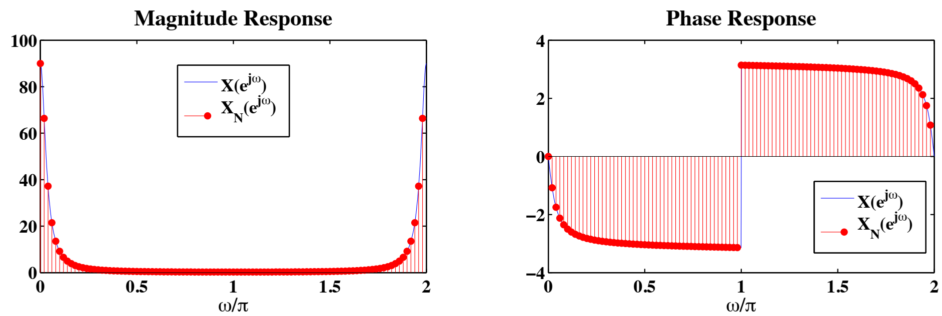


FIGURE 7.8: Magnitude and phase responses of $\tilde{X}(\mathrm{e}^{\mathrm{j}\omega})$ and $\tilde{X}_N(\mathrm{e}^{\mathrm{j}\omega})$ when $N = 100$.

```
stem(wk,angle(XN),'filled','color','red');
xlabel('\omega/\pi','fontsize',LFS)
title('Phase Response','fontsize',TFS)
legend('X(e^{j\omega})','X_N(e^{j\omega})','location','best')
```

4. (a) Solution:
The $N \times N$ DFT matrix is:

$$\boldsymbol{W}_N = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & W_N & \cdots & W_N^{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{N-1} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix}$$

The kth column of $\boldsymbol{W}_N$ is $\boldsymbol{w}_k = [1 W_N^k \dots W_N^{N-1} k]^T$.
The i, jth element of $\boldsymbol{W}_N^2$ is:

$$\begin{aligned}\left(\boldsymbol{W}_N^2\right)_{i,j} &= \left(\boldsymbol{W}_N^T \boldsymbol{W}_N\right)_{i,j} = \boldsymbol{w}_i^T \boldsymbol{w}_j \\ &= \begin{cases} 0, & i+j \neq N \\ N, & i+j = N \end{cases}\end{aligned}$$

Hence, we proved that

$$\boldsymbol{W}_N^2 = \begin{bmatrix} 0 & \cdots & 0 & N \\ \vdots & \iddots & N & 0 \\ 0 & \iddots & \iddots & \vdots \\ N & 0 & \cdots & 0 \end{bmatrix} = N \boldsymbol{J}_N$$

The effect of the flip matrix on $\boldsymbol{J}_N\boldsymbol{x}$ product is rearranging the elements in column vector $\boldsymbol{x}$ in reverse order (that is flip $\boldsymbol{x}$ upside down).

(b) Solution:

$$\boldsymbol{W}_N^4 = (N\boldsymbol{J}_N)(N\boldsymbol{J}_N) = N^2(\boldsymbol{J}_N \cdot \boldsymbol{J}_N) = N^2\boldsymbol{I}_N$$

(c) Solution:
The multiplicity depends on the value of N modulo 4 and can be summarized in the following table.

size N	$\lambda = 1$	$\lambda = -1$	$\lambda = -\mathrm{j}$	$\lambda = \mathrm{j}$
$4m$	$m+1$	m	m	$m-1$
$4m+1$	$m+1$	m	m	m
$4m+2$	$m+1$	$m+1$	m	m
$4m+3$	$m+1$	$m+1$	$m+1$	m

TABLE 7.1: Multiplicity of the eigenvalues of the DFT matrix $\boldsymbol{W}_N$.

MATLAB script:

```
% P0704: Investigate the eigenvalues multiplicity
%        of the DFT matrix WN
close all; clc
N = 4:10;
for ii = 1:length(N)
    WN = dftmtx(N(ii))/sqrt(N(ii));
    eig(WN)
end
```

5. (a) Solution:
The DFT of $x[n]$ is:

$$\begin{aligned} X[k] &= \sum_{n=0}^{N-1} x[n]W_N^{kn} = \sum_{n=0}^{7}(4-n)\cdot \mathrm{e}^{-\mathrm{j}\frac{2\pi}{8}kn} \\ &= 4 + 3(\mathrm{e}^{-\mathrm{j}\frac{2\pi}{8}} - \mathrm{e}^{\mathrm{j}\frac{2\pi}{8}}) + 2(\mathrm{e}^{-\mathrm{j}\frac{2\pi}{8}2} - \mathrm{e}^{\mathrm{j}\frac{2\pi}{8}2}) + (\mathrm{e}^{-\mathrm{j}\frac{2\pi}{8}3} - \mathrm{e}^{\mathrm{j}\frac{2\pi}{8}3}) \\ &= 4 - 6\mathrm{j}\sin\left(\frac{k\pi}{4}\right) - 4\mathrm{j}\sin\left(\frac{k\pi}{2}\right) - 2\mathrm{j}\sin\left(\frac{3k\pi}{4}\right) \end{aligned}$$

(b) Solution:
The DFT of $x[n]$ is:

$$\begin{aligned}
X[k] &= \sum_{n=0}^{N-1} x[n]W_N^{kn} = \sum_{n=0}^{9} \frac{2}{\mathrm{j}}\left(\mathrm{e}^{\mathrm{j}\frac{2\pi}{10}n} - \mathrm{e}^{-\mathrm{j}\frac{2\pi}{10}n}\right)\mathrm{e}^{-\mathrm{j}\frac{2\pi}{10}nk} \\
&= \frac{2}{\mathrm{j}}\sum_{n=0}^{9} \mathrm{e}^{\mathrm{j}\frac{2\pi}{10}n}\cdot\mathrm{e}^{-\mathrm{j}\frac{2\pi}{10}nk} - \frac{2}{\mathrm{j}}\sum_{n=0}^{9} \mathrm{e}^{-\mathrm{j}\frac{2\pi}{10}n}\cdot\mathrm{e}^{-\mathrm{j}\frac{2\pi}{10}nk} \\
&= -20\mathrm{j}\delta[k-1] + 20\mathrm{j}\delta[k-9]
\end{aligned}$$

(c) Solution:
The DFT of $x[n]$ is:

$$\begin{aligned}
X[k] &= \sum_{n=0}^{N-1} x[n]W_N^{kn} = \sum_{n=0}^{9}\left(3 + \frac{3}{2}\mathrm{e}^{\mathrm{j}\frac{2\pi}{10}n2} - \frac{3}{2}\mathrm{e}^{-\mathrm{j}\frac{2\pi}{10}n2}\right)\mathrm{e}^{-\mathrm{j}\frac{2\pi}{10}nk} \\
&= 30\delta[k] + 15\delta[k-2] - 15\delta[k-8]
\end{aligned}$$

(d) Solution:
The DFT of $x[n]$ is:

$$\begin{aligned}
X[k] &= \sum_{n=0}^{N-1} x[n]W_N^{kn} = \sum_{n=0}^{15} 5(0.8)^n\mathrm{e}^{-\mathrm{j}\frac{2\pi}{16}nk} \\
&= 5\sum_{n=0}^{15}\left(0.8\mathrm{e}^{-\mathrm{j}\frac{2\pi}{16}k}\right)^n = 5\cdot\frac{1-0.8^{16}}{1-0.8\mathrm{e}^{-\mathrm{j}\frac{2\pi}{16}k}}
\end{aligned}$$

(e) Solution:
The DFT of $x[n]$ is:

$$\begin{aligned}
X[k] &= \sum_{n=0}^{N-1} x[n]W_N^{kn} = \sum_{n=0}^{19} x[n]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{20}nk} \\
&= \sum_{m=0}^{9} 3\mathrm{e}^{-\mathrm{j}\frac{2\pi}{20}k\cdot 2m} + \sum_{m=0}^{9}(-2)\mathrm{e}^{-\mathrm{j}\frac{2\pi}{20}k\cdot(2m+1)} \\
&= 3\sum_{m=0}^{9}\left(\mathrm{e}^{-\mathrm{j}\frac{\pi}{5}m}\right)^m - 2\mathrm{e}^{-\mathrm{j}\frac{\pi}{10}k}\sum_{m=0}^{9}\left(\mathrm{e}^{-\mathrm{j}\frac{\pi}{5}m}\right)^m \\
&= 10\delta[k] + 50\delta[k-10]
\end{aligned}$$

6. Proof:

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{nk} = \begin{bmatrix} \boldsymbol{w}_1^T \\ \boldsymbol{w}_2^T \\ \vdots \\ \boldsymbol{w}_{N-1}^T \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} \boldsymbol{w}_1^T \boldsymbol{x} \\ \boldsymbol{w}_2^T \boldsymbol{x} \\ \vdots \\ \boldsymbol{w}_{N-1}^T \boldsymbol{x} \end{bmatrix}$$

Hence, we proved that the DFT coefficients $X[k]$ are the projections of the signal $x[n]$ on the DFT (basis) vectors $\{\boldsymbol{w}_k\}$.

7. (a) Solution:
The DFT of $\tilde{x}[n] = 2\cos(\pi n/4)$ is:

$$\begin{aligned} X[k] &= \sum_{n=0}^{7} \left(\mathrm{e}^{\mathrm{j}\frac{2\pi}{8}n2} + \mathrm{e}^{-\mathrm{j}\frac{2\pi}{8}n2} \right) \mathrm{e}^{-\mathrm{j}\frac{2\pi}{8}nk} \\ &= 8\delta[k-2] + 8\delta[k-6] \end{aligned}$$

The DFS of $\tilde{x}[n] = 2\cos(\pi n/4)$ is:

$$\tilde{X}[k] = 8\delta[\langle k \rangle_8 - 2] + 8\delta[\langle k \rangle_8 - 6]$$

(b) Solution:
The DFT of $\tilde{x}[n] = 3\sin(0.25\pi n) + 4\cos(0.75\pi n)$ is:

$$\begin{aligned} X[k] &= \sum_{n=0}^{7} \left[\frac{3}{2\mathrm{j}} \left(\mathrm{e}^{\mathrm{j}\frac{2\pi}{8}n2} - \mathrm{e}^{-\mathrm{j}\frac{2\pi}{8}n2} \right) + 2\left(\mathrm{e}^{\mathrm{j}\frac{2\pi}{8}n3} + \mathrm{e}^{-\mathrm{j}\frac{2\pi}{8}n3} \right) \right] \mathrm{e}^{-\mathrm{j}\frac{2\pi}{8}nk} \\ &= -12\delta[k-2] + 12\delta[k-6] + 16\delta[k-3] + 16\delta[n-5] \end{aligned}$$

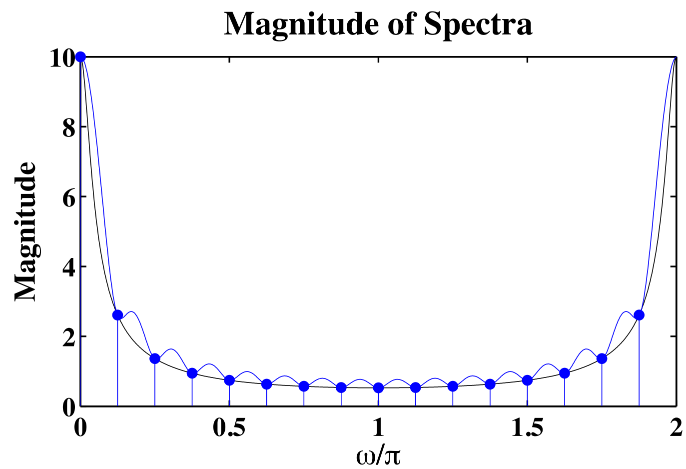
The DFS of $\tilde{x}[n] = 3\sin(0.25\pi n) + 4\cos(0.75\pi n)$ is:

$$\tilde{X}[k] = -12\delta[\langle k \rangle_8 - 2] + 12\delta[\langle k \rangle_8 - 6] + 16\delta[\langle k \rangle_8 - 3] + 16\delta[\langle k \rangle_8 - 5]$$

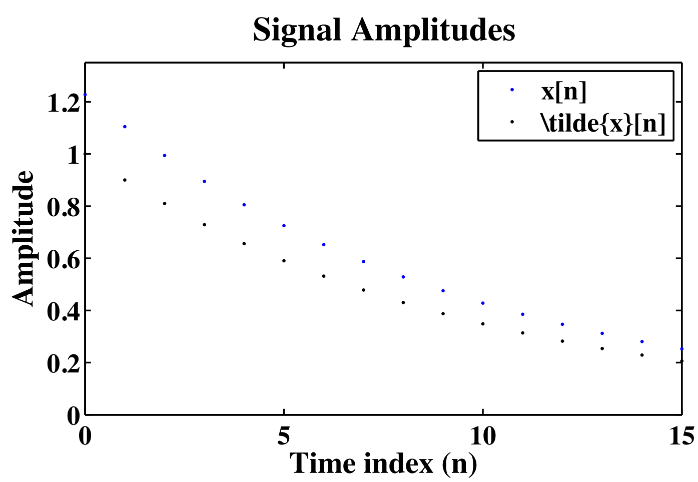
8. (a) See plot below.
   (b) See plot below.
   (c) See plot below.

   MATLAB script:

```
% P0708: Regenerate Figure~7.5 and Example 7.3
close all; clc
N = 16; a = 0.9; % Part (a)
% N = 8; a = 0.8; % Part (b)
% N = 64; a = 0.8; % Part (c)
wk = 2*pi/N*(0:N-1);
Xk = 1./(1-a*exp(-j*wk));
xn = real(ifft(Xk));
w = linspace(0,2,1000)*pi;
X = fft(xn,length(w));
X_ref = 1./(1-a*exp(-j*w));
n = 0:N-1;
xn_ref = a.^n;
%% Plot:
hfa = figconfg('P0708a','small');
plot(w/pi,abs(X_ref),'color','black'); hold on
plot(w/pi,abs(X))
stem(wk/pi,abs(Xk),'filled');
ylim([0 max(abs(X))])
xlabel('\omega/\pi','fontsize',LFS)
ylabel('Magnitude','fontsize',LFS)
title('Magnitude of Spectra','fontsize',TFS)
hfb = figconfg('P0708b','small');
plot(n,xn,'.'); hold on
plot(n,xn_ref,'.','color','black')
ylim([0 1.1*max(xn)])
xlim([0 N-1])
xlabel('Time index (n)','fontsize',LFS)
ylabel('Amplitude','fontsize',LFS)
title('Signal Amplitudes','fontsize',TFS)
legend('x[n]','\tilde{x}[n]','location','northeast')
```

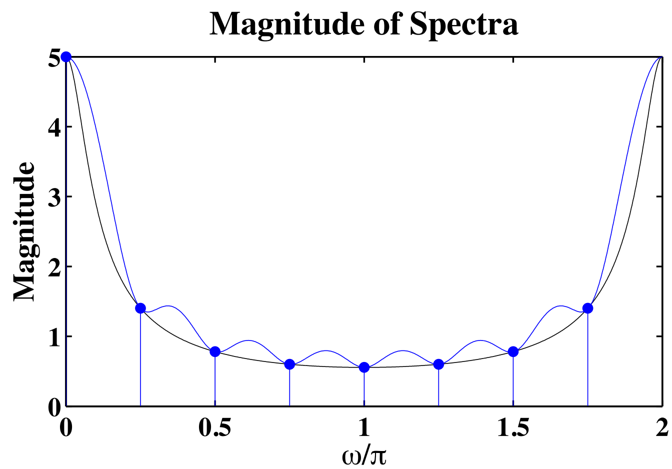


(a)

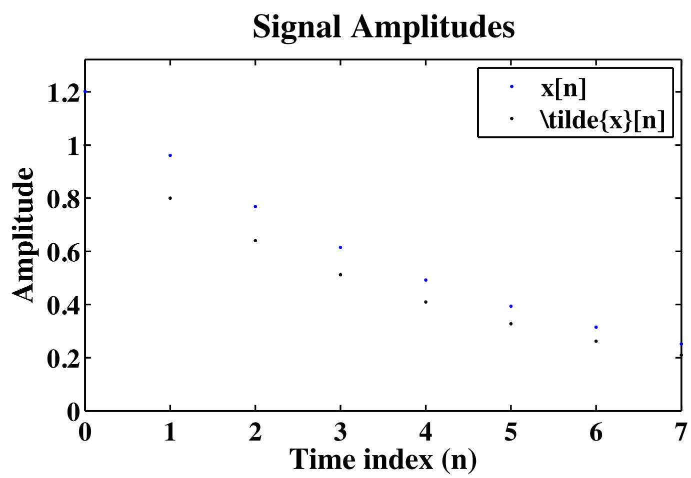


(b)

FIGURE 7.9: (a) Magnitude response of the DTFT signal. (b) Time sequence and reconstructed time sequence.



(a)



(b)

FIGURE 7.10: (a) Magnitude response of the DTFT signal and (b) time sequence and reconstructed time sequence for $a = 0.8$ and $N = 8$.

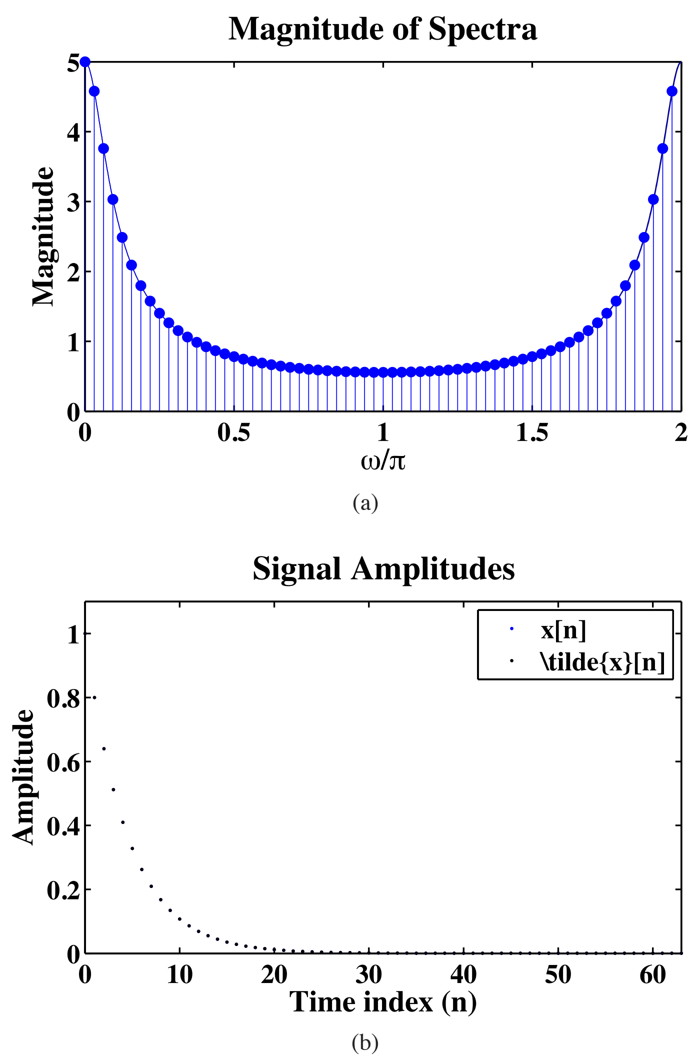


FIGURE 7.11: (a) Magnitude response of the DTFT signal and (b) time sequence and reconstructed time sequence for $a = 0.8$ and $N = 64$.

9. Proof:

$$\begin{aligned}\tilde{x}[n] &= \sum_{\ell=-\infty}^{\infty} x[n-\ell N] = \sum_{\ell=-\infty}^{\infty} a^{n-\ell N} u[n-\ell N] \\ &= \sum_{\ell=-\infty}^{0} a^{n-\ell N} = a^n \sum_{\ell=0}^{\infty} \left(a^N\right)^{\ell} = \frac{x[n]}{1-a^N}\end{aligned}$$

Hence, as $a \to 0$ or $N \to \infty$, we have $1 - a^N \to 1$, that is $\tilde{x}[n]$ tends to $x[n]$.

10. Proof:

$$X(z) = \frac{1-z^{-N}}{N} \sum_{k=0}^{N-1} \frac{\tilde{X}\left(\mathrm{e}^{\mathrm{j}\frac{2\pi}{N}k}\right)}{1-\mathrm{e}^{\mathrm{j}\frac{2\pi}{N}k} z^{-1}} \tag{7.72}$$

$$\tilde{X}(\mathrm{e}^{\mathrm{j}\omega}) = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}\left(\mathrm{e}^{\mathrm{j}\frac{2\pi}{N}k}\right) \tilde{P}_N\left[\mathrm{e}^{\mathrm{j}(\omega-\frac{2\pi}{N}k)}\right] \tag{7.62}$$

$$\tilde{P}_N(\mathrm{e}^{\mathrm{j}\omega}) = \frac{\sin(\omega N/2)}{N\sin(\omega/2)} \mathrm{e}^{-\mathrm{j}\omega(N-1)/2} \tag{7.63}$$

Substitute $z = \mathrm{e}^{j\omega}$ into (7.72), we have

$$\begin{aligned}X(z)|_{z=\mathrm{e}^{\mathrm{j}\omega}} &= \frac{1-\mathrm{e}^{-\mathrm{j}\omega N}}{N} \sum_{k=0}^{N-1} \frac{\tilde{X}\left(\mathrm{e}^{\mathrm{j}\frac{2\pi}{N}k}\right)}{1-\mathrm{e}^{\mathrm{j}\frac{2\pi}{N}k}\mathrm{e}^{-\mathrm{j}\omega}} \\ &= \frac{1}{N} \sum_{k=0}^{N-1} \frac{\tilde{X}\left(\mathrm{e}^{\mathrm{j}\frac{2\pi}{N}k}\right)(1-\mathrm{e}^{-\mathrm{j}\omega N}\cdot \mathrm{e}^{\mathrm{j}\frac{2\pi}{N}Nk})}{\mathrm{e}^{\mathrm{j}(\frac{2\pi}{N}k-\omega)\frac{1}{2}}\left[\mathrm{e}^{\mathrm{j}(\omega-\frac{2\pi}{N}k)\frac{1}{2}} - \mathrm{e}^{-\mathrm{j}(\omega-\frac{2\pi}{N}k)\frac{1}{2}}\right]} \\ &= \frac{1}{N} \sum_{k=0}^{N-1} \frac{\tilde{X}\left(\mathrm{e}^{\mathrm{j}\frac{2\pi}{N}k}\right)\mathrm{e}^{-\mathrm{j}(\omega-\frac{2\pi}{N}k)N/2}(\mathrm{e}^{\mathrm{j}(\omega-\frac{2\pi}{N}k)N/2} - \mathrm{e}^{-\mathrm{j}(\omega-\frac{2\pi}{N}k)N/2})}{\mathrm{e}^{\mathrm{j}(\frac{2\pi}{N}k-\omega)\frac{1}{2}}\left[\mathrm{e}^{\mathrm{j}(\omega-\frac{2\pi}{N}k)\frac{1}{2}} - \mathrm{e}^{-\mathrm{j}(\omega-\frac{2\pi}{N}k)\frac{1}{2}}\right]} \\ &= \frac{1}{N} \sum_{k=0}^{N-1} \frac{\tilde{X}\left(\mathrm{e}^{\mathrm{j}\frac{2\pi}{N}k}\right)\sin[(\omega-\frac{2\pi}{N}k)\frac{N}{2}]}{\sin[(\omega-\frac{2\pi}{N}k)\frac{1}{2}]} \mathrm{e}^{-\mathrm{j}(\omega-\frac{2\pi}{N}k)\frac{N-1}{2}} \\ &= \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}\left(\mathrm{e}^{\mathrm{j}\frac{2\pi}{N}k}\right) \tilde{P}_N\left[\mathrm{e}^{\mathrm{j}(\omega-\frac{2\pi}{N}k)}\right]\end{aligned}$$

11. Proof:

$$
\begin{aligned}
\sum_{n=0}^{N-1} x[\langle -n\rangle_N] \mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}nk} &= x[0] + \sum_{n=1}^{N-1} x[N-n]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}nk} \\
&= x[0] + \sum_{n=1}^{N-1} x[n]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}(N-n)k} = x[0] + \sum_{n=1}^{N-1} x[n]\mathrm{e}^{\mathrm{j}\frac{2\pi}{N}nk} \\
&= \sum_{n=0}^{N-1} x[n]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}n(-k)} = \sum_{n=0}^{N-1} x[n]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}n\langle -k\rangle_N} \\
&= X[\langle -k\rangle_N]
\end{aligned}
$$

12. (a) Proof:

$$x[n] \overset{\text{DFT}}{\longleftrightarrow} X_1[k] + \mathrm{j}X_2[k]$$

$$x^*[\langle -n\rangle_N] \overset{\text{DFT}}{\longleftrightarrow} X_1[k] - \mathrm{j}X_2[k]$$

$$
\begin{aligned}
x^{\text{cce}}[n] &= \frac{1}{2}x[n] + \frac{1}{2}x^*[\langle -n\rangle_N] \\
X^{\text{cce}}[k] &= \frac{1}{2}(X_1[k] + \mathrm{j}X_2[k]) + \frac{1}{2}(X_1[k] - \mathrm{j}X_2[k]) \\
&= X_1[k]
\end{aligned}
$$

$$
\begin{aligned}
x^{\text{cco}}[n] &= \frac{1}{2}x[n] - \frac{1}{2}x^*[\langle -n\rangle_N] \\
X^{\text{cco}}[k] &= \frac{1}{2}(X_1[k] + \mathrm{j}X_2[k]) - \frac{1}{2}(X_1[k] - \mathrm{j}X_2[k]) \\
&= \mathrm{j}X_2[k]
\end{aligned}
$$

(b) MATLAB script:

```
function [X1 X2] = tworealDFTs(x1,x2)
% Compute the DFTs of two real sequences using one DFT
xc = x1 + j*x2;
X = fft(xc);
XX = conj([X(1) fliplr(X(2:end))]);
X1 = (X+XX)/2;
X2 = (X-XX)/2/j;
```

(c) MATLAB script:

```
% P0712: Matlab Verification of function 'towrealDFTs'
close all; clc
n = 0:49;
N = length(n);
x1 = 0.9.^n;
x2 = 1 - 0.8.^n;
[X1 X2] = tworealDFTs(x1,x2);
% Verification:
X1_ref = fft(x1);
X2_ref = fft(x2);
```

13. (a) Proof:
If k is even and N is even, the correspondent DFT is:

$$
\begin{aligned}
X[k] &= \sum_{n=0}^{N-1} x[n]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}nk} = \sum_{n=0}^{N/2-1} x[n]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}nk} + \sum_{n=N/2}^{N-1} x[n]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}nk} \\
&= \sum_{n=0}^{N/2-1} \left(x[n]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}nk} + x[n+\frac{N}{2}]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}(n+\frac{N}{2})k} \right) \\
&= \sum_{n=0}^{N/2-1} \left(x[n] - x[n] \right) \mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}nk} = 0
\end{aligned}
$$

(b) Proof:
If $N = 4m, k = 4\ell$, the correspondent DFT is:

$$
\begin{aligned}
X[4\ell] &= \sum_{n=0}^{N-1} x[n]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}n(4\ell)} = \sum_{n=0}^{\frac{N}{4}-1} x[n]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}n(4\ell)} \\
&+ \sum_{n=\frac{N}{4}}^{\frac{2N}{4}-1} x[n]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}n(4\ell)} + \sum_{n=\frac{2N}{4}}^{\frac{3N}{4}-1} x[n]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}n(4\ell)} + \sum_{n=\frac{3N}{4}}^{N-1} x[n]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}n(4\ell)} \\
&= \left(\sum_{n=0}^{\frac{N}{4}-1} x[n]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}n(4\ell)} + \sum_{n=0}^{\frac{N}{4}-1} x[n+\frac{N}{4}]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}(n+\frac{N}{4})(4\ell)} \right) \\
&+ \left(\sum_{n=\frac{2N}{4}}^{\frac{3N}{4}-1} x[n]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}n(4\ell)} + \sum_{n=\frac{2N}{4}}^{\frac{3N}{4}-1} x[n+\frac{N}{4}]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}(n+\frac{N}{4})(4\ell)} \right) \\
&= \sum_{n=0}^{\frac{N}{4}-1} \left(x[n] + x[n+\frac{N}{4}] \right) \mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}n(4\ell)} + \sum_{n=\frac{2N}{4}}^{\frac{3N}{4}-1} \left(x[n] + x[n+\frac{N}{4}] \right) \mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}n(4\ell)} \\
&= 0
\end{aligned}
$$

14. (a) Solution:
Solving the circular convolution using hand calculation:

$$
\begin{bmatrix} 2 & 0 & -1 & 1 & -1 \\ -1 & 2 & 0 & -1 & 1 \\ 1 & -1 & 2 & 0 & -1 \\ -1 & 1 & -1 & 2 & 0 \\ 0 & -1 & 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ 0 \\ 6 \\ 7 \end{bmatrix}
$$

(b) See script below.

(c) See script below.

MATLAB script:

```
% P0714: Circular convolution
close all; clc
xn1 = 1:5;
xn2 = [2 -1 1 -1];
%% Part (b):
xn = circonv(xn1',[xn2 0]');
%% Part (c):
N = max(length(xn1),length(xn2));
Xk1 = fft(xn1,N);
Xk2 = fft(xn2,N);
Xk = Xk1.*Xk2;
xn_dft = ifft(Xk);
```

15. (a) Proof:
$X_4[K]$ can be obtained by frequency sampling of $X_3[k]$, hence in the time domain, according the aliasing equation, we have

$$x_4[n] = \sum_{\ell=-\infty}^{\infty} x_3[n + \ell N]$$

(b) Proof:
When $N \geq L$, there is no time aliasing, we conclude

$$x_4[n] = x_3[n], \quad \text{for} \quad 0 \leq n \leq L$$

When $\max(N_1, N_2) \leq N < L$, since $L = N_1 + N_2 - 1 \leq 2N - 1$, we conclude that

$$x_4[n] = x_3[n] + x_3[n + N], \quad \text{for} \quad 0 \leq n \leq N - 1$$

Hence, we proved the equation (**??**).

(c) MATLAB script:

```
% P0715: Verify formula in Problem 0715
close all; clc
xn1 = 1:4;
xn2 = 4:-1:1;
```

```
% N = 5;
N = 8;
n = 0:N-1;
xn3 = conv(xn1,xn2);
xn4 = circonv([xn1 zeros(1,N-4)]',[xn2 zeros(1,N-4)]')';
if N<7
xn3_N = xn3(N+1:end);
xn3_N = [xn3_N zeros(1,N)];
else
    xn3_N = zeros(1,N);
end
Nind = min(N,7);
xn4_ref = xn3(1:Nind) + xn3_N(1:Nind);
```

16. Proof:
The DFT of circular correlation $r_{xy}[\ell]$ is defined as

$$\begin{aligned}
R_{xy}[k] &= \sum_{\ell=0}^{N-1} r_{xy}[\ell]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}k\ell} = \sum_{\ell=0}^{N-1}\left(\sum_{n=0}^{N-1} x[n]y[\langle n-\ell\rangle_N]\right)\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}k\ell} \\
&= \sum_{n=0}^{N-1} x[n]\left(\sum_{\ell=0}^{N-1} y[\langle n-\ell\rangle_N]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}k\ell}\right) \\
&= \sum_{n=0}^{N-1} x[n]\left(\sum_{\ell=0}^{N-1} y[\langle n-\ell\rangle_N]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}k(n-\ell)}\right)^{*}\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}kn} \\
&= \left(\sum_{n=0}^{N-1} x[n]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}kn}\right)\left(\sum_{m=0}^{N-1} y[m]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}km}\right)^{*} \\
&= X[k]Y^{*}[k]
\end{aligned}$$

If $y[n]$ is real sequency.

17. Proof:

$$x^{(L)}[n] = \begin{cases} x[n/L], & n = 0, L, \cdots, (N-1)L \\ 0, & \text{otherwise} \end{cases}$$

$$x^{(L)}[n] \overset{\text{DFT}}{\longleftrightarrow} \tilde{X}[k] = X[\langle k\rangle_N] \tag{7.140}$$

$$\frac{1}{L}x[\langle n\rangle_N] = \frac{1}{L}\tilde{x}[n] \overset{\text{DFT}}{\longleftrightarrow} X^{(L)}[k] \tag{7.141}$$

$$\begin{aligned}
X^{(L)}[k] &= \sum_{n=0}^{NL-1} x^{(L)}[n]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{NL}nk} = \sum_{m=0}^{N-1} x^{(L)}[mL]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{NL}mLk} \\
&= \sum_{m=0}^{N-1} x[m]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}mk} = \sum_{m=0}^{N-1} x[m]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}m\langle k\rangle_N} \\
&= X[\langle k\rangle_N]
\end{aligned}$$

$$\begin{aligned}
x[n] &= \frac{1}{NL}\sum_{k=0}^{NL-1} X^{(L)}[k]\mathrm{e}^{\mathrm{j}\frac{2\pi}{NL}nk} = \frac{1}{NL}\sum_{m=0}^{N-1} X^{(L)}[mL]\mathrm{e}^{\mathrm{j}\frac{2\pi}{NL}nmL} \\
&= \frac{1}{NL}\sum_{n=0}^{N-1} X[m]\mathrm{e}^{\mathrm{j}\frac{2\pi}{N}mn} = \frac{1}{L}\left(\frac{1}{N}\sum_{m=0}^{N-1} X[m]\mathrm{e}^{\mathrm{j}\frac{2\pi}{N}mn}\right) \\
&= \frac{1}{L}\left(\frac{1}{N}\sum_{m=0}^{N-1} X[m]\mathrm{e}^{\mathrm{j}\frac{2\pi}{N}m\langle n\rangle_N}\right) = \frac{1}{L}x[\langle n\rangle_N]
\end{aligned}$$

18. Proof:

$$x_{(M)}[n] = x[nM], \qquad 0 \le n \le \frac{N}{M} - 1$$

$$x_{(M)}[n] \overset{\text{DFT}}{\longleftrightarrow} \frac{1}{M}\sum_{m=0}^{M-1} X[k + m\frac{N}{M}] \tag{7.143}$$

$$\frac{1}{M}\sum_{m=0}^{M-1} x[n + m\frac{N}{M}] \overset{\text{DFT}}{\longleftrightarrow} X_{(M)}[k] \tag{7.144}$$

We first prove equation (7.143):

$$\begin{aligned}
\frac{1}{M}\sum_{m=0}^{M-1} X[k + m\frac{N}{M}] &= \frac{1}{M}\sum_{m=0}^{M-1}\left(\sum_{n=0}^{N-1} x[n]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}n(k+m\frac{N}{M})}\right) \\
&= \frac{1}{M}\sum_{n=0}^{N-1} x[n]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}nk}\left(\sum_{m=0}^{M-1}\mathrm{e}^{-\mathrm{j}\frac{2\pi}{M}mn}\right) \\
&= \frac{1}{M}\sum_{n=0}^{N-1} x[n]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}nk}\cdot M\delta[\langle n\rangle_N] \\
&= \sum_{m=0}^{N/M-1} x[mM]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N/M}mk} = \text{DFT}(x_{(M)}[n])
\end{aligned}$$

We then prove equation (7.144):

$$\begin{aligned}
\frac{1}{M}\sum_{m=0}^{M-1} x[n+m\frac{N}{M}] &= \frac{1}{M}\sum_{m=0}^{M-1}\left(\frac{1}{N}\sum_{k=0}^{N-1} X[k]\mathrm{e}^{\mathrm{j}\frac{2\pi}{N}k(n+m\frac{N}{M})}\right) \\
&= \frac{1}{MN}\sum_{k=0}^{N-1} X[k]\mathrm{e}^{\mathrm{j}\frac{2\pi}{N}kn}\left(\sum_{m=0}^{M-1}\mathrm{e}^{\mathrm{j}\frac{2\pi}{M}km}\right) \\
&= \frac{1}{MN}\sum_{k=0}^{N-1} X[k]\mathrm{e}^{\mathrm{j}\frac{2\pi}{N}kn}\cdot M\delta[\langle k\rangle_N] \\
&= \frac{1}{N}\sum_{m=0}^{N/M-1} X[mM]\mathrm{e}^{\mathrm{j}\frac{2\pi}{N/M}mn} \\
&= \frac{1}{M}\mathrm{IDFT}(X_{(M)}[k])
\end{aligned}$$

19. (a) Proof:

$$w[n]x[n] \overset{\text{DFT}}{\longleftrightarrow} \frac{1}{N} W[k] \circledN X[k] \tag{7.148}$$

$$\begin{aligned}
\sum_{n=0}^{N-1} w[n]x[n]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}nk} &= \sum_{n=0}^{N-1} w[n]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}nk} \left(\frac{1}{N} \sum_{m=0}^{N-1} X[m]\mathrm{e}^{\mathrm{j}\frac{2\pi}{N}mn} \right) \\
&= \frac{1}{N} \sum_{m=0}^{N-1} X[m] \left(\sum_{n=0}^{N-1} w[n]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}n(k-m)} \right) \\
&= \frac{1}{N} \sum_{m=0}^{N-1} X[m]W[k-m] \\
&= \frac{1}{N} \sum_{m=0}^{N-1} X[m]W[\langle k-m \rangle_N] \\
&= \frac{1}{N} W[k] \circledN X[k]
\end{aligned}$$

(b) Proof:

$$\begin{aligned}
\frac{1}{N} W[k] \circledN X[k] &= \frac{1}{N} \sum_{m=0}^{N-1} X[m]W[\langle k-m \rangle_N] \\
&= \frac{1}{N} \sum_{m=0}^{N-1} \left(\sum_{n=0}^{N-1} x[n]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}nm} \right) W[\langle k-m \rangle_N] \\
&= \sum_{n=0}^{N-1} x[n] \left(\frac{1}{N} \sum_{m=0}^{N-1} W[\langle k-m \rangle_N]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}nm} \right) \\
&= \sum_{n=0}^{N-1} x[n]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}nk} \left(\frac{1}{N} \sum_{m=0}^{N-1} W[\langle k-m \rangle_N]\mathrm{e}^{\mathrm{j}\frac{2\pi}{N}n(k-m)} \right) \\
&= \sum_{n=0}^{N-1} x[n]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}nk} \left(\frac{1}{N} \sum_{m=0}^{N-1} W[\langle k-m \rangle_N]\mathrm{e}^{\mathrm{j}\frac{2\pi}{N}n\langle k-m \rangle_N} \right) \\
&= \sum_{n=0}^{N-1} (x[n]w[n])\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}nk} = \text{DFT}(x[n]w[n])
\end{aligned}$$

20. (a) Proof:

$$\tilde{x}_c(t) = w_c(t)x_c(t) \tag{7.161}$$

Linearity.

$$\begin{aligned} w_c(t)\cdot(a_1x_{c1}(t)+a_2x_{c2}(t)) &= a_1w_c(t)x_{c1}(t)+a_2w_c(t)x_{c2}(t)\\ &= a_1\tilde{x}_{c1}(t)+a_2\tilde{x}_{c2}(t)\end{aligned}$$

Time-varying.

$$\text{In general,}\qquad w_c(t-\tau)x_c(t-\tau)\neq w_c(t)x_c(t)$$

Hence, $\tilde{x}_c(t-\tau)\neq\tilde{x}_c(t)$.

(b) Proof:

$$\tilde{x}[n]=w[n]x[n] \tag{7.160}$$

If $0\leq t\leq T_0$, and $0\leq n\leq L$, we have

$$\begin{aligned}&\tilde{x}_c(nT)=w_c(nT)x_c(nT)=x_c(nT)=x[n]\\ &\tilde{x}[n]=w[n]x[n]=x[n]\end{aligned}$$

If $t>T_0$, and $N>L$, we have

$$\tilde{x}_c(nT)=\tilde{x}[n]=0$$

21. Proof:

$$\hat{X}_c(\mathrm{j}\Omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X_c(\mathrm{j}\theta)W_c(j(\Omega-\theta))\mathrm{d}\theta \tag{7.170}$$

The CTFT of $\hat{x}_c(t)$ is:

$$\begin{aligned}\hat{X}_c(\mathrm{j}\Omega) &= \int_{-\infty}^{\infty}\hat{x}_c(t)\mathrm{e}^{-\mathrm{j}\Omega t}\mathrm{d}t=\int_{-\infty}^{\infty}w_c(t)x_c(t)\mathrm{e}^{-\mathrm{j}\Omega t}\mathrm{d}t\\ &= \int_{-\infty}^{\infty}w_c(t)\mathrm{e}^{-\mathrm{j}\Omega t}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}X_c(\mathrm{j}\theta)\mathrm{e}^{\mathrm{j}\theta t}\mathrm{d}\theta\right)\mathrm{d}t\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}X_c(\mathrm{j}\theta)\left(\int_{-\infty}^{\infty}w_c(t)\mathrm{e}^{-\mathrm{j}(\Omega-\theta)t}\right)\mathrm{d}\theta\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}X_c(\mathrm{j}\theta)W_c(j(\Omega-\theta))\mathrm{d}\theta\end{aligned}$$

22. Proof:

$$\text{Scaling Property:}\quad x_c(at)\overset{\text{CTFT}}{\longleftrightarrow}\frac{1}{|a|}X_c\left(\frac{\mathrm{j}\Omega}{a}\right) \tag{7.172}$$

The CTFT of $x_c(at)$ is:

$$\int_{-\infty}^{\infty}x_c(at)\mathrm{e}^{-\mathrm{j}\Omega t}\mathrm{d}t=\frac{1}{a}\int_{-\infty}^{\infty}x_c(at)\mathrm{e}^{-\mathrm{j}\frac{\Omega}{a}at}\mathrm{d}at$$

If $a > 0$, we have

$$\frac{1}{a}\int_{-\infty}^{\infty} x_c(at)\mathrm{e}^{-\mathrm{j}\frac{\Omega}{a}at}\mathrm{d}at = \frac{1}{a}\int_{-\infty}^{\infty} x_c(t)\mathrm{e}^{-\mathrm{j}\frac{\Omega}{a}t}\mathrm{d}t = \frac{1}{a}X_c\left(\frac{\mathrm{j}\Omega}{a}\right)$$

If $a < 0$, we have

$$\frac{1}{a}\int_{\infty}^{-\infty} x_c(at)\mathrm{e}^{-\mathrm{j}\frac{\Omega}{a}at}\mathrm{d}at = \frac{1}{a}\int_{\infty}^{-\infty} x_c(t)\mathrm{e}^{-\mathrm{j}\frac{\Omega}{a}t}\mathrm{d}t = -\frac{1}{a}X_c\left(\frac{\mathrm{j}\Omega}{a}\right)$$

Hence, we proved the scaling property.

23. Proof:

$$\left|\int_{-\infty}^{\infty} x_{c1}(t)x_{c2}(t)\mathrm{d}t\right|^2 \leq \int_{-\infty}^{\infty} |x_{c1}(t)|^2\mathrm{d}t \int_{-\infty}^{\infty} |x_{c2}(t)|^2\mathrm{d}t \tag{7.179}$$

Suppose a is a real number, define function $p(a)$ as

$$p(a) = \int_{-\infty}^{\infty} \left(a \cdot x_{c1}(t) + x_{c2}(t)\right)^2 \mathrm{d}t = Aa^2 + 2Ba + C \geq 0$$

where

$$A = \int_{-\infty}^{\infty} x_{c1}^2(t)\mathrm{d}t, \quad B = \int_{-\infty}^{\infty} x_{c1}(t)x_{c2}(t)\mathrm{d}t, \quad C = \int_{-\infty}^{\infty} x_{c2}^2(t)\mathrm{d}t.$$

Since we have $4B^2 - 4AC \leq 0$, that is $B^2 \leq AC$,

$$\left(\int_{-\infty}^{\infty} x_{c1}(t)x_{c2}(t)\mathrm{d}t\right)^2 \leq \int_{-\infty}^{\infty} x_{c1}^2(t)\mathrm{d}t \int_{-\infty}^{\infty} x_{c2}^2(t)\mathrm{d}t$$

24. Proof:
The CTFT of generic window is:

$$W(\mathrm{j}\Omega) = aW_{\mathrm{R}}(\mathrm{j}\Omega) + bW_{\mathrm{R}}(\mathrm{j}(\Omega - 2\pi)/T_0) + bW_{\mathrm{R}}(\mathrm{j}(\Omega + 2\pi)/T_0) \tag{7.189}$$

The ICTFT is:

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-\infty}^{\infty} \left[aW_{\mathrm{R}}(\mathrm{j}\Omega) + bW_{\mathrm{R}}(\mathrm{j}(\Omega - 2\pi)/T_0) + bW_{\mathrm{R}}(\mathrm{j}(\Omega + 2\pi)/T_0)\right] \mathrm{e}^{\mathrm{j}\Omega t}\mathrm{d}\Omega \\
&\quad = aw_R(t) + b\mathrm{e}^{\mathrm{j}\frac{2\pi}{T_0}t}w_R(t) + b\mathrm{e}^{-\mathrm{j}\frac{2\pi}{T_0}t}w_R(t) \\
&\quad = \left[a + 2b\cos\left(\frac{2\pi}{T_0}t\right)\right] w_R(t)
\end{aligned}$$

If $a = b = 0.5$, we can prove:

$$w_{\mathrm{Han}}(t) = \left[0.5 + \cos\left(\frac{2\pi}{T_0}t\right)\right] w_R(t) \tag{7.190}$$

If $a = 0.54$ and $b = 0.23$, we can prove:

$$w_{\mathrm{Ham}}(t) = \left[0.5 + 0.46\cos\left(\frac{2\pi}{T_0}t\right)\right] w_R(t) \tag{7.191}$$

25. tba

26. (a) Proof:

$$X[k, ell] = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} x[m,n]W_M^{mk}W_N^{n\ell} = \sum_{m=0}^{M-1}\left(\sum_{n=0}^{N-1} x[m,n]W_N^{n\ell}\right)W_M^{mk}$$

(b) See figure below.

(c) Proof:

$$\begin{aligned} X[k,\ell] &= \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} x_1[m]x_2[n]W_M^{mk}W_N^{n\ell} \\ &= \left(\sum_{m=0}^{M-1} x_1[m]W_M^{mk}\right)\left(\sum_{n=0}^{N-1} x_2[n]W_N^{n\ell}\right) \end{aligned}$$

(d) See figure below.

MATLAB script:

```
% P0726: 2D FFT and 1D FFT
close all; clc
M = 100; N = 100;
m = 0:M-1; n = 0:N-1;
% Part (b):
[NN MM] = meshgrid(n,m);
xmn = 0.9.^(MM+NN);
X = fftshift(fft2(xmn));
X_mag = abs(X);
% Part (d)
Xm = abs(fft(0.9.^m));
Xn = abs(fft(0.9.^n));
```

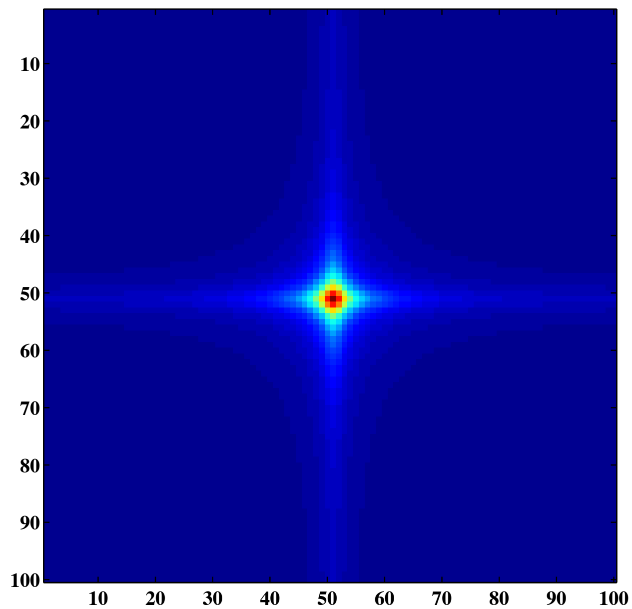


FIGURE 7.12: Magnitude response of the image using `fft2` function to compute 2D-DFT.

```
X_mag2 = fftshift(Xm(:)*Xn);
% Plot:
hfa = figure;
imagesc(X_mag); axis square
hfb = figure;
imagesc(X_mag2); axis square
```

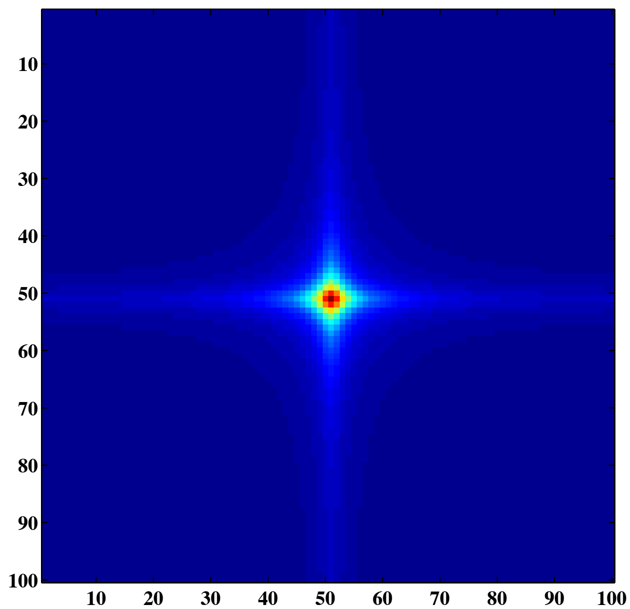


FIGURE 7.13: Magnitude response of the image using `fft2` function to compute 2D-DFT.