CHAPTER 3

The z-Transform

Tutorial Problems

1. (a) Solution:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^n (u[n] - u[n-10])z^{-n}$$
$$= \sum_{n=0}^{9} \left(\frac{1}{2}\right)^n z^{-n} = \frac{1 - \left(\frac{1}{2}z^{-1}\right)^{10}}{1 - \frac{1}{2}z^{-1}} \quad \text{ROC: } |z| > 0$$

(b) Solution:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^{|n|} z^{-n} = \sum_{n=-\infty}^{-1} \left(\frac{1}{2}\right)^{-n} z^{-n} + \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^{-n}$$
$$= \sum_{n=1}^{\infty} \left(\frac{z}{2}\right)^n + \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^{-n} = \frac{\frac{z}{2}}{1 - \frac{1}{2}z} + \frac{1}{1 - \frac{1}{2}z^{-1}}$$
$$= \frac{-1}{1 - 2z^{-1}} + \frac{1}{1 - \frac{1}{2}z^{-1}} = \frac{-\frac{3}{2}z^{-1}}{1 - \frac{5}{2}z^{-1} + z^{-2}} \quad \text{ROC: } \frac{1}{2} < |z| < 2$$

(c) Solution:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=-\infty}^{\infty} 5^{|n|} z^{-n} = \sum_{n=-\infty}^{-1} 5^{-n} z^{-n} + \sum_{n=0}^{\infty} 5^n z^{-n}$$
$$= \frac{5z}{1 - 5z} + \frac{1}{1 - 5z^{-1}} = \frac{\frac{24}{5}z^{-1}}{1 - \frac{26}{5}z^{-1} + z^{-2}} \quad \text{ROC: } |z| \in \phi$$

(d) Solution:

$$x[n] = \left(\frac{1}{2}\right)^n \cos(\pi n/3)u[n] = x[n] = \left(\frac{1}{2}\right)^n (\frac{1}{2}\mathrm{e}^{j\pi n/3}+\frac{1}{2}\mathrm{e}^{-j\pi n/3})u[n]$$

$$\begin{aligned}
X(z) &= \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{1}{2}\mathrm{e}^{j\pi/3}z^{-1}\right)^n + \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{1}{2}\mathrm{e}^{-j\pi/3}z^{-1}\right)^n \\
&= \frac{1}{2}\frac{1}{1-\frac{1}{2}\mathrm{e}^{j\pi/3}z^{-1}} + \frac{1}{2}\frac{1}{1-\frac{1}{2}\mathrm{e}^{-j\pi/3}z^{-1}} \\
&= \frac{1-\frac{1}{2}\cos\left(\frac{\pi}{3}\right)z^{-1}}{1-\cos\left(\frac{\pi}{3}\right)z^{-1}+\frac{1}{4}z^{-2}} = \frac{1-\frac{1}{4}z^{-1}}{1-\frac{1}{2}z^{-1}+\frac{1}{4}z^{-2}} \quad \text{ROC: } |z| > \frac{1}{2}
\end{aligned}$$

2. (a) Proof:
The input in `x=filter(b,a,[1,zeros(1,N)])` is actually impulse signal $\delta[n]$. The z-transform of impulse response is 1. Hence, $Y(z) = X(z)$.

(b) Solution:

$$\begin{aligned}
X(z) &= \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=-\infty}^{\infty}\left[\left(\frac{1}{2}\right)^n + \left(-\frac{1}{3}\right)^n\right]u[n]z^{-n} \\
&= \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n z^{-n} + \sum_{n=0}^{\infty}\left(-\frac{1}{3}\right)^n z^{-n} \\
&= \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{1+\frac{1}{3}z^{-1}} = \frac{2-\frac{1}{6}z^{-1}}{1-\frac{1}{6}z^{-1}-\frac{1}{6}z^{-2}} \quad \text{ROC: } |z| > \frac{1}{2}
\end{aligned}$$

(c) MATLAB script:

```
% P0302: verify z-transform expression of a causal
%        sequence using function 'filter'
close all; clc
n = 0:20;
xn = (1/2).^n+(-1/3).^n;
b = [2 -1/6];
a = [1 -1/6 -1/6];
xnz = filter(b,a,[1,zeros(1,length(n)-1)]);
% Plot
hf = figconfg('P0302');
```

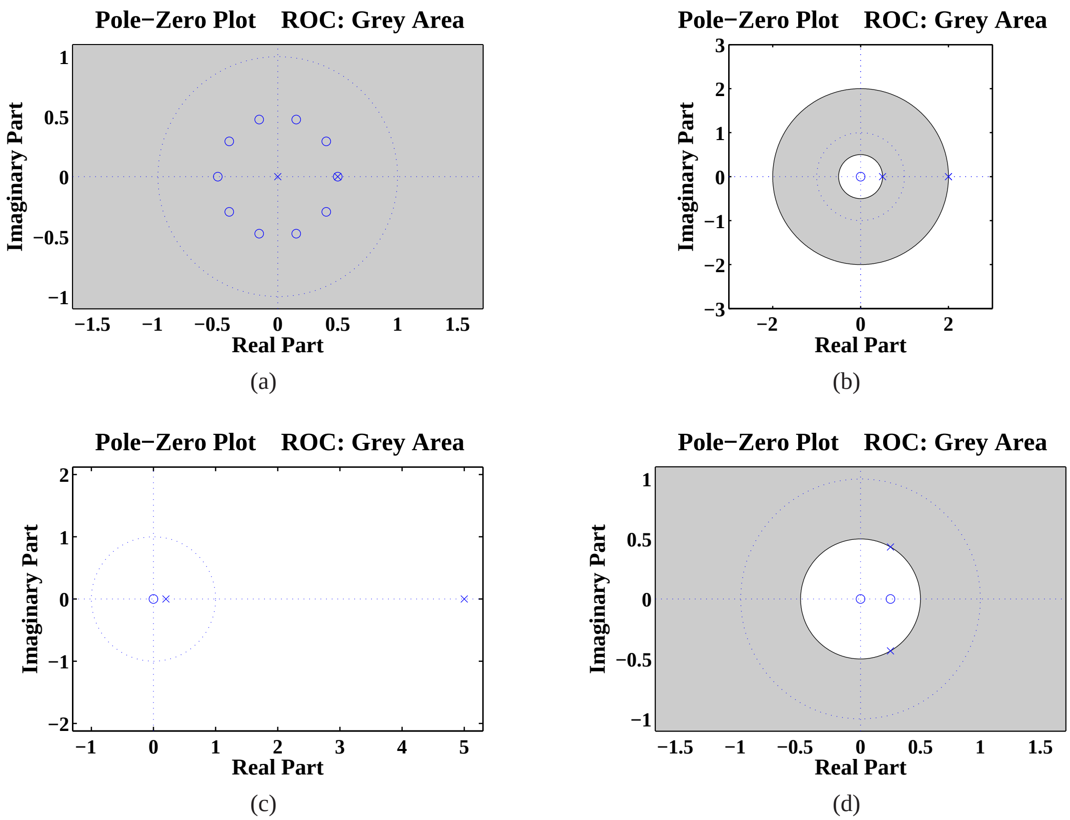


FIGURE 3.1: Pole-zero plot and ROC of (a) $x[n] = \left(\frac{1}{2}\right)^n (u[n] - u[n-10])$. (b) $x[n] = \left(\frac{1}{2}\right)^{|n|}$. (c) $x[n] = 5^{|n|}$. (d) $x[n] = \left(\frac{1}{2}\right)^n \cos(\pi n/3)u[n]$.

```
subplot(211)
stem(n,xn,'filled')
ylabel('x[n]','fontsize',LFS)
title('Original Sequence','fontsize',TFS)
subplot(212)
stem(n,xnz,'filled')
xlabel('Time Index n','fontsize',LFS)
ylabel('x[n]','fontsize',LFS)
title('Sequence Computed from z-transform','fontsize',TFS)
```

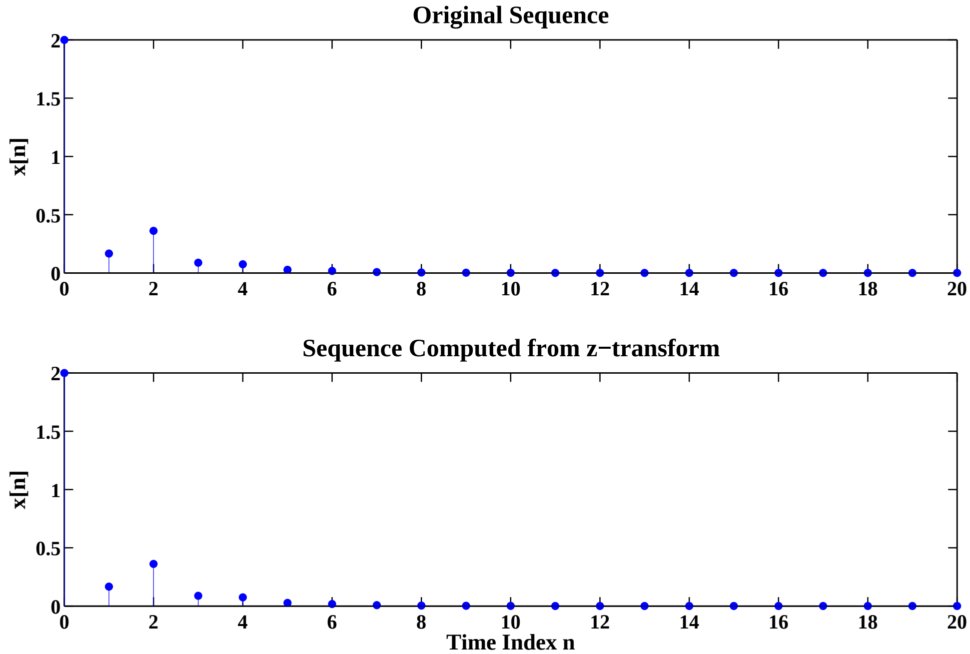


FIGURE 3.2: MATLAB verification of z-transform expression using“x=filter(b,a,[1,zeros(1,N)])”.

3. Proof:

$$x[n] = (r^n \sin\omega_0 n)u[n] = r^n\left(\frac{1}{2j}e^{j\omega_0 n} - \frac{1}{2j}e^{-j\omega_0 n}\right)u[n]$$

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \frac{1}{2j}\sum_{n=0}^{\infty}\left(r e^{j\omega_0}z^{-1}\right)^n - \frac{1}{2j}\sum_{n=0}^{\infty}\left(r e^{-j\omega_0}z^{-1}\right)^n \\ &= \frac{1}{2j}\frac{1}{1-re^{j\omega_0}z^{-1}} - \frac{1}{2j}\frac{1}{1-re^{-j\omega_0}z^{-1}} \\ &= \frac{r(\sin\omega_0)z^{-1}}{1-2(r\cos\omega_0)z^{-1}+r^2z^{-2}} \quad \text{ROC: } |z|>r \end{aligned}$$

4. (a) Solution:

$$X(z) = \frac{1-\frac{1}{3}z^{-1}}{(1-z^{-1})(1+2z^{-1})} = \frac{\frac{2}{9}}{1-z^{-1}} + \frac{\frac{7}{9}}{1+2z^{-1}}$$

$$\text{ROC: } |z| > 2 \quad x[n] = \frac{2}{9}u[n] + \frac{7}{9}(-2)^n u[n]$$
$$\text{ROC: } |z| < 1 \quad x[n] = -\frac{2}{9}u[-n-1] - \frac{7}{9}(-2)^n u[-n-1]$$
$$\text{ROC: } 1 < |z| < 2 \quad x[n] = \frac{2}{9}u[n] - \frac{7}{9}(-2)^n u[-n-1]$$

(b) Solution:

$$X(z) = \frac{1-z^{-1}}{1-\frac{1}{4}z^{-1}} = 4 - \frac{3}{1-\frac{1}{4}z^{-1}} \quad \text{ROC: } |z| > \frac{1}{4}$$
$$x[n] = 4\delta[n] - 3\left(\frac{1}{4}\right)^n u[n]$$

(c) Solution:

$$X(z) = \frac{2}{1-0.5z^{-1}} + \frac{-1}{1-0.25z^{-1}}, \quad \text{ROC: } |z| > 0.5$$
$$x[n] = 2(0.5)^n u[n] - (0.25)^n u[n]$$

5. Solution:

$$X(z) = z^2(1-\frac{1}{3}z^{-1})(1-z^{-1})(1+2z^{-2}) = z^2 - \frac{4}{3}z + \frac{7}{3} - \frac{8}{3}z^{-1} + \frac{2}{3}z^{-2}$$
$$x[n] = \delta[n+2] - \frac{4}{3}\delta[n+1] + \frac{7}{3}\delta[n] - \frac{8}{3}\delta[n-1] + \frac{2}{3}\delta[n-2]$$

6. (a) Solution:

$$\text{Time-shifting:} \quad Y(z) = z^{-3}X(z) = \frac{z^{-3}}{1-2z^{-1}}, \quad \text{ROC: } |z| < 2$$

(b) Solution:

$$\text{Scaling:} \quad Y(z) = X(3z) = \frac{1}{1-\frac{2}{3}z^{-1}}, \quad \text{ROC: } |z| < \frac{2}{3}$$

(c) Solution:

$$\text{Folding and convolution:} \quad Y(z) = X(z)X(1/z) = \frac{1}{1-2z^{-1}} \cdot \frac{1}{1-2z}$$
$$= \frac{-\frac{1}{2}}{1-\frac{5}{2}z^{-1}+z^{-2}}, \quad \text{ROC: } \frac{1}{2} < |z| < 2$$

(d) Solution:

Differentiation: $$Y(z) = -z\frac{d\,X(z)}{dz} = \frac{2z^{-1}}{1-4z^{-1}+4z^{-2}}, \quad \text{ROC: } |z| < 2$$

(e) Solution:

Time-shifting and linearity: $$\begin{aligned} Y(z) &= z^{-1}X(z) + z^2X(z) \\ &= \frac{z^{-1}}{1-2z^{-1}} + \frac{z^2}{1-2z^{-1}} \\ &= \frac{1+z^{-3}}{z^{-2}-2z^{-3}}, \quad \text{ROC: } 0 < |z| < 2 \end{aligned}$$

(f) Solution:

Time-shifting and convolution: $$\begin{aligned} Y(z) &= X(z)X(z)z^{-2} \\ &= \frac{z^{-2}}{(1-2z^{-1})^{-2}}, \quad \text{ROC: } |z| < 2 \end{aligned}$$

7. Solution:
$$x[n] = \left(\frac{1}{2}\right)^n u[n]$$

(a) $Y(z) = X(1/z)$

Folding: $$y[n] = x[-n] = 2^n u[-n]$$

(b) $Y(z) = dX(z)/dz$

Differentiation and time-shifting: $$\begin{aligned} y[n] &= -(n-1)x[n-1] \\ &= -(n-1)\left(\frac{1}{2}\right)^{n-1} u[n-1] \end{aligned}$$

(c) $Y(z) = X^2(z)$

Convolution: $$y[n] = x[n] * x[n] = (n+1)\left(\frac{1}{2}\right)^n u[n]$$

8. (a) Proof:

$$\begin{aligned} \sum_{n=-\infty}^{\infty} x^*[n]z^{-n} &= \sum_{n=-\infty}^{\infty} (x[n](z^*)^{-n})^* = \left(\sum_{n=-\infty}^{\infty} x[n](z^*)^{-n}\right)^* \\ &= X^*(z^*) \end{aligned}$$

(b) Proof:

$$\sum_{n=-\infty}^{\infty} x[-n]z^{-n} = \sum_{n=-\infty}^{\infty} x[n]z^{n} = \sum_{n=-\infty}^{\infty} x[n](z^{-1})^{-n} = X(1/z)$$

(c) Proof:

$$\begin{aligned}\sum_{n=-\infty}^{\infty} x_R[n]z^{-n} &= \sum_{n=-\infty}^{\infty} \frac{1}{2}(x[n] + x^*[n])z^{-n} \\ &= \frac{1}{2}\sum_{n=-\infty}^{\infty} x[n]z^{-n} + \frac{1}{2}\sum_{n=-\infty}^{\infty} x^*[n]z^{-n} = \frac{1}{2}[X(z) + X^*(z^*)]\end{aligned}$$

(d) Proof:

$$\begin{aligned}\sum_{n=-\infty}^{\infty} x_I[n]z^{-n} &= \sum_{n=-\infty}^{\infty} \frac{1}{2\mathrm{j}}(x[n] - x^*[n])z^{-n} \\ &= \frac{1}{2\mathrm{j}}\sum_{n=-\infty}^{\infty} x[n]z^{-n} - \frac{1}{2\mathrm{j}}\sum_{n=-\infty}^{\infty} x^*[n]z^{-n} = \frac{1}{2\mathrm{j}}[X(z) - X^*(z^*)]\end{aligned}$$

9. Solution:

$$\begin{aligned}X(z) &= \frac{z}{(z+\frac{3}{4})(z-\frac{1}{2}(1+\mathrm{j}))(z-\frac{1}{2}(1-\mathrm{j}))} \\ &= \frac{z}{z^3 - \frac{1}{4}z^2 - \frac{1}{4}z + \frac{3}{8}} \quad \text{ROC: } |z| > \frac{3}{4}\end{aligned}$$

$$Y(z) = z^{-3}X(1/z) = \frac{z^{-4}}{\frac{3}{8} - \frac{1}{4}z^{-1} - \frac{1}{4}z^{-2} + z^{-3}} \quad \text{ROC: } 0 < |z| < \frac{4}{3}$$

10. Solution:

$$H(z) = \frac{1}{1-az^{-1}}, \quad \text{ROC: } |z| > |a|$$

$$X(z) = \frac{-1}{1-z^{-1}}, \quad \text{ROC: } |z| < 1$$

$$\begin{aligned}Y(z) = H(z)X(z) &= \frac{-1}{(1-az^{-1})(1-z^{-1})} \\ &= \frac{\frac{a}{1-a}}{1-az^{-1}} + \frac{\frac{-1}{1-a}}{1-z^{-1}} \quad \text{ROC: } |a| < |z| < 1 \\ y[n] &= \left(\frac{a}{1-a}\right)a^n u[n] + \frac{1}{1-a}u[-n-1]\end{aligned}$$

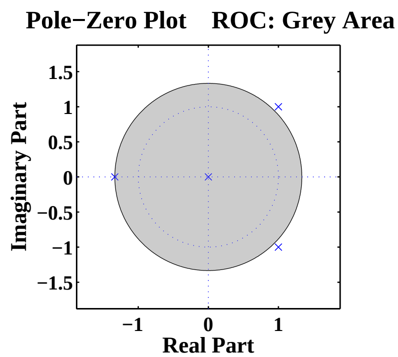


FIGURE 3.3: Pole-zero plot and ROC of $y[n]$.

11. (a) Solution:

$$H(z) = \frac{1}{1 - az^{-1}}, \quad \text{ROC: } |z| > |a|$$

$$X(z) = \frac{1}{1 - bz^{-1}}, \quad \text{ROC: } |z| > |b|$$

$$\begin{aligned} Y(z) = H(z)X(z) &= \frac{1}{1 - az^{-1}} \cdot \frac{1}{1 - bz^{-1}} \\ &= \frac{\frac{a}{a-b}}{1 - az^{-1}} + \frac{\frac{-b}{a-b}}{1 - bz^{-1}}, \quad \text{ROC: } |z| > \max\{|a|, |b|\} \end{aligned}$$

$$y[n] = \frac{a}{a-b} \cdot a^n u[n] + \frac{b}{b-a} \cdot b^n u[n]$$

(b) Solution:

$$H(z) = X(z) = \frac{1}{1 - az^{-1}}, \quad \text{ROC: } |z| > |a|$$

$$\begin{aligned} Y(z) = H(z)X(z) &= \frac{1}{(1 - az^{-1})^2} \\ &= \frac{1}{1 - az^{-1}} + \frac{az^{-1}}{1 - az^{-1}}, \quad \text{ROC: } |z| > |a| \end{aligned}$$

$$y[n] = a^n u[n] + na^n u[n]$$

(c) Solution:

$$x[n] = a^{-n}u[-n] = \delta[n] + (a^{-1})^n u[-n-1]$$

$$X(z) = 1 + \frac{-1}{1-a^{-1}z^{-1}}, \quad \text{ROC: } |z| < |a|^{-1}$$

$$\begin{aligned} Y(z) = H(z)X(z) &= \left(\frac{1}{1-az^{-1}}\right)\cdot\left(1+\frac{-1}{1-a^{-1}z^{-1}}\right) \\ &= \frac{\frac{1}{1-a^2}}{1-az^{-1}} + \frac{\frac{-1}{1-a^2}}{1-a^{-1}z^{-1}}, \quad \text{ROC: } |a| < |z| < |a|^{-1} \end{aligned}$$

$$y[n] = \frac{a^n}{1-a^2}u[n] + \frac{a^{-n}}{1-a^2}u[-n-1] = \frac{a^{|n|}}{1-a^2}$$

12. (a) Proof:

$$r_{xx}[\ell] \triangleq \sum_n x[n]x[n-\ell] = \sum_n x[n+\ell]x[n] = x[l] * x[-l]$$

By applying the folding property, the z-transform of sequence $x[-l]$ is $X(z^{-1})$ with ROC $\beta^{-1} < |z| < \alpha^{-1}$. Hence, we proved

$$R_{xx}(z) = X(z)X\left(z^{-1}\right) \quad \text{ROC: } \max\{\alpha, \beta^{-1}\} < |z| < \min\{\beta, \alpha^{-1}\}$$

(b) Solution:

$$X(z) = \frac{1}{1-az^{-1}}, \quad \text{ROC: } |z| > |a|$$

$$X(z^{-1}) = \frac{1}{1-az} = \frac{-az^{-1}}{1-a^{-1}z^{-1}}, \quad \text{ROC: } |z| < |a|^{-1}$$

$$\begin{aligned} R_{xx}(z) = X(z)X\left(z^{-1}\right) &= \left(\frac{1}{1-az^{-1}}\right)\left(\frac{-az^{-1}}{1-a^{-1}z^{-1}}\right) \\ &= \frac{-az^{-1}}{1-(a+a^{-1})z^{-1}+z^{-2}}, \quad \text{ROC: } |a| < |z| < |a|^{-1} \end{aligned}$$

(c) Solution:

$$R_{xx}(z) = \frac{\frac{a^2}{1-a^2}}{1-az^{-1}} + \frac{\frac{-a^2}{1-a^2}}{1-a^{-1}z^{-1}}, \quad \text{ROC: } |a| < |z| < |a|^{-1}$$

$$r_{xx}[\ell] = \left(\frac{a^2}{1-a^2}\right)a^l u[l] + \left(\frac{a^2}{1-a^2}\right)a^{-l}u[-l-1] = \left(\frac{a^2}{1-a^2}\right)a^{|l|}$$

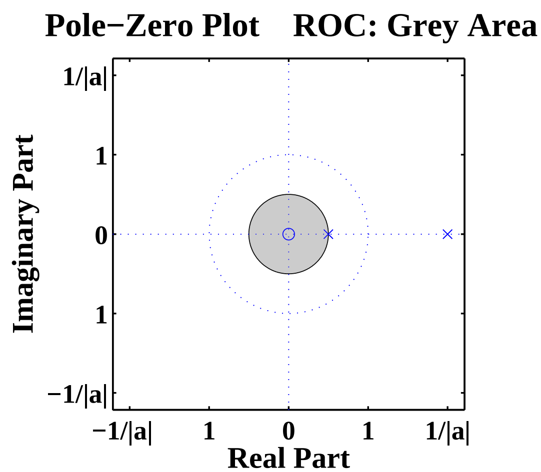


FIGURE 3.4: Pole-zero plot and ROC of $R_{xx}(z)$.

13. Solution:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-1}}{1 - \frac{5}{2}z^{-1} + z^{-2}} = \frac{\frac{2}{3}}{1 - 2z^{-1}} + \frac{-\frac{2}{3}}{1 - \frac{1}{2}z^{-1}}$$

ROC: $|z| > 2$

$$h[n] = \frac{2}{3} \cdot 2^n u[n] - \frac{2}{3} \cdot \left(\frac{1}{2}\right)^n u[n]$$

ROC: $|z| < \frac{1}{2}$

$$h[n] = -\frac{2}{3} \cdot 2^n u[-n-1] + \frac{2}{3} \cdot \left(\frac{1}{2}\right)^n u[-n-1]$$

ROC: $\frac{1}{2} < |z| < 2$

$$h[n] = -\frac{2}{3} \cdot 2^n u[-n-1] - \frac{2}{3} \cdot \left(\frac{1}{2}\right)^n u[n]$$

14. Solution:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - \frac{1}{2}z^{-1}}, \quad \text{ROC: } |z| > \frac{1}{2}$$

(a) $x[n] = \mathrm{e}^{\mathrm{j}(\pi/4)n}, \quad -\infty < n < \infty$

$$y[n] = \mathrm{e}^{\mathrm{j}(\pi/4)n} \cdot H(z)\mid_{z=\mathrm{e}^{\mathrm{j}(\pi/4)}} = \frac{1}{1-\frac{1}{2}e^{-\mathrm{j}(\pi/4)}} e^{\mathrm{j}(\pi/4)n}$$

(b) $x[n] = \mathrm{e}^{\mathrm{j}(\pi/4)n}u[n]$

$$X(z) = \frac{1}{1-\mathrm{e}^{\mathrm{j}(\pi/4)}z^{-1}}, \quad \text{ROC: } |z| > 1$$

$$\begin{aligned} Y(z) = X(z)H(z) &= \left(\frac{1}{1-\frac{1}{2}z^{-1}}\right)\left(\frac{1}{1-\mathrm{e}^{\mathrm{j}(\pi/4)}z^{-1}}\right) \\ &= \frac{\frac{1}{1-2\mathrm{e}^{\mathrm{j}(\pi/4)}}}{1-\frac{1}{2}z^{-1}} + \frac{\frac{1}{1-2\mathrm{e}^{-\mathrm{j}(\pi/4)}}}{1-\mathrm{e}^{\mathrm{j}(\pi/4)}z^{-1}}, \quad \text{ROC: } |z| > 1 \end{aligned}$$

$$y[n] = \frac{1}{1-2\mathrm{e}^{\mathrm{j}(\pi/4)}}\left(\frac{1}{2}\right)^n u[n] + \frac{1}{1-2\mathrm{e}^{-\mathrm{j}(\pi/4)}}\mathrm{e}^{\mathrm{j}(\pi/4)n}u[n]$$

(c) $x[n] = (-1)^n, \quad -\infty < n < \infty$

$$y[n] = (-1)^n \cdot H(z)\mid_{z=-1} = \frac{2}{3}(-1)^n$$

(d) $x[n] = (-1)^n u[n]$

$$X(z) = \frac{1}{1+z^{-1}}, \quad \text{ROC: } |z| > 1$$

$$\begin{aligned} Y(z) = X(z)H(z) &= \left(\frac{1}{1-\frac{1}{2}z^{-1}}\right)\left(\frac{1}{1+z^{-1}}\right) \\ &= \frac{\frac{1}{3}}{1-\frac{1}{2}z^{-1}} + \frac{\frac{2}{3}}{1+z^{-1}}, \quad \text{ROC: } |z| > 1 \end{aligned}$$

$$y[n] = \frac{1}{3}\left(\frac{1}{2}\right)^n u[n] + \frac{2}{3}(-1)^n u[n]$$

15. (a) Solution:

$$\begin{aligned} H(z) = \frac{Y(z)}{X(z)} &= \frac{1}{1-\frac{3}{4}z^{-1}-\frac{1}{8}z^{-1}} \\ &= \frac{1}{\left(1-\frac{1}{4}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)}, \quad \text{ROC: } |z| > \frac{1}{2} \end{aligned}$$

The system is stable if it is causal.

(b) Solution:

$$H(z) = \frac{-1}{1-\frac{1}{4}z^{-1}} + \frac{2}{1-\frac{1}{2}z^{-1}}, \quad \text{ROC: } |z| > \frac{1}{2}$$

$$h[n] = -\left(\frac{1}{4}\right)^n u[n] + 2\left(\frac{1}{2}\right)^n u[n]$$

(c) Solution:

$$U(z) = \frac{1}{1-z^{-1}}, \quad \text{ROC: } |z| > 1$$

$$\begin{aligned} S(z) = U(z)X(z) &= \frac{1}{\left(1-\frac{1}{4}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)\left(1-z^{-1}\right)}, \\ &= \frac{\frac{1}{3}}{1-\frac{1}{4}z^{-1}} + \frac{-2}{1-\frac{1}{2}z^{-1}} + \frac{\frac{8}{3}}{1-z^{-1}}, \quad \text{ROC: } |z| > \frac{1}{2} \end{aligned}$$

$$s[n] = \frac{1}{3}\left(\frac{1}{4}\right)^n u[n] - 2\left(\frac{1}{2}\right)^n u[n] + \frac{8}{3}u[n]$$

(d) MATLAB script:

```
% P0315: verify the calculated impulse and step
%        response sequences using function 'filter'
close all; clc
n = 0:10;
%% Impluse Response:
hn = -(1/4).^n + 2*(1/2).^n ;
b = 1;
a = [1 -3/4 1/8];
hnz = filter(b,a,[1,zeros(1,length(n)-1)]);
% Plot
hf1 = figconfg('P0315a');
subplot(211)
stem(n,hn,'filled')
ylabel('h[n]','fontsize',LFS)
title('Impulse Response Expression Sequence','fontsize',TFS)
subplot(212)
stem(n,hnz,'filled')
xlabel('Time Index n','fontsize',LFS)
ylabel('h[n]','fontsize',LFS)
title('Sequence Computed from z-transform','fontsize',TFS)
```

```
%% Step Response:
un = 1/3*(1/4).^n - 2*(1/2).^n +8/3;
b = 1;
a = [1 -7/4 7/8 -1/8];
unz = filter(b,a,[1,zeros(1,length(n)-1)]);
% Plot
hf2 = figconfg('P0315b');
subplot(211)
stem(n,un,'filled')
ylabel('s[n]','fontsize',LFS)
title('Step Response Expression Sequence','fontsize',TFS)
subplot(212)
stem(n,unz,'filled')
xlabel('Time Index n','fontsize',LFS)
ylabel('s[n]','fontsize',LFS)
title('Sequence Computed from z-transform','fontsize',TFS)
```

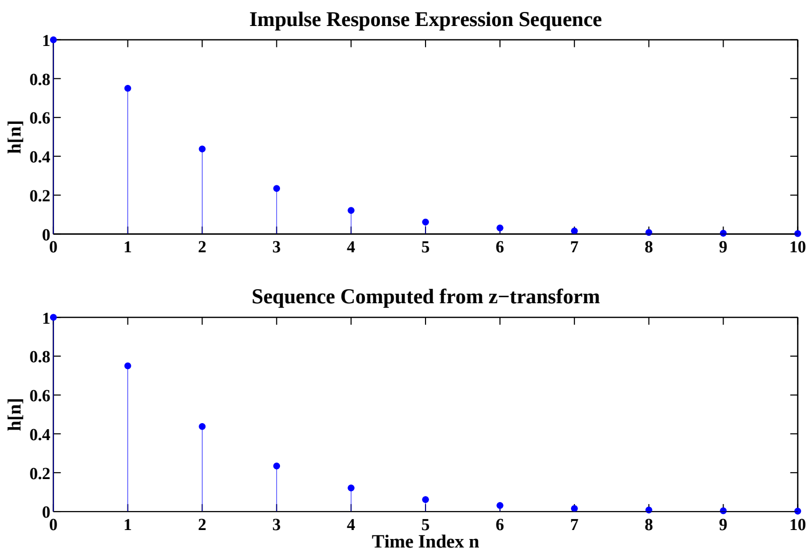


FIGURE 3.5: MATLAB verification of the impulse response expression obtained in part (b).

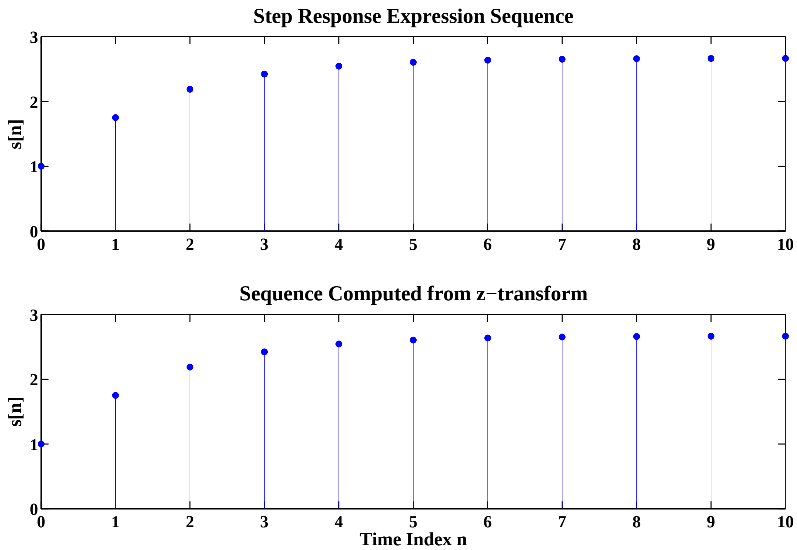


FIGURE 3.6: MATLAB verification of the step response expression obtained in part (c).

16. (a) Solution:

$$X(z) = \frac{1}{1-z^{-1}}, \quad \text{ROC: } |z| > 1$$
$$Y(z) = \frac{2}{1-\frac{1}{3}z^{-1}}, \quad \text{ROC: } |z| > \frac{1}{3}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{2(1-z^{-1})}{1-\frac{1}{3}z^{-1}} = 6 + \frac{-4}{1-\frac{1}{3}z^{-1}}, \quad \text{ROC: } |z| > \frac{1}{3}$$
$$h[n] = 6\delta[n] - 4\left(\frac{1}{3}\right)^n u[n]$$

(b) Solution:

$$X(z) = \frac{1}{1-\frac{1}{2}z^{-1}}, \quad \text{ROC: } |z| > \frac{1}{2}$$

$$\begin{aligned} Y(z) = X(z)H(z) &= \frac{2(1-z^{-1})}{\left(1-\frac{1}{3}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)} \\ &= \frac{8}{1-\frac{1}{3}z^{-1}} + \frac{-6}{1-\frac{1}{2}z^{-1}}, \quad \text{ROC: } |z| > \frac{1}{2} \end{aligned}$$

$$y[n] = 8\left(\frac{1}{3}\right)u[n] - 6\left(\frac{1}{2}\right)u[n]$$

(c) MATLAB script:

```
% P0316: verify the calculated response sequence
%        expressions using function 'filter'
close all; clc
n = 0:10;
%% Impluse Response:
hn = -4*(1/3).^n; hn(1) = hn(1) + 6;
b = [2 -2];
a = [1 -1/3];
hnz = filter(b,a,[1,zeros(1,length(n)-1)]);
% Plot
hf1 = figconfg('P0316a');
subplot(211)
stem(n,hn,'filled')
ylabel('h[n]','fontsize',LFS)
title('Impulse Response Expression Sequence','fontsize',TFS)
subplot(212)
stem(n,hnz,'filled')
xlabel('Time Index n','fontsize',LFS)
ylabel('h[n]','fontsize',LFS)
title('Sequence Computed from z-transform','fontsize',TFS)
%% Response y[n]:
yn = 8*(1/3).^n - 6*(1/2).^n;
b = [2 -2];
a = [1 -5/6 1/6];
ynz = filter(b,a,[1,zeros(1,length(n)-1)]);
% Plot
hf2 = figconfg('P0316b');
subplot(211)
stem(n,yn,'filled')
ylabel('y[n]','fontsize',LFS)
title('Response Expression Sequence','fontsize',TFS)
subplot(212)
stem(n,ynz,'filled')
xlabel('Time Index n','fontsize',LFS)
ylabel('y[n]','fontsize',LFS)
title('Sequence Computed from z-transform','fontsize',TFS)
```

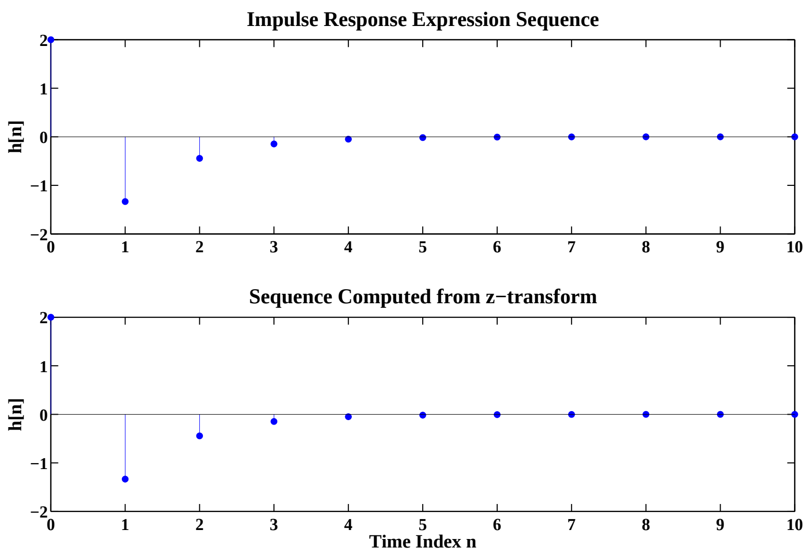


FIGURE 3.7: MATLAB verification of the impulse response expression obtained in part (a).

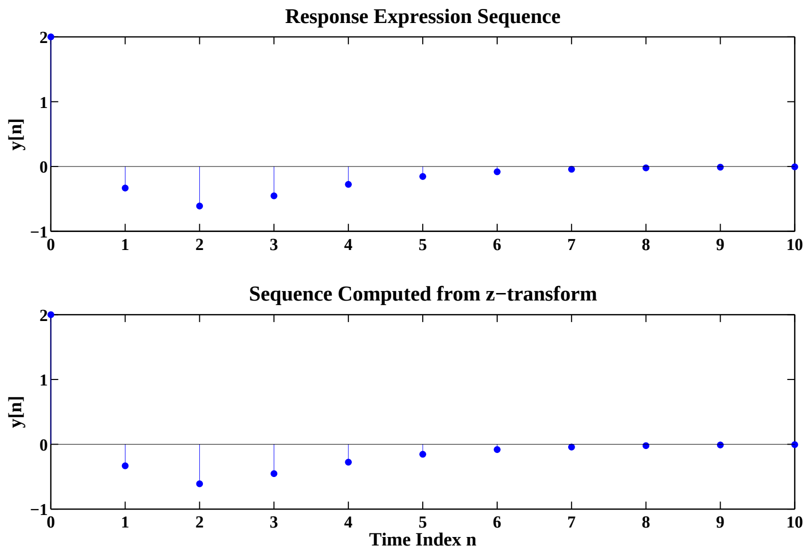


FIGURE 3.8: MATLAB verification of the expression of response $y[n]$ obtained in part (b).

17. Solution:
    We repeat the formulas for the ease of comparison as follows.

$$H(z) = \frac{b_0 + b_1 z^{-1}}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

$$h[n] = a|A|r^n \cos(\omega_0 n + \theta)u[n]$$

$$(r^n \cos\omega_0 n)u[n] \stackrel{\mathcal{Z}}{\longleftrightarrow} \frac{1 - (r\cos\omega_0)z^{-1}}{1 - 2(r\cos\omega_0)z^{-1} + r^2 z^{-2}}, \quad \text{ROC: } |z| > r$$

$$(r^n \sin\omega_0 n)u[n] \stackrel{\mathcal{Z}}{\longleftrightarrow} \frac{(r\sin\omega_0)z^{-1}}{1 - 2(r\cos\omega_0)z^{-1} + r^2 z^{-2}}, \quad \text{ROC: } |z| > r$$

18. Solution:
    We first repeat equation (3.97)

$$H(z) = \frac{b_0 + b_1 z^{-1}}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

When the system has two real and equal poles, we have

$$a_1^2 - 4a_2 = 0, \quad p_{1,2} = -\frac{a_1}{2}$$

$$H(z) = \frac{b_0 + b_1 z^{-1}}{\left(1 + \frac{a_1}{2} z^{-1}\right)^2} = \frac{b_0}{1 + \frac{a_1}{2} z^{-1}} + \left(b_0 - \frac{2b_1}{a_1}\right) \frac{-\frac{a_1}{2} z^{-1}}{\left(1 + \frac{a_1}{2} z^{-1}\right)^2} \quad \text{ROC: } |z| > \frac{|a_1|}{2}$$

$$h[n] = b_0 \left(-\frac{a_1}{2}\right)^n u[n] + \left(b_0 - \frac{2b_1}{a_1}\right) \left(-\frac{a_1}{2}\right)^n n u[n]$$

If $|a_1| < 2$, the system $h[n]$ is stable and its shape is decaying.
If $|a_1| = 2$, the system $h[n]$ is unstable and its envelop is constant.
If $|a_1| > 2$, the system $h[n]$ is unstable and its shape is growing.

MATLAB script:

```
% P0318: Plot time sequence and its z-transform
%        plot-zero plot and ROC
close all; clc
n = 0:20;
a1 = 1;
% a1 = 2;
% a1 = 3;
a2 = a1^2/4;
b0 = 1; b1=1;
%% Impluse Response:
hn = b0*(-a1/2).^n+(b0-2*b1/a1).*n.*(-a1/2).^n;
b = [b0 b1];
a = [1 a1 a2];
hnz = filter(b,a,[1,zeros(1,length(n)-1)]);
% Plot
hf1 = figconfg('P0318a','small');
stem(n,hnz,'filled')
xlabel('Time Index n','fontsize',LFS)
ylabel('h[n]','fontsize',LFS)
title('Time Sequence','fontsize',TFS)
%% Pole-zero plot:
hf2 = figconfg('P0318b','small');
p = -a1/2;
zplane(b,a)
xlabel('Real Part','fontsize',LFS)
ylabel('Imaginary Part','fontsize',LFS)
title('Pole-Zero Plot','fontsize',TFS)
```

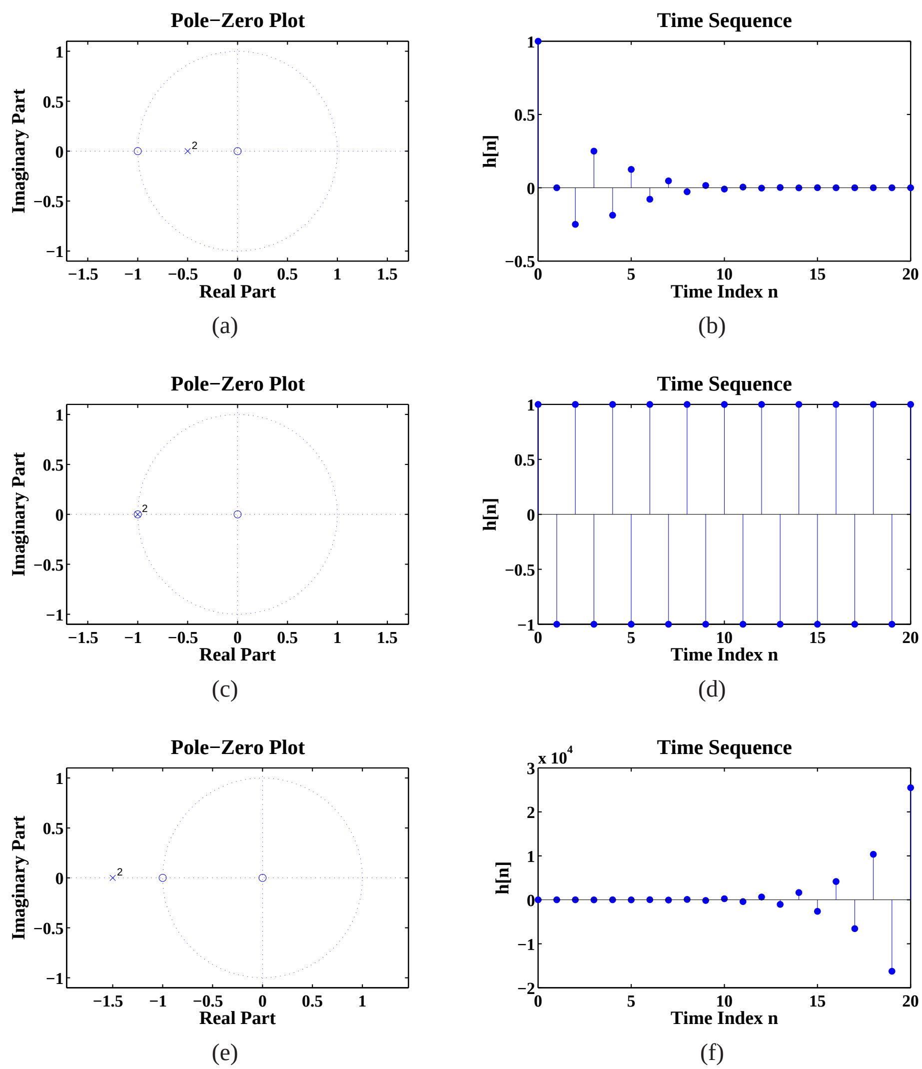


FIGURE 3.9: (a) Poles inside unit circle. (b) Decaying sequence. (c) Poles on unit circle. (b) Constant envelop sequence. (e) Poles outside unit circle. (b) Growing sequence.

19. (a) Solution:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - \frac{1}{1024}z^{-10}}{1 - \frac{1}{2}z^{-1}}$$

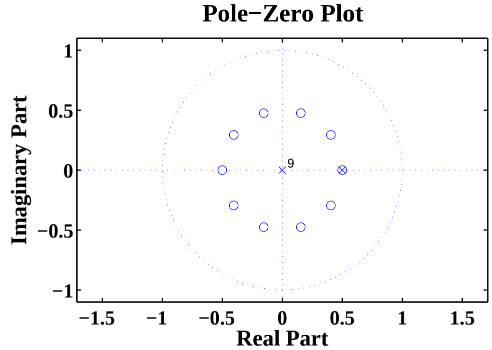


FIGURE 3.10: Pole-zero pattern of the system.

(b) Impulse response.

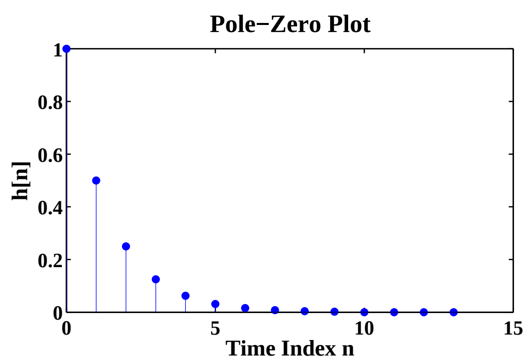


FIGURE 3.11: Impulse response $h[n]$.

(c) Comments: The pole at $z = \frac{1}{2}$ is canceled and the ROC is all the z plane. The corresponding time sequence should be of finite length.

(d) Solution:

$$H(z) = \sum_{m=0}^{9} 2^{-m}x[n-m]$$

20. (a) Solution:

$$X(z) = \frac{-\frac{1}{3}}{1-\frac{1}{2}z^{-1}} + \frac{\frac{4}{3}}{1-2z^{-1}} = \frac{1}{(1-\frac{1}{2}z^{-1})(1-2z^{-1})}, \quad \text{ROC: } \frac{1}{2} < |z| < 2$$

(b) Solution:

$$H(z) = \frac{Y(z)}{X(z)} = 1 - z^{-2}$$
$$h[n] = \delta[n] - \delta[n-2]$$

21. (a) Pole-zero plot.

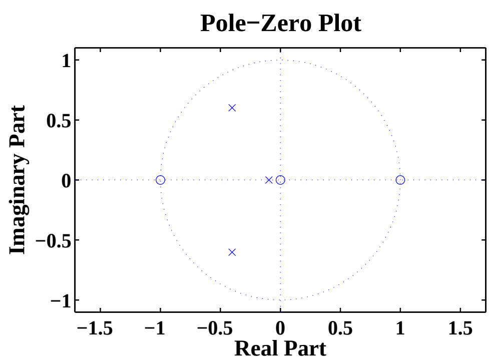


FIGURE 3.12: MATLAB verification of the impulse response expression obtained in part (a).

(b) Impulse response.

(c) Solution:

$$h[n] = -2.1760(-0.0956)^n u[n] + 2\times 1.5942\cos(2.1605n - 0.0885)0.7233^n u[n]$$

(d) See plot below.

MATLAB script:

```
% P0321: Plot plot-zero pattern and compute
%        impulse response
close all; clc
n = 0:20;
b = [1 0 -1];
a = [1 0.9 0.6 0.05];
```

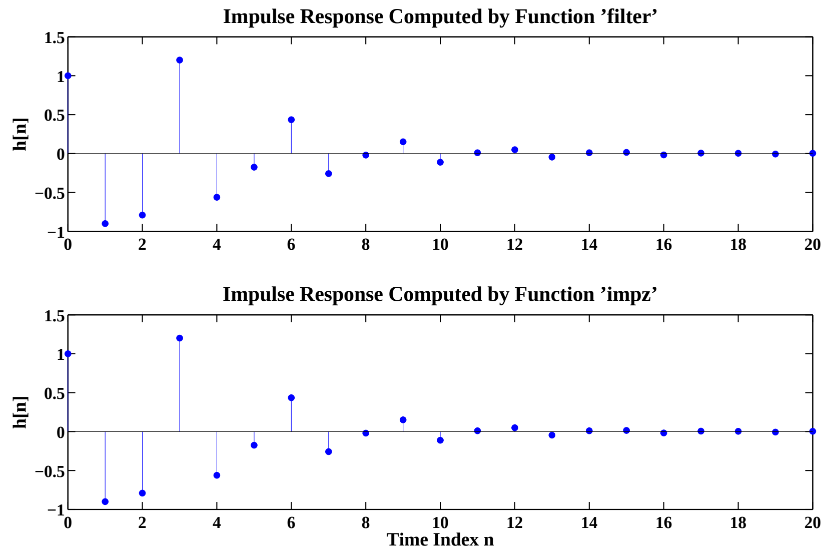


FIGURE 3.13: Impulse responses comparison using the functions `filter` and `impz`.

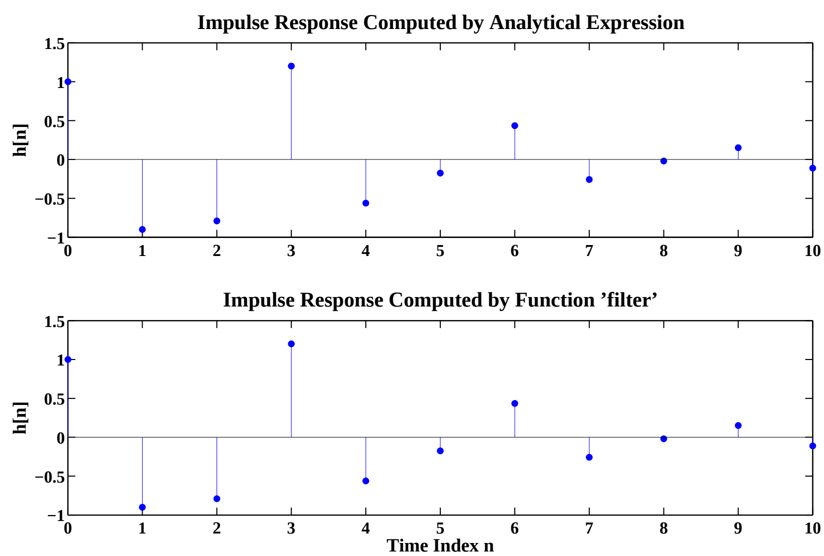


FIGURE 3.14: MATLAB verification of the analytical expression of system impulse responses obtained in part (c).

```
%% Part (a):
hf1 = figconfg('P0321a','small');
zplane(b,a)
xlabel('Real Part','fontsize',LFS)
ylabel('Imaginary Part','fontsize',LFS)
title('Pole-Zero Plot','fontsize',TFS)
%% Part (b):
hn1 = filter(b,a,[1,zeros(1,length(n)-1)]);
hn2 = impz(b,a,length(n));
% Plot
hf2 = figconfg('P0321b');
subplot(211)
stem(n,hn1,'filled')
ylabel('h[n]','fontsize',LFS)
title('Impulse Response Computed by Function ''filter''',...
    'fontsize',TFS)
subplot(212)
stem(n,hn2,'filled')
xlabel('Time Index n','fontsize',LFS)
ylabel('h[n]','fontsize',LFS)
title('Impulse Response Computed by Function ''impz''',...
    'fontsize',TFS)
%% Part (c):
[r,p,k] = residuez(b,a);
% [b2 a2] = residuez(r(1:2),p(1:2),k);
%% Part (d):
n = 0:10;
hn = r(3)*p(3).^n + 2*abs(r(1))*cos(angle(p(1)).*n...
    +angle(r(1))).*(abs(p(1)).^n);
hn_ref = filter(b,a,[1,zeros(1,length(n)-1)]);
% Plot
hf3 = figconfg('P0321c');
subplot(211)
stem(n,hn,'filled')
ylabel('h[n]','fontsize',LFS)
title('Impulse Response Computed by Analytical Expression',...
    'fontsize',TFS)
subplot(212)
stem(n,hn_ref,'filled')
xlabel('Time Index n','fontsize',LFS)
```

```
ylabel('h[n]','fontsize',LFS)
title('Impulse Response Computed by Function ''filter''',...
    'fontsize',TFS)
```

22. Solution:

$$h[n] = -\frac{2}{3}\left(-\frac{1}{2}\right)^n u[n] + \frac{5}{3}\left(\frac{1}{4}\right)^n u[n]$$

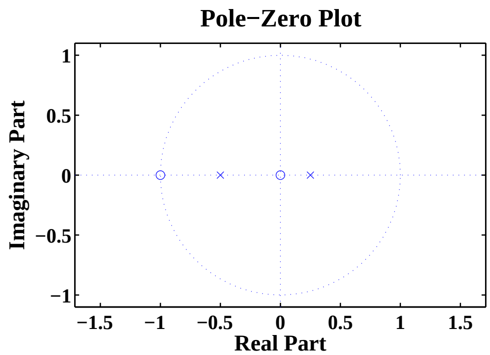


FIGURE 3.15: MATLAB verification of the impulse response expression obtained in part (a).

MATLAB script:

```
% P0322: Plot plot-zero pattern and compute
%        impulse response
close all; clc
n = 0:20;
b = [1 1];
a = [1 1/4 -1/8];
%% Part (a):
hf1 = figconfg('P0322a','small');
zplane(b,a)
xlabel('Real Part','fontsize',LFS)
ylabel('Imaginary Part','fontsize',LFS)
title('Pole-Zero Plot','fontsize',TFS)
%% Part (b):
hn1 = filter(b,a,[1,zeros(1,length(n)-1)]);
hn2 = impz(b,a,length(n));
```

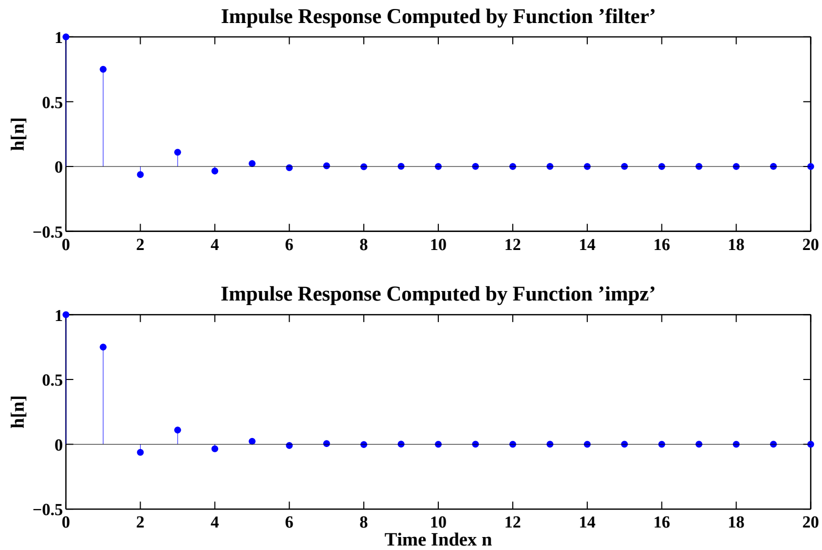


FIGURE 3.16: Impulse responses comparison using the functions `filter` and `impz`.

```
% Plot
hf2 = figconfg('P0322b');
subplot(211)
stem(n,hn1,'filled')
ylabel('h[n]','fontsize',LFS)
title('Impulse Response Computed by Function ''filter''',...
    'fontsize',TFS)
subplot(212)
stem(n,hn2,'filled')
xlabel('Time Index n','fontsize',LFS)
ylabel('h[n]','fontsize',LFS)
title('Impulse Response Computed by Function ''impz''',...
    'fontsize',TFS)
%% Part (c):
[r,p,k] = residuez(b,a);
% [b2 a2] = residuez(r(1:2),p(1:2),k);
%% Part (d):
n = 0:10;
```

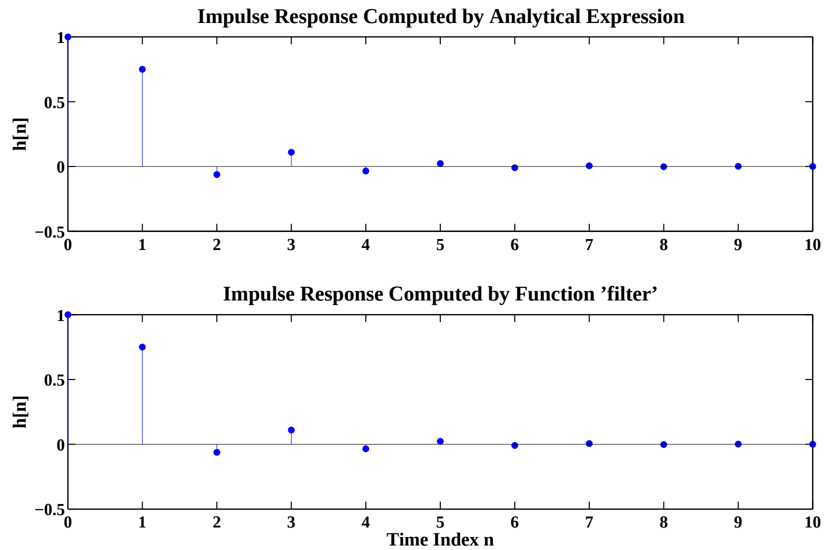
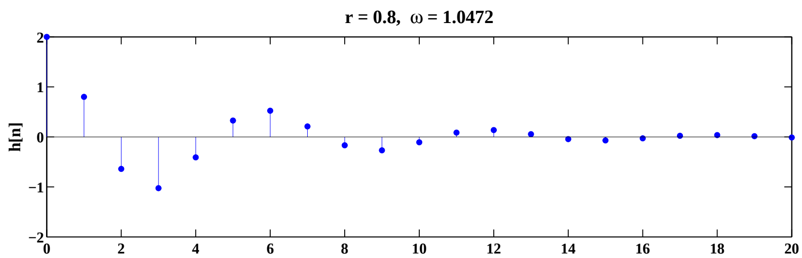


FIGURE 3.17: MATLAB verification of the analytical expression of system impulse responses obtained in part (c).

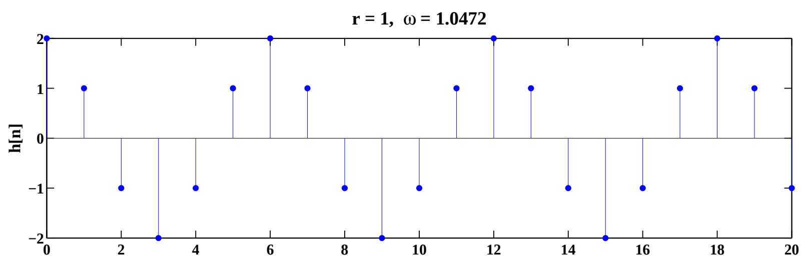
```
hn = r(1)*p(1).^n + r(2)*p(2).^n;
hn_ref = filter(b,a,[1,zeros(1,length(n)-1)]);
% Plot
hf3 = figconfg('P0322c');
subplot(211)
stem(n,hn,'filled')
ylabel('h[n]','fontsize',LFS)
title('Impulse Response Computed by Analytical Expression',...
    'fontsize',TFS)
subplot(212)
stem(n,hn_ref,'filled')
xlabel('Time Index n','fontsize',LFS)
ylabel('h[n]','fontsize',LFS)
title('Impulse Response Computed by Function ''filter''',...
    'fontsize',TFS)
```

23. MATLAB script:

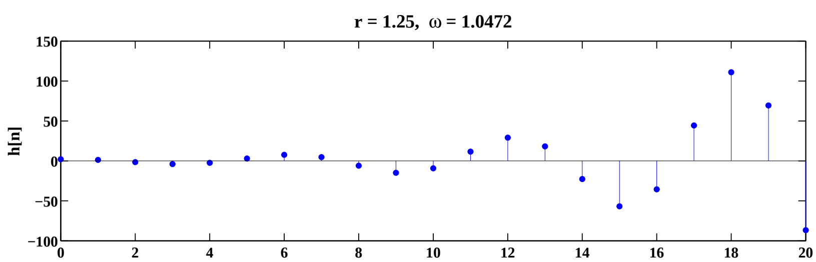
```
% P0323: Script to generate plots shown in Figure3.11
%       h[n] = 2|A|r^n*cos(w0*n+theta)*u[n]
close all; clc
w0 = pi/3;
r = [0.8 1 1.25];
n = 0:20;
A = 1; theta = 0;
rlen = length(r);
hn = zeros(rlen,length(n));
for ii = 1:rlen
hn(ii,:) = 2*A*r(ii).^n.*cos(w0.*n+theta);
end
% Plot
hf1 = figconfg('P0323a','long');
stem(n,hn(1,:),'filled')
ylabel('h[n]','fontsize',LFS)
title(['r = ',num2str(r(1)),',  \omega = ',num2str(w0)],...
    'fontsize',TFS)
hf2 = figconfg('P0323b','long');
stem(n,hn(2,:),'filled')
ylabel('h[n]','fontsize',LFS)
title(['r = ',num2str(r(2)),',  \omega = ',num2str(w0)],...
    'fontsize',TFS)
hf3 = figconfg('P0323c','long');
stem(n,hn(3,:),'filled')
ylabel('h[n]','fontsize',LFS)
title(['r = ',num2str(r(3)),',  \omega = ',num2str(w0)],...
    'fontsize',TFS)
```



(a)



(b)



(c)

FIGURE 3.18: (a) Stable system. (b) Marginally stable system. (c) Unstable system.

24. (a) Solution:

$$y[n] = b_0x[n] + b_1x[n-1] + b_2x[n-2] - a_1y[n-1] - a_2y[n-2]$$

Apply one-sided z-transform, we have

$$\begin{aligned}Y^+(z) &= \frac{b_0 + b_1z^{-1} + b_2z^{-2}}{1 + a_1z^{-1} + a_2z^{-2}}X^+(z)\\ &+ \frac{(b_2x[-1] - a_2y[-1])z^{-1} + (b_1x[-1] + b_2x[-2] - a_1y[-1] - a_2y[-2])}{1 + a_1z^{-1} + a_2z^{-2}}\end{aligned}$$

By substituting the specification in problem and using MATLAB, we have

$$\begin{aligned}y[n] &= 2 \times 3.9473\cos(1.0472n - 1.5560)u[n]\\ &+ 2 \times 3.7578\cos(1.0472n + 1.5420)(0.95)^n u[n]\\ &+ 2 \times 1.8760\cos(1.0472n - 1.1670)(0.95)^n u[n]\end{aligned}$$

(b) MATLAB script:

```
% P0324: Verify system response using filter
close all; clc
n = 0:20;
xn = cos(pi/3*n);
b = 1/3*ones(1,3);
a = [1 -0.95 0.9025];
bx = [1 -cos(pi/3)];
ax = [1 -2*cos(pi/3) 1];
yi = [-2 -3];
xi = [1 1];
bzi = [b(2)*xi(1)+b(3)*xi(2)-a(2)*yi(1)-a(3)*yi(2),...
    b(3)*xi(1)-a(3)*yi(1)];
bzs = conv(b,bx);
bz = bzs + conv(ax,bzi);
azs = conv(a,ax);
[rzi pzi kzi] = residuez(bzi,a);
[rzs pzs kzs] = residuez(bzs,azs);
ynzi = 2*abs(rzi(1))*cos(angle(pzi(1)).*n...
    +angle(rzi(1))).*(abs(pzi(1)).^n);
ynzs = 2*abs(rzs(1))*cos(angle(pzs(1)).*n...
    +angle(rzs(1))).*(abs(pzs(1)).^n) + ...
```

```
    2*abs(rzs(3))*cos(angle(pzs(3)).*n...
    +angle(rzs(3))).*(abs(pzs(3)).^n);
yn1 = ynzi + ynzs;
% Matlab verification:
zi = filtic(b,a,yi,xi);
yn2 = filter(b,a,xn,zi);
% Plot
hf = figconfg('P0324');
subplot(211)
stem(n,yn1,'filled')
ylabel('y[n]','fontsize',LFS)
title('System Response Computed by Analytical Expressoin',...
    'fontsize',TFS)
subplot(212)
stem(n,yn2,'filled')
xlabel('Time Index n','fontsize',LFS)
ylabel('y[n]','fontsize',LFS)
title('System Response Computed by Function ''filter''',...
    'fontsize',TFS)
```

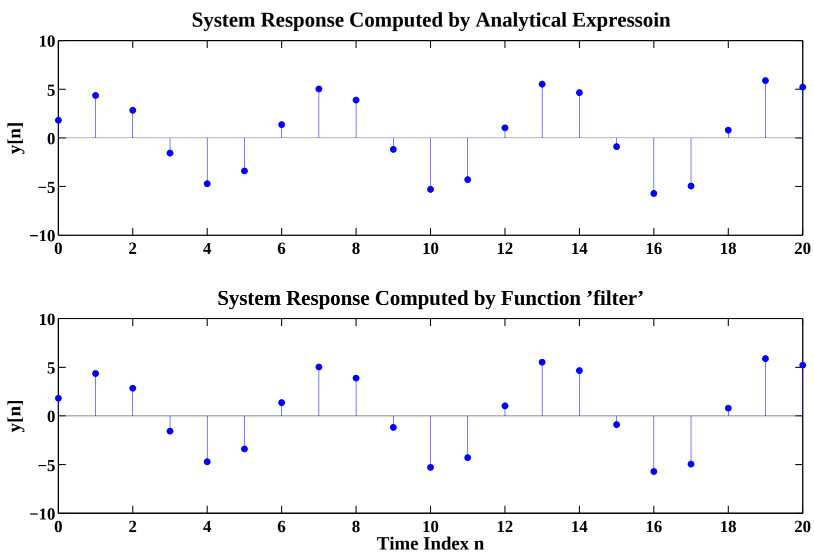


FIGURE 3.19: MATLAB verification of the analytical expression obtained in part (a).

Basic Problems

25. (a) Solution:

$$X(z) = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{1-\frac{1}{3}z^{-1}} = \frac{2-\frac{5}{6}z{-1}}{1-\frac{5}{6}z^{-1}+\frac{1}{6}z^{-2}} \quad \text{ROC: } |z| > \frac{1}{2}$$

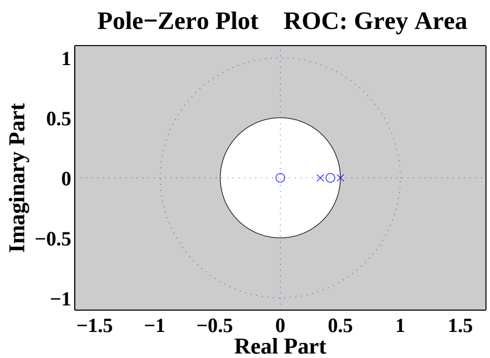


FIGURE 3.20: Pole-zero plot and ROC of $x[n] = (1/2)^n u[n] + (1/3)^n u[n]$.

(b) Solution:

$$X(z) = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{-1}{1-\frac{1}{3}z^{-1}} = \frac{\frac{1}{6}z{-1}}{1-\frac{5}{6}z^{-1}+\frac{1}{6}z^{-2}} \quad \text{ROC: } |z| \in \phi$$

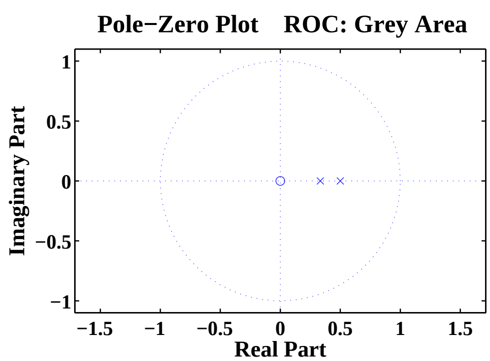


FIGURE 3.21: Pole-zero plot and ROC of $x[n] = (1/2)^n u[n] + (1/3)^n u[-n-1]$.

(c) Solution:

$$X(z) = \frac{-1}{1-\frac{1}{2}z^{-1}} + \frac{1}{1-\frac{1}{3}z^{-1}} = \frac{-\frac{1}{6}z^{-1}}{1-\frac{5}{6}z^{-1}+\frac{1}{6}z^{-2}} \quad \text{ROC: } \frac{1}{3} < |z| < \frac{1}{2}$$

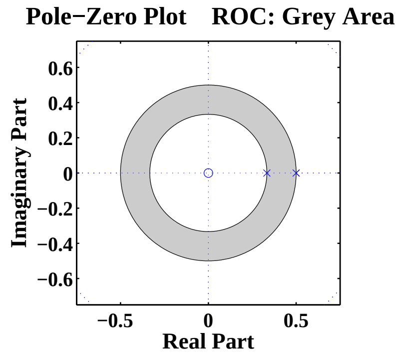


FIGURE 3.22: Pole-zero plot and ROC of $x[n] = (1/3)^n u[n] + (1/2)^n u[-n-1]$.

26. Proof:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=-\infty}^{\infty} a^{|n|}z^{-n} = \sum_{n=-\infty}^{-1} a^{-n}z^{-n} + \sum_{n=0}^{\infty} a^n z^{-n}$$
$$= \sum_{n=-1}^{\infty} a^n z^n + \sum_{n=0}^{\infty} a^n z^{-n} = \frac{az}{1-az} + \frac{1}{1-az^{-1}}$$
$$= \frac{(a-a^{-1})z^{-1}}{(1-az^{-1})(1-a^{-1}z^{-1})} \quad \text{ROC: } |a| < |z| < 1/|a|$$

If $|a| > 1$, $1/|a| < |a|$, the ROC is empty, the z-transform does not exist.

27. (a) Proof:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$
$$Y(z) = \sum_{n=-\infty}^{\infty} y[n]z^{-n} = \sum_{n=0,\pm2,\pm4,\ldots} x[n/2]z^{-n} \xrightarrow{n=2m}$$
$$= \sum_{m=-\infty}^{\infty} x[m]z^{-2m} = \sum_{m=-\infty}^{\infty} x[m](z^2)^{-m} = X(z^2)$$

(b) Solution:

$$X(z) = \frac{1}{1-0.8z^{-1}}, \quad \text{ROC: } |z| > 0.8$$
$$Y(z) = X(z^2) = \frac{1}{1-0.8z^{-2}}, \quad \text{ROC: } |z| > \frac{2}{\sqrt{5}}$$

(c) See plot below.

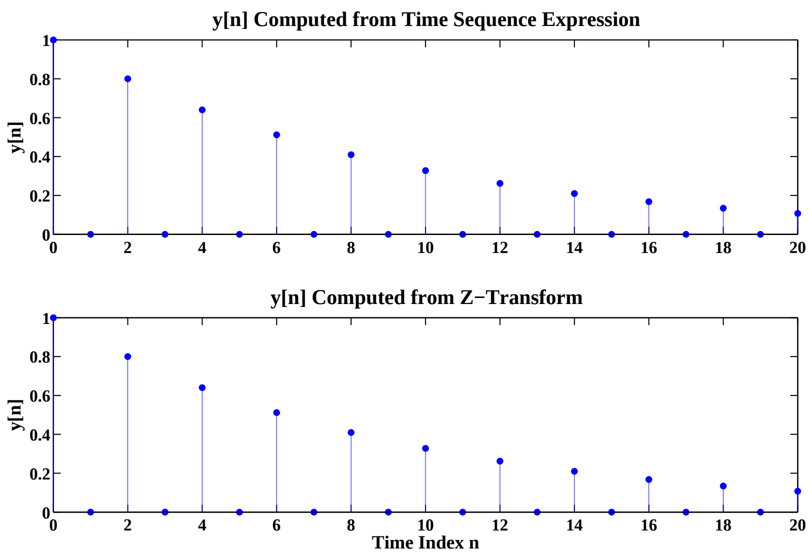


FIGURE 3.23: MATLAB verification of the z-transform of $y[n]$.

28. (a) Solution:

$$X(z) = -6.25 + \frac{0.5383}{z-3.7443} + \frac{6.7117z - 2.7313}{z^2 - 0.2557z + 0.0427}$$

If ROC: $|z| > 3.7443$

$$x[n] = \delta[n] + 0.5383(3.7443)^n u[n] + 2 \times 6.6714 \cos(0.9041n + 1.0437)(0.2067)^n u[n]$$

If ROC: $|z| < 0.2067$

$$x[n] = \delta[n] - 0.5383(3.7443)^n u[-n-1] - 2 \times 6.6714 \cos(0.9041n + 1.0437)(0.2067)^n u[-n-1]$$

If ROC: $0.2067 < |z| < 3.7443$

$$x[n] = \delta[n] - 0.5383(3.7443)^n u[-n-1] + 2 \times 6.6714 \cos(0.9041n + 1.0437)(0.2067)^n u[n]$$

(b) Solution:

$$X(z) = \frac{4}{1+z^{-1}} + \frac{-4}{1+\frac{1}{2}z^{-1}} - 4\frac{-\frac{1}{2}z^{-1}}{\left(1+\frac{1}{2}z^{-1}\right)^2}$$

$$x[n] = 4(-1)^n u[n] - 4\left(-\frac{1}{2}\right)^n u[n] - 4n\left(-\frac{1}{2}\right)^n u[n]$$

(c) tba

29. (a) Solution:

$$X(z) = 2\frac{1-\frac{1}{2}z^{-1}}{1-z^{-1}+0.81z^{-2}} + \frac{20}{\sqrt{14}}\frac{\frac{\sqrt{14}}{5}z^{-1}}{1-z^{-1}+0.81z^{-2}}$$

$$x[n] = 2(0.9)^n \cos(\omega_0 n)u[n] + \frac{20}{\sqrt{14}}(0.9)^n \sin(\omega_0 n)u[n], \ \cos\omega_0 = \frac{5}{9}$$

(b) See plot below.

30. (a) Solution:

$$\text{Time shifting:} \quad Y(z) = z^{-2}X(z) = \frac{z^{-2}}{1-\frac{1}{3}z^{-1}} \quad \text{ROC: } |z| > \frac{1}{3}$$

(b) Solution:

$$\text{Scaling:} \quad Y(z) = X(z/2) = \frac{1}{1-\frac{2}{3}z^{-1}} \quad \text{ROC: } |z| > \frac{2}{3}$$

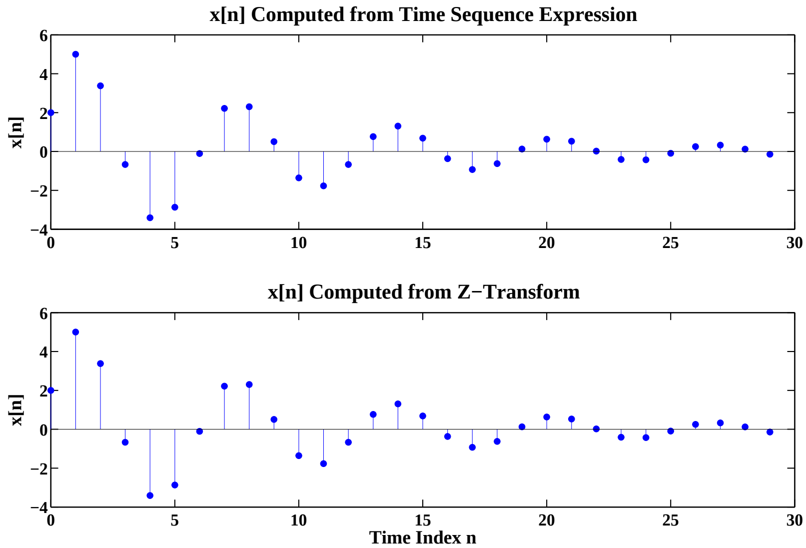


FIGURE 3.24: MATLAB verification of $x[n]$ expression obtained in part (a).

(c) Solution:

Convolution, time shifting and folding:

$$Y(z) = X(z)X(1/z)z = \frac{-3}{1-3z^{-1}}\frac{1}{1-\frac{1}{3}z^{-1}} = \frac{-3}{1-\frac{10}{3}+z^{-2}}$$

ROC: $\dfrac{1}{3} < |z| < \dfrac{2}{3}$

(d) Solution:

$$\text{Differentiation:} \quad Y(z) = \frac{\frac{1}{3}z\left(z+\frac{1}{3}\right)}{z-\frac{1}{3}} \quad \text{ROC: } |z| > \frac{1}{3}$$

(e) Solution:

Linearity and time shifting:

$$Y(z) = \frac{2z}{1-\frac{1}{3}z^{-1}} + \frac{3z^{-3}}{1-\frac{1}{3}z^{-1}} = \frac{2z+3z^{-3}}{1-\frac{1}{3}z^{-1}} \quad \text{ROC: } |z| > \frac{1}{3}$$

(f) Solution:

Convolution and time shifting:

$$Y(z) = \frac{z^{-1}}{1-\frac{1}{3}z^{-1}} \cdot \frac{1}{1-\frac{1}{3}z^{-1}} = \frac{z^{-1}}{\left(1-\frac{1}{3}z^{-1}\right)^2} \quad \text{ROC: } |z| > \frac{1}{3}$$

31. Solution:

$$x[n] = 0.8^n u[n]$$

(a) $Y(z) = X(1/z)$
Solution:

$$\text{Folding} \quad y[n] = 0.8^{-n} u[-n]$$

(b) $Y(z) = dX(z)/dz$
Solution:

$$\text{Differentiation and shifting} \quad y[n] = -n0.8^{n-1} u[n-1]$$

(c) $Y(z) = X^2(z)$
Solution:

$$\text{Convolution} \quad y[n] = x[n] * x[n] = 0.8^n (n+1) u[n]$$

32. Solution:

$$y[n] = \sum_{k=-n}^{+n} a^{|k|} u[n] = \left(1 + 2\sum_{k=1}^{+n} a^k\right) u[n] = u[n] + \frac{2a(1-a^n)}{1-a} u[n]$$
$$= \frac{1+a}{1-a} u[n] - \frac{2a}{1-a} a^n u[n]$$

$$Y(z) = \frac{1+a}{1-a} \frac{1}{1-z^{-1}} - \frac{2a}{1-a} \frac{1}{1-az^{-1}} \quad \text{ROC: } |z| > 1$$

33. Solution:

$$X(z) = \frac{A(z-\mathrm{j})(z+\mathrm{j})}{(z-0.8)(z+0.8)(z-\mathrm{j}0.8)(z+\mathrm{j}0.8)} = A\frac{z^2+1}{z^4-0.8^4}$$

$$X(1) = 1 \implies A = 0.2952$$

$$\text{System stable} \implies \text{ROC: } |z| > 0.8$$

$$Y(z) = z^{-2} X(z) = 0.2952 \frac{z^{-2}+1}{z^4-0.8^4} \quad \text{ROC: } |z| > 0.8$$

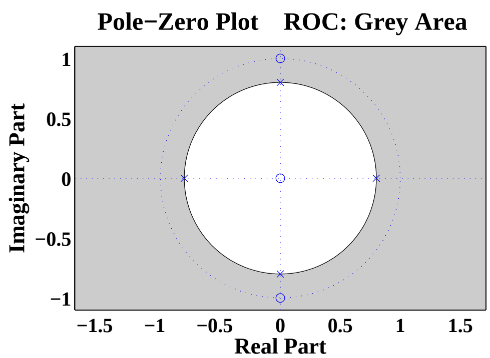


FIGURE 3.25: Pole-zero plot and ROC of signal $y[n]$.

34. Compute $y[n] = h[n] * x[n]$ for $h[n] = (1/2)^n u[n]$ and $x[n] = 3^n u[-n]$.✔
Solution:

$$h[n] = (1/2)^n u[n] \implies H(z) = \frac{1}{1 - \frac{1}{2}z^{-1}}, \quad \text{ROC: } |z| > \frac{1}{2}$$

$$x[n] = \delta[n] + 3^n u[-n-1] \implies X(z) = 1 - \frac{1}{1 - 3z^{-1}}, \quad \text{ROC: } |z| < 3$$

$$Y(z) = X(z)H(z) = \frac{\frac{6}{5}}{1 - \frac{1}{2}z^{-1}} + \frac{-\frac{6}{5}}{1 - 2z^{-1}}, \quad \text{ROC: } \frac{1}{2} < |z| < 3$$

$$y[n] = \frac{6}{5}\left(\frac{1}{2}\right)^n u[n] + \frac{6}{5} \cdot 3^n u[-n-1]$$

35. (a) Solution:

$$H(z) = \frac{1}{1 - \frac{4}{5}z^{-1}}, \quad \text{ROC: } |z| > \frac{4}{5}$$

$$X(z) = \delta[n] + \frac{-1}{1 - \frac{6}{5}z^{-1}}, \quad \text{ROC: } |z| < \frac{6}{5}$$

$$Y(z) = H(z)X(z) = \frac{3}{1 - \frac{4}{5}z^{-1}} + \frac{-3}{1 - \frac{6}{5}z^{-1}}, \quad \text{ROC: } \frac{4}{5} < |z| < \frac{6}{5}$$

$$y[n] = 3\left(\frac{4}{5}\right)^n u[n] + 3\left(\frac{6}{5}\right)^n u[-n-1]$$

(b) Solution:

$$H(z) = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{-1}{1-3z^{-1}}, \quad \text{ROC: } \frac{1}{2} < |z| < 3$$

$$X(z) = \frac{1}{1-\frac{3}{4}z^{-1}}, \quad \text{ROC: } |z| > \frac{3}{4}$$

$$Y(z) = H(z)X(z) = \frac{-\frac{4}{3}}{1-3z^{-1}} + \frac{\frac{10}{3}}{1-\frac{3}{4}z^{-1}} + \frac{-2}{1-\frac{1}{2}z^{-1}}, \quad \text{ROC: } \frac{3}{4} < |z| < 3$$

$$y[n] = \frac{4}{3}\cdot 3^n u[-n-1] + \frac{10}{3}\left(\frac{3}{4}\right)^n u[n] - 2\left(\frac{1}{2}\right)^n u[n]$$

(c) Solution:

$$\begin{aligned} H(z) &= \frac{1}{1-0.8z^{-1}} + \frac{\frac{5}{6}}{1-\frac{5}{6}z^{-1}} \\ &= \frac{\frac{11}{6} - \frac{28}{15}z^{-1}}{1-2z^{-1}+\frac{24}{25}z^{-2}}, \quad \text{ROC: } \frac{9}{10} < |z| < \frac{3}{2} \end{aligned}$$

$$\begin{aligned} X(z) &= \frac{1}{1-0.9z^{-1}} + 1 + \frac{-1}{1-\frac{3}{2}z^{-1}} \\ &= \frac{1-3z^{-1}+\frac{27}{20}z^{-2}}{\left(1-\frac{9}{10}z^{-1}\right)\left(1-\frac{3}{2}z^{-1}\right)}, \quad \text{ROC: } \frac{4}{5} < |z| < \frac{6}{5} \end{aligned}$$

$$Y(z) = X(z)H(z) = \frac{-\frac{265}{42}}{1-\frac{3}{2}z^{-1}} + \frac{\frac{15}{2}}{1-\frac{6}{5}z^{-1}} + \frac{-\frac{13}{2}}{1-\frac{9}{10}z^{-1}} + \frac{-\frac{41}{7}}{1-\frac{4}{5}z^{-1}},$$
$$\text{ROC: } \frac{9}{10} < |z| < \frac{6}{5}$$

$$\begin{aligned} y[n] =& \frac{265}{42}\left(\frac{3}{2}\right)^n u[-n-1] - \frac{15}{2}\left(\frac{6}{5}\right)^n u[-n-1] \\ &+ \frac{13}{2}\left(\frac{9}{10}\right)^n u[n] - \frac{41}{7}\left(\frac{4}{5}\right)^n u[n] \end{aligned}$$

36. (a) Proof:

$$r_{xx}[\ell] \triangleq \sum_n x[n]x^*[n-\ell] = \sum_n x[n+\ell]x^*[n] = x[\ell] * x^*[\ell]$$

Hence, we proved

$R_{xx}(z) = X(z)X^*\left(1/z^*\right)$ ROC: $\max(\alpha, \beta^{-1}) < |z| < \min(\beta, \alpha^{-1})$

(b) Solution:

$$X(z) = \frac{1}{1 - r\mathrm{e}^{\mathrm{j}\theta}z^{-1}}, \quad \text{ROC: } |z| > r$$
$$X^*(1/z^*) = \frac{1}{1 - r\mathrm{e}^{-\mathrm{j}\theta}z}, \quad \text{ROC: } |z| < r^{-1}$$

$$R_{xx}(z) = X(z)X^*(1/z^*) = \frac{1}{(1 - r\mathrm{e}^{\mathrm{j}\theta}z^{-1})(1 - r\mathrm{e}^{-\mathrm{j}\theta}z)},$$
$$\text{ROC: } r < |z| < r^{-1}$$

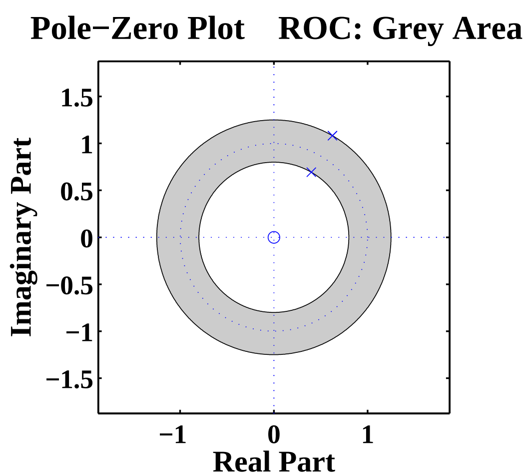


FIGURE 3.26: Pole-zero plot and ROC of $R_{xx}(z)$.

(c) Solution:

$$R_{xx}(z) = \frac{\frac{-\mathrm{e}^{2\mathrm{j}\theta}}{1-\mathrm{e}^{2\mathrm{j}\theta}}}{1 - r\mathrm{e}^{\mathrm{j}\theta}z^{-1}} + \frac{\frac{-r^{-2}\mathrm{e}^{2\mathrm{j}\theta}}{1-\mathrm{e}^{2\mathrm{j}\theta}}}{1 - r^{-1}\mathrm{e}^{\mathrm{j}\theta}z^{-1}}$$
$$r_{xx}[\ell] = \frac{-\mathrm{e}^{2\mathrm{j}\theta}}{1 - \mathrm{e}^{2\mathrm{j}\theta}} r^l \mathrm{e}^{\mathrm{j}\theta l} u[l] + \frac{r^2\mathrm{e}^{2\mathrm{j}\theta}}{1 - \mathrm{e}^{2\mathrm{j}\theta}} r^{-l}\mathrm{e}^{\mathrm{j}\theta l} u[-l-1]$$

37. Solution:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-1} + z^{-2}}{1 + 0.2z^{-1} - 0.18z^{-2} + 0.891z^{-3}}$$
$$= \frac{-0.0332}{1 + 1.1z^{-1}} + \frac{0.0332 + 0.9336z^{-1}}{1 - 0.9z^{-1} + 0.81z^{-2}}$$

If ROC: $|z| > 1.1$

$$h[n] = -0.0332 \cdot 1.1^n u[n] + 2 \cdot 0.6087 \cdot 0.9^n \cos(1.0472n - 1.5435)u[n]$$

If ROC: $|z| < 0.9$

$$h[n] = 0.0332 \cdot 1.1^n u[-n-1] - 2 \cdot 0.6087 \cdot 0.9^n \cos(1.0472n - 1.5435)u[-n-1]$$

If ROC: $0.9 < |z| < 1.1$

$$h[n] = 0.0332 \cdot 1.1^n u[-n-1] + 2 \cdot 0.6087 \cdot 0.9^n \cos(1.0472n - 1.5435)u[n]$$

38. Solution:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-1} + z^{-2}}{1 - 0.8z^{-1} + 0.81z^{-2}}, \quad \text{ROC: } |z| > 0.9$$

(a) $x[n] = \mathrm{e}^{\mathrm{j}(\pi/3)n}, \quad -\infty < n < \infty$

Solution:

$$y[n] = \mathrm{e}^{\mathrm{j}(\pi/3)n} H(z)\,|_{z=\mathrm{e}^{\mathrm{j}(\pi/3)}} = (0.3937 - 8.8648\mathrm{j})\mathrm{e}^{\mathrm{j}(\pi/3)n}$$

(b) $x[n] = \mathrm{e}^{\mathrm{j}(\pi/3)n} u[n]$

Solution:

$$X(z) = \frac{1}{1 - \mathrm{e}^{\mathrm{j}(\pi/3)} z^{-1}}, \quad \text{ROC: } |z| > 1$$

MATLAB script:

```
    % P0338: Compute the system response
close all; clc
bh = [0 1 1];
ah = [1 -0.8 0.81];
%% Part (b)
bx = 1;
ax = [1 -exp(j*pi/3)];
[r p k] = residuez(conv(bx,bh),conv(ax,ah))
```

The results in the command window are:

```
    r =
   0.3937 - 8.8648i
   0.1415 + 8.5978i
  -0.5352 + 0.2670i
   p =
```

```
0.5000 + 0.8660i
0.4000 + 0.8062i
0.4000 - 0.8062i
k =
   []
```

(c) $x[n] = 1, \quad -\infty < n < \infty$
Solution:

$$y[n] = H(z)\,|_{z=1} = 1.9802$$

(d) $x[n] = (-1)^n u[n]$
Solution:

$$X(z) = \frac{1}{1+z^{-1}}, \quad \text{ROC: } |z| > 1$$

$$Y(z) = X(z)H(z) = \frac{z^{-1}}{1-0.8z^{-1}+0.81z^{-2}}, \quad \text{ROC: } |z| > 0.9$$

$$y[n] = 2 \cdot 0.6202 \cos(1.1102n - 1.5708)u[n]$$

39. Solution:

$$X(z) = \frac{1}{1-z^{-1}}, \quad \text{ROC: } |z| > 1$$

$$Y(z) = \frac{1}{4}z + \frac{2}{1-\frac{1}{2}z^{-1}}, \quad \text{ROC: } |z| > \frac{1}{2}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{4}z + \frac{15}{4} + \frac{-2}{1-\frac{1}{2}z^{-1}}, \quad \text{ROC: } |z| > \frac{1}{2}$$

$$h[n] = \frac{1}{4}\delta[n+1] + \frac{15}{4}\delta[n] - 2 \cdot \left(\frac{1}{2}\right) u[n]$$

The system is NOT causal but is stable.

40. (a) Solution:

$$X(z) = \frac{1}{1-\frac{1}{4}z^{-1}}, \quad \text{ROC: } |z| > \frac{1}{4}$$

$$Y(z) = \frac{5}{1-\frac{3}{4}z^{-1}}, \quad \text{ROC: } |z| > \frac{3}{4}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{5}{3} + \frac{\frac{10}{3}}{1-\frac{3}{4}z^{-1}}, \quad \text{ROC: } |z| > \frac{3}{4}$$

$$h[n] = \frac{5}{3}\delta[n] + \frac{10}{3}\left(\frac{3}{4}\right)^n u[n]$$

(b) Solution:

$$X(z) = \frac{1}{1 - \frac{1}{3}z^{-1}}, \quad \text{ROC: } |z| > \frac{1}{3}$$

$$Y(z) = H(z)X(z) = \frac{-1}{1 - \frac{1}{3}z^{-1}} + \frac{6}{1 - \frac{3}{4}z^{-1}}, \quad \text{ROC: } |z| > \frac{3}{4}$$

$$y[n] = -\left(\frac{1}{3}\right)^n u[n] + 6 \cdot \left(\frac{3}{4}\right)^n u[n]$$

(c) MATLAB script:

```
% P0340: Verification of time sequences and
%        their z-tranforms
close all; clc
n = 0:20;
bh = 5*[1 -1/4];
ah = [1 -3/4];
hn1 = 10/3*(3/4).^n;
hn1(1) = hn1(1)+5/3;
hn2 = filter(bh,ah,[1,zeros(1,length(n)-1)]);
by = bh;
ay = conv([1 -1/3],[1 -3/4]);
yn1 = -(1/3).^n+6*(3/4).^n;
yn2 = filter(by,ay,[1,zeros(1,length(n)-1)]);
% Plot
hf1 = figconfg('P0340a');
subplot(211)
stem(n,hn1,'filled')
ylabel('h[n]','fontsize',LFS)
title('h[n] Computed from Time Sequence Expression',...
    'fontsize',TFS)
subplot(212)
stem(n,hn2,'filled')
xlabel('Time Index n','fontsize',LFS)
ylabel('h[n]','fontsize',LFS)
title('h[n] Computed from Z-Transform',...
    'fontsize',TFS)
hf2 = figconfg('P0340b');
subplot(211)
stem(n,yn1,'filled')
ylabel('y[n]','fontsize',LFS)
```

```
title('y[n] Computed from Time Sequence Expression',...
    'fontsize',TFS)
subplot(212)
stem(n,yn2,'filled')
xlabel('Time Index n','fontsize',LFS)
ylabel('y[n]','fontsize',LFS)
title('y[n] Computed from Z-Transform',...
    'fontsize',TFS)
```

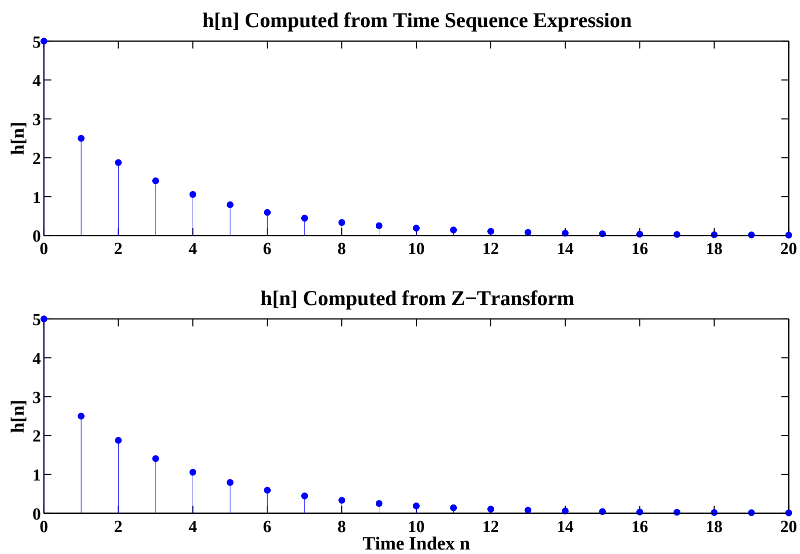


FIGURE 3.27: MATLAB verification of impulse response $h[n]$.

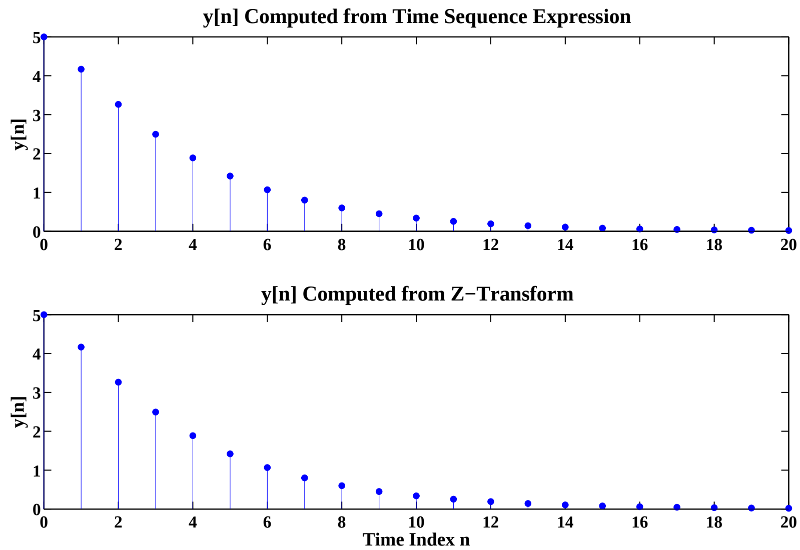


FIGURE 3.28: MATLAB verification of system response $y[n]$.

41. (a) Solution:

$$X(z) = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{-1}{1-2z^{-1}}, \quad \text{ROC: } \frac{1}{2} < |z| < 2$$

$$Y(z) = \frac{6}{1-\frac{1}{2}z^{-1}} + \frac{-6}{1-\frac{3}{4}z^{-1}}, \quad \text{ROC: } |z| > \frac{3}{4}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{8}{3} + \frac{\frac{5}{3}}{1-\frac{3}{4}z^{-1}}, \quad \text{ROC: } |z| > \frac{3}{4}$$

$$h[n] = \frac{8}{3}\delta[n] + \frac{5}{3}\cdot\left(\frac{3}{4}\right)^n u[n]$$

(b) Solution:

$$X(z) = \frac{1}{1+3z^{-1}}, \quad \text{ROC: } |z| > 3$$

$$Y(z) = \frac{4}{1-2z^{-1}} + \frac{-1}{1-\frac{1}{2}z^{-1}}, \quad \text{ROC: } |z| > 2$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{10}{1-2z^{-1}} + \frac{-7}{1-\frac{1}{2}z^{-1}}, \quad \text{ROC: } |z| > 2$$

$$h[n] = 10 \cdot 2^n u[n] - 7 \cdot \left(\frac{1}{2}\right)^n u[n]$$

42. (a) Solution:

$$X(z) = \frac{3}{1 - \frac{1}{3}z^{-1}} + \frac{4}{1 - 4z^{-1}}, \quad \text{ROC: } \frac{1}{3} < |z| < 4$$

(b) Solution:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + z^{-1} - z^{-2}}{7 - \frac{40}{3}z^{-1}} = -0.0356 + 0.075z^{-1} + \frac{0.1785}{1 - 1.9048z^{-1}}$$

The system is causal, hence we choose the ROC region as:

$$|z| > \frac{40}{3}$$

Thus, the system impulse response is:

$$h[n] = -0.0356 \cdot \delta[n] + 0.075 \cdot \delta[n-1] + 0.1785 \cdot 1.9048^n \cdot u[n]$$

43. (a) Solution:

$$x[n] = (0.7)^n u[n+1] \implies x[-1] = \frac{10}{7}$$

Recall the one-sided z-transform to the general second order LCCDE in Problem 24, we have

$$\begin{aligned} Y^+(z) = {} & \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}} X^+(z) \\ & + \frac{(b_2 x[-1] - a_2 y[-1])z^{-1} + (b_1 x[-1] + b_2 x[-2] - a_1 y[-1] - a_2 y[-2])}{1 + a_1 z^{-1} + a_2 z^{-2}} \end{aligned}$$

In this problem, the specifications are:

$$b_0 = 1; b_1 = -1; b_2 = 0; x[-2] = 0; x[-1] = \frac{10}{7}$$

$$a_1 = 0; a_2 = -0.81; y[-2] = y[-1] = 2;$$

$$X^+(z) = \frac{1}{1 - 0.7z^{-1}}, \quad \text{ROC: } |z| > 0.7$$

Using MATLAB, we compute the system response is

$$y[n] = 0.96 \cdot 0.9^n u[n] + 0.0038 \cdot (-0.9)^n u[n] + 0.6563 \cdot 0.7^n u[n]$$

(b) MATLAB script:

```
% P0343: Verify system response using filter
close all; clc
n = 0:20;
xn = 0.7.^n;
b = [1 -1];
a = [1 0 -0.81];
bx = 1;
ax = [1 -0.7];
yi = [2 2];
xi = [1 0];
bzi = [b(2)*xi(1)-a(2)*yi(1)-a(3)*yi(2),-a(3)*yi(1)];
bzs = conv(b,bx);
azs = conv(a,ax);
[rzi pzi kzi] = residuez(bzi,a);
[rzs pzs kzs] = residuez(bzs,azs);
ynzi = rzi(1)*(pzi(1).^n) + rzi(2)*(pzi(2).^n);
ynzs = rzs(1)*(pzs(1).^n) + rzs(2)*(pzs(2).^n)...
    + rzs(3)*(pzs(3).^n);
yn1 = ynzi + ynzs;
% Matlab verification:
zi = filtic(b,a,yi,xi);
yn2 = filter(b,a,xn,zi);
% Plot
hf = figconfg('P0343');
subplot(211)
stem(n,yn1,'filled')
ylabel('y[n]','fontsize',LFS)
title('System Response Computed by Analytical Expressoin',...
    'fontsize',TFS)
subplot(212)
stem(n,yn2,'filled')
xlabel('Time Index n','fontsize',LFS)
ylabel('y[n]','fontsize',LFS)
title('System Response Computed by Function ''filter''',...
    'fontsize',TFS)
```

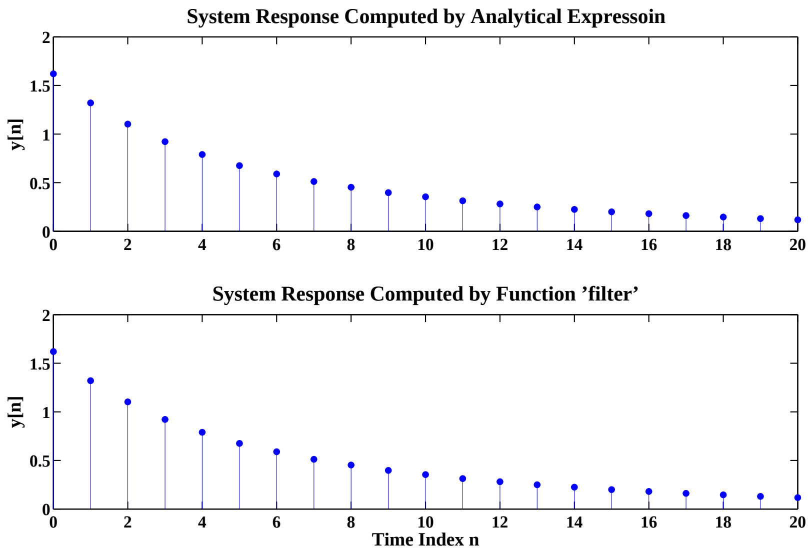


FIGURE 3.29: MATLAB verification of system response $y[n]$.

44. Solution:

$$Y^{+}(z) = \frac{1}{4}(y[-1] + z^{-1}Y^{+}(z)) + X^{+}(z) + 3(x[-1] + z^{-1}X^{+}(z))$$

$$Y^{+}(z) = \frac{1}{4}(2 + z^{-1}Y^{+}(z)) + X^{+}(z) + 3(0 + z^{-1}X^{+}(z))$$

$$Y^{+}(z) = \frac{\frac{1}{2}}{1 - \frac{1}{4}z^{-1}} + \frac{1 + 3z^{-1}}{1 - \frac{1}{4}z^{-1}}X^{+}(z)$$

$$X^{+}(z) = \frac{1}{1 - \mathrm{e}^{\mathrm{j}\pi/4}z^{-1}}, \quad \text{ROC: } |z| > 1$$

$$y[n] = \frac{1}{2}\cdot\left(\frac{1}{4}\right).^{n}u[n] + (3.0955 - 3.2416\mathrm{j})\mathrm{e}^{\mathrm{j}\pi n/4}u[n] + (-2.0955 + 3.2416\mathrm{j})\left(\frac{1}{4}\right).^{n}u[n]$$

zero-input response:

$$\frac{1}{2}\cdot\left(\frac{1}{4}\right).^{n}u[n]$$

zero-state response

$$(3.0955 - 3.2416\mathrm{j})\mathrm{e}^{\mathrm{j}\pi n/4}u[n] + (-2.0955 + 3.2416\mathrm{j})\left(\frac{1}{4}\right).^{n}u[n]$$

transient response:

$$\frac{1}{2} \cdot \left(\frac{1}{4}\right).^{n} u[n] + (-2.0955 + 3.2416\,\mathrm{j}) \left(\frac{1}{4}\right).^{n} u[n]$$

steady-state response:

$$(3.0955 - 3.2416\,\mathrm{j})\mathrm{e}^{\mathrm{j}\pi n/4} u[n]$$

Assessment Problems

45. (a) Solution:

$$X(z) = \frac{1}{1-2z^{-1}} + \frac{3}{1-\frac{1}{2}z^{-1}} = \frac{4-\frac{13}{2}z^{-1}}{1-\frac{5}{2}z^{-1}+z^{-2}}, \quad \text{ROC: } |z| > 2$$

(b) Solution

$$\begin{aligned} X(z) &= 2z + \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{-1}{1-3z^{-1}} \\ &= 2z + \frac{-\frac{5}{2}z^{-1}}{1-\frac{7}{2}z^{-1}+\frac{3}{2}z^{-2}}, \quad \text{ROC: } \frac{1}{2} < |z| < 3 \end{aligned}$$

(c) Solution:

$$X(z) = \frac{(\frac{1}{3}\sin\frac{\pi}{4})z^{-1}}{1-(\frac{2}{3}\cos\frac{\pi}{4})z^{-1}+\frac{1}{9}z^{-2}}, \quad \text{ROC: } |z| > \frac{1}{3}$$

(d) Solution

$$\begin{aligned} X(z) &= \frac{2z^{-1}}{(1-2z^{-1})^2} + \frac{\frac{1}{2}z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)^2} \\ &= \frac{\frac{5}{2}z^{-1} - 4z^{-2} + \frac{5}{2}z^{-3}}{(1-2z^{-1})^2\left(1-\frac{1}{2}z^{-1}\right)^2}, \quad \text{ROC: } \frac{1}{2} < |z| < 2 \end{aligned}$$

46. (a) Solution:
Use MATLAB function ‘residuez’ we have

$$X(z) = -16 + \frac{-10}{1-\frac{1}{2}z^{-1}} + \frac{27}{1-\frac{1}{4}z^{-1}}, \quad \text{ROC: } |z| > \frac{1}{2}$$

$$x[n] = -16\delta[n] - 10\cdot\left(\frac{1}{2}\right)^n u[n] + 27\cdot\left(\frac{1}{4}\right)^n u[n]$$

(b) Solution:
Use MATLAB function ‘residuez’ we have

$$X(z) = -6.25 + \frac{0.5383}{1-3.7443z^{-1}} + \frac{6.7117-2.7313z^{-1}}{1-0.2257z^{-1}+0.0427z^{-2}}, \quad \text{ROC: } |z| < 0.2067$$

$$\begin{aligned} x[n] &= -6.25\delta[n] - 0.5383\cdot(3.7443)^n\, u[-n-1] \\ &\quad - 2\cdot 6.6714\cdot 0.2067^n\cos(0.9041n+1.0437)u[-n-1] \end{aligned}$$

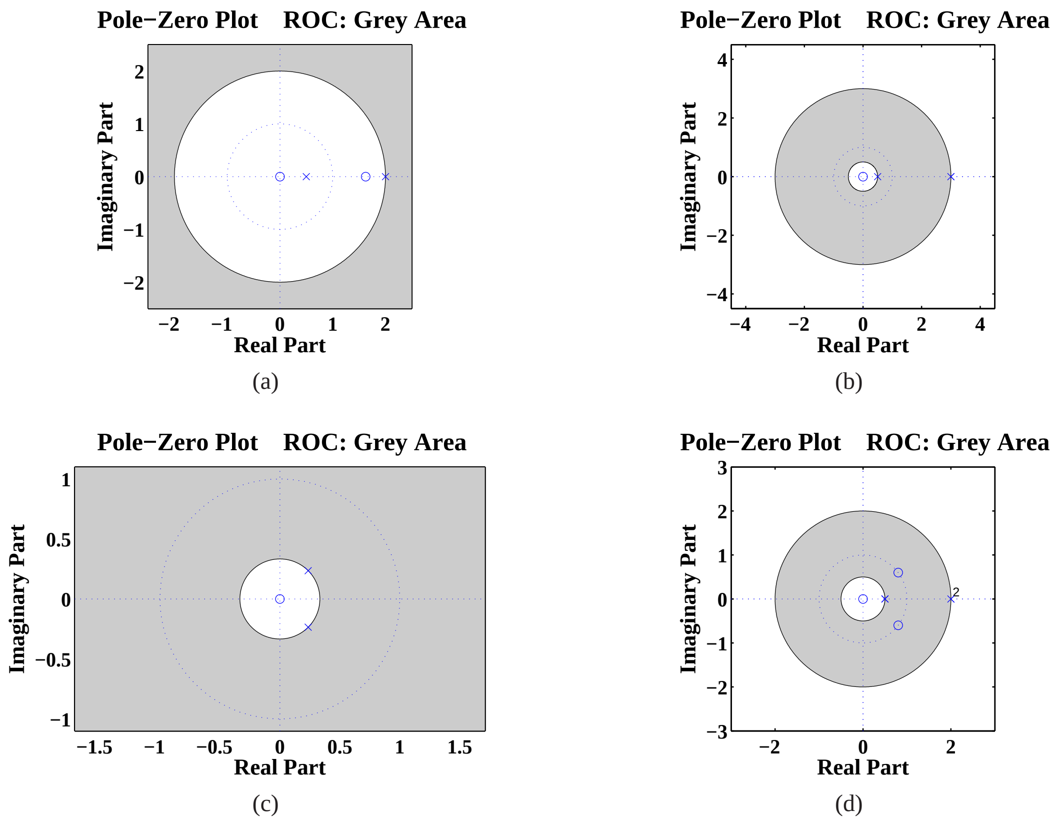


FIGURE 3.30: Pole-zero plot and ROC of (a) $x[n] = 2^n u[n] + 3(1/2)^n u[n]$. (b) $x[n] = (1/2)^n u[n+1] + 3^n u[-n-1]$. (c) $x[n] = (1/3)^n \sin(\pi n/4) u[n]$. (d) $x[n] = |n|(1/2)^{|n|}$.

(c) Solution:
Use MATLAB function 'residuez' we have

$$X(z) = \frac{4}{1+z^{-1}} + \frac{-4}{1+0.25z^{-1}} + (-4)\frac{-0.25z^{-1}}{1+z^{-1}+0.25z^{-2}}$$

$$x[n] = 4 \cdot (-1)^n u[n] - 4 \cdot (-0.25)^n u[n] - 4n(-0.25)^n u[n]$$

(d) Solution:

Use MATLAB function 'residuez' we have

$$\begin{aligned} X(z) &= \frac{2+2z^{-1}}{(1+0.5z^{-1})^2} + \frac{-2+2z^{-1}}{(1-0.5z^{-1})^2} \\ &= \frac{2}{1+0.5z^{-1}} - 2\frac{-0.5z^{-1}}{(1+0.5z^{-1})^2} + \frac{-2}{1-0.5z^{-1}} + 2\frac{0.5z^{-1}}{(1-0.5z^{-1})^2} \end{aligned}$$

$$x[n] = 2 \cdot (-0.5)^n u[n] - 2n(-0.5)^n u[n] - 2 \cdot 0.5^n u[n] + 2n0.5^n u[n]$$

47. (a) Solution:

Time shifting: $Y(z) = z^2 X(z) = z/(1+0.8z^{-1})$, ROC: $|z| > 0.8$

(b) Solution:

Time shifting and folding:

$$Y(z) = z^{-3} X(1/z) = z^{-2}/(1+0.8z), \quad \text{ROC: } |z| < 1.25$$

(c) Solution:

Scaling: $Y(z) = X(4z/5) = 1.25z^{-1}/(1+z^{-1})$, ROC: $|z| > 1$

(d) Solution:

Differentiation, time shifting and linearity

$$Y(z) = z^{-1}(-z)dX(z)/dz + 2z^{-1}X(z) = \frac{2z^{-1}+0.8z^{-2}}{(1+0.8z^{-1})^2}, \text{ ROC: } |z| > 0.8$$

(e) Solution:
Convolution, time shifting and folding:

$$Y(z) = X(z) \cdot z^{-2} X(1/z) = \frac{z^{-2}}{(1+0.8z)(1+0.8z^{-1})}, \quad \text{ROC: } 0.8 < |z| < 1.25$$

(f) Solution:
Linearity, time shifting and folding:

$$\begin{aligned} Y(z) &= z^2 X(z) + z^{-3} X(1/z) \\ &= \frac{z}{1+0.8z^{-1}} + \frac{z^{-2}}{1+0.8z}, \quad \text{ROC: } 0.8 < |z| < 1.25 \end{aligned}$$

48. (a) Solution:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=0}^{N-1} z^{-n} = \frac{1-z^{-N}}{1-z^{-1}}, \quad \text{ROC: } |z| > 0$$

(b) Solution:

$$y[n] = \sum_{m=-\infty}^{\infty} x[m]x[n-m] = \begin{cases} n+1, & 0 \le n \le N-1 \\ 2N-1-n, & N-1 \le n \le 2N-1-n \\ 0, & \text{elsewhere} \end{cases}$$

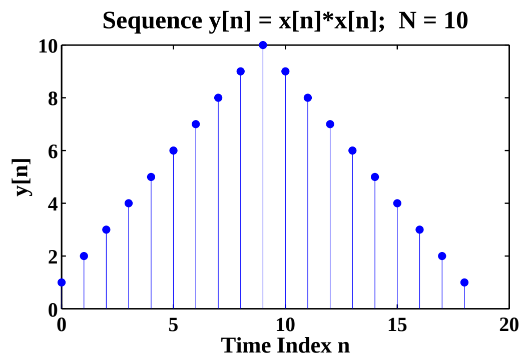


FIGURE 3.31: Sequence $y[n] = x[n] * x[n]$.

(c) Solution:

$$Y(z) = X(z)X(z) = \left(\frac{1-z^{-N}}{1-z^{-1}}\right)^2, \quad \text{ROC: } |z| > 0$$

49. (a) Proof:

$$X(z^{-1}) = \sum_{n=-\infty}^{\infty} x[n]z^{n} = \sum_{n=-\infty}^{\infty} x[-n]z^{-n} = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = X(z)$$

(b) Proof:

$$X(z^{-1}) = \sum_{n=-\infty}^{\infty} x[n]z^{n} = \sum_{n=-\infty}^{\infty} x[-n]z^{-n} = -\sum_{n=-\infty}^{\infty} x[n]z^{-n} = -X(z)$$

(c) Proof:

$$X(z^{-1}) = -X(z) \xrightarrow{z=1} X(1) = -X(1) \implies X(z)\,|_{z=1} = 0$$

50. Proof:
Assume the ROCs of $x_1[n]$ and $x_2[n]$ intersect, we have in the z domain

$$\sum_{n=-\infty}^{\infty} x_3[n]z^{-n} = \left(\sum_{n=-\infty}^{\infty} x_1[n]z^{-n}\right)\left(\sum_{n=-\infty}^{\infty} x_2[n]z^{-n}\right)$$

Suppose the ROC contains unit circle, otherwise the infinite sum is divergent, we conclude

$$\sum_{n=-\infty}^{\infty} x_3[n] = \left(\sum_{n=-\infty}^{\infty} x_1[n]\right)\left(\sum_{n=-\infty}^{\infty} x_2[n]\right)$$

51. Solution:

$$X(z) = \frac{A(1+z^{-1})}{(1-\frac{4}{5}z^{-1})(1-\frac{2}{3}z^{-1}+\frac{2}{9}z^{-2})}, \quad \text{ROC: } |z| > \frac{4}{5}$$

$$Y(z) = z^{-2}X(1/z) = \frac{z^{-2}A(1+z)}{(1-\frac{4}{5}z)(1-\frac{2}{3}z+\frac{2}{9}z^{2})}, \quad \text{ROC: } |z| < \frac{5}{4}$$

52. Solution:

$$H(z) = \frac{0.8z^{-1}}{(1-0.8z^{-1})^2}, \quad \text{ROC: } |z| > 0.8$$

$$X(z) = 1 + \frac{-1}{1-2z^{-1}}, \quad \text{ROC: } |z| < 2$$

$$\begin{aligned} Y(z) = H(z)X(z) &= \frac{35/9}{1-2z^{-1}} + \frac{-26/9 + 92/45z^{-1}}{(1-\frac{4}{5}z^{-1})^2} \\ &= \frac{35/9}{1-2z^{-1}} - \frac{\frac{26}{9}}{1-\frac{4}{5}z^{-1}} - \frac{1}{3}\frac{\frac{4}{5}z^{-1}}{(1-\frac{4}{5}z^{-1})^2} \quad \text{ROC: } 0.8 < |z| < 2 \end{aligned}$$

$$y[n] = -\left(\frac{35}{9}\right)\cdot 2^n u[-n-1] - \left(\frac{26}{9}\right)\left(\frac{4}{5}\right)^n u[n] - \left(\frac{1}{3}\right) n \left(\frac{4}{5}\right)^n u[n]$$

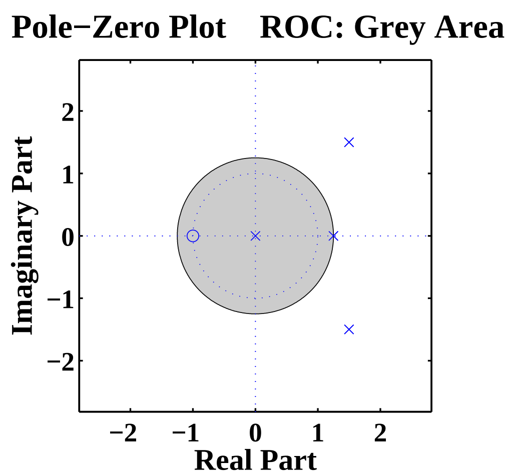


FIGURE 3.32: Pole-zero plot and ROC of $y[n] = x[-n+2]$.

53. (a) Solution:

$$H(z) = \frac{1}{1-\frac{1}{2}z^{-1}}, \quad \text{ROC: } |z| > \frac{1}{2}$$

$$X(z) = 1 + \frac{-1}{1-2z^{-1}}, \quad \text{ROC: } |z| < 2$$

$$Y(z) = H(z)X(z) = \frac{-4/5}{1-2z^{-1}} + \frac{4/5}{1-\frac{1}{2}z^{-1}}, \quad \text{ROC: } \frac{1}{2} < |z| < 2$$

$$y[n] = \left(\frac{4}{5}\right) n2^n u[-n-1] + \left(\frac{4}{5}\right)\left(\frac{1}{2}\right)^n u[n]$$

(b) Solution:

$$H(z) = \frac{1}{1-\frac{1}{3}z^{-1}} + \frac{-1}{1-3z^{-1}}, \quad \text{ROC: } \frac{1}{3} < |z| < 3$$

$$X(z) = \frac{1}{1-\frac{1}{2}z^{-1}}, \quad \text{ROC: } |z| > \frac{1}{2}$$

$$Y(z) = H(z)X(z) = \frac{3/20}{1-3z^{-1}} + \frac{13/5}{1-\frac{1}{2}z^{-1}} + \frac{-7/4}{1-\frac{1}{3}z^{-1}}, \quad \text{ROC: } \frac{1}{2} < |z| < 3$$

$$y[n] = -\left(\frac{3}{20}\right)\cdot 3^n u[-n-1] + \left(\frac{13}{5}\right)\left(\frac{1}{2}\right)^n u[n] - \left(\frac{7}{4}\right)\left(\frac{1}{3}\right)^n u[n]$$

(c) Solution:

$$H(z) = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{1-2z^{-1}}, \quad \text{ROC: } \frac{1}{2} < |z| < 2$$

$$X(z) = \frac{1}{1-\frac{1}{4}z^{-1}} + \frac{1}{1-4z^{-1}}, \quad \text{ROC: } \frac{1}{4} < |z| < 4$$

$$\begin{aligned} Y(z) = H(z)X(z) = & \frac{-2/35}{1-4z^{-1}} + \frac{-3/7}{1-2z^{-1}} + \frac{20/7}{1-\frac{1}{2}z^{-1}} \\ & + \frac{-48/35}{1-\frac{1}{4}z^{-1}}, \quad \text{ROC: } \frac{1}{2} < |z| < 2 \end{aligned}$$

$$\begin{aligned} y[n] = & \left(\frac{2}{35}\right) \cdot 4^n u[-n-1] + \left(\frac{3}{7}\right) \cdot 2^n u[-n-1] \\ & + \left(\frac{20}{7}\right)\left(\frac{1}{2}\right)^n u[n] + \left(-\frac{48}{35}\right)\left(\frac{1}{4}\right)^n u[n] \end{aligned}$$

54. (a) Solution:

$$X(z) = \frac{-1}{1-bz^{-1}}, \quad \text{ROC: } |z| < |b|$$

$$X(z^{-1}) = \frac{-1}{1-bz}, \quad \text{ROC: } |z| > |b|^{-1}$$

$$R_{xx}(z) = X(z)X(1/z) = \frac{-b^{-1}z^{-1}}{(1-bz^{-1})(1-b^{-1}z^{-1})}, \quad \text{ROC: } |b|^{-1} < |z| < |b|$$

(b) See plot below.

(c) Solution:

$$R_{xx}(z) = \frac{\frac{-b^{-2}}{1-b^{-2}}}{1-bz^{-1}} + \frac{\frac{b^{-2}}{1-b^{-2}}}{1-b^{-1}z^{-1}}, \quad \text{ROC: } |b|^{-1} < |z| < |b|$$

$$\begin{aligned} r_{xx}[\ell] &= \left(\frac{b^{-2}}{1-b^{-2}}\right) b^{\ell} u[-\ell-1] + \left(\frac{b^{-2}}{1-b^{-2}}\right) b^{-\ell} u[\ell] \\ &= \left(\frac{b^{-2}}{1-b^{-2}}\right) b^{-|\ell|} \end{aligned}$$

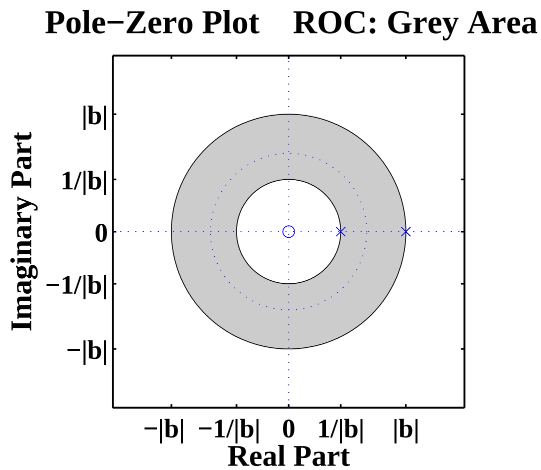


FIGURE 3.33: Pole-zero plot and ROC of $R_{xx}(z)$.

55. (a) Solution:

$$X(z) = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{-1}{1-3z^{-1}}, \quad \text{ROC: } \frac{1}{2} < |z| < 3$$

$$X(z^{-1}) = \frac{1}{1-\frac{1}{2}z} + \frac{-1}{1-3z}, \quad \text{ROC: } \frac{1}{3} < |z| < 2$$

$$R_{xx}(z) = X(z)X(1/z) = \frac{-\frac{5}{2}z^{-1}}{1-\frac{7}{2}z^{-1}+\frac{3}{2}z^{-2}} \cdot \frac{-\frac{5}{3}z^{-1}}{1-\frac{7}{3}z^{-1}+\frac{2}{3}z^{-2}},$$

$$\text{ROC: } \frac{1}{2} < |z| < 2$$

(b) See plot below.

(c) Solution:

$$R_{xx}(z) = \frac{15/8}{1-3z^{-1}} + \frac{-10/3}{1-2z^{-1}} + \frac{10/3}{1-\frac{1}{2}z^{-1}} + \frac{-15/8}{1-\frac{1}{3}z^{-1}}$$

$$\begin{aligned} r_{xx}[\ell] &= \left(-\frac{15}{8}\right) \cdot 3^{\ell}u[-\ell-1] + \left(\frac{10}{3}\right) \cdot 2^{\ell}u[-\ell-1] \\ &\quad + \left(\frac{10}{3}\right) \cdot 2^{-\ell}u[\ell] + \left(-\frac{15}{8}\right) \cdot 3^{-\ell}u[\ell] \\ &= \left(-\frac{15}{8}\right) \cdot 3^{-|\ell|} + \left(\frac{10}{3}\right) \cdot 2^{-|\ell|} \end{aligned}$$

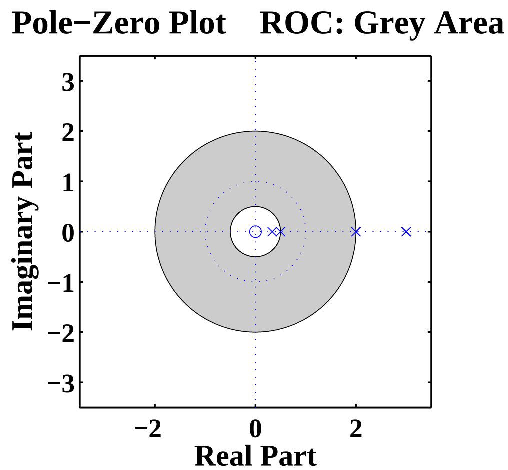


FIGURE 3.34: Pole-zero plot and ROC of $R_{xx}(z)$.

56. (a) Solution:

$$X(z) = \frac{1}{1-0.9\mathrm{e}^{\mathrm{j}\pi/3}z^{-1}}, \quad \text{ROC: } |z| > 0.9$$

$$X^*(1/z^*) = \frac{1}{1-0.9\mathrm{e}^{-\mathrm{j}\pi/3}z} = \frac{-\frac{10}{9}\mathrm{e}^{\mathrm{j}\pi/3}z^{-1}}{1-\frac{10}{9}\mathrm{e}^{\mathrm{j}\pi/3}z^{-1}}, \quad \text{ROC: } |z| < \frac{10}{9}$$

$$R_{xx}(z) = X(z)X^*(1/z^*) = \frac{-\frac{10}{9}\mathrm{e}^{\mathrm{j}\pi/3}z^{-1}}{(1-0.9\mathrm{e}^{\mathrm{j}\pi/3}z^{-1})(1-\frac{10}{9}\mathrm{e}^{\mathrm{j}\pi/3}z^{-1})},$$
$$\text{ROC: } \frac{9}{10} < |z| < \frac{10}{9}$$

(b) See plot below.

(c) Determine the autocorrelation $r_{xx}[\ell]$.
Solution:

$$R_{xx}(z) = \frac{-100/19}{1-\frac{10}{9}\mathrm{e}^{\mathrm{j}\pi/3}z^{-1}} + \frac{100/19}{1-\frac{9}{10}\mathrm{e}^{\mathrm{j}\pi/3}z^{-1}}, \quad \text{ROC: } \frac{9}{10} < |z| < \frac{10}{9}$$

$$r_{xx}[\ell] = \left(\frac{100}{19}\right)\left(\frac{10}{9}\mathrm{e}^{\mathrm{j}\pi/3}\right)^{\ell} u[-\ell-1] + \left(\frac{100}{19}\right)\left(\frac{9}{10}\mathrm{e}^{\mathrm{j}\pi/3}\right)^{\ell} u[\ell]$$

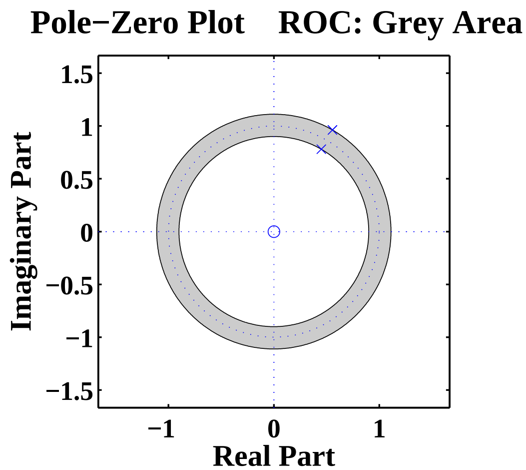


FIGURE 3.35: Pole-zero plot and ROC of $R_{xx}(z)$.

57. Solution:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{2}{1 + \frac{11}{6}z^{-1} + \frac{1}{2}z^{-2}} = \frac{18/7}{1 + \frac{3}{2}z^{-1}} + \frac{-4/7}{1 + \frac{1}{3}z^{-1}}$$

If ROC $|z| > \frac{3}{2}$,

$$h[n] = \left(\frac{18}{7}\right)\left(-\frac{3}{2}\right)^n u[n] + \left(-\frac{4}{7}\right)\left(-\frac{1}{3}\right)^n u[n]$$

If ROC $|z| < \frac{1}{3}$,

$$h[n] = \left(-\frac{18}{7}\right)\left(-\frac{3}{2}\right)^n u[-n-1] + \left(\frac{4}{7}\right)\left(-\frac{1}{3}\right)^n u[-n-1]$$

If ROC $\frac{1}{3} < |z| < \frac{3}{2}$,

$$h[n] = \left(-\frac{18}{7}\right)\left(-\frac{3}{2}\right)^n u[-n-1] + \left(-\frac{4}{7}\right)\left(-\frac{1}{3}\right)^n u[n]$$

58. Solution:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 + \frac{4}{5}z^{-1}}, \quad \text{ROC: } |z| > \frac{4}{5}$$

(a) $x[n] = (1+\mathrm{j})^n, \quad -\infty < n < \infty$
Solution:

$$y[n] = (1+\mathrm{j})^n H(z)\mid_{z=1+\mathrm{j}} = (0.6604 + 0.1887\mathrm{j})(1+\mathrm{j})^n$$

which is also steady-state response.

(b) $x[n] = \cos(\pi n/4)u[n]$
Solution:

$$X(z) = \frac{1-\cos(\pi/4)z^{-1}}{1-2\cos(\pi/4)z^{-1}+z^{-2}}, \quad \text{ROC: } |z| > 1$$

$$Y(z) = H(z)X(z) = \frac{1}{1+\frac{4}{5}z^{-1}} \cdot \frac{1-\cos(\pi/4)z^{-1}}{1-2\cos(\pi/4)z^{-1}+z^{-2}}, \quad \text{ROC: } |z| > 1$$

$$y[n] = 0.4351 \cdot \left(-\frac{4}{5}\right)^n u[n] + 2 * 0.3003\cos(0.7854n + 0.3467)u[n]$$

where

$$y_{tr}[n] = 0.4351 \cdot \left(-\frac{4}{5}\right)^n u[n]$$

$$y_{ss}[n] = 2 * 0.3003\cos(0.7854n + 0.3467)u[n]$$

(c) $x[n] = (-1)^n, \quad -\infty < n < \infty$
Solution:

$$y[n] = (-1)^n H(z)\mid_{z=-1} = 5 \cdot (-1)^n$$

which is also steady-state response.

(d) $x[n] = (-1)^n u[n]$
Solution:

$$X(z) = \frac{1}{1+z^{-1}}, \quad \text{ROC: } |z| > 1$$

$$Y(z) = H(z)X(z) = \frac{-4}{1+\frac{4}{5}z^{-1}} + \frac{5}{1+z^{-1}}, \quad \text{ROC: } |z| > 1$$

$$y[n] = -4 \cdot \left(-\frac{4}{5}\right)^n u[n] + 5 \cdot (-1)^n u[n]$$

where

$$y_{tr}[n] = -4 \cdot \left(-\frac{4}{5}\right)^n u[n]$$

$$y_{ss}[n] = 5 \cdot (-1)^n u[n]$$

59. (a) Solution:

$$X(z) = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{-1}{1-2z^{-1}} = \frac{-\frac{3}{2}z^{-1}}{1-\frac{5}{2}z^{-1}+z^{-2}}, \quad \text{ROC: } \frac{1}{2} < |z| < 2$$

$$Y(z) = \frac{3}{1-0.7z^{-1}}, \quad \text{ROC: } |z| > 0.7$$

$$H(z) = \frac{Y(z)}{X(z)} = -2z + \frac{20}{7} + \frac{26/35}{1-0.7z^{-1}}, \quad \text{ROC: } |z| > 0.7$$

$$h[n] = -2\delta[n+1] + \frac{20}{7}\delta[n] + \frac{26}{35}(0.7)^n u[n]$$

(b) Solution:

$$X(z) = 1 + \frac{-1}{1-0.9z^{-1}}, \quad \text{ROC: } |z| < 0.9$$

$$y[n] = -\frac{20}{9}\delta[n+1] + \frac{20}{7}\delta[n] + \left(\frac{117}{35}\right)(0.9)^n u[-n-1] + \left(\frac{117}{35}\right)(0.7)^n u[n]$$

60. (a) Solution:

$$X(z) = \frac{1}{1-\frac{1}{3}z^{-1}} + \frac{-1}{1-3z^{-1}} = \frac{-\frac{8}{3}z^{-1}}{1-\frac{10}{3}z^{-1}+z^{-2}}, \quad \text{ROC: } \frac{1}{3} < |z| < 3$$

$$\begin{aligned} Y(z) &= \frac{2}{1-\frac{1}{3}z^{-1}} - 4 + \frac{4}{1-4z^{-1}} \\ &= \frac{2+8z^{-1}-\frac{16}{3}z^{-2}}{(1-\frac{1}{3}z^{-1})(1-4z^{-1})}, \quad \text{ROC: } \frac{1}{3} < |z| < 4 \end{aligned}$$

$$H(z) = \frac{Y(z)}{X(z)} = -\frac{3}{4}z - \frac{19}{8} + \frac{3}{2}z^{-1} + \frac{-\frac{5}{4}}{1-4z^{-1}}, \quad \text{ROC: } |z| < 4$$

$$h[n] = -\frac{3}{4}\delta[n+1] - \frac{19}{8}\delta[n] + \frac{3}{2}\delta[n-1] + \left(\frac{5}{4}\right) \cdot 4^n u[-n-1]$$

(b) Solution:

$$X(z) = \frac{1}{1-\frac{3}{4}z^{-1}}, \quad \text{ROC: } |z| > \frac{3}{4}$$

$$Y(z) = \frac{2-\frac{19}{4}z^{-1}}{(1-\frac{3}{4}z^{-1})(1-4z^{-1})}, \quad \text{ROC: } \frac{3}{4} < |z| < 4$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{19}{16} + \frac{\frac{13}{16}}{1 - 4z^{-1}}, \quad \text{ROC: } |z| < 4$$

$$h[n] = \frac{19}{16}\delta[n] + \left(-\frac{13}{16}\right) \cdot 4^n u[-n-1]$$

61. (a) Solution:

$$X(z) = \frac{-\frac{7}{6} + \frac{31}{18}z^{-1}}{1 - \frac{10}{3}z^{-1} + z^{-2}}, \quad \text{ROC: } \frac{1}{3} < |z| < 3$$

(b) Soluton:

$$H(z) = \frac{Y(z)}{X(z)} = -\frac{\frac{20}{11}}{1 - 2z^{-1}} + \frac{\frac{759}{260}}{1 - \frac{31}{21}z^{-1}} - \frac{\frac{10}{41}}{1 - \frac{1}{2}z^{-1}}$$

If ROC $|z| > 2$,

$$h[n] = -\frac{20}{11} \cdot 2^n u[n] + \frac{759}{260}\left(\frac{31}{21}\right)^n u[n] - \frac{10}{41}\left(\frac{1}{2}\right)^n u[n]$$

If ROC $\frac{31}{21} < |z| < 2$,

$$h[n] = \frac{20}{11} \cdot 2^n u[-n-1] + \frac{759}{260}\left(\frac{31}{21}\right)^n u[n] - \frac{10}{41}\left(\frac{1}{2}\right)^n u[n]$$

If ROC $\frac{1}{2} < |z| < \frac{31}{21}$,

$$h[n] = \frac{20}{11} \cdot 2^n u[-n-1] - \frac{759}{260}\left(\frac{31}{21}\right)^n u[-n-1] - \frac{10}{41}\left(\frac{1}{2}\right)^n u[n]$$

If ROC $|z| < \frac{1}{2}$,

$$h[n] = \frac{20}{11} \cdot 2^n u[-n-1] - \frac{759}{260}\left(\frac{31}{21}\right)^n u[-n-1] + \frac{10}{41}\left(\frac{1}{2}\right)^n u[-n-1]$$

62. (a) Solution:

$$\begin{aligned} H(z) &= \frac{A(1 + \mathrm{j}z^{-1})(1 - \mathrm{j}z^{-1})}{(1 - \frac{-1+\mathrm{j}}{2}z^{-1})(1 - \frac{-1-\mathrm{j}}{2}z^{-1})} \\ &= \frac{A(1 + z^{-2})}{1 - z^{-1} + \frac{1}{2}z^{-2}}, \quad \text{ROC: } |z| > \frac{\sqrt{2}}{2} \end{aligned}$$

$$H(1) = 0.8 \implies A = \frac{1}{5}$$

(b) Solution:

$$y[n] = \frac{1}{5}(x[n] + x[n-2]) + y[n-1] + \frac{1}{2}y[n-2]$$

(c) Solution:

$$X(z) = \frac{1}{\sqrt{2}}\frac{z^{-1}}{1+z^{-2}}, \quad \text{ROC: } |z| > 1$$

$$Y(z) = H(z)X(z) = \frac{1}{5\sqrt{2}}\frac{z^{-1}}{1-z^{-1}+\frac{1}{2}z^{-2}}, \quad \text{ROC: } |z| > \frac{\sqrt{2}}{2}$$

$$y[n] = 2 \cdot 0.1414\cos(0.7854n - 1.5708)(0.7071)^n u[n]$$

which is also its transient response.

63. (a) Proof:

Applying one-sided z-transform to the difference equation:

$$Y^+(z) = 2\cos(\omega_0)(y[-1]+z^{-1}Y^+(z)) - (y[-2]+y[-1]z^{-1}+z^(-2)Y^+(z))$$

$$Y^+(z) = \frac{A\sin(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$$

$$y[n] = A\sin(n\omega_0)u[n]$$

(b) MATLAB script:

```
% P0363: Verify the operation of digital oscillator
close all; clc
n=0:100;
A = 2;
w0 = 0.1*pi;
yn1 = A*sin(n*w0);
b = [0 A*sin(w0)];
a = [1 -2*cos(w0) 1];
yn2 = filter(b,a,[1 zeros(1,length(n)-1)]);
% Plot:
hf = figconfg('P0363');
subplot(211)
stem(n,yn1,'filled')
ylabel('y[n]','fontsize',LFS)
title('System Response by Analytical Expression','fontsize',TFS)
subplot(212)
stem(n,yn2,'filled')
```

```
ylim([min(yn2) max(yn2)])
ylabel('y[n]','fontsize',LFS)
title('System Response by Z-Transform','fontsize',TFS)
```

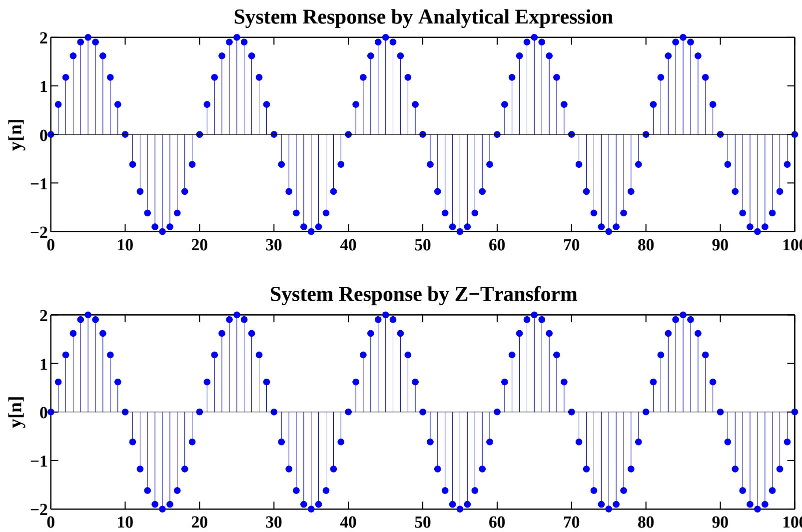


FIGURE 3.36: MATLAB verification of the operation of the above digital oscillator.

Review Problems

64. (a) Solution:

$$Y(z) = \frac{1}{1-0.9z^{-1}}, \quad \text{ROC: } |z| > 0.9$$

$$X(z) = \frac{Y(z)}{H(z)} = \frac{1-\frac{1}{2}z^{-1}}{(1-0.9z^{-1})(1-2z^{-1})} = \frac{15/11}{1-2z^{-1}} + \frac{-4/11}{1-0.9z^{-1}}$$

If ROC $|z| > 2$

$$x[n] = \frac{15}{11} \cdot 2^n u[n] - \frac{4}{11} \cdot 0.9^n u[n]$$

If ROC $0.9 < |z| < 2$

$$x[n] = -\frac{15}{11} \cdot 2^n u[-n-1] - \frac{4}{11} \cdot 0.9^n u[n]$$

(b) Solution:

$$x[n] = -\frac{15}{11} \cdot 2^n u[-n-1] - \frac{4}{11} \cdot 0.9^n u[n]$$

(c) Solution:

$$H(z) = \frac{X(z)}{Y(z)} = \frac{1-\frac{1}{2}z^{-1}}{1-2z^{-1}} = \frac{1}{4} + \frac{\frac{3}{4}}{1-2z^{-1}}, \quad \text{ROC: } |z| < 2$$

$$h[n] = \frac{1}{4}\delta[n] - \frac{3}{4} \cdot 2^n u[-n-1]$$

65. (a) Solution:

$$H(z) = \frac{1+z^{-1}}{1-0.5z^{-1}} = -2 + \frac{3}{1-0.5z^{-1}}, \quad \text{ROC: } |z| > 0.5$$

(i) the impulse response

$$h[n] = -2\delta[n] + 3 \cdot 0.5^n u[n]$$

(ii) difference equation

$$y[n] = 0.5y[n-1] + x[n] + x[n-1]$$

(iii) pole-zero plot

(iv) output $y[n]$

$$Y(z) = H(z)X(z) = \frac{6}{1-0.5z^{-1}} + \frac{-5}{1-\frac{1}{4}z^{-1}}, \quad \text{ROC: } |z| > \frac{1}{2}$$

$$y[n] = 6 \cdot 0.5^n u[n] - 5 \cdot \left(\frac{1}{4}\right)^n u[n]$$

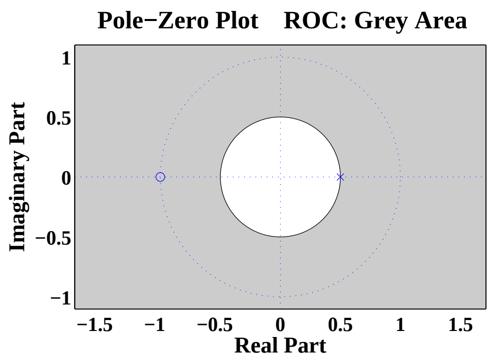


FIGURE 3.37: Pole-zero plot and ROC of $H(z) = (z+1)/(z-0.5)$, causal system.

(b) Solution:

$$H(z) = -4 + \frac{0.9348}{1+0.809z^{-1}} + \frac{4.0652}{1-0.309z^{-1}}, \quad \text{ROC: } |z| > 0.809$$

(i) the impulse response

$$h[n] = -4\delta[n] + 0.9348 \cdot (-0.809)^n u[n] + 4.0652 \cdot 0.309^n u[n]$$

(ii) difference equation

$$y[n] = -0.5y[n-1] + 0.25y[n-2] + x[n] + x[n-1] + x[n-2]$$

(iii) pole-zero plot

(iv) output $y[n]$

$$Y(z) = H(z)X(z) = \frac{0.7141}{1+0.809z^{-1}} + \frac{21.2859}{1-0.309z^{-1}} + \frac{-21}{1-\frac{1}{4}z^{-1}}, \quad \text{ROC: } |z| > 0.809$$

$$y[n] = 0.7141 \cdot (-0.809)^n u[n] + 21.2859 \cdot 0.309^n u[n] - 21 \cdot \left(\frac{1}{4}\right)^n u[n]$$

(c) Solution:

$$H(z) = -\frac{1}{9} + \frac{10/9}{1-3z^{-1}} + \frac{8}{9}\frac{3z^{-1}}{(1-3z^{-1})^2}, \quad \text{ROC: } |z| < 3$$

(i) the impulse response

$$h[n] = -\frac{1}{9}\delta[n] - \frac{10}{9} \cdot 3^n u[-n-1] - \frac{8}{9} \cdot n3^n u[-n-1]$$

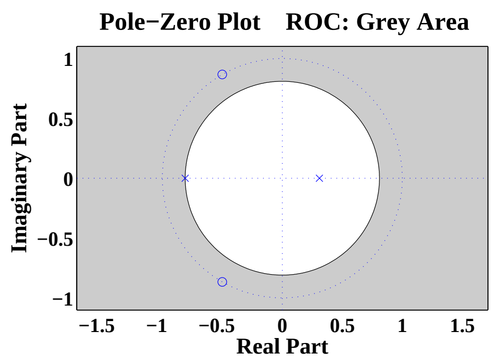


FIGURE 3.38: Pole-zero plot and ROC of $H(z) = \dfrac{1 + z^{-1} + z^{-2}}{1 + 0.5z^{-1} - 0.25z^{-2}}$, stable system.

(ii) difference equation

$$y[n] = 6y[n-1] - 9y[n-2] + x[n] - x[n-2]$$

(iii) pole-zero plot

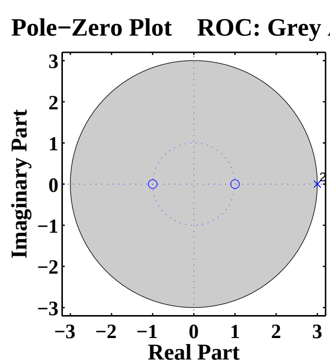


FIGURE 3.39: Pole-zero plot and ROC of $H(z) = (z^2 - 1)/(z - 3)^2$, anticausal system.

(iv) output $y[n]$

$$Y(z) = H(z)X(z) = \frac{1.124}{1 - 3z^{-1}} + 0.9697\frac{3z^{-1}}{(1 - 3z^{-1})^2} + \frac{-0.124}{1 - \frac{1}{4}z^{-1}},$$

$$\text{ROC: } \frac{1}{4} < |z| < 3$$

$$y[n] = -1.124 \cdot 3^n u[-n-1] - 0.9697 \cdot n \cdot 3^n u[-n-1] - 0.124 \cdot \left(\frac{1}{4}\right)^n u[n]$$

(d) Solution:

$$H(z) = 1 + 2z^{-1} + 3z^{-2} + 2z^{-3} + z^{-4}, \quad \text{ROC: } |z| > 0$$

(i) the impulse response

$$h[n] = \delta[n] + 2\delta[n-1] + 3\delta[n-2] + 2\delta[n-3] + \delta[n-4]$$

(ii) difference equation

$$y[n] = x[n] + 2x[n-1] + 3x[n-2] + 2x[n-3] + x[n-4]$$

(iii) pole-zero plot

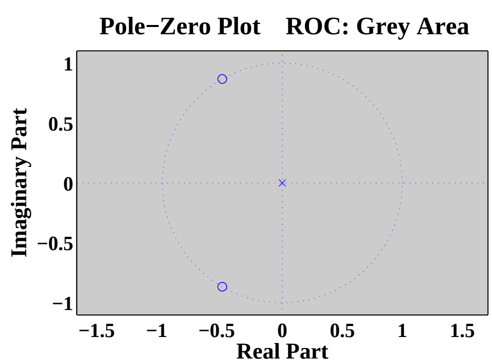


FIGURE 3.40: Pole-zero plot and ROC of $H(z) = (1+z^{-1}+z^{-2})^2$, causal system.

(iv) output $y[n]$

$$Y(z) = H(z)X(z) = -440 - 108z^{-1} - 24z^{-2} - 4z^{-3} + \frac{441}{1 - \frac{1}{4}z^{-1}},$$

$$\text{ROC: } |z| > \frac{1}{4}$$

$$y[n] = -440\delta[n] - 108\delta[n-1] - 24\delta[n-2] - 4\delta[n-3] + 441 \cdot \left(\frac{1}{4}\right)^n u[n]$$