CHAPTER 5:

DISCRETE FOURIER TRANSFORM (DFT)

Lecture 9: DFT and Inverse DFT

Lecture 10: Fast Fourier Transform (FFT)

Duration: 4 hrs

Lecture 9

DFT and Inverse DFT

- **Duration:** **2** hrs
- **Outline:**

1. Review of DTFT of DT periodic signals

2. DFT and Inverse DFT

3. Frequency resolution

4. DFT properties

Procedure to calculate DTFT of periodic signals

Step 1:

Start with $\mathbf{x_0(n)}$ – one period of $\mathbf{x(n)}$**,** with zero everywhere else

Step 2:

Find the DTFT $\mathbf{X_0(\Omega)}$ of the signal $x_0[n]$ above

Step 3:

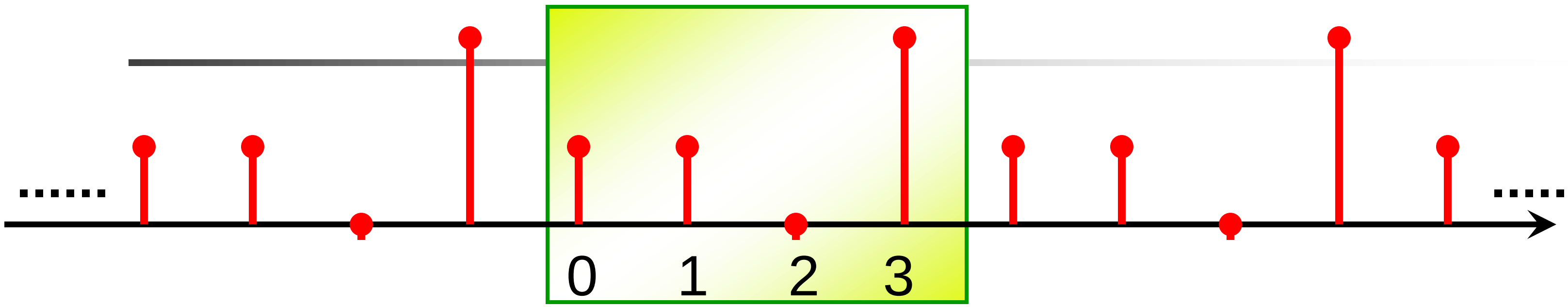
Find $X_0(\Omega)$ at **N** equally spacing frequency points $\mathbf{X_0(k2\pi/N)}$

Step 4:

Obtain the DTFT of x(n):

$$\boxed{X(\Omega) = \frac{2\pi}{N} \sum_k X_0(k\frac{2\pi}{N})\delta(\Omega - k\frac{2\pi}{N})}$$

Example of calculating DTFT of periodic signals



$$X_0(\Omega) = \sum_{n=0}^{3} x_0(n) e^{-j\Omega n} = 1 + e^{-j\Omega} + 2e^{-j3\Omega}$$

$$X_o\left(\frac{2\pi k}{4}\right) = 1 + e^{-j\frac{2\pi k}{4}} + 2e^{-3j\frac{2\pi k}{4}} \qquad k = 0, 1, 2, 3$$

k = 0 → $X_0(0) = 4$; k = 1 → $X_0(1) = 1+j$

k = 2 → $X_0(2) = -2$; k = 3 → $X_0(3) = 1-j$

Example (cont)

$$X(\Omega) = \frac{2\pi}{N} \sum_{k=-\infty}^{\infty} X_o(\frac{2\pi k}{N})\delta(\Omega - \frac{2\pi k}{N})$$

$$= \frac{2\pi}{4} \sum_{k=-\infty}^{\infty} X_o(\frac{2\pi k}{4})\delta(\Omega - \frac{2\pi k}{4})$$

For one period $0 \le \Omega < 2\pi$

$$\frac{\pi}{2}\left\{4\delta(\Omega) + (1+j)\delta(\Omega - \frac{\pi}{2}) - 2\delta(\Omega - \pi) + (1-j)\delta(\Omega - \frac{3\pi}{2})\right\}$$

Lecture 9

DFT and Inverse DFT

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4. Applications

DFT to the rescue!

Could we calculate the **frequency spectrum** of a signal using a **digital computer** with **CTFT/DTFT**?

- ➢ Both CTFT and DTFT produce continuous function of frequency → can't calculate an infinite continuum of frequencies using a computer
- ➢ Most real-world data is not in the simple form such as $a^n u(n)$

DFT can be used as a FT approximation that can calculate a **finite set of discrete-frequency spectrum values** from a finite set of discrete-time samples of an analog signal

Building the DFT formula

Continuous time signal **x(t)**

sample

Discrete time signal **x(n)**

window

Discrete time signal **x_0(n)**

Finite length

"Window" x(n) is like multiplying the signal by the finite length rectangular window

$$w_R = \begin{cases} 1 & n = 0, 1, \cdots N-1 \\ 0, & otherwise \end{cases}$$

$$\boxed{x_o[n] = x[n]\, w_R[n]}$$

Building the DFT formula (cont)

Continuous time signal **x(t)**

sample

Discrete time signal **x(n)**

window

Discrete time signal $\mathbf{x_0(n)}$

Finite length

DTFT

$$X_0(\Omega) = \sum_{n=-\infty}^{\infty} x_0[n] e^{-j\Omega n} = \sum_{n=0}^{N-1} x_0[n] e^{-j\Omega n}$$

Discrete Time Fourier Transform (DTFT), $X_o(\Omega)$

(periodic over $[0, 2\pi)$)

Building the DFT formula (cont)

Continuous time signal **$x(t)$**

sample

Discrete time signal **$x(n)$**

window

Discrete time signal **$x_0(n)$**

Finite length

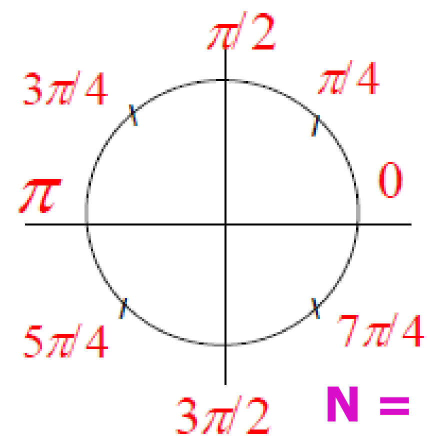
DFT

DTFT

Discrete Fourier Transform DFT **$X(k)$**

Discrete + periodic with period **N**

Sample at N values around the unit circle



N = 8

$X_0(\Omega)$

Continuous + periodic with period 2π

DFT and inverse DFT formulas

Notation: $W_N = e^{-j\frac{2\pi}{N}}$

$$X[k] = \sum_{n=0}^{N-1} x[n]\, W_N^{kn}$$

k = 0,1,…,N-1

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]\, W_N^{-kn}$$

n = 0,1,…,N-1

Note that the DFT is a sequence of N numbers (in the frequency domain), just like $x_o[n]$ is a sequence of N numbers in the time domain

You only have to store N points

Examples of calculation DFT and IDFT

Ex.1. Find the DFT of x(n) = 1, n = 0, 1, 2, ..., (N-1)

$$X[k] = \sum_{n=0}^{N-1} x[n]\, W_N^{kn} = \sum_{n=0}^{N-1} W^{kn} = \frac{1-W^{kN}}{1-W^{k}}$$

k = 0,1,...,N-1

k = 0 → X(k) = X(0) = N

k ≠ 0 → X(k) = 0

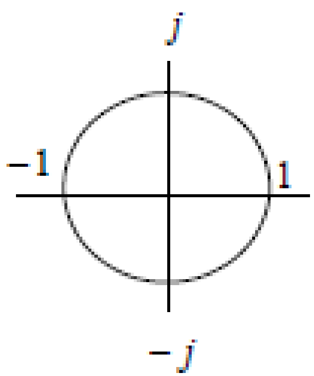
$$\Rightarrow \boxed{X[k] = N\delta[k]}$$

Examples of calculation DFT and IDFT

Ex.2. Given y(n) = δ(n-2) and N = 8, find Y(k)

$$Y[k] = \sum_{n=0}^{N-1} y[n] W_N^{kn} = \sum_{n=0}^{7} \delta[n-2] e^{-j2\pi kn/N}$$

Using N=8



$$= e^{-j4\pi k/8} = e^{-j\pi k/2} = (-j)^k$$

$$= [1, -j, -1, j, 1, -j, -1, j]$$

Examples of calculation DFT and IDFT

Ex.3. Find the IDFT of X(k) = 1, k = 0, 1, ..., 7.

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, W_N^{-kn}$$

$$\boxed{x[n] = \frac{1}{8}\sum_{k=0}^{7} W_8^{-kn} = \tfrac{1}{8} N\delta[n] = \delta[n]}$$

Examples of calculation DFT and IDFT

Ex.4. Given x(n) = δ(n) + 2δ(n-1) +3δ(n-2) + δ(n-3) and N = 4. Find X(k).

$$X[k] = 1 + 2e^{-j\pi k/2} + 3e^{-j\pi k} + e^{-j3\pi k/2}$$

$$= 1 + 2(-j)^k + 3(-1)^k + (j)^k$$

$$X[0] = 7$$

$$X[1] = 1 - 2j - 3 + j = -2 - j$$

$$X[2] = 1 - 2 + 3 - 1 = 1$$

$$X[3] = 1 + 2j - 3 - j = -2 + j$$

- x = [1 2 3 1];
- >> X = fft(x)

- X = 7.0000 -2.0000 - 1.0000i 1.0000 -2.0000 + 1.0000i
- 7.0000 1.7071 - 5.1213i -2.0000 - 1.0000i 0.2929 + 0.8787i

- Columns 5 through 8

- 1.0000 0.2929 - 0.8787i -2.0000 + 1.0000i 1.7071 + 5.1213i

Lecture 9

DFT and Inverse DFT

- **Duration:** **2** hrs
- **Outline:**

1. Review of DTFT of DT periodic signals

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3. Frequency resolution

4. DFT properties

Frequency resolution of the DFT

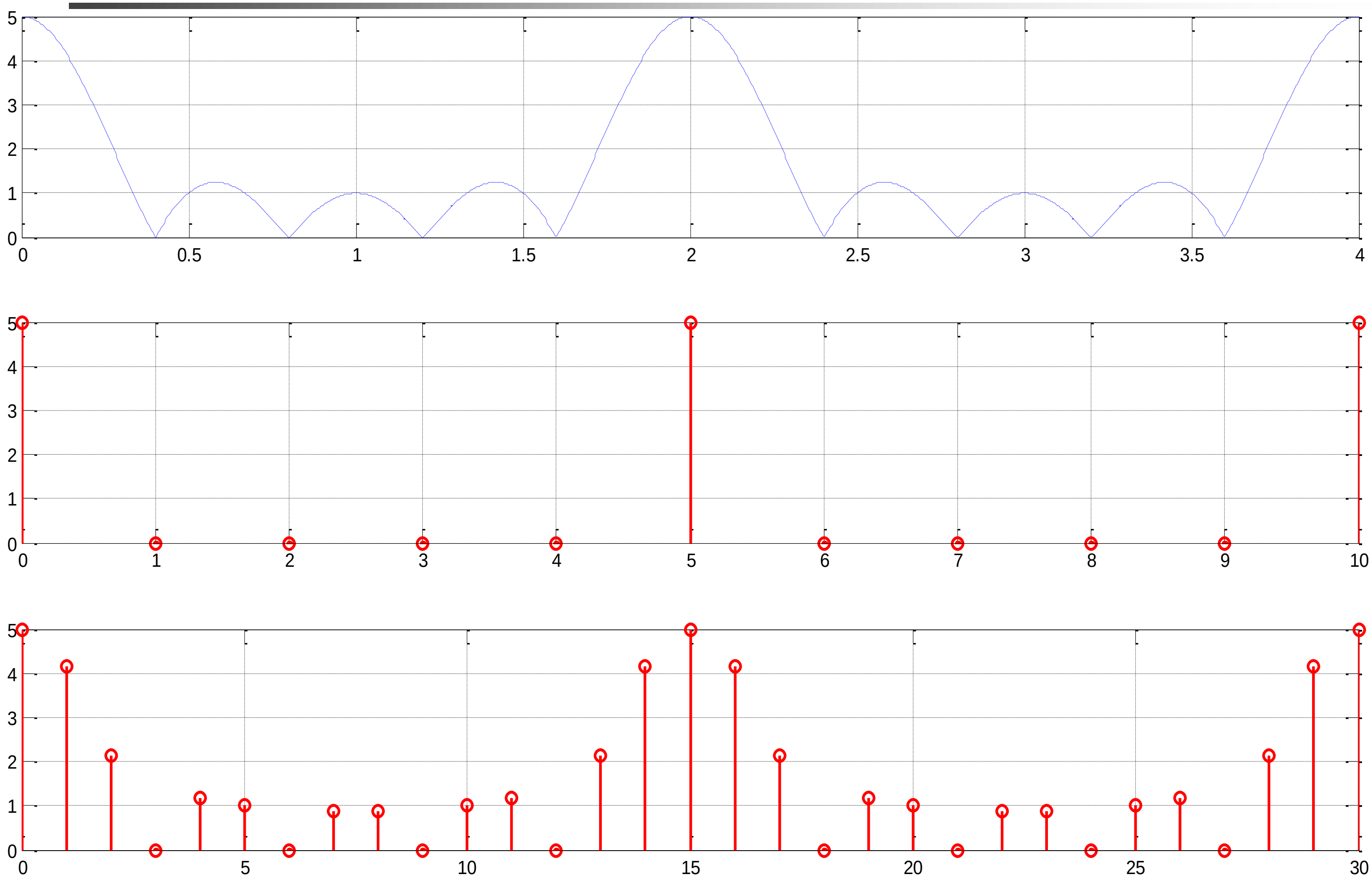
Discrete frequency spectrum computed from DFT has the spacing between frequency samples of:

$$\boxed{\Delta f = \frac{f_s}{N}} \qquad \boxed{\Delta\Omega = \frac{2\pi}{N}}$$

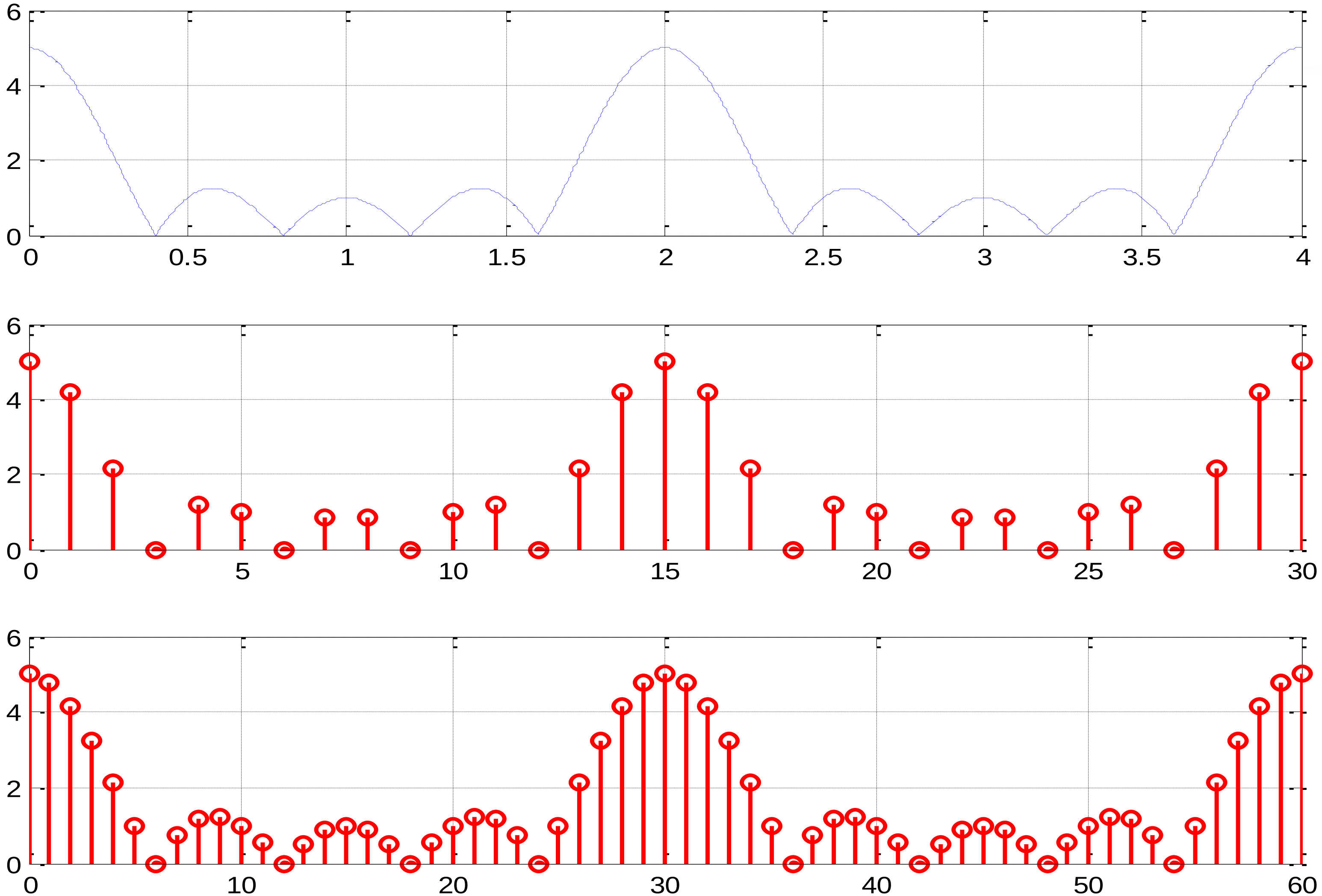
→ The choice of N determines the resolution of the frequency spectrum, or vice-versa

→ To obtain the adequate resolution, some zeros can be appended to the signal (*zero padding*)

Examples of N = 5 and N = 15



Examples of N = 15 and N = 30



Lecture 9

DFT and Inverse DFT

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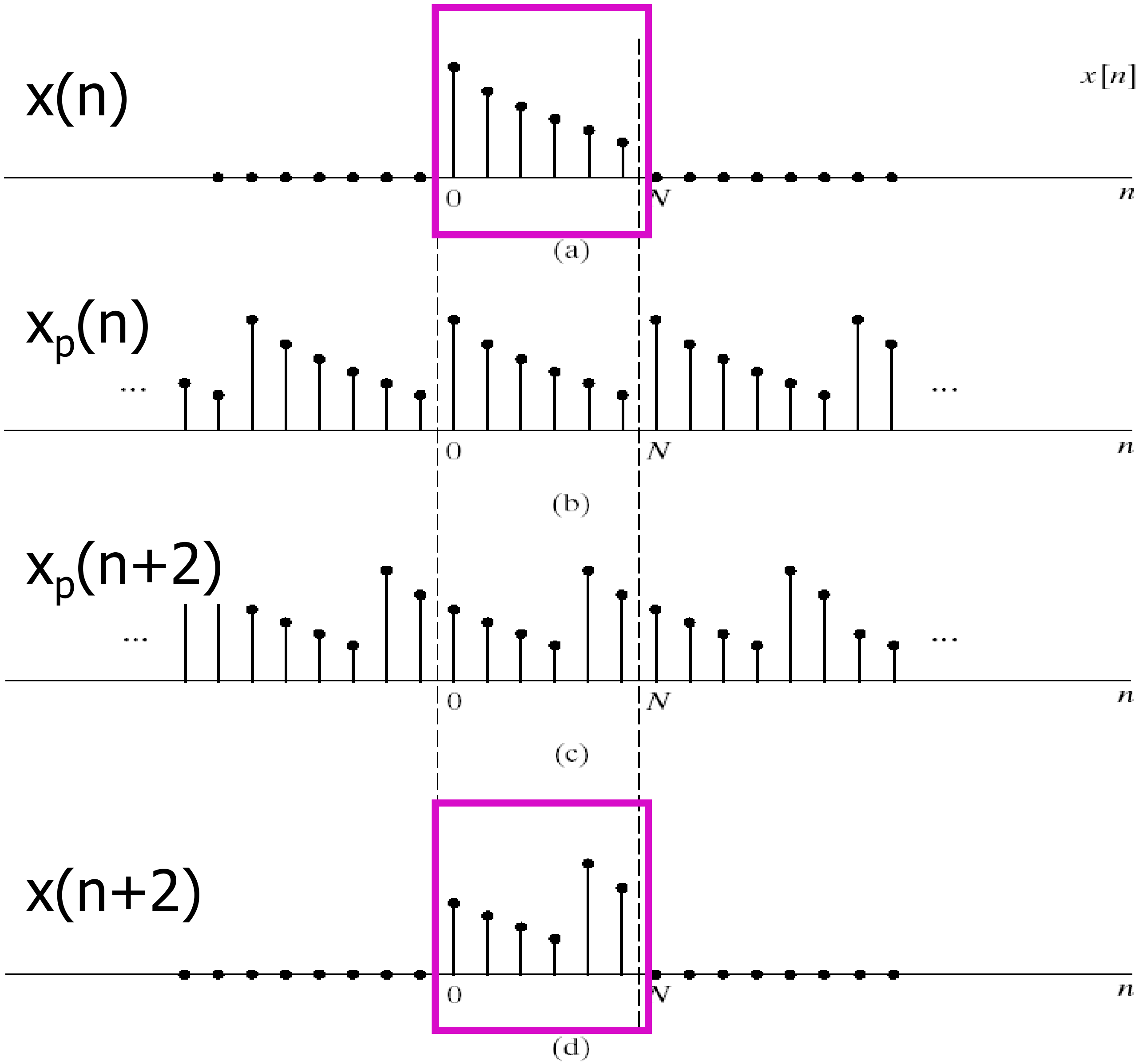
Circular shift property of the DFT

$x(n)$	N periodic extension $\longrightarrow$	$x_p(n)$
Finite length	$\longleftarrow$ truncate	Infinite length, periodic

$x(n)$ is one period of signal $x_p(n)$

$$\boxed{x[n-m] \overset{\text{DFT}}{\longleftrightarrow} W^{km} X[k]}$$

Circular shift property of the DFT



Circular shift by ***m*** is the same as a shift by ***m modulo N***

Recall linear convolution

$$y[n] = x_1[n] * x_2[n] = \sum_{p=-\infty}^{\infty} x_1[p] x_2[n-p]$$

- N_1: the non-zero length of $x_1(n)$; N_2: the non-zero length of $x_2(n)$; $N_y = N_1 + N_2 - 1$
- The shift operation is the regular shift
- The flip operation is the regular flip

Circular convolution of the DFT

$$y[n] = x_1[n] \otimes x_2[n] = \sum_{p=0}^{N-1} x_1[p] x_2[n-p]_{\mathrm{mod}\, N}$$

- The non-zero length of $x_1(n)$, $x_2(n)$ and y(n) can be no longer than N
- The shift operation is circular shift
- The flip operation is circular flip

$$x_1[n] \otimes x_2[n] \overset{DFT}{\leftrightarrow} X_1[k] X_2[k]$$

Direct method to calculate circular convolution

1. Draw a circle with *N* values of *x(n)* with *N* equally spaced angles in a counterclockwise direction.

2. Draw a smaller radius circle with *N* values of *h(n)* with equally spaced angles in a clockwise direction. Superimpose the centers of 2 circles, and have *h(0)* in front of *x(0).*

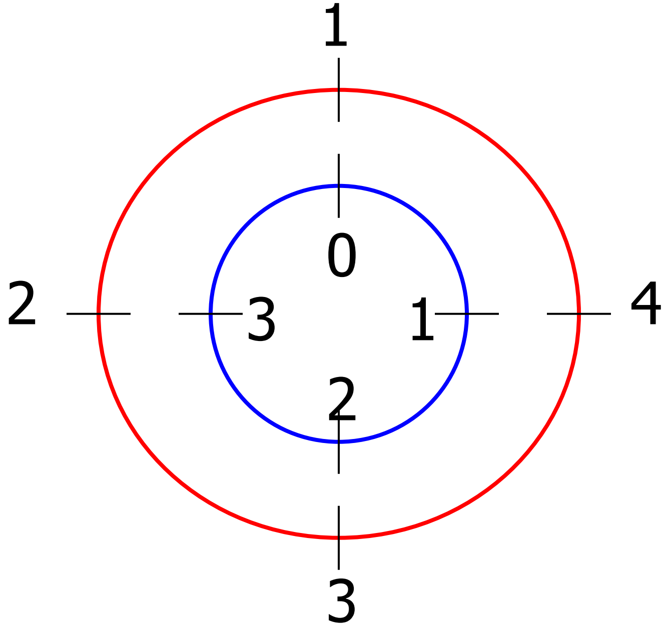
3. Calculate *y(0)* by multiplying the corresponding values on each radial line, and then adding the products.

4. Find succeeding values of *y(n)* in the same way after rotating the inner disk counterclockwise through the angle $2\pi k/N$

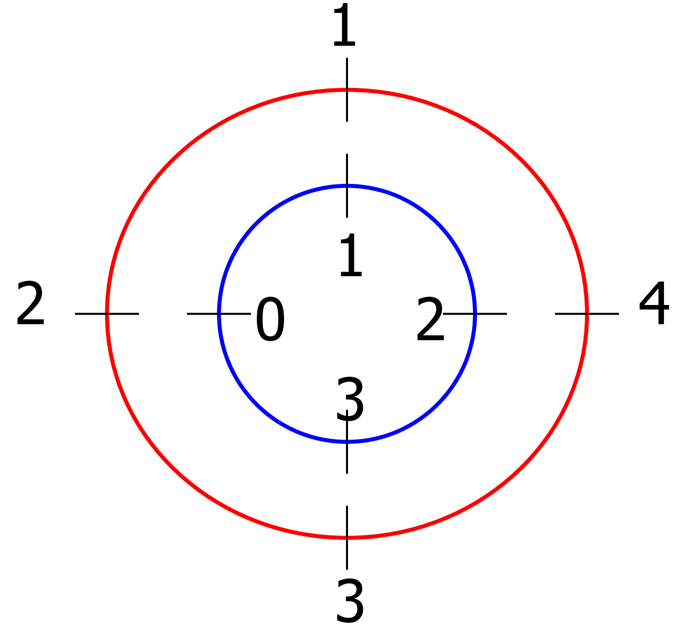
Example to calculate circular convolution

Evaluate the circular convolution, *y(n)* of 2 signals:

$x_1(n) = [\ 1\ 2\ 3\ 4\ ];\ x_2(n) = [\ 0\ 1\ 2\ 3\ ]$

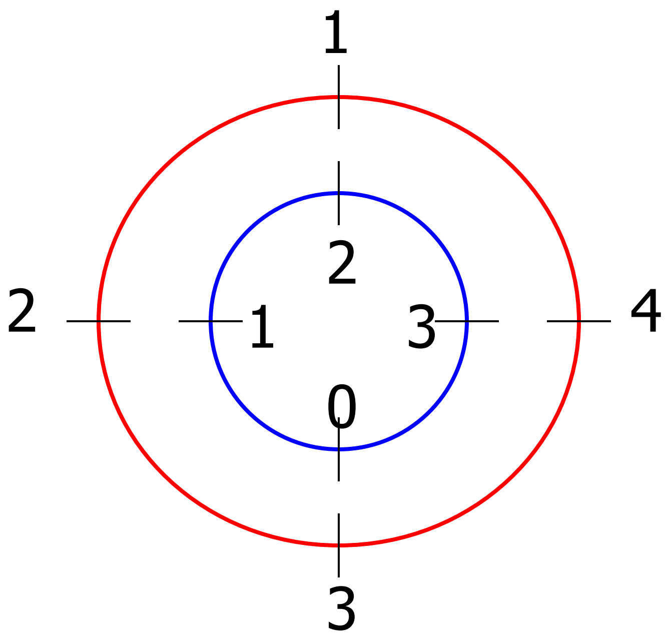


y(0) = 16

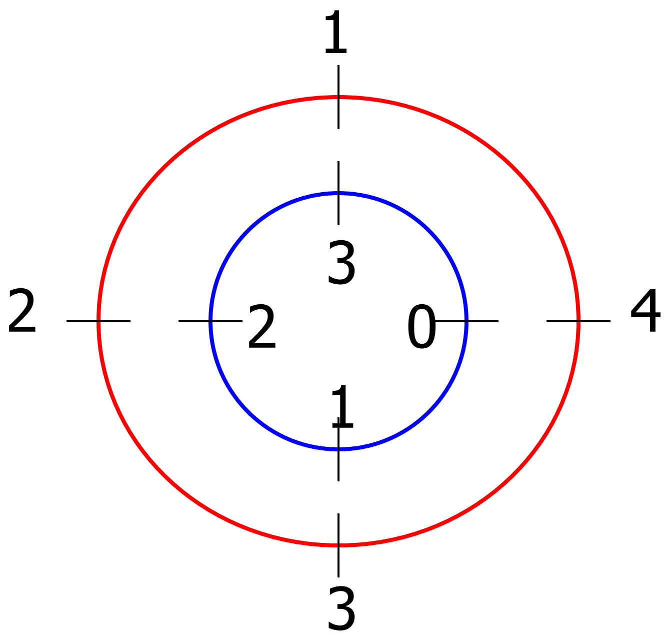


y(1) = 18

Example (cont)

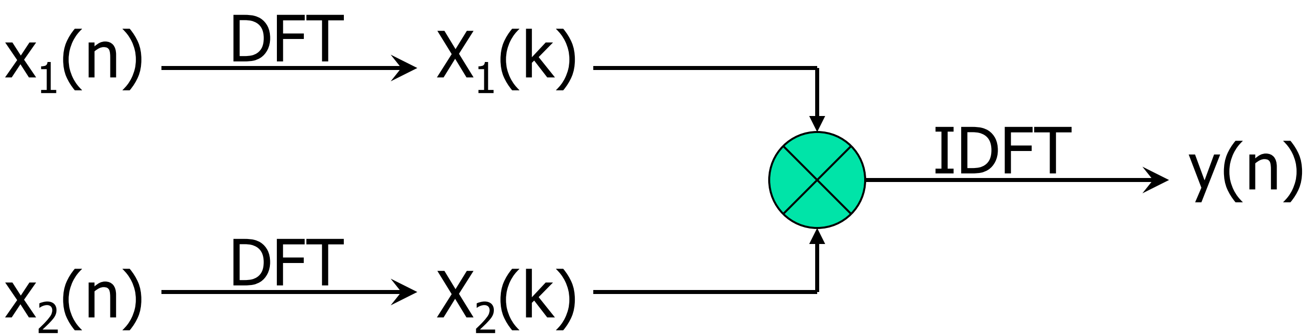


y(2) = 16



y(3) = 10

Another method to calculate circular convolution



Ex. $x_1(n) = [\ 1\ 2\ 3\ 4\ ]$; $x_2(n) = [\ 0\ 1\ 2\ 3\ ]$

$X_1(k) = [\ 10, -2+j2, -2, -2-j2\ ]$;

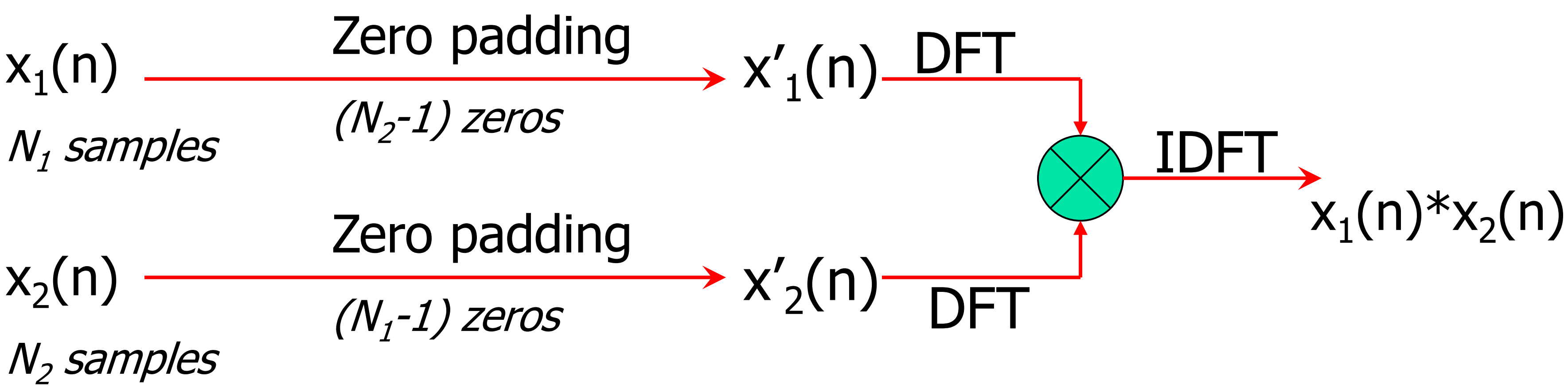
$X_2(k) = [\ 6, -2+j2, -2, -2-j2\ ]$;

$Y(k) = X_1(k).X_2(k) = [\ 60, -j8, 4, j8\ ]$

$y(n) = [\ 16, 18, 16, 10\ ]$

Calculation of the linear convolution

The circular convolution of 2 sequences of length N_1 and N_2 can be made **equal to** the linear convolution of 2 sequences by zero padding both sequences so that they both consists of N_1+N_2-1 samples.



Example of calculation the linear convolution

1 2 3 4]; $x_2(n)$ = [0 1 2 3]

$x'_1(n)$ = [1 2 3 4 0 0 0]; $x'_2(n)$ = [0 1 2 3 0 0 0]

$X'_1(k)$ = [10, -2.0245-j6.2240, 0.3460+j2.4791, 0.1784-j2.4220, 0.1784+j2.4220, 0.3460-j2.4791, -2.0245-j6.2240];

$X'_2(k)$ = [6, -2.5245-j4.0333, -0.1540+j2.2383, -0.3216-j1.7950, -0.3216+j1.7950, -0.1540-j2.2383, -2.5245+j4.0333];

Y'(k) = [60, -19.9928+j23.8775, -5.6024+j0.3927, -5.8342-j0.8644, -4.4049+j0.4585, -5.6024-j0.3927, -19.9928 +j23.8775]

IDFT{Y'(k)} = y'(n) = [0 1 4 10 16 17 12]

HW

Prob.1

Compute the DFT with N time samples:

$$\text{(a)} \quad x[n] = \delta[n]$$

$$\text{(b)} \quad x[n] = a^{n}\{u[n] - u[n-N]\}$$

$$\text{(c)} \quad x[n] = \begin{cases} 1 & n \text{ even} \\ 0 & n \text{ odd} \quad 0 \le n \le N-1 \end{cases}$$

HW

Prob.2

Given the two four-point sequences:

$$x(n) = [\ 1 \ \ 0.75 \ \ 0.5 \ \ 0.25\ ]$$

$$y(n) = [\ 0.75 \ \ 0.5 \ \ 0.25 \ \ 1]$$

Express the DFT $Y(k)$ in terms of the DFT $X(k)$

HW

Prob.3

Given signals below and their DFT-5

(a) $x_1[n] = \delta[n-1] + 2\delta[n-2] + 3\delta[n-3] + 4\delta[n-4]$

(b) $x_2[n] = \delta[n-1]$

(c) $s[n] = \delta[n]$

1. Find y[n] so that $Y[k] = X_1[k].X_2[k]$

2. Does $x_3[n]$ exist, if $S[k] = X_1[k].X_3[k]$?

HW

Prob.4 Given x(n) and its 8-point DFT, X(k)

$$x[n]=\begin{cases}1, & 0\le n\le 3\\ 0, & 4\le n\le 7\end{cases}$$

Express the DFTs of the signals below in terms of X(k).

$$\text{(a)}\quad x_1[n]=\begin{cases}1, & n=0\\ 0, & 1\le n\le 4\\ 1, & 5\le n\le 7\end{cases}$$

$$\text{(b)}\quad x_2[n]=\begin{cases}0, & 0\le n\le 1\\ 1, & 2\le n\le 5\\ 0, & 6\le n\le 7\end{cases}$$

HW

Prob.5 The 8-point DFTs of x(n) and h(n) are:

$$X(k) = [0, -j0.707, -j, -j0.707, 0, j0.707, j, j0.707]$$

and

$$H(k) = [3, 2.414, 1, -0.414, -1, -0.414, 1, 2.414]$$

Find the value of y(2), where y(n) is the circular convolution of x(n) and h(n).

Lecture 10

Fast Fourier Transform (FFT)

- **Duration: 2** hrs
- **Outline:**

1. What is FFT?

2. The decomposition-in-time Fast Fourier Transform algorithm

Recall DFT and IDFT definition

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{nk} \quad 0 \le k \le N-1$$

$$x[n] = \frac{1}{N} \sum_{n=0}^{N-1} X[k] W_N^{-nk} \quad 0 \le n \le N-1$$

$$W_N = e^{-j\frac{2\pi}{N}}$$

DFT plays an important role in the analysis, design and implementation of the DT signal processing algorithms and systems.

Major reason: existence of efficient algorithms for computing DFT called **FFT**

Direct computation of the DFT

$$\boxed{X[k]=\sum_{n=0}^{N-1}x[n]W_N^{nk} \quad 0\le k\le N-1}$$

The direct computation of the DFT requires:

1. ***N*** complex multiplications for each of *k*
2. ***N²*** complex multiplications for all *N* points of *X(k)*
3. ***(N-1)*** complex summations for each of *k*
4. ***N(N-1)*** complex summations for all *N* points of *X(k)*

→ **FFT** optimize computational processes (1) & (2) in different algorithms

Decomposition in time FFT (DIT-FFT)

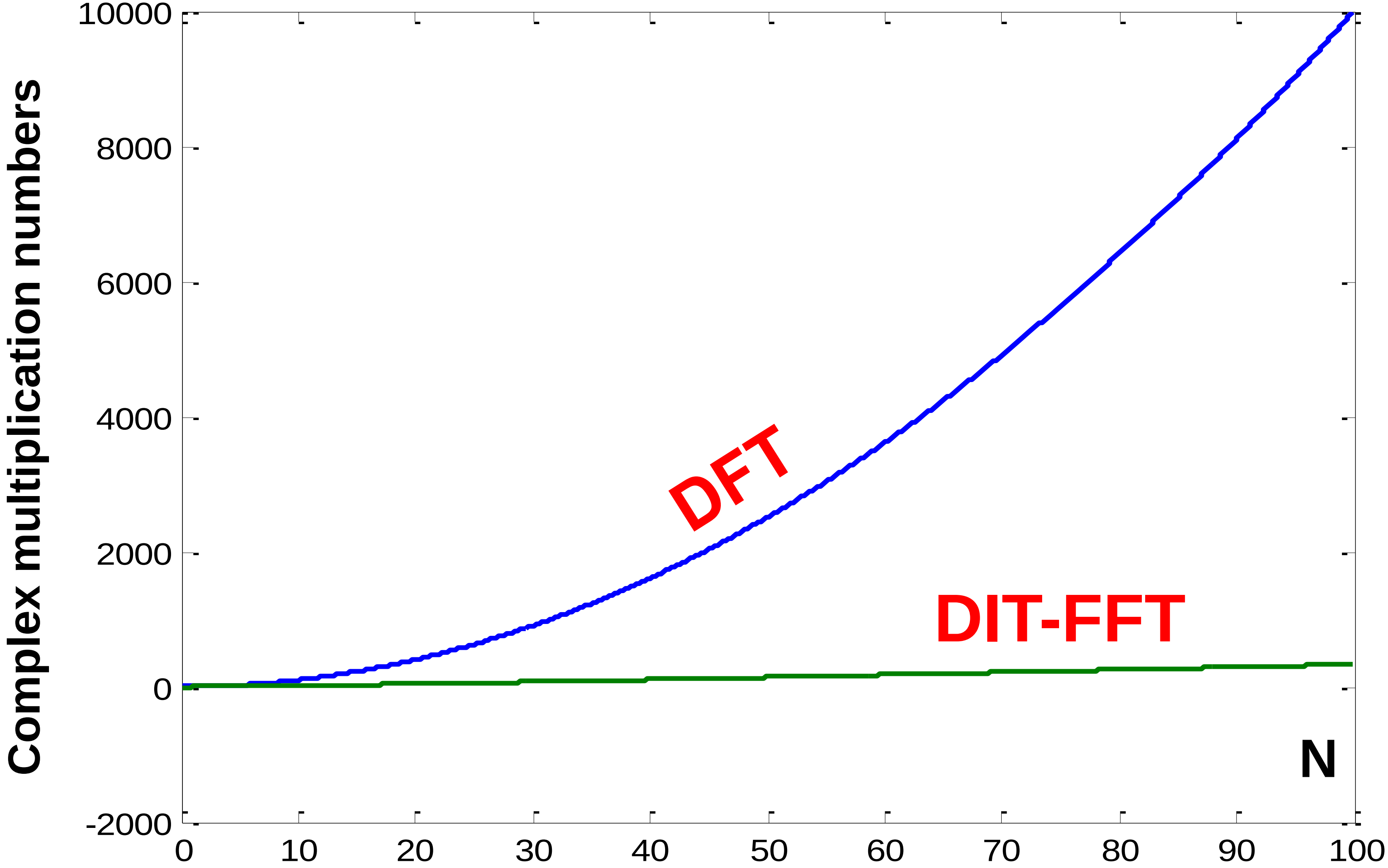
Breaking an N-point DFT into smaller DFTs

→ Fewer calculations with the same output

For example: N is radix-2 number

$$\boxed{N^2 \Rightarrow \frac{N}{2}\log_2 N \text{ complex multiplications}}$$

Comparing DFT and FFT efficiency



Computation of DFT

- Some properties of $\left\{W_N^{nk}\right\}$ can be exploited

$$W_N^{k(N-n)} = W_N^{-kn} = \left(W_N^{kn}\right)^*, \text{complex conjugate symmetry}$$

$$W_N^{kn} = W_N^{k(n+N)} = W_N^{(k+N)n}, \text{periodicity in n and k}$$

- Other useful properties:

$$W_N^{(k+\frac{N}{2})n} = W_N^{kn} W_N^{\frac{nN}{2}} = W_N^{kn} e^{-jn\pi} = \begin{cases} W_N^{kn}, & \text{if n even} \\ -W_N^{kn}, & \text{if n odd} \end{cases}$$

$$W_N^{2kn} = W_{\frac{N}{2}}^{kn}$$

Lecture 10

Fast Fourier Transform (FFT)

- **Duration: 2** hrs
- **Outline:**

1. What is FFT?

2. The decomposition-in-time Fast Fourier Transform algorithm

DIT-FFT with N as a 2-radix number

- *G(k)* is N/2 points DFT of the even numbered data: x(0), x(2), x(4), …., x(N-2).
- *H(k)* is the N/2 points DFT of the odd numbered data: x(1), x(3), …, x(N-1).

$$X[k] = \sum_{n\,\mathrm{even}} x[n]W^{kn} + \sum_{n\,\mathrm{odd}} x[n]W^{kn}$$

DIT-FFT (cont)

$$X[k] = \sum_{m=0}^{\frac{N}{2}-1} x[2m]W^{2mk} + \sum_{m=0}^{\frac{N}{2}-1} x[2m+1]W^{k(2m+1)}$$

$$= \sum_{m=0}^{\frac{N}{2}-1} x[2m](W^2)^{mk} + W^k \sum_{m=0}^{\frac{N}{2}-1} x[2m+1](W^2)^{mk} =$$

$$\mathrm{W}_{\mathrm{N}}^{2} = \left(\mathrm{e}^{-\mathrm{j}2\pi/\mathrm{N}}\right)^{2} = \mathrm{e}^{-\mathrm{j}2\pi/(\mathrm{N}/2)} = \mathrm{W}_{\mathrm{N}/2}$$

$$\boxed{\mathrm{X(k)} = \mathrm{G(k)} + \mathrm{W}_{\mathrm{N}}^{\mathrm{k}}\mathrm{H(k)}, \quad \mathrm{k} = 0,1,\ldots,\mathrm{N}-1}$$

G(k) and *H(k)* are of length N/2; *X(k)* is of length *N*

G(k)=G(k+N/2) and *H(k)=H(k+N/2)*

8-point FFT

$$X[k]_8 = G[k]_4 + W_8^k H[k]_4$$

$$X[0] = G[0] + W_8^0 H[0]$$

$$X[1] = G[1] + W_8^1 H[1]$$

$$X[2] = G[2] + W_8^2 H[2]$$

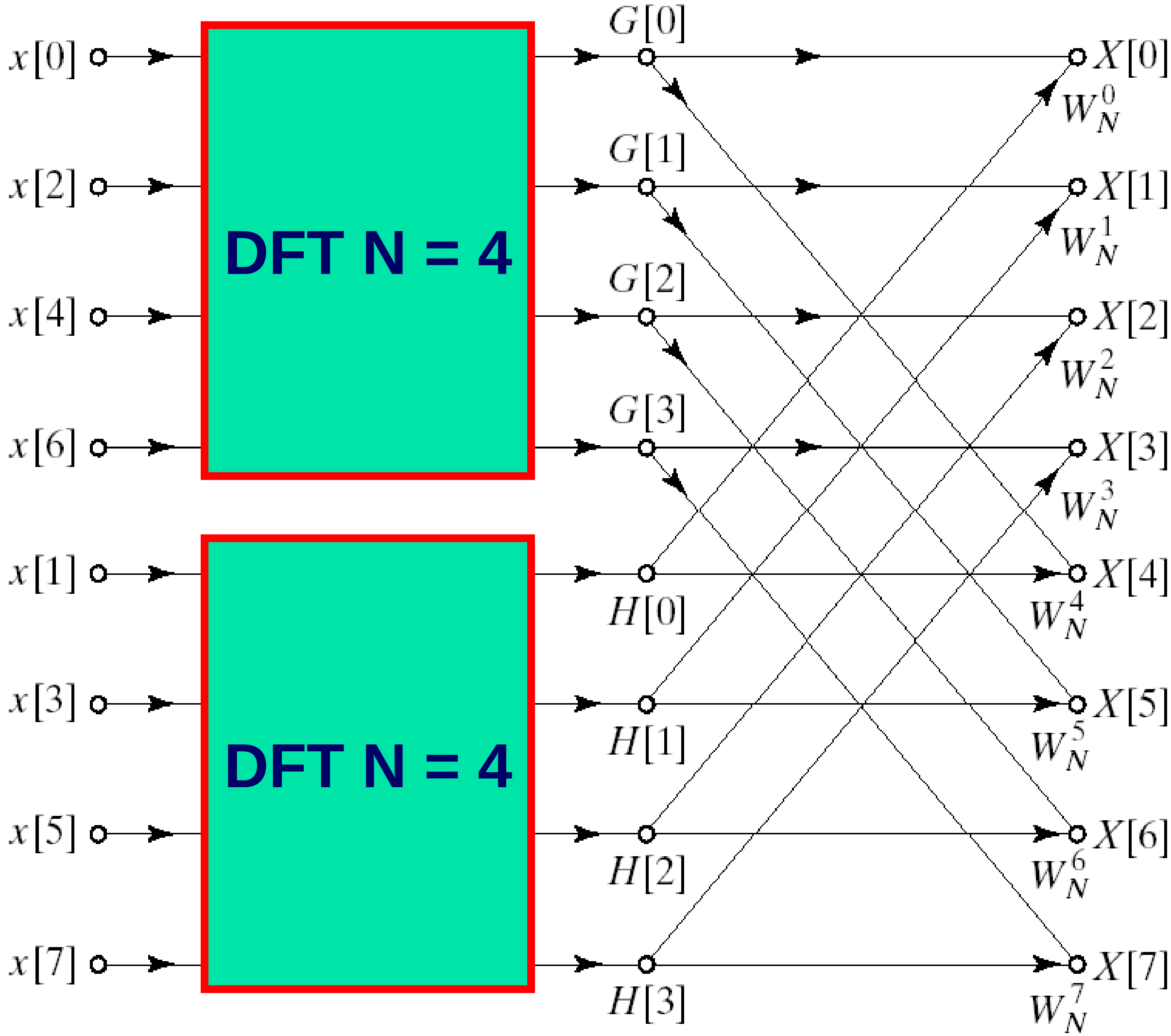
$$X[3] = G[3] + W_8^3 H[3]$$

$$X[4] = G[0] + W_8^4 H[0]$$

$$X[5] = G[1] + W_8^5 H[1]$$

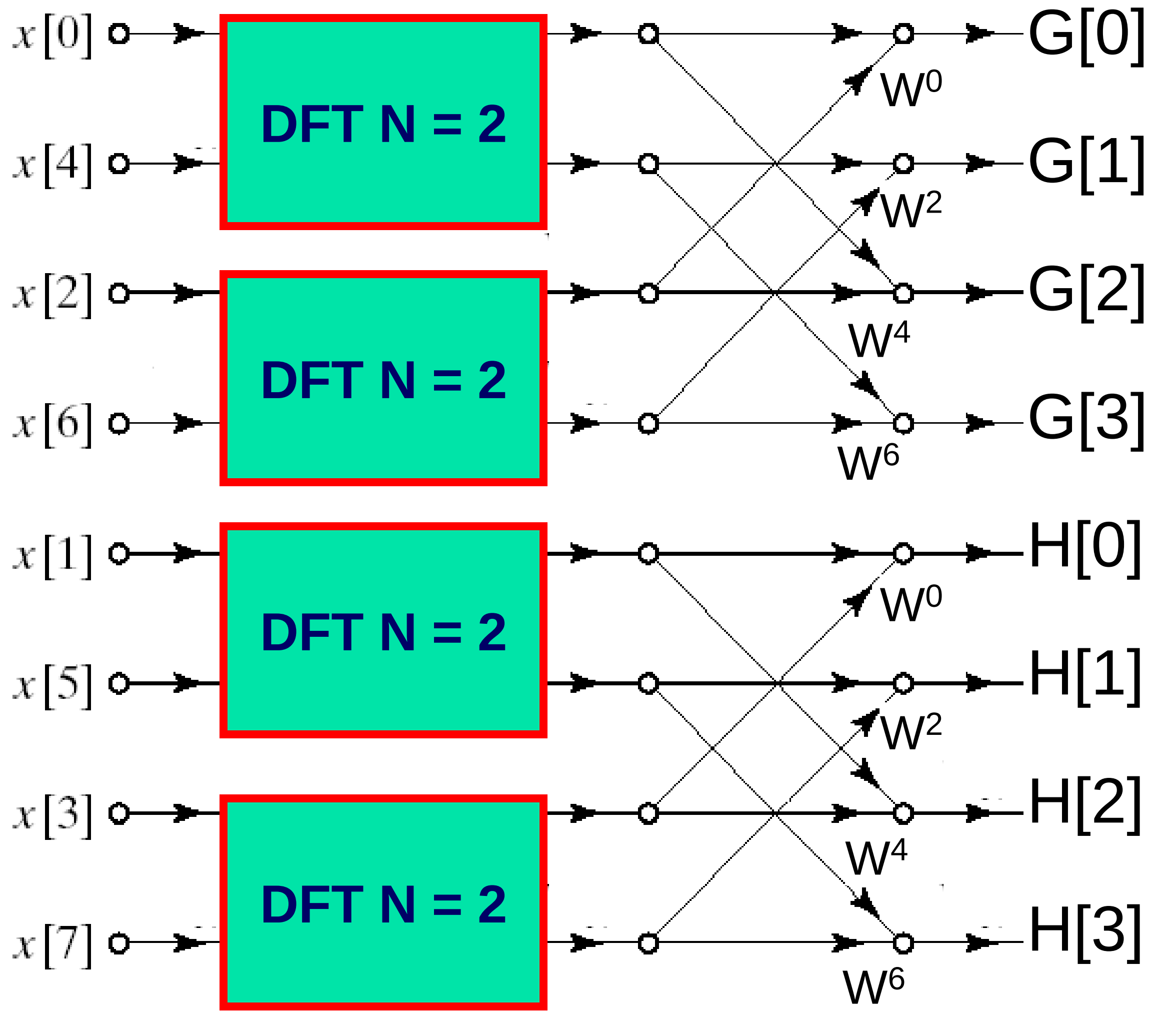
$$X[6] = G[2] + W_8^6 H[2]$$

$$X[7] = G[3] + W_8^7 H[3]$$

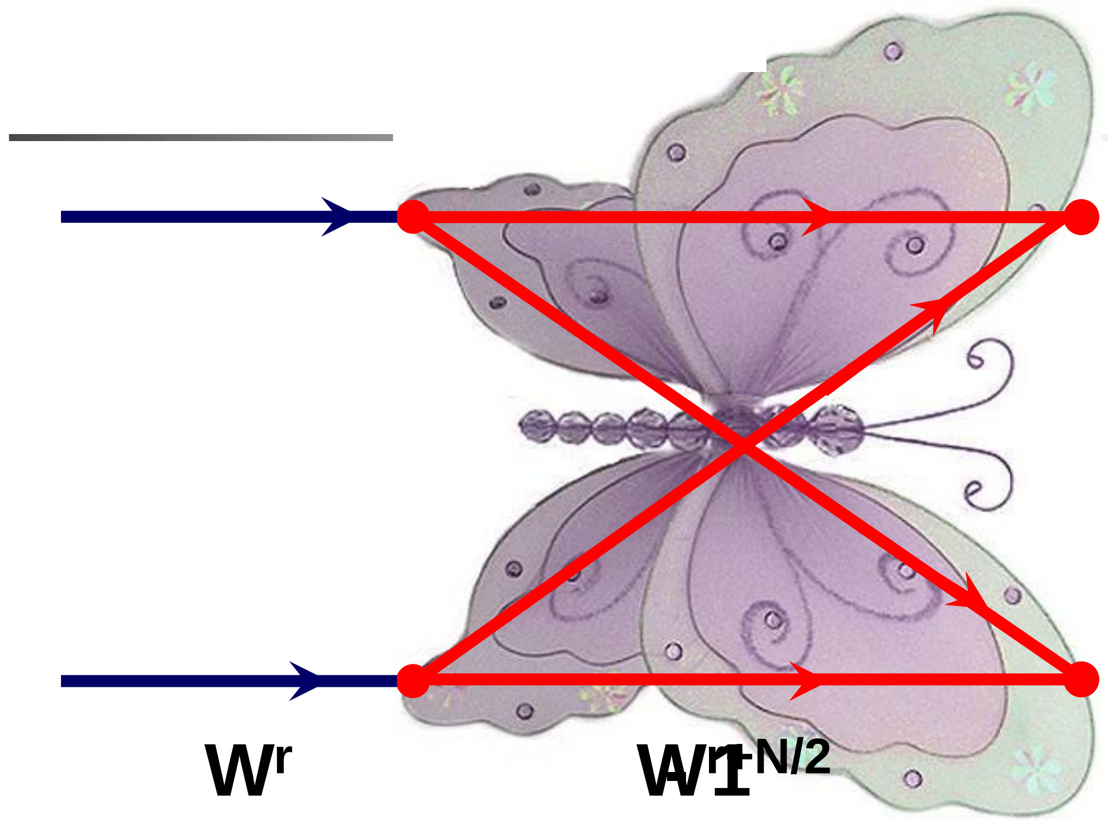


The new computation counts are reduced

8-point FFT (cont)

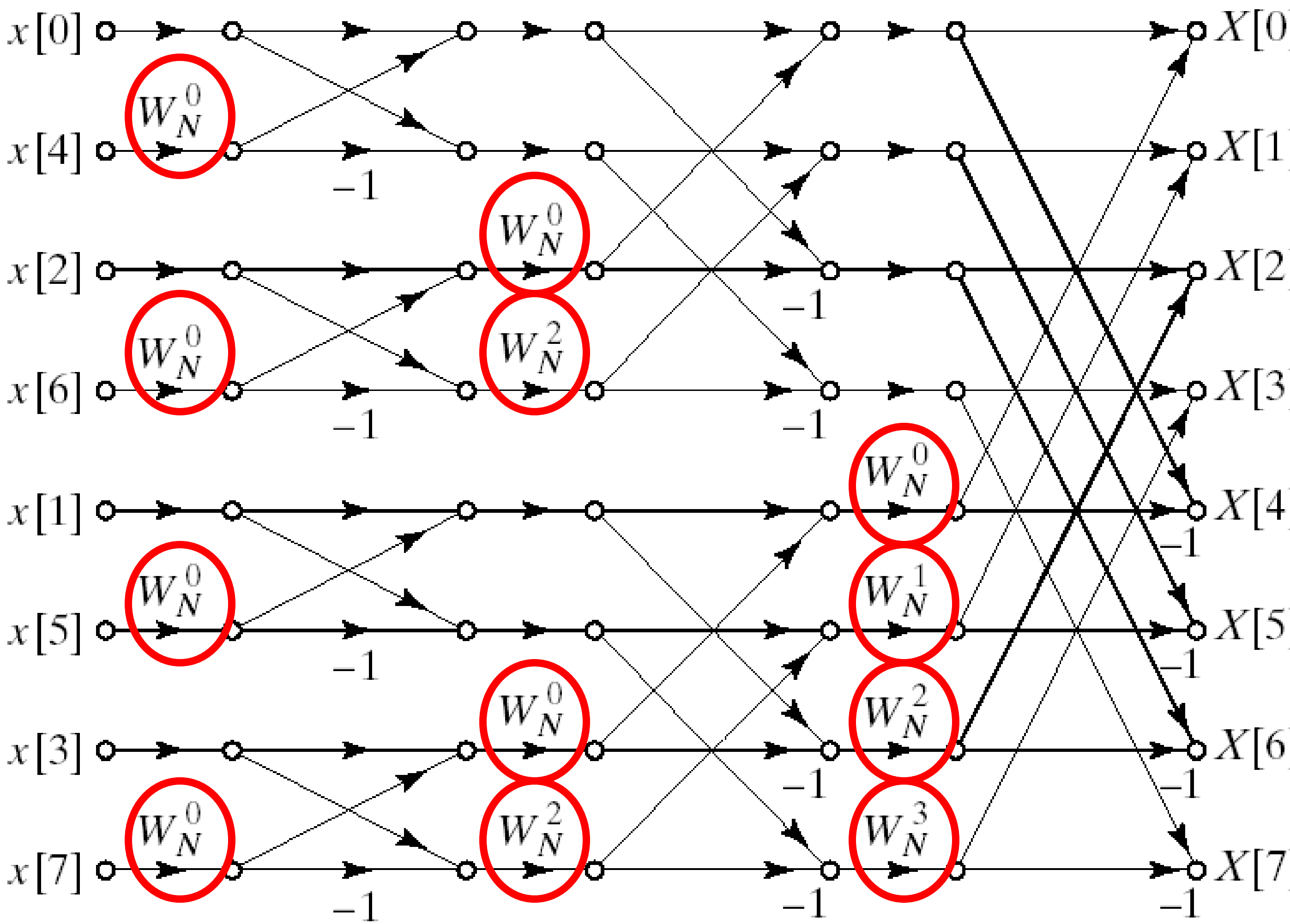


2-point FFT



$$\boxed{W^{r+N/2} = -W^r}$$

8-point FFT



Now the overall computation is reduced to:

$$N^2 \Rightarrow \frac{N}{2}\log_2 N \text{ complex multiplications}$$

HW

Prob.6

(a) Draw an eight-point DIT FFT signal-flow diagram, and use it to solve for the DFT of the sequence x(n)

$$\mathbf{x(n) = (0.5)^{n}\,[u(n) - u(n-8)]}$$

(b) Use Matlab (*fft*) to confirm the result of part (a)