CHAPTER 7

The Discrete Fourier Transform

Tutorial Problems

1. (a) Solution:
   The CTFT of $x_c(t)$ is:

$$X_c(j2\pi F) = \int_{-\infty}^{\infty} x_c(t)e^{-j\Omega t}\,dt = \int_0^{\infty} 5e^{-10t}\sin(20\pi t)e^{-j\Omega t}\,dt$$
$$= \frac{100\pi}{(20\pi)^2 + (j\Omega + 10)^2} = \frac{100\pi}{(20\pi)^2 + (j20\pi F + 10)^2}$$

   (b) See plot below.

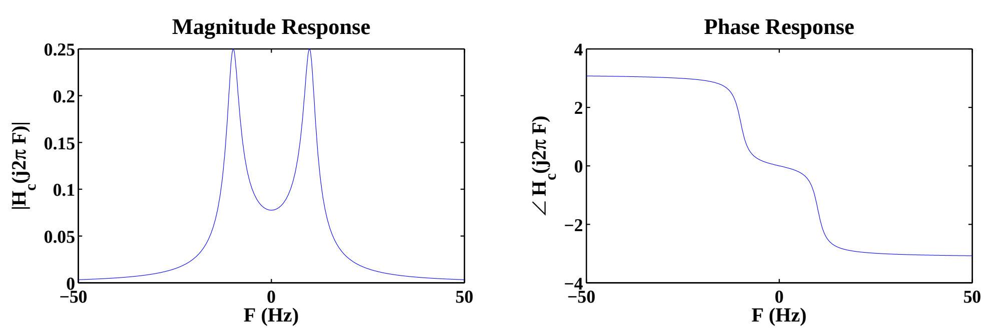


FIGURE 7.1: Magnitude and phase responses of $X_c(j2\pi F)$ over $-50 \leq F \leq 50$ Hz.

   (c) See plot below.

MATLAB script:

```
% P0701: Numerical DFT approximation of CTFT
```

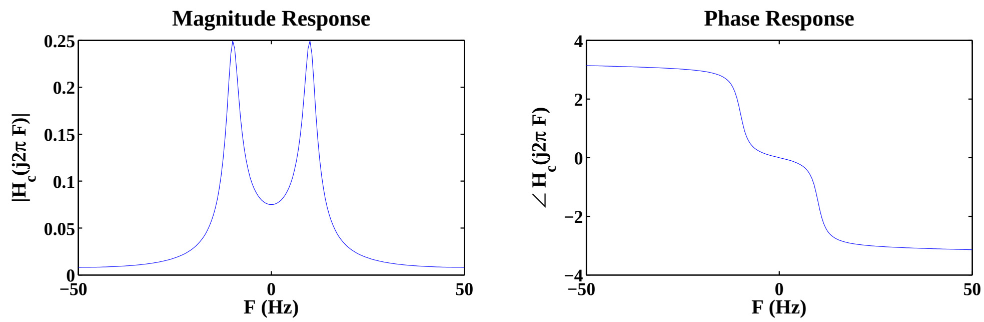


FIGURE 7.2: Approximated magnitude and phase responses of $X_c(j2\pi F)$ over $-50 \leq F \leq 50$ Hz using `fft` function.

```
close all; clc
%% Part (b):
t1 = 0; t2 = 2;
dF = 0.1;
F = -50:dF:50;
Xc = 100*pi./((20*pi)^2+(j*2*pi*F+10).^2);
%% Part (c):
Fs = 100;
T = Fs\1;
nT = t1:T:t2;
N = length(nT);
xn = 5*exp(-10*nT).*sin(20*pi*nT);
X = fftshift(fft(xn));
w = linspace(-pi,pi,N);
Xc_approx = T*X;
%% Plot:
hfa = figconfg('P0701a','long');
subplot(121)
plot(F,abs(Xc))
xlabel('F (Hz)','fontsize',LFS)
ylabel('|H_c(j2\pi F)|','fontsize',LFS)
title('Magnitude Response','fontsize',TFS)
subplot(122)
plot(F,angle(Xc))
xlabel('F (Hz)','fontsize',LFS)
ylabel('\angle H_c(j2\pi F)','fontsize',LFS)
```

```
title('Phase Response','fontsize',TFS)

hfb = figconfg('P0701b','long');
subplot(121)
plot(w/T/2/pi,abs(Xc_approx))
xlabel('F (Hz)','fontsize',LFS)
ylabel('|H_c(j2\pi F)|','fontsize',LFS)
title('Magnitude Response','fontsize',TFS)
subplot(122)
plot(w/T/2/pi,angle(Xc_approx))
xlabel('F (Hz)','fontsize',LFS)
ylabel('\angle H_c(j2\pi F)','fontsize',LFS)
title('Phase Response','fontsize',TFS)
```

2. (a) Solution:
$T_0 = 5$, the fundamental period $\Omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{5}$.
The CTFS of $\tilde{x}_c(t)$ is:

$$c_k = \frac{1}{T_0}\int_0^{T_0} \tilde{x}_c(t)\mathrm{e}^{-\mathrm{j}k\Omega_0 t}\mathrm{d}t = \frac{1}{5}\int_0^{5} \mathrm{e}^{-t}\cdot \mathrm{e}^{-\mathrm{j}k\Omega_0 t}dt$$
$$= \frac{1-\mathrm{e}^{-5}}{5+\mathrm{j}2\pi k}$$

(b) See plot below.

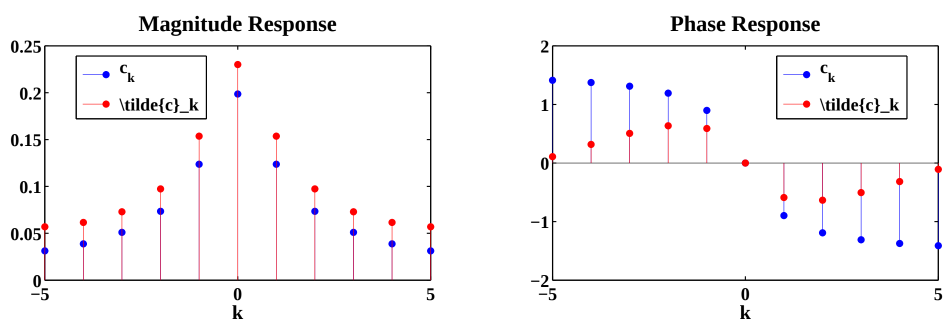


FIGURE 7.3: Magnitude and phase responses of c_k and $\hat{c}_k$ when sampling interval $T = 0.5$s.

(c) See plot below.

(d) See plot below.

MATLAB script:

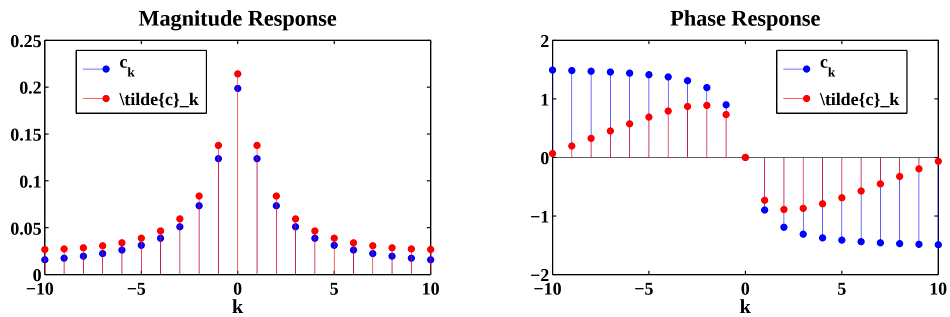


FIGURE 7.4: Magnitude and phase responses of c_k and $\hat{c}_k$ when sampling interval $T = 0.25$s.

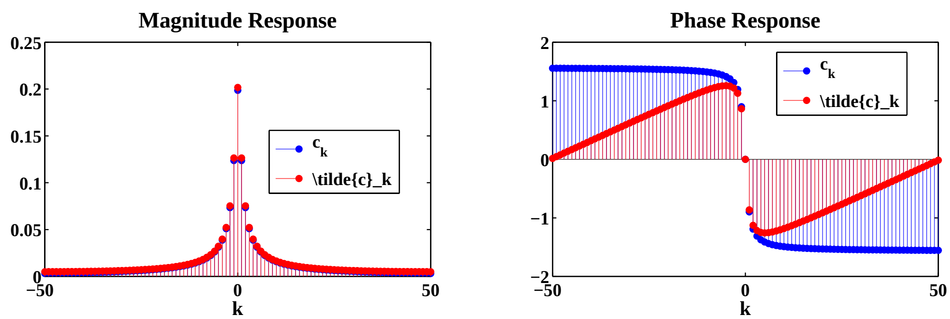


FIGURE 7.5: Magnitude and phase responses of c_k and $\hat{c}_k$ when sampling interval $T = 0.05$s.

```
% P0702: Numerical DFT approximation of CTFS
close all; clc
t1 = 0; t2 = 5;
T = 0.5; % Part (b)
% T = 0.25; % Part (c)
% T = 0.05; % Part (d)
N = t2/T;
k = -N/2:N/2;
ck = (1-exp(-5))./(5+j*2*pi*k);
nT = t1:T:t2;
xn = exp(-nT);
ck_approx = length(xn)\fftshift(fft(xn));
%% Plot:
```

```
hfa = figconfg('P0702a','long');
subplot(121)
stem(k,abs(ck),'filled');hold on
stem(k,abs(ck_approx),'filled','color','red');
xlabel('k','fontsize',LFS)
title('Magnitude Response','fontsize',TFS)
legend('c_k','\tilde{c}_k','location','best')
subplot(122)
stem(k,angle(ck),'filled');hold on
stem(k,angle(ck_approx),'filled','color','red');
xlabel('k','fontsize',LFS)
title('Phase Response','fontsize',TFS)
legend('c_k','\tilde{c}_k','location','best')
```

3. (a) Solution:
   The DTFT of $(0.9)^n u[n]$ is:

$$\frac{1}{1-0.9\mathrm{e}^{-\mathrm{j}\omega}}$$

   The DTFT of $x[n]$ is:

$$\begin{aligned}\tilde{X}(\mathrm{e}^{\mathrm{j}\omega}) &= (-\mathrm{j})\frac{\mathrm{d}}{\mathrm{d}\omega}\left(\frac{1}{1-0.9\mathrm{e}^{-\mathrm{j}\omega}}\right)\\ &= \frac{0.9\mathrm{e}^{-\mathrm{j}\omega}}{(1-0.9\mathrm{e}^{-\mathrm{j}\omega})^2}\end{aligned}$$

   (b) See plot below.

   (c) See plot below.

   (d) See plot below.

MATLAB script:

```
% P0703: Numerical DFT approximation of DTFT
close all; clc
w = linspace(0,2,1000)*pi;
X = 0.9*exp(-j*w)./(1-0.9*exp(-j*w)).^2;
N = 20; % Part (b)
% N = 50; % Part (c)
% N = 100; % Part (d)
n = 0:N-1;
```

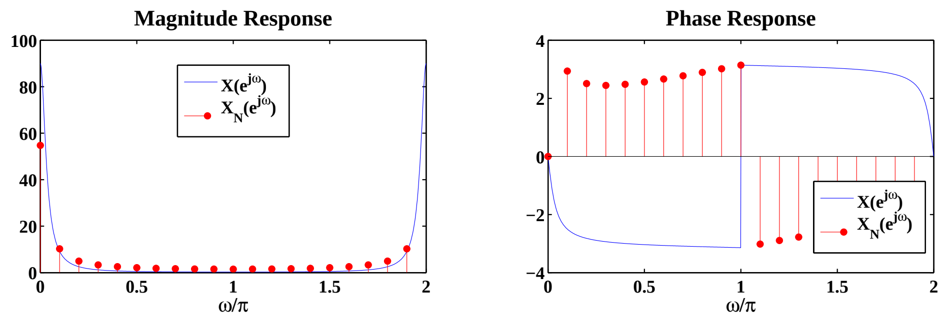


FIGURE 7.6: Magnitude and phase responses of $\tilde{X}(\mathrm{e}^{\mathrm{j}\omega})$ and $\tilde{X}_N(\mathrm{e}^{\mathrm{j}\omega})$ when $N = 20$.

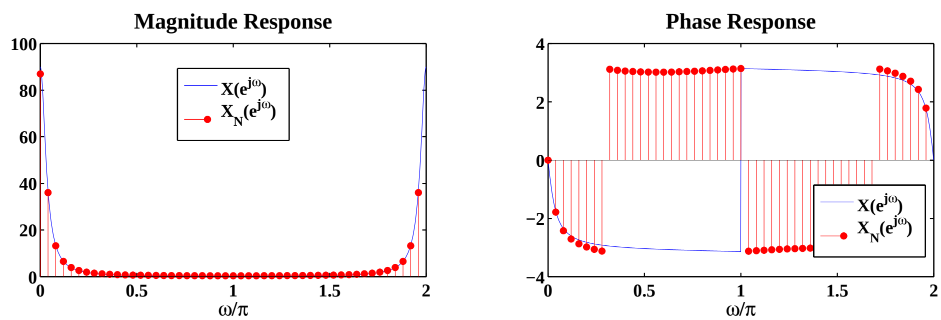


FIGURE 7.7: Magnitude and phase responses of $\tilde{X}(\mathrm{e}^{\mathrm{j}\omega})$ and $\tilde{X}_N(\mathrm{e}^{\mathrm{j}\omega})$ when $N = 50$.

```
xn = n.*0.9.^n;
XN = fft(xn);
wk = 2/N*(0:N-1);
%% Plot:
hfa = figconfg('P0703a','long');
subplot(121)
plot(w/pi,abs(X));hold on
stem(wk,abs(XN),'filled','color','red');
xlabel('\omega/\pi','fontsize',LFS)
title('Magnitude Response','fontsize',TFS)
legend('X(e^{j\omega})','X_N(e^{j\omega})','location','best')
subplot(122)
plot(w/pi,angle(X));hold on
```

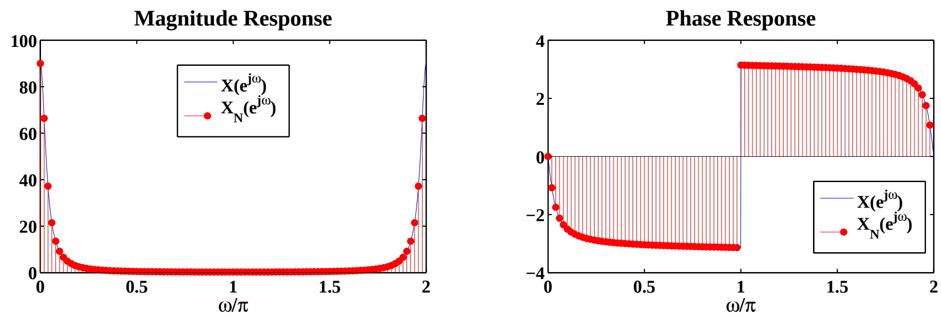


FIGURE 7.8: Magnitude and phase responses of $\tilde{X}(\mathrm{e}^{\mathrm{j}\omega})$ and $\tilde{X}_N(\mathrm{e}^{\mathrm{j}\omega})$ when $N = 100$.

```
stem(wk,angle(XN),'filled','color','red');
xlabel('\omega/\pi','fontsize',LFS)
title('Phase Response','fontsize',TFS)
legend('X(e^{j\omega})','X_N(e^{j\omega})','location','best')
```

4. (a) Solution:
The $N \times N$ DFT matrix is:

$$\boldsymbol{W}_N = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & W_N & \cdots & W_N^{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{N-1} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix}$$

The kth column of $\boldsymbol{W}_N$ is $\boldsymbol{w}_k = [1 W_N^k \ldots W_N^{N-1} k]^T$.
The i, jth element of $\boldsymbol{W}_N^2$ is:

$$\begin{aligned}\left(\boldsymbol{W}_N^2\right)_{i,j} &= \left(\boldsymbol{W}_N^T \boldsymbol{W}_N\right)_{i,j} = \boldsymbol{w}_i^T \boldsymbol{w}_j \\ &= \begin{cases} 0, & i+j \neq N \\ N, & i+j = N \end{cases}\end{aligned}$$

Hence, we proved that

$$\boldsymbol{W}_N^2 = \begin{bmatrix} 0 & \cdots & 0 & N \\ \vdots & \iddots & N & 0 \\ 0 & \iddots & \iddots & \vdots \\ N & 0 & \cdots & 0 \end{bmatrix} = N\boldsymbol{J}_N$$

The effect of the flip matrix on $\boldsymbol{J}_N\boldsymbol{x}$ product is rearranging the elements in column vector $\boldsymbol{x}$ in reverse order (that is flip $\boldsymbol{x}$ upside down).

(b) Solution:

$$\boldsymbol{W}_N^4 = (N\boldsymbol{J}_N)(N\boldsymbol{J}_N) = N^2(\boldsymbol{J}_N \cdot \boldsymbol{J}_N) = N^2\boldsymbol{I}_N$$

(c) Solution:
The multiplicity depends on the value of N modulo 4 and can be summarized in the following table.

size N	$\lambda = 1$	$\lambda = -1$	$\lambda = -\mathrm{j}$	$\lambda = \mathrm{j}$
$4m$	$m+1$	m	m	$m-1$
$4m+1$	$m+1$	m	m	m
$4m+2$	$m+1$	$m+1$	m	m
$4m+3$	$m+1$	$m+1$	$m+1$	m

TABLE 7.1: Multiplicity of the eigenvalues of the DFT matrix $\boldsymbol{W}_N$.

MATLAB script:

```
% P0704: Investigate the eigenvalues multiplicity
%        of the DFT matrix WN
close all; clc
N = 4:10;
for ii = 1:length(N)
    WN = dftmtx(N(ii))/sqrt(N(ii));
    eig(WN)
end
```

5. (a) Solution:
The DFT of $x[n]$ is:

$$\begin{aligned} X[k] &= \sum_{n=0}^{N-1} x[n]W_N^{kn} = \sum_{n=0}^{7}(4-n)\cdot \mathrm{e}^{-\mathrm{j}\frac{2\pi}{8}kn} \\ &= 4 + 3(\mathrm{e}^{-\mathrm{j}\frac{2\pi}{8}} - \mathrm{e}^{\mathrm{j}\frac{2\pi}{8}}) + 2(\mathrm{e}^{-\mathrm{j}\frac{2\pi}{8}2} - \mathrm{e}^{\mathrm{j}\frac{2\pi}{8}2}) + (\mathrm{e}^{-\mathrm{j}\frac{2\pi}{8}3} - \mathrm{e}^{\mathrm{j}\frac{2\pi}{8}3}) \\ &= 4 - 6\mathrm{j}\sin\left(\frac{k\pi}{4}\right) - 4\mathrm{j}\sin\left(\frac{k\pi}{2}\right) - 2\mathrm{j}\sin\left(\frac{3k\pi}{4}\right) \end{aligned}$$

(b) Solution:
The DFT of $x[n]$ is:

$$\begin{aligned} X[k] &= \sum_{n=0}^{N-1} x[n]W_N^{kn} = \sum_{n=0}^{9} \frac{2}{\mathrm{j}}\left(\mathrm{e}^{\mathrm{j}\frac{2\pi}{10}n} - \mathrm{e}^{-\mathrm{j}\frac{2\pi}{10}n}\right)\mathrm{e}^{-\mathrm{j}\frac{2\pi}{10}nk} \\ &= \frac{2}{\mathrm{j}}\sum_{n=0}^{9} \mathrm{e}^{\mathrm{j}\frac{2\pi}{10}n}\cdot \mathrm{e}^{-\mathrm{j}\frac{2\pi}{10}nk} - \frac{2}{\mathrm{j}}\sum_{n=0}^{9} \mathrm{e}^{-\mathrm{j}\frac{2\pi}{10}n}\cdot \mathrm{e}^{-\mathrm{j}\frac{2\pi}{10}nk} \\ &= -20\mathrm{j}\delta[k-1] + 20\mathrm{j}\delta[k-9] \end{aligned}$$

(c) Solution:
The DFT of $x[n]$ is:

$$\begin{aligned} X[k] &= \sum_{n=0}^{N-1} x[n]W_N^{kn} = \sum_{n=0}^{9}\left(3 + \frac{3}{2}\mathrm{e}^{\mathrm{j}\frac{2\pi}{10}n2} - \frac{3}{2}\mathrm{e}^{-\mathrm{j}\frac{2\pi}{10}n2}\right)\mathrm{e}^{-\mathrm{j}\frac{2\pi}{10}nk} \\ &= 30\delta[k] + 15\delta[k-2] - 15\delta[k-8] \end{aligned}$$

(d) Solution:
The DFT of $x[n]$ is:

$$\begin{aligned} X[k] &= \sum_{n=0}^{N-1} x[n]W_N^{kn} = \sum_{n=0}^{15} 5(0.8)^n\mathrm{e}^{-\mathrm{j}\frac{2\pi}{16}nk} \\ &= 5\sum_{n=0}^{15}\left(0.8\mathrm{e}^{-\mathrm{j}\frac{2\pi}{16}k}\right)^n = 5\cdot\frac{1-0.8^{16}}{1-0.8\mathrm{e}^{-\mathrm{j}\frac{2\pi}{16}k}} \end{aligned}$$

(e) Solution:
The DFT of $x[n]$ is:

$$\begin{aligned} X[k] &= \sum_{n=0}^{N-1} x[n]W_N^{kn} = \sum_{n=0}^{19} x[n]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{20}nk} \\ &= \sum_{m=0}^{9} 3\mathrm{e}^{-\mathrm{j}\frac{2\pi}{20}k\cdot 2m} + \sum_{m=0}^{9}(-2)\mathrm{e}^{-\mathrm{j}\frac{2\pi}{20}k\cdot(2m+1)} \\ &= 3\sum_{m=0}^{9}\left(\mathrm{e}^{-\mathrm{j}\frac{\pi}{5}m}\right)^m - 2\mathrm{e}^{-\mathrm{j}\frac{\pi}{10}k}\sum_{m=0}^{9}\left(\mathrm{e}^{-\mathrm{j}\frac{\pi}{5}m}\right)^m \\ &= 10\delta[k] + 50\delta[k-10] \end{aligned}$$

6. Proof:

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{nk} = \begin{bmatrix} \boldsymbol{w}_1^T \\ \boldsymbol{w}_2^T \\ \vdots \\ \boldsymbol{w}_{N-1}^T \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} \boldsymbol{w}_1^T \boldsymbol{x} \\ \boldsymbol{w}_2^T \boldsymbol{x} \\ \vdots \\ \boldsymbol{w}_{N-1}^T \boldsymbol{x} \end{bmatrix}$$

Hence, we proved that the DFT coefficients $X[k]$ are the projections of the signal $x[n]$ on the DFT (basis) vectors $\{\boldsymbol{w}_k\}$.

7. (a) Solution:
The DFT of $\tilde{x}[n] = 2\cos(\pi n/4)$ is:

$$\begin{aligned} X[k] &= \sum_{n=0}^{7} \left(\mathrm{e}^{\mathrm{j}\frac{2\pi}{8}n2} + \mathrm{e}^{-\mathrm{j}\frac{2\pi}{8}n2} \right) \mathrm{e}^{-\mathrm{j}\frac{2\pi}{8}nk} \\ &= 8\delta[k-2] + 8\delta[k-6] \end{aligned}$$

The DFS of $\tilde{x}[n] = 2\cos(\pi n/4)$ is:

$$\tilde{X}[k] = 8\delta[\langle k \rangle_8 - 2] + 8\delta[\langle k \rangle_8 - 6]$$

(b) Solution:
The DFT of $\tilde{x}[n] = 3\sin(0.25\pi n) + 4\cos(0.75\pi n)$ is:

$$\begin{aligned} X[k] &= \sum_{n=0}^{7} \left[\frac{3}{2\mathrm{j}} \left(\mathrm{e}^{\mathrm{j}\frac{2\pi}{8}n2} - \mathrm{e}^{-\mathrm{j}\frac{2\pi}{8}n2} \right) + 2 \left(\mathrm{e}^{\mathrm{j}\frac{2\pi}{8}n3} + \mathrm{e}^{-\mathrm{j}\frac{2\pi}{8}n3} \right) \right] \mathrm{e}^{-\mathrm{j}\frac{2\pi}{8}nk} \\ &= -12\delta[k-2] + 12\delta[k-6] + 16\delta[k-3] + 16\delta[n-5] \end{aligned}$$

The DFS of $\tilde{x}[n] = 3\sin(0.25\pi n) + 4\cos(0.75\pi n)$ is:

$$\tilde{X}[k] = -12\delta[\langle k \rangle_8 - 2] + 12\delta[\langle k \rangle_8 - 6] + 16\delta[\langle k \rangle_8 - 3] + 16\delta[\langle k \rangle_8 - 5]$$

8. (a) See plot below.
   (b) See plot below.
   (c) See plot below.

   MATLAB script:

```
% P0708: Regenerate Figure~7.5 and Example 7.3
close all; clc
N = 16; a = 0.9; % Part (a)
% N = 8; a = 0.8; % Part (b)
% N = 64; a = 0.8; % Part (c)
wk = 2*pi/N*(0:N-1);
Xk = 1./(1-a*exp(-j*wk));
xn = real(ifft(Xk));
w = linspace(0,2,1000)*pi;
X = fft(xn,length(w));
X_ref = 1./(1-a*exp(-j*w));
n = 0:N-1;
xn_ref = a.^n;
%% Plot:
hfa = figconfg('P0708a','small');
plot(w/pi,abs(X_ref),'color','black'); hold on
plot(w/pi,abs(X))
stem(wk/pi,abs(Xk),'filled');
ylim([0 max(abs(X))])
xlabel('\omega/\pi','fontsize',LFS)
ylabel('Magnitude','fontsize',LFS)
title('Magnitude of Spectra','fontsize',TFS)
hfb = figconfg('P0708b','small');
plot(n,xn,'.'); hold on
plot(n,xn_ref,'.','color','black')
ylim([0 1.1*max(xn)])
xlim([0 N-1])
xlabel('Time index (n)','fontsize',LFS)
ylabel('Amplitude','fontsize',LFS)
title('Signal Amplitudes','fontsize',TFS)
legend('x[n]','\tilde{x}[n]','location','northeast')
```

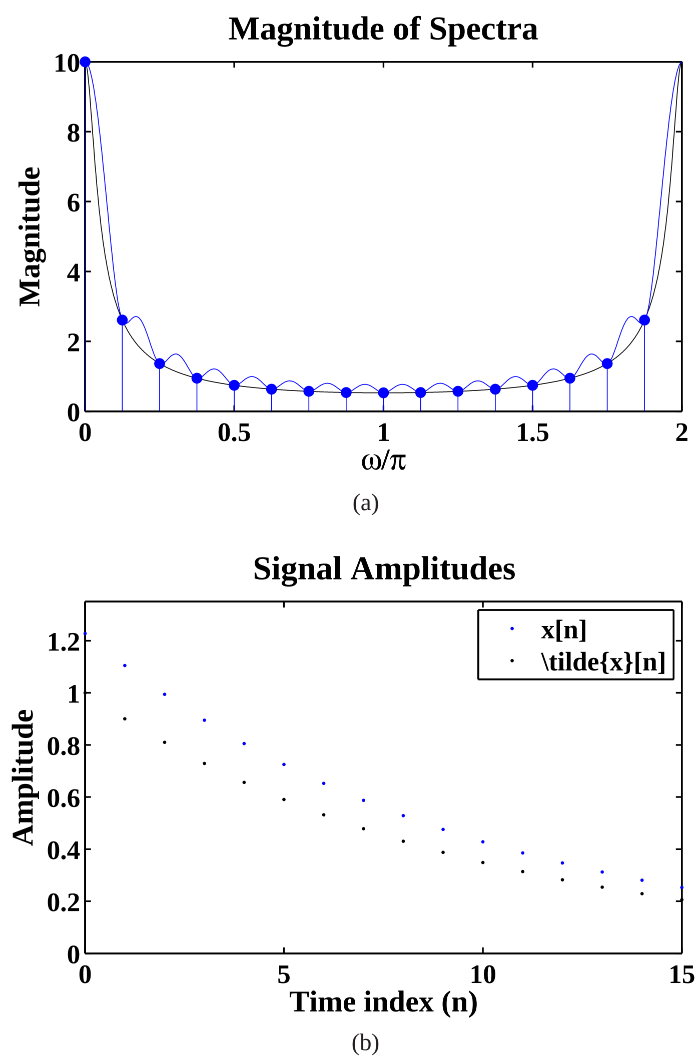


FIGURE 7.9: (a) Magnitude response of the DTFT signal. (b) Time sequence and reconstructed time sequence.

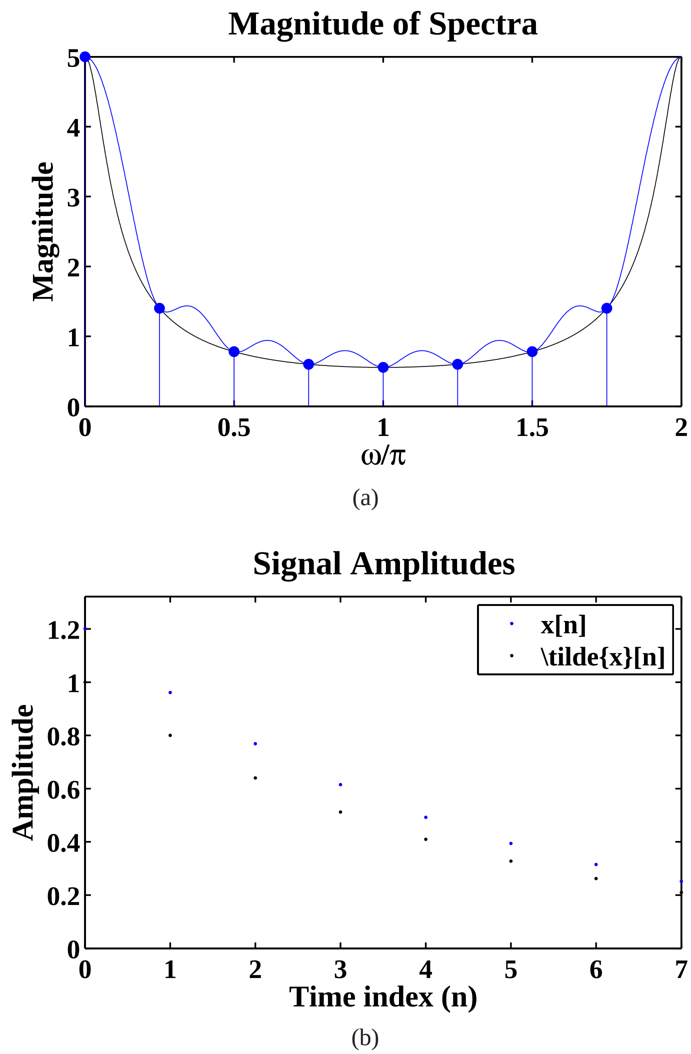
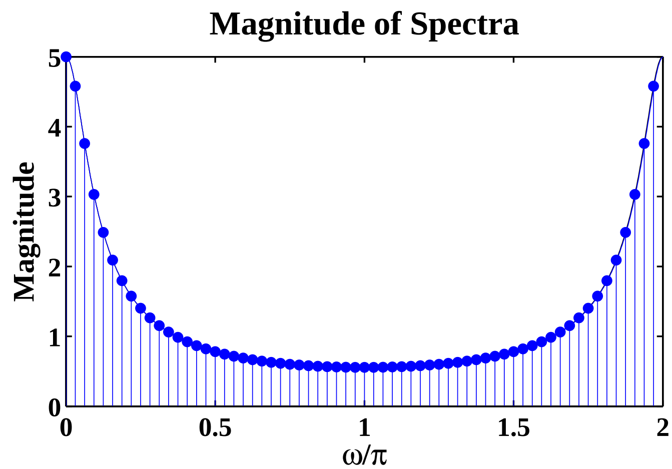
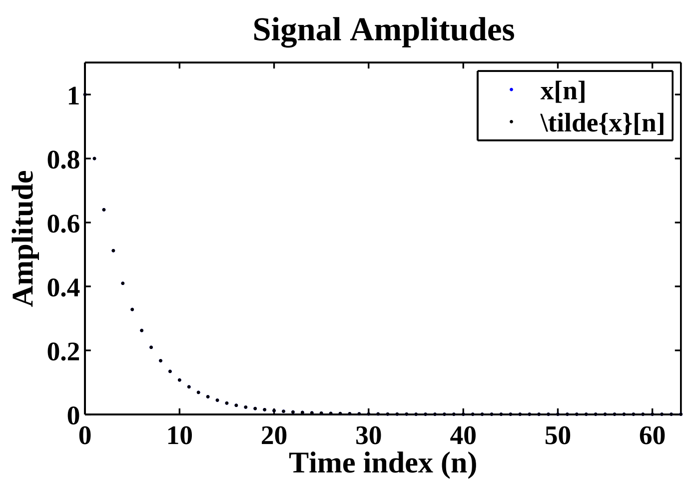


FIGURE 7.10: (a) Magnitude response of the DTFT signal and (b) time sequence and reconstructed time sequence for $a = 0.8$ and $N = 8$.



(a)



(b)

FIGURE 7.11: (a) Magnitude response of the DTFT signal and (b) time sequence and reconstructed time sequence for $a = 0.8$ and $N = 64$.

9. Proof:

$$\begin{aligned}\tilde{x}[n] &= \sum_{\ell=-\infty}^{\infty} x[n-\ell N] = \sum_{\ell=-\infty}^{\infty} a^{n-\ell N} u[n-\ell N] \\ &= \sum_{\ell=-\infty}^{0} a^{n-\ell N} = a^n \sum_{\ell=0}^{\infty} \left(a^N\right)^{\ell} = \frac{x[n]}{1-a^N}\end{aligned}$$

Hence, as $a \to 0$ or $N \to \infty$, we have $1 - a^N \to 1$, that is $\tilde{x}[n]$ tends to $x[n]$.

10. Proof:

$$X(z) = \frac{1-z^{-N}}{N} \sum_{k=0}^{N-1} \frac{\tilde{X}\left(\mathrm{e}^{\mathrm{j}\frac{2\pi}{N}k}\right)}{1-\mathrm{e}^{\mathrm{j}\frac{2\pi}{N}k} z^{-1}} \tag{7.72}$$

$$\tilde{X}(\mathrm{e}^{\mathrm{j}\omega}) = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}\left(\mathrm{e}^{\mathrm{j}\frac{2\pi}{N}k}\right) \tilde{P}_N\left[\mathrm{e}^{\mathrm{j}(\omega-\frac{2\pi}{N}k)}\right] \tag{7.62}$$

$$\tilde{P}_N(\mathrm{e}^{\mathrm{j}\omega}) = \frac{\sin(\omega N/2)}{N\sin(\omega/2)} \mathrm{e}^{-\mathrm{j}\omega(N-1)/2} \tag{7.63}$$

Substitute $z = \mathrm{e}^{j\omega}$ into (7.72), we have

$$\begin{aligned}X(z)|_{z=\mathrm{e}^{\mathrm{j}\omega}} &= \frac{1-\mathrm{e}^{-\mathrm{j}\omega N}}{N} \sum_{k=0}^{N-1} \frac{\tilde{X}\left(\mathrm{e}^{\mathrm{j}\frac{2\pi}{N}k}\right)}{1-\mathrm{e}^{\mathrm{j}\frac{2\pi}{N}k}\mathrm{e}^{-\mathrm{j}\omega}} \\ &= \frac{1}{N}\sum_{k=0}^{N-1} \frac{\tilde{X}\left(\mathrm{e}^{\mathrm{j}\frac{2\pi}{N}k}\right)(1-\mathrm{e}^{-\mathrm{j}\omega N}\cdot \mathrm{e}^{\mathrm{j}\frac{2\pi}{N}Nk})}{\mathrm{e}^{\mathrm{j}(\frac{2\pi}{N}k-\omega)\frac{1}{2}}\left[\mathrm{e}^{\mathrm{j}(\omega-\frac{2\pi}{N}k)\frac{1}{2}} - \mathrm{e}^{-\mathrm{j}(\omega-\frac{2\pi}{N}k)\frac{1}{2}}\right]} \\ &= \frac{1}{N}\sum_{k=0}^{N-1} \frac{\tilde{X}\left(\mathrm{e}^{\mathrm{j}\frac{2\pi}{N}k}\right)\mathrm{e}^{-\mathrm{j}(\omega-\frac{2\pi}{N}k)N/2}(\mathrm{e}^{\mathrm{j}(\omega-\frac{2\pi}{N}k)N/2} - \mathrm{e}^{-\mathrm{j}(\omega-\frac{2\pi}{N}k)N/2})}{\mathrm{e}^{\mathrm{j}(\frac{2\pi}{N}k-\omega)\frac{1}{2}}\left[\mathrm{e}^{\mathrm{j}(\omega-\frac{2\pi}{N}k)\frac{1}{2}} - \mathrm{e}^{-\mathrm{j}(\omega-\frac{2\pi}{N}k)\frac{1}{2}}\right]} \\ &= \frac{1}{N}\sum_{k=0}^{N-1} \frac{\tilde{X}\left(\mathrm{e}^{\mathrm{j}\frac{2\pi}{N}k}\right)\sin[(\omega-\frac{2\pi}{N}k)\frac{N}{2}]}{\sin[(\omega-\frac{2\pi}{N}k)\frac{1}{2}]}\mathrm{e}^{-\mathrm{j}(\omega-\frac{2\pi}{N}k)\frac{N-1}{2}} \\ &= \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}\left(\mathrm{e}^{\mathrm{j}\frac{2\pi}{N}k}\right)\tilde{P}_N\left[\mathrm{e}^{\mathrm{j}(\omega-\frac{2\pi}{N}k)}\right]\end{aligned}$$

11. Proof:

$$\begin{aligned}\sum_{n=0}^{N-1} x[\langle -n\rangle_N]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}nk} &= x[0] + \sum_{n=1}^{N-1} x[N-n]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}nk}\\ &= x[0] + \sum_{n=1}^{N-1} x[n]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}(N-n)k} = x[0] + \sum_{n=1}^{N-1} x[n]\mathrm{e}^{\mathrm{j}\frac{2\pi}{N}nk}\\ &= \sum_{n=0}^{N-1} x[n]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}n(-k)} = \sum_{n=0}^{N-1} x[n]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}n\langle -k\rangle_N}\\ &= X[\langle -k\rangle_N]\end{aligned}$$

12. (a) Proof:

$$x[n] \overset{\text{DFT}}{\longleftrightarrow} X_1[k] + \mathrm{j}X_2[k]$$
$$x^*[\langle -n\rangle_N] \overset{\text{DFT}}{\longleftrightarrow} X_1[k] - \mathrm{j}X_2[k]$$

$$\begin{aligned}x^{\text{cce}}[n] &= \frac{1}{2}x[n] + \frac{1}{2}x^*[\langle -n\rangle_N]\\ X^{\text{cce}}[k] &= \frac{1}{2}(X_1[k] + \mathrm{j}X_2[k]) + \frac{1}{2}(X_1[k] - \mathrm{j}X_2[k])\\ &= X_1[k]\end{aligned}$$

$$\begin{aligned}x^{\text{cco}}[n] &= \frac{1}{2}x[n] - \frac{1}{2}x^*[\langle -n\rangle_N]\\ X^{\text{cco}}[k] &= \frac{1}{2}(X_1[k] + \mathrm{j}X_2[k]) - \frac{1}{2}(X_1[k] - \mathrm{j}X_2[k])\\ &= \mathrm{j}X_2[k]\end{aligned}$$

(b) MATLAB script:

```
function [X1 X2] = tworealDFTs(x1,x2)
% Compute the DFTs of two real sequences using one DFT
xc = x1 + j*x2;
X = fft(xc);
XX = conj([X(1) fliplr(X(2:end))]);
X1 = (X+XX)/2;
X2 = (X-XX)/2/j;
```

(c) MATLAB script:

```
% P0712: Matlab Verification of function 'towrealDFTs'
close all; clc
n = 0:49;
N = length(n);
x1 = 0.9.^n;
x2 = 1 - 0.8.^n;
[X1 X2] = tworealDFTs(x1,x2);
% Verification:
X1_ref = fft(x1);
X2_ref = fft(x2);
```

13. (a) Proof:
If k is even and N is even, the correspondent DFT is:

$$
\begin{aligned}
X[k] &= \sum_{n=0}^{N-1} x[n]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}nk} = \sum_{n=0}^{N/2-1} x[n]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}nk} + \sum_{n=N/2}^{N-1} x[n]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}nk} \\
&= \sum_{n=0}^{N/2-1} \left(x[n]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}nk} + x[n+\frac{N}{2}]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}(n+\frac{N}{2})k} \right) \\
&= \sum_{n=0}^{N/2-1} \left(x[n] - x[n] \right) \mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}nk} = 0
\end{aligned}
$$

(b) Proof:
If $N = 4m$, $k = 4\ell$, the correspondent DFT is:

$$
\begin{aligned}
X[4\ell] &= \sum_{n=0}^{N-1} x[n]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}n(4\ell)} = \sum_{n=0}^{\frac{N}{4}-1} x[n]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}n(4\ell)} \\
&+ \sum_{n=\frac{N}{4}}^{\frac{2N}{4}-1} x[n]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}n(4\ell)} + \sum_{n=\frac{2N}{4}}^{\frac{3N}{4}-1} x[n]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}n(4\ell)} + \sum_{n=\frac{3N}{4}}^{N-1} x[n]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}n(4\ell)} \\
&= \left(\sum_{n=0}^{\frac{N}{4}-1} x[n]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}n(4\ell)} + \sum_{n=0}^{\frac{N}{4}-1} x[n+\frac{N}{4}]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}(n+\frac{N}{4})(4\ell)} \right) \\
&+ \left(\sum_{n=\frac{2N}{4}}^{\frac{3N}{4}-1} x[n]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}n(4\ell)} + \sum_{n=\frac{2N}{4}}^{\frac{3N}{4}-1} x[n+\frac{N}{4}]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}(n+\frac{N}{4})(4\ell)} \right) \\
&= \sum_{n=0}^{\frac{N}{4}-1} \left(x[n] + x[n+\frac{N}{4}] \right) \mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}n(4\ell)} + \sum_{n=\frac{2N}{4}}^{\frac{3N}{4}-1} \left(x[n] + x[n+\frac{N}{4}] \right) \mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}n(4\ell)} \\
&= 0
\end{aligned}
$$

14. (a) Solution:
Solving the circular convolution using hand calculation:

$$
\begin{bmatrix} 2 & 0 & -1 & 1 & -1 \\ -1 & 2 & 0 & -1 & 1 \\ 1 & -1 & 2 & 0 & -1 \\ -1 & 1 & -1 & 2 & 0 \\ 0 & -1 & 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ 0 \\ 6 \\ 7 \end{bmatrix}
$$

(b) See script below.

(c) See script below.

MATLAB script:

```
% P0714: Circular convolution
close all; clc
xn1 = 1:5;
xn2 = [2 -1 1 -1];
%% Part (b):
xn = circonv(xn1',[xn2 0]');
%% Part (c):
N = max(length(xn1),length(xn2));
Xk1 = fft(xn1,N);
Xk2 = fft(xn2,N);
Xk = Xk1.*Xk2;
xn_dft = ifft(Xk);
```

15. (a) Proof:
$X_4[K]$ can be obtained by frequency sampling of $X_3[k]$, hence in the time domain, according the aliasing equation, we have

$$x_4[n] = \sum_{\ell=-\infty}^{\infty} x_3[n + \ell N]$$

(b) Proof:
When $N \geq L$, there is no time aliasing, we conclude

$$x_4[n] = x_3[n], \quad \text{for} \quad 0 \leq n \leq L$$

When $\max(N_1, N_2) \leq N < L$, since $L = N_1 + N_2 - 1 \leq 2N - 1$, we conclude that

$$x_4[n] = x_3[n] + x_3[n + N], \quad \text{for} \quad 0 \leq n \leq N - 1$$

Hence, we proved the equation (**??**).

(c) MATLAB script:

```
% P0715: Verify formula in Problem 0715
close all; clc
xn1 = 1:4;
xn2 = 4:-1:1;
```

```
% N = 5;
N = 8;
n = 0:N-1;
xn3 = conv(xn1,xn2);
xn4 = circonv([xn1 zeros(1,N-4)]',[xn2 zeros(1,N-4)]')';
if N<7
xn3_N = xn3(N+1:end);
xn3_N = [xn3_N zeros(1,N)];
else
    xn3_N = zeros(1,N);
end
Nind = min(N,7);
xn4_ref = xn3(1:Nind) + xn3_N(1:Nind);
```

16. Proof:
The DFT of circular correlation $r_{xy}[\ell]$ is defined as

$$\begin{aligned}
R_{xy}[k] &= \sum_{\ell=0}^{N-1} r_{xy}[\ell] \mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}k\ell} = \sum_{\ell=0}^{N-1} \left(\sum_{n=0}^{N-1} x[n] y[\langle n-\ell \rangle_N] \right) \mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}k\ell} \\
&= \sum_{n=0}^{N-1} x[n] \left(\sum_{\ell=0}^{N-1} y[\langle n-\ell \rangle_N] \mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}k\ell} \right) \\
&= \sum_{n=0}^{N-1} x[n] \left(\sum_{\ell=0}^{N-1} y[\langle n-\ell \rangle_N] \mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}k(n-\ell)} \right)^* \mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}kn} \\
&= \left(\sum_{n=0}^{N-1} x[n] \mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}kn} \right) \left(\sum_{m=0}^{N-1} y[m] \mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}km} \right)^* \\
&= X[k] Y^*[k]
\end{aligned}$$

If $y[n]$ is real sequency.

17. Proof:

$$x^{(L)}[n] = \begin{cases} x[n/L], & n = 0, L, \cdots, (N-1)L \\ 0, & \text{otherwise} \end{cases}$$

$$x^{(L)}[n] \overset{\mathrm{DFT}}{\longleftrightarrow} \tilde{X}[k] = X[\langle k \rangle_N] \tag{7.140}$$

$$\frac{1}{L} x[\langle n \rangle_N] = \frac{1}{L} \tilde{x}[n] \overset{\mathrm{DFT}}{\longleftrightarrow} X^{(L)}[k] \tag{7.141}$$

$$\begin{aligned}
X^{(L)}[k] &= \sum_{n=0}^{NL-1} x^{(L)}[n]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{NL}nk} = \sum_{m=0}^{N-1} x^{(L)}[mL]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{NL}mLk} \\
&= \sum_{m=0}^{N-1} x[m]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}mk} = \sum_{m=0}^{N-1} x[m]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}m\langle k\rangle_N} \\
&= X[\langle k\rangle_N]
\end{aligned}$$

$$\begin{aligned}
x[n] &= \frac{1}{NL}\sum_{k=0}^{NL-1} X^{(L)}[k]\mathrm{e}^{\mathrm{j}\frac{2\pi}{NL}nk} = \frac{1}{NL}\sum_{m=0}^{N-1} X^{(L)}[mL]\mathrm{e}^{\mathrm{j}\frac{2\pi}{NL}nmL} \\
&= \frac{1}{NL}\sum_{n=0}^{N-1} X[m]\mathrm{e}^{\mathrm{j}\frac{2\pi}{N}mn} = \frac{1}{L}\left(\frac{1}{N}\sum_{m=0}^{N-1} X[m]\mathrm{e}^{\mathrm{j}\frac{2\pi}{N}mn}\right) \\
&= \frac{1}{L}\left(\frac{1}{N}\sum_{m=0}^{N-1} X[m]\mathrm{e}^{\mathrm{j}\frac{2\pi}{N}m\langle n\rangle_N}\right) = \frac{1}{L}x[\langle n\rangle_N]
\end{aligned}$$

18. Proof:

$$x_{(M)}[n] = x[nM], \qquad 0 \le n \le \frac{N}{M} - 1$$

$$x_{(M)}[n] \overset{\text{DFT}}{\longleftrightarrow} \frac{1}{M}\sum_{m=0}^{M-1} X[k + m\frac{N}{M}] \tag{7.143}$$

$$\frac{1}{M}\sum_{m=0}^{M-1} x[n + m\frac{N}{M}] \overset{\text{DFT}}{\longleftrightarrow} X_{(M)}[k] \tag{7.144}$$

We first prove equation (7.143):

$$\begin{aligned}
\frac{1}{M}\sum_{m=0}^{M-1} X[k + m\frac{N}{M}] &= \frac{1}{M}\sum_{m=0}^{M-1}\left(\sum_{n=0}^{N-1} x[n]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}n(k+m\frac{N}{M})}\right) \\
&= \frac{1}{M}\sum_{n=0}^{N-1} x[n]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}nk}\left(\sum_{m=0}^{M-1}\mathrm{e}^{-\mathrm{j}\frac{2\pi}{M}mn}\right) \\
&= \frac{1}{M}\sum_{n=0}^{N-1} x[n]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}nk}\cdot M\delta[\langle n\rangle_N] \\
&= \sum_{m=0}^{N/M-1} x[mM]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N/M}mk} = \mathrm{DFT}(x_{(M)}[n])
\end{aligned}$$

We then prove equation (7.144):

$$
\begin{aligned}
\frac{1}{M}\sum_{m=0}^{M-1} x[n+m\frac{N}{M}] &= \frac{1}{M}\sum_{m=0}^{M-1}\left(\frac{1}{N}\sum_{k=0}^{N-1} X[k]\mathrm{e}^{\mathrm{j}\frac{2\pi}{N}k(n+m\frac{N}{M})}\right) \\
&= \frac{1}{MN}\sum_{k=0}^{N-1} X[k]\mathrm{e}^{\mathrm{j}\frac{2\pi}{N}kn}\left(\sum_{m=0}^{M-1}\mathrm{e}^{\mathrm{j}\frac{2\pi}{M}km}\right) \\
&= \frac{1}{MN}\sum_{k=0}^{N-1} X[k]\mathrm{e}^{\mathrm{j}\frac{2\pi}{N}kn}\cdot M\delta[\langle k\rangle_N] \\
&= \frac{1}{N}\sum_{m=0}^{N/M-1} X[mM]\mathrm{e}^{\mathrm{j}\frac{2\pi}{N/M}mn} \\
&= \frac{1}{M}\text{IDFT}(X_{(M)}[k])
\end{aligned}
$$

19. (a) Proof:

$$w[n]x[n] \overset{\text{DFT}}{\longleftrightarrow} \frac{1}{N}W[k] \circledN X[k] \tag{7.148}$$

$$\begin{aligned}
\sum_{n=0}^{N-1} w[n]x[n]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}nk} &= \sum_{n=0}^{N-1} w[n]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}nk}\left(\frac{1}{N}\sum_{m=0}^{N-1} X[m]\mathrm{e}^{\mathrm{j}\frac{2\pi}{N}mn}\right) \\
&= \frac{1}{N}\sum_{m=0}^{N-1} X[m]\left(\sum_{n=0}^{N-1} w[n]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}n(k-m)}\right) \\
&= \frac{1}{N}\sum_{m=0}^{N-1} X[m]W[k-m] \\
&= \frac{1}{N}\sum_{m=0}^{N-1} X[m]W[\langle k-m\rangle_N] \\
&= \frac{1}{N}W[k] \circledN X[k]
\end{aligned}$$

(b) Proof:

$$\begin{aligned}
\frac{1}{N}W[k] \circledN X[k] &= \frac{1}{N}\sum_{m=0}^{N-1} X[m]W[\langle k-m\rangle_N] \\
&= \frac{1}{N}\sum_{m=0}^{N-1}\left(\sum_{n=0}^{N-1} x[n]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}nm}\right)W[\langle k-m\rangle_N] \\
&= \sum_{n=0}^{N-1} x[n]\left(\frac{1}{N}\sum_{m=0}^{N-1} W[\langle k-m\rangle_N]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}nm}\right) \\
&= \sum_{n=0}^{N-1} x[n]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}nk}\left(\frac{1}{N}\sum_{m=0}^{N-1} W[\langle k-m\rangle_N]\mathrm{e}^{\mathrm{j}\frac{2\pi}{N}n(k-m)}\right) \\
&= \sum_{n=0}^{N-1} x[n]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}nk}\left(\frac{1}{N}\sum_{m=0}^{N-1} W[\langle k-m\rangle_N]\mathrm{e}^{\mathrm{j}\frac{2\pi}{N}n\langle k-m\rangle_N}\right) \\
&= \sum_{n=0}^{N-1}(x[n]w[n])\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}nk} = \text{DFT}(x[n]w[n])
\end{aligned}$$

20. (a) Proof:

$$\tilde{x}_c(t) = w_c(t)x_c(t) \tag{7.161}$$

Linearity.

$$\begin{aligned} w_c(t)\cdot(a_1x_{c1}(t)+a_2x_{c2}(t)) &= a_1w_c(t)x_{c1}(t)+a_2w_c(t)x_{c2}(t) \\ &= a_1\tilde{x}_{c1}(t)+a_2\tilde{x}_{c2}(t) \end{aligned}$$

Time-varying.

$$\text{In general,} \qquad w_c(t-\tau)x_c(t-\tau) \neq w_c(t)x_c(t)$$

Hence, $\tilde{x}_c(t-\tau)\neq\tilde{x}_c(t)$.

(b) Proof:

$$\tilde{x}[n]=w[n]x[n] \tag{7.160}$$

If $0\le t\le T_0$, and $0\le n\le L$, we have

$$\begin{aligned} &\tilde{x}_c(nT)=w_c(nT)x_c(nT)=x_c(nT)=x[n] \\ &\tilde{x}[n]=w[n]x[n]=x[n] \end{aligned}$$

If $t>T_0$, and $N>L$, we have

$$\tilde{x}_c(nT)=\tilde{x}[n]=0$$

21. Proof:

$$\hat{X}_c(\mathrm{j}\Omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X_c(\mathrm{j}\theta)W_c(j(\Omega-\theta))\mathrm{d}\theta \tag{7.170}$$

The CTFT of $\hat{x}_c(t)$ is:

$$\begin{aligned} \hat{X}_c(\mathrm{j}\Omega) &= \int_{-\infty}^{\infty}\hat{x}_c(t)\mathrm{e}^{-\mathrm{j}\Omega t}\mathrm{d}t=\int_{-\infty}^{\infty}w_c(t)x_c(t)\mathrm{e}^{-\mathrm{j}\Omega t}\mathrm{d}t \\ &= \int_{-\infty}^{\infty}w_c(t)\mathrm{e}^{-\mathrm{j}\Omega t}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}X_c(\mathrm{j}\theta)\mathrm{e}^{\mathrm{j}\theta t}\mathrm{d}\theta\right)\mathrm{d}t \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}X_c(\mathrm{j}\theta)\left(\int_{-\infty}^{\infty}w_c(t)\mathrm{e}^{-\mathrm{j}(\Omega-\theta)t}\right)\mathrm{d}\theta \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}X_c(\mathrm{j}\theta)W_c(j(\Omega-\theta))\mathrm{d}\theta \end{aligned}$$

22. Proof:

$$\text{Scaling Property:} \quad x_c(at)\overset{\text{CTFT}}{\longleftrightarrow}\frac{1}{|a|}X_c\left(\frac{\mathrm{j}\Omega}{a}\right) \tag{7.172}$$

The CTFT of $x_c(at)$ is:

$$\int_{-\infty}^{\infty}x_c(at)\mathrm{e}^{-\mathrm{j}\Omega t}\mathrm{d}t=\frac{1}{a}\int_{-\infty}^{\infty}x_c(at)\mathrm{e}^{-\mathrm{j}\frac{\Omega}{a}at}\mathrm{d}at$$

If $a > 0$, we have

$$\frac{1}{a}\int_{-\infty}^{\infty} x_c(at)\mathrm{e}^{-\mathrm{j}\frac{\Omega}{a}at}\mathrm{d}at = \frac{1}{a}\int_{-\infty}^{\infty} x_c(t)\mathrm{e}^{-\mathrm{j}\frac{\Omega}{a}t}\mathrm{d}t = \frac{1}{a}X_c\left(\frac{\mathrm{j}\Omega}{a}\right)$$

If $a < 0$, we have

$$\frac{1}{a}\int_{\infty}^{-\infty} x_c(at)\mathrm{e}^{-\mathrm{j}\frac{\Omega}{a}at}\mathrm{d}at = \frac{1}{a}\int_{\infty}^{-\infty} x_c(t)\mathrm{e}^{-\mathrm{j}\frac{\Omega}{a}t}\mathrm{d}t = -\frac{1}{a}X_c\left(\frac{\mathrm{j}\Omega}{a}\right)$$

Hence, we proved the scaling property.

23. Proof:

$$\left|\int_{-\infty}^{\infty} x_{c1}(t)x_{c2}(t)\mathrm{d}t\right|^2 \leq \int_{-\infty}^{\infty}|x_{c1}(t)|^2\mathrm{d}t\int_{-\infty}^{\infty}|x_{c2}(t)|^2\mathrm{d}t \qquad (7.179)$$

Suppose a is a real number, define function $p(a)$ as

$$p(a) = \int_{-\infty}^{\infty}\left(a\cdot x_{c1}(t) + x_{c2}(t)\right)^2\mathrm{d}t = Aa^2 + 2Ba + C \geq 0$$

where

$$A = \int_{-\infty}^{\infty} x_{c1}^2(t)\mathrm{d}t, \quad B = \int_{-\infty}^{\infty} x_{c1}(t)x_{c2}(t)\mathrm{d}t, \quad C = \int_{-\infty}^{\infty} x_{c2}^2(t)\mathrm{d}t.$$

Since we have $4B^2 - 4AC \leq 0$, that is $B^2 \leq AC$,

$$\left(\int_{-\infty}^{\infty} x_{c1}(t)x_{c2}(t)\mathrm{d}t\right)^2 \leq \int_{-\infty}^{\infty} x_{c1}^2(t)\mathrm{d}t\int_{-\infty}^{\infty} x_{c2}^2(t)\mathrm{d}t$$

24. Proof:
The CTFT of generic window is:

$$W(\mathrm{j}\Omega) = aW_{\mathrm{R}}(\mathrm{j}\Omega) + bW_{\mathrm{R}}(\mathrm{j}(\Omega-2\pi)/T_0) + bW_{\mathrm{R}}(\mathrm{j}(\Omega+2\pi)/T_0) \qquad (7.189)$$

The ICTFT is:

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[aW_{\mathrm{R}}(\mathrm{j}\Omega) + bW_{\mathrm{R}}(\mathrm{j}(\Omega-2\pi)/T_0) + bW_{\mathrm{R}}(\mathrm{j}(\Omega+2\pi)/T_0)\right]\mathrm{e}^{\mathrm{j}\Omega t}\mathrm{d}\Omega\\
&\quad = aw_R(t) + b\mathrm{e}^{\mathrm{j}\frac{2\pi}{T_0}t}w_R(t) + b\mathrm{e}^{-\mathrm{j}\frac{2\pi}{T_0}t}w_R(t)\\
&\quad = \left[a + 2b\cos\left(\frac{2\pi}{T_0}t\right)\right]w_R(t)
\end{aligned}$$

If $a = b = 0.5$, we can prove:

$$w_{\text{Han}}(t) = \left[0.5 + \cos\left(\frac{2\pi}{T_0}t\right)\right] w_R(t) \tag{7.190}$$

If $a = 0.54$ and $b = 0.23$, we can prove:

$$w_{\text{Ham}}(t) = \left[0.5 + 0.46\cos\left(\frac{2\pi}{T_0}t\right)\right] w_R(t) \tag{7.191}$$

25. tba

26. (a) Proof:

$$X[k, ell] = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} x[m,n] W_M^{mk} W_N^{n\ell} = \sum_{m=0}^{M-1}\left(\sum_{n=0}^{N-1} x[m,n] W_N^{n\ell}\right) W_M^{mk}$$

(b) See figure below.

(c) Proof:

$$\begin{aligned} X[k,\ell] &= \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} x_1[m] x_2[n] W_M^{mk} W_N^{n\ell} \\ &= \left(\sum_{m=0}^{M-1} x_1[m] W_M^{mk}\right)\left(\sum_{n=0}^{N-1} x_2[n] W_N^{n\ell}\right) \end{aligned}$$

(d) See figure below.

MATLAB script:

```
% P0726: 2D FFT and 1D FFT
close all; clc
M = 100; N = 100;
m = 0:M-1; n = 0:N-1;
% Part (b):
[NN MM] = meshgrid(n,m);
xmn = 0.9.^(MM+NN);
X = fftshift(fft2(xmn));
X_mag = abs(X);
% Part (d)
Xm = abs(fft(0.9.^m));
Xn = abs(fft(0.9.^n));
```

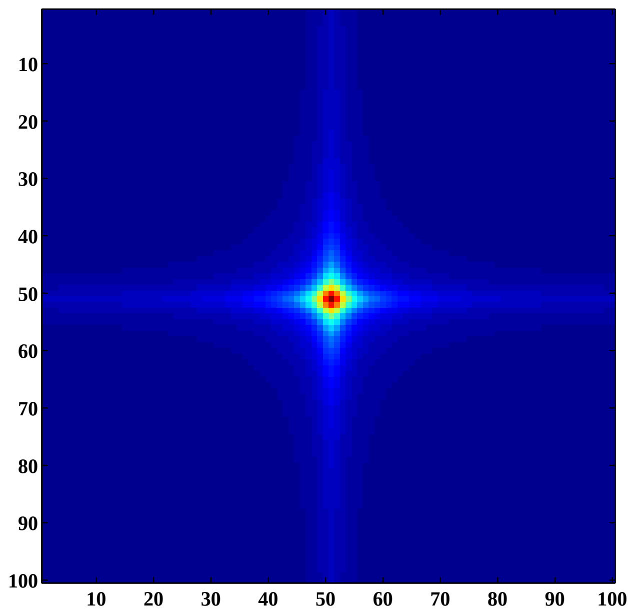


FIGURE 7.12: Magnitude response of the image using `fft2` function to compute 2D-DFT.

```
X_mag2 = fftshift(Xm(:)*Xn);
% Plot:
hfa = figure;
imagesc(X_mag); axis square
hfb = figure;
imagesc(X_mag2); axis square
```

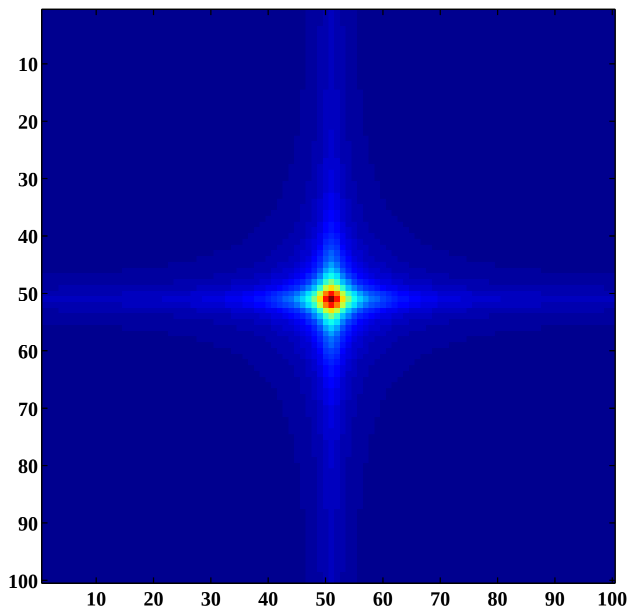


FIGURE 7.13: Magnitude response of the image using `fft2` function to compute 2D-DFT.

Basic Problems

27. (a) Solution:
The CTFT of $\mathrm{e}^{-20t}\cos(20\pi t)u(t)$ is:

$$\frac{20+\mathrm{j}\Omega}{(20+\mathrm{j}\Omega)^2+(20\pi)^2}$$

The CTFT of $x_\mathrm{c}(t)$ is:

$$10\mathrm{j}\frac{\mathrm{d}}{\mathrm{d}\Omega}\left(\frac{20+\mathrm{j}\Omega}{(20+\mathrm{j}\Omega)^2+(20\pi)^2}\right)=\frac{-10\mathrm{j}\times[-(20+\mathrm{j}2\pi F)^2]+(20\pi)^2}{[(20+\mathrm{j}2\pi F)^2]+(20\pi)^2]^2}$$

(b) See plot below.

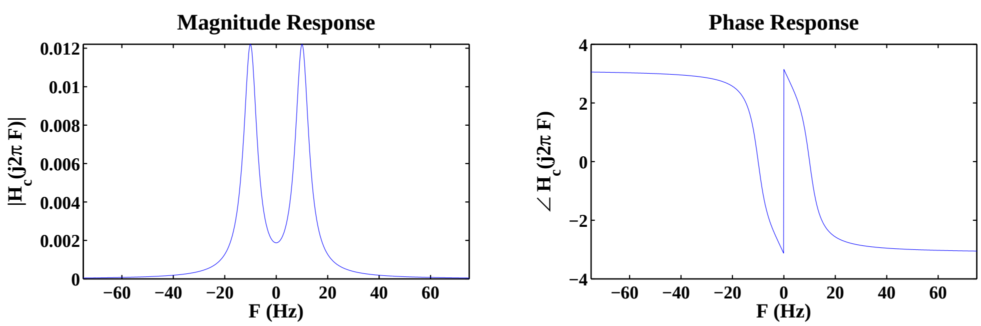


FIGURE 7.14: Magnitude and phase responses of $X_\mathrm{c}(\mathrm{j}2\pi F)$ over $-75\le F\le 75$ Hz.

(c) See plot below.

MATLAB script:

```
% P0727: Numerical DFT approximation of CTFT
close all; clc
%% Part (b):
t1 = 0; t2 = 2;
dF = 0.1;
F = -75:dF:75;
Xc = -10*((20*pi)^2-(j*2*pi*F+20).^2)...
    ./((20*pi)^2+(j*2*pi*F+20).^2).^2;
%% Part (c):
Fs = 150;
```

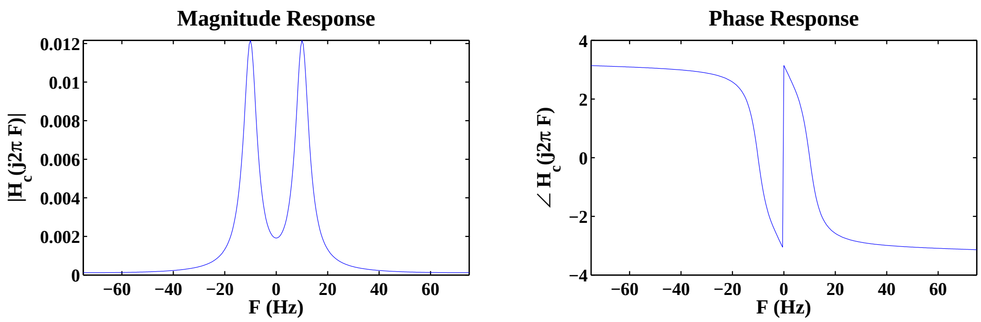


FIGURE 7.15: Approximated magnitude and phase responses of $X_c(j2\pi F)$ over $-75 \le F \le 75$ Hz using `fft` function.

```
T = Fs\1;
nT = t1:T:t2;
N = length(nT);
xn = 10*nT.*exp(-20*nT).*cos(20*pi*nT);
X = fftshift(fft(xn));
w = linspace(-pi,pi,N);
Xc_approx = T*X;
%% Plot:
hfa = figconfg('P0727a','long');
subplot(121)
plot(F,abs(Xc))
xlim([F(1) F(end)])
ylim([0 max(abs(Xc))])
xlabel('F (Hz)','fontsize',LFS)
ylabel('|H_c(j2\pi F)|','fontsize',LFS)
title('Magnitude Response','fontsize',TFS)
subplot(122)
plot(F,angle(Xc))
xlim([F(1) F(end)])
xlabel('F (Hz)','fontsize',LFS)
ylabel('\angle H_c(j2\pi F)','fontsize',LFS)
title('Phase Response','fontsize',TFS)

hfb = figconfg('P0727b','long');
subplot(121)
plot(w/T/2/pi,abs(Xc_approx))
```

```
xlim([F(1) F(end)])
ylim([0 max(Xc_approx)])
xlabel('F (Hz)','fontsize',LFS)
ylabel('|H_c(j2\pi F)|','fontsize',LFS)
title('Magnitude Response','fontsize',TFS)
subplot(122)
plot(w/T/2/pi,angle(Xc_approx))
xlim([F(1) F(end)])
xlabel('F (Hz)','fontsize',LFS)
ylabel('\angle H_c(j2\pi F)','fontsize',LFS)
title('Phase Response','fontsize',TFS)
```

28. (a) Solution:
$T_0 = 5$, the fundamental period $\Omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{5}$.
The CTFS of $\tilde{x}_c(t)$ is:

$$c_k = \frac{1}{T_0}\int_0^{T_0} \tilde{x}_c(t)\mathrm{e}^{-\mathrm{j}k\Omega_0 t}\mathrm{d}t = \frac{1}{5}\int_0^5 t\mathrm{e}^{-t}\cdot \mathrm{e}^{-\mathrm{j}k\Omega_0 t}dt$$
$$= \frac{1}{5}\times\frac{\mathrm{e}^{-2.5}}{-0.5-\mathrm{j}k\frac{2\pi}{5}} + \frac{1}{5}\times\frac{1-\mathrm{e}^{-2.5}}{(0.5+\mathrm{j}k\frac{2\pi}{5})^2}$$

(b) See plot below.

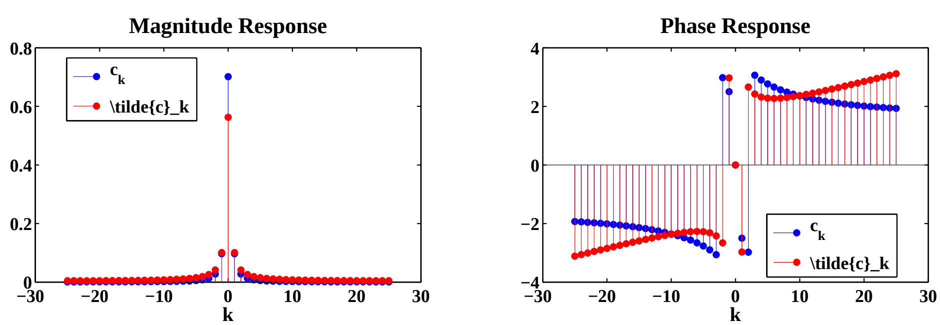


FIGURE 7.16: Magnitude and phase responses of c_k and $\hat{c}_k$ when the sampling interval is $T = 0.1$s.

(c) See plot below.

(d) See plot below.

MATLAB script:

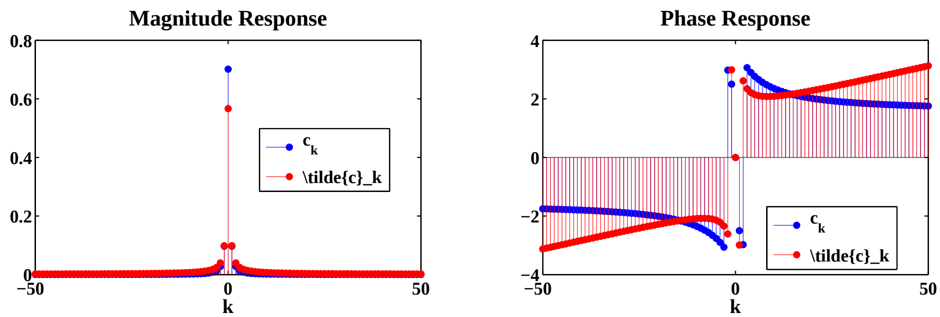


FIGURE 7.17: Magnitude and phase responses of c_k and $\hat{c}_k$ when the sampling interval is $T = 0.05$s.

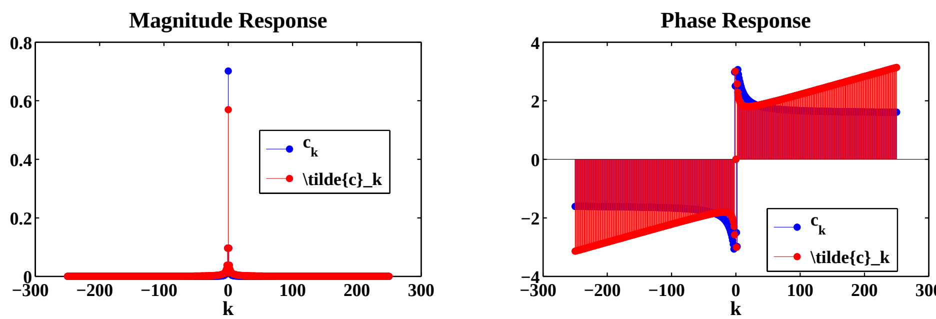


FIGURE 7.18: Magnitude and phase responses of c_k and $\hat{c}_k$ when the sampling interval is $T = 0.01$s.

```
% P0728: Numerical DFT approximation of CTFS
close all; clc
t1 = 0; t2 = 5;
T = 0.1; % Part (b)
% T = 0.05; % Part (c)
% T = 0.01; % Part (d)
T0 = t2-t1; Omega0 = 2*pi/T0;
N = t2/T;
k = -N/2:N/2;
ck = 1/T0*exp(-0.5*t2)./(-0.5-j*Omega0*k) + ...
    1/T0*(1-exp(-0.5*t2))./(0.5+j*Omega0*k).^2;
nT = t1:T:t2;
xn = nT.*exp(-0.5*nT);
```

```
ck_approx = length(xn)\fftshift(fft(xn));
%% Plot:
hfa = figconfg('P0728a','long');
subplot(121)
stem(k,abs(ck),'filled');hold on
stem(k,abs(ck_approx),'filled','color','red');
xlabel('k','fontsize',LFS)
title('Magnitude Response','fontsize',TFS)
legend('c_k','\tilde{c}_k','location','best')
subplot(122)
stem(k,angle(ck),'filled');hold on
stem(k,angle(ck_approx),'filled','color','red');
xlabel('k','fontsize',LFS)
title('Phase Response','fontsize',TFS)
legend('c_k','\tilde{c}_k','location','best')
```

29. (a) Solution:
The DTFT of $x[n]$ is:

$$\begin{aligned}
\tilde{X}(\mathrm{e}^{\mathrm{j}\omega}) &= \frac{10}{2\mathrm{j}}\sum_{n=0}^{\infty}(0.5)^n(\mathrm{e}^{\mathrm{j}0.1\pi n} - \mathrm{e}^{-\mathrm{j}0.1\pi n})\mathrm{e}^{-\mathrm{j}\omega n} \\
&= \frac{10}{2\mathrm{j}}\left(\sum_{n=0}^{\infty}\left[0.5\mathrm{e}^{-\mathrm{j}(\omega-0.1\pi)}\right]^n - \sum_{n=0}^{\infty}\left[0.5\mathrm{e}^{-\mathrm{j}(\omega+0.1\pi)}\right]^n\right) \\
&= \frac{10}{2\mathrm{j}}\left(\frac{1}{1-0.5\mathrm{e}^{-\mathrm{j}(\omega-0.1\pi)}} - \frac{1}{1-0.5\mathrm{e}^{-\mathrm{j}(\omega+0.1\pi)}}\right) \\
&= \frac{5\sin(0.1\pi)\mathrm{e}^{-\mathrm{j}\omega}}{1-\cos(0.1\pi)\mathrm{e}^{-\mathrm{j}\omega}+0.25\mathrm{e}^{-2\mathrm{j}\omega}}
\end{aligned}$$

(b) See plot below.

(c) See plot below.

(d) See plot below.

MATLAB script:

```
% P0729: Numerical DFT approximation of DTFT
close all; clc
w = linspace(0,2,1000)*pi;
X = 5*sin(0.1*pi)*exp(-j*w)./...
    (1-cos(0.1*pi)*exp(-j*w)+0.25*exp(-j*2*w));
```

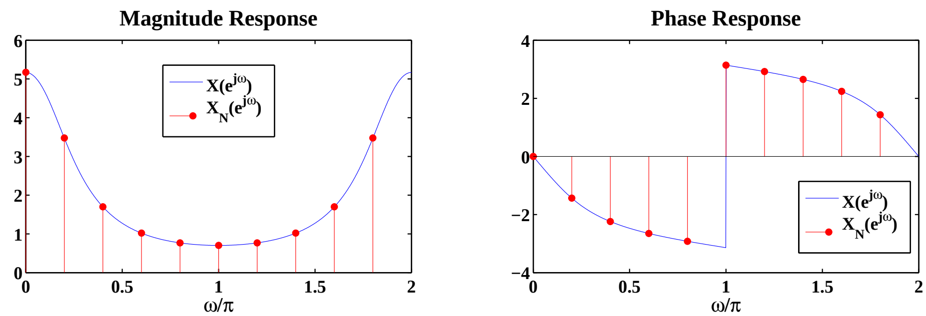


FIGURE 7.19: Magnitude and phase responses of $\tilde{X}(\mathrm{e}^{\mathrm{j}\omega})$ and $\tilde{X}_N(\mathrm{e}^{\mathrm{j}\omega})$ when $N = 10$.

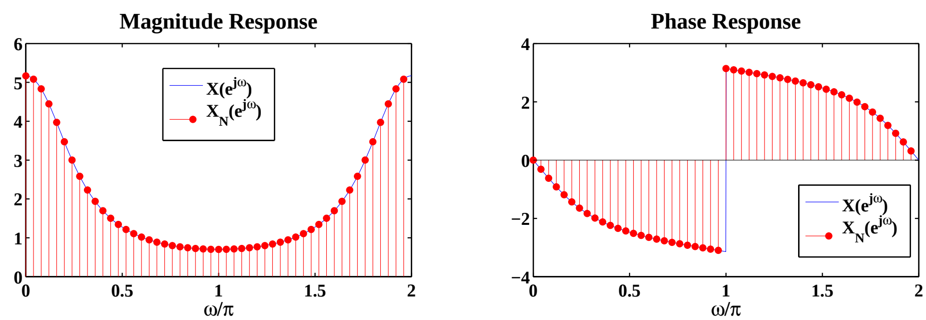


FIGURE 7.20: Magnitude and phase responses of $\tilde{X}(\mathrm{e}^{\mathrm{j}\omega})$ and $\tilde{X}_N(\mathrm{e}^{\mathrm{j}\omega})$ when $N = 50$.

```
N = 10; % Part (b)
% N = 50; % Part (c)
% N = 100; % Part (d)
n = 0:N-1;
xn = 10*(0.5.^n).*sin(0.1*pi*n);
XN = fft(xn);
wk = 2/N*(0:N-1);
%% Plot:
hfa = figconfg('P0729a','long');
subplot(121)
plot(w/pi,abs(X));hold on
stem(wk,abs(XN),'filled','color','red');
xlabel('\omega/\pi','fontsize',LFS)
```

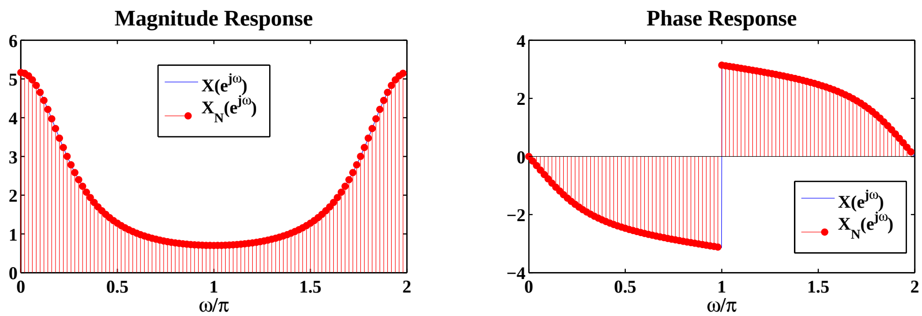


FIGURE 7.21: Magnitude and phase responses of $\tilde{X}(\mathrm{e}^{\mathrm{j}\omega})$ and $\tilde{X}_N(\mathrm{e}^{\mathrm{j}\omega})$ when $N = 100$.

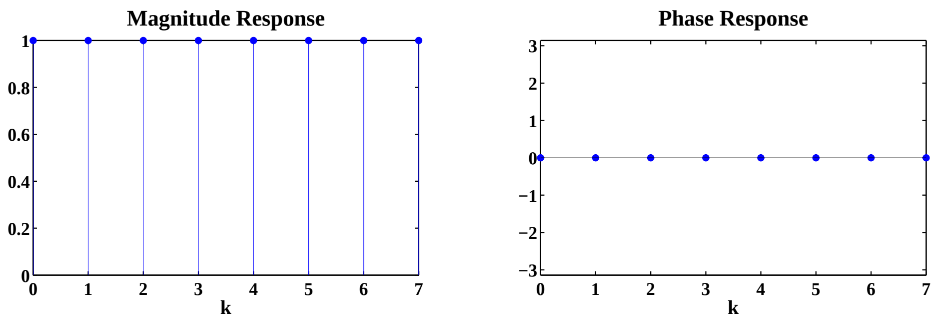
```
title('Magnitude Response','fontsize',TFS)
legend('X(e^{j\omega})','X_N(e^{j\omega})','location','best')
subplot(122)
plot(w/pi,angle(X));hold on
stem(wk,angle(XN),'filled','color','red');
xlabel('\omega/\pi','fontsize',LFS)
title('Phase Response','fontsize',TFS)
legend('X(e^{j\omega})','X_N(e^{j\omega})','location','best')
```

30. tba

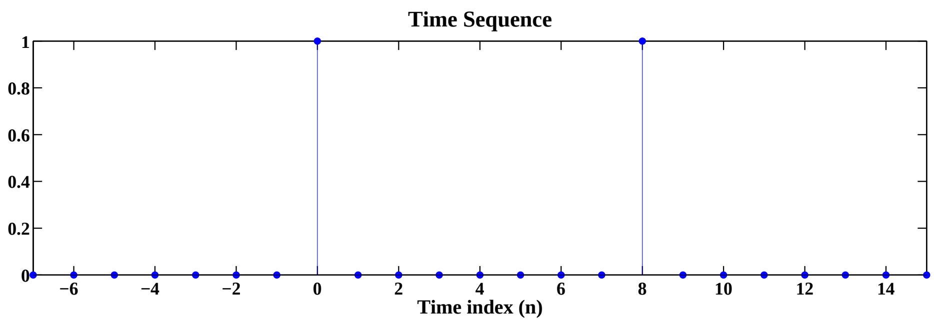
31. (a) See plot below.
    (b) See plot below.
    (c) See plot below.
    (d) See plot below.

    MATLAB script:

```
% P0731: Compute and plot DFT and IDFT
close all; clc
%% Part (a):
N = 8;
n = 0:N-1;
xn = zeros(size(n));
xn(1) = 1;

%% Part (b):
```



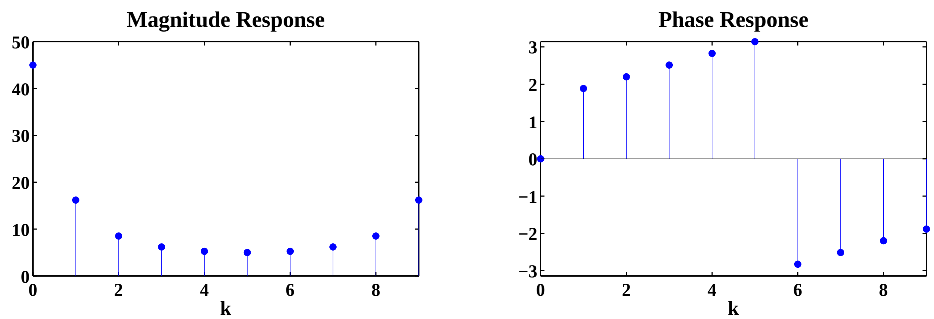
(a)



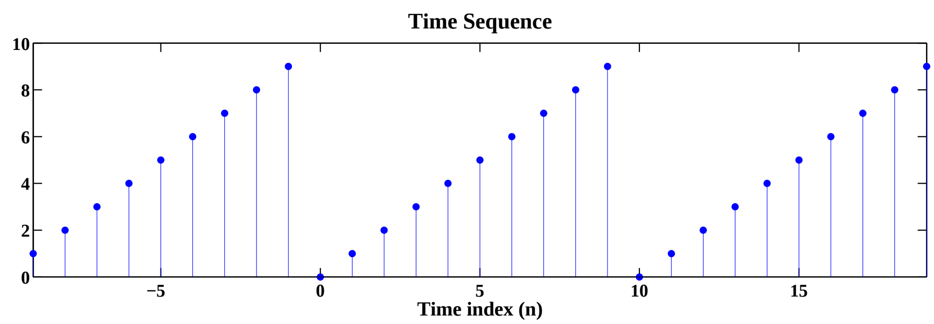
(b)

FIGURE 7.22: N-point (a) DFT and (b) IDFT of $x[n] = \delta[n]$, $N = 8$ in the range $-(N-1) \leq n \leq (2N-1)$.

```
% N = 10;
% n = 0:N-1;
% xn = n;

%% Part (c):
% N = 30;
% n = 0:N-1;
% xn = cos(6*pi*n/15);

%% Part (d):
% N = 30;
% n = 0:N-1;
% xn = cos(0.1*pi*n);

Xk = fft(xn);
```



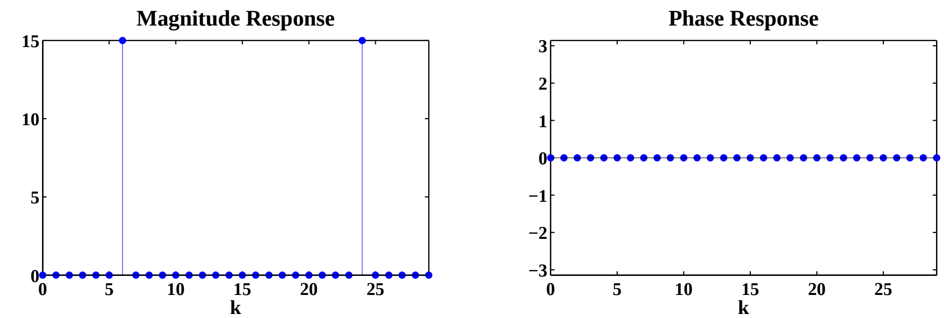
(a)



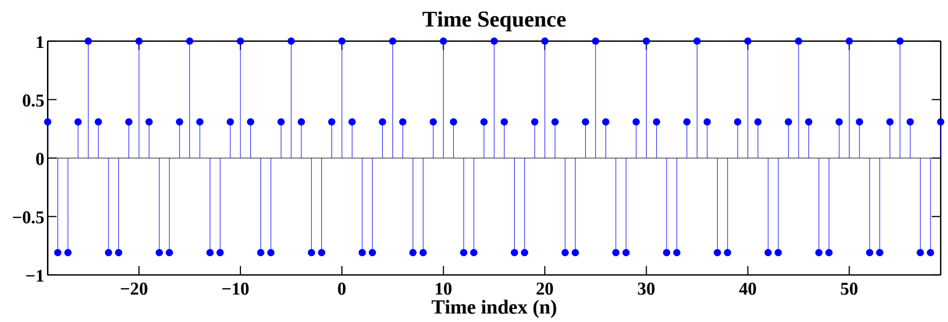
(b)

FIGURE 7.23: N-point (a) DFT and (b) IDFT of $x[n] = n$, $N = 10$ in the range $-(N-1) \leq n \leq (2N-1)$.

```
ind = abs(Xk) < 1e-10;
Xk(ind) = 0;
xn_ref = ifft(Xk);
nn = -(N-1):2*N-1;
xn_plot = xn_ref(mod(nn,N)+1);

%% Plot:
hfa = figconfg('P0731a','long');
subplot(121)
stem(n,abs(Xk),'filled');
xlim([0 N-1])
xlabel('k','fontsize',LFS)
title('Magnitude Response','fontsize',TFS)
subplot(122)
stem(n,angle(Xk),'filled');
```



(a)



(b)

FIGURE 7.24: N-point (a) DFT and (b) IDFT of $x[n] = \cos(6\pi n/15)$, $N = 30$ in the range $-(N-1) \leq n \leq (2N-1)$.

```
xlim([0 N-1])
ylim([-pi pi])
xlabel('k','fontsize',LFS)
title('Phase Response','fontsize',TFS)

hfb = figconfg('P0731b','long');
stem(nn,xn_plot,'filled')
xlim([nn(1) nn(end)])
xlabel('Time index (n)','fontsize',LFS)
title('Time Sequence','fontsize',TFS)
```

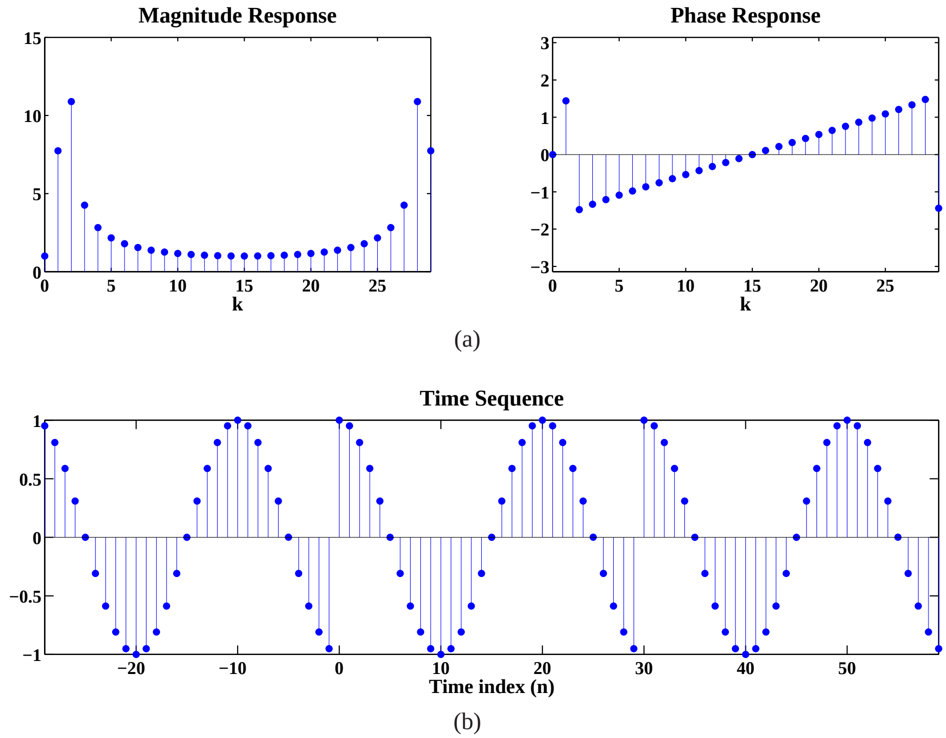


FIGURE 7.25: N-point (a) DFT and (b) IDFT of $x[n] = \cos(0.1\pi n)$, $N = 30$ in the range $-(N-1) \leq n \leq (2N-1)$.

32. (a) Solution: The DFS of $\tilde{x}[n]$ and $\tilde{x}_3[n]$ can be written as:

$$\tilde{X}[k] = X[\langle k\rangle_N] = \sum_{n=0}^{N-1} \tilde{x}[n]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}n\langle k\rangle_N}$$

$$\tilde{X}_3[k] = X_3[\langle k\rangle_{3N}] = \sum_{n=0}^{3N-1} \tilde{x}[n]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{3N}n\langle k\rangle_{3N}}$$

We have

$$
\begin{aligned}
\tilde{X}_3[k] &= \sum_{n=0}^{3N-1} \tilde{x}[n] \mathrm{e}^{-\mathrm{j}\frac{2\pi}{3N} n \langle k \rangle_{3N}} \\
&= \sum_{n=0}^{N-1} \tilde{x}[n] \mathrm{e}^{-\mathrm{j}\frac{2\pi}{3N} n \langle k \rangle_{3N}} + \sum_{n=N}^{2N-1} \tilde{x}[n] \mathrm{e}^{-\mathrm{j}\frac{2\pi}{3N} n \langle k \rangle_{3N}} + \sum_{n=2N}^{3N-1} \tilde{x}[n] \mathrm{e}^{-\mathrm{j}\frac{2\pi}{3N} n \langle k \rangle_{3N}} \\
&= \sum_{n=0}^{N-1} \tilde{x}[n] \mathrm{e}^{-\mathrm{j}\frac{2\pi}{3N} n \langle k \rangle_{3N}} + \sum_{n=0}^{N-1} \tilde{x}[n+N] \mathrm{e}^{-\mathrm{j}\frac{2\pi}{3N} (n+N) \langle k \rangle_{3N}} \\
&\quad + \sum_{n=0}^{N-1} \tilde{x}[n+2N] \mathrm{e}^{-\mathrm{j}\frac{2\pi}{3N} (n+2N) \langle k \rangle_{3N}} \\
&= \left(\sum_{n=0}^{N-1} \tilde{x}[n] \mathrm{e}^{-\mathrm{j}\frac{2\pi}{N} n \langle k \rangle_{N}} \right) \cdot \left(1 + \mathrm{e}^{-\mathrm{j}\frac{2\pi}{3} \langle k \rangle_{3N}} + \mathrm{e}^{-\mathrm{j}\frac{4\pi}{3} \langle k \rangle_{3N}} \right) \\
&= 3\tilde{X}[k/3]
\end{aligned}
$$

(b) MATLAB script:

```
% P0732: Matlab verification
close all; clc
xn = [1 3 1 3 1 3];
Xk = fft(xn(1:2));
X3k = fft(xn);
```

33. Solution:
`plot(dftmtx(16))` plots each complex vector $\boldsymbol{w}_k$ within DFT matrix W_{16}. It plots the elements of $\boldsymbol{w}_k$ with real part versus imaginary part and then connect these points from the first one to the last with solid line. Different line color corresponds to different k value.

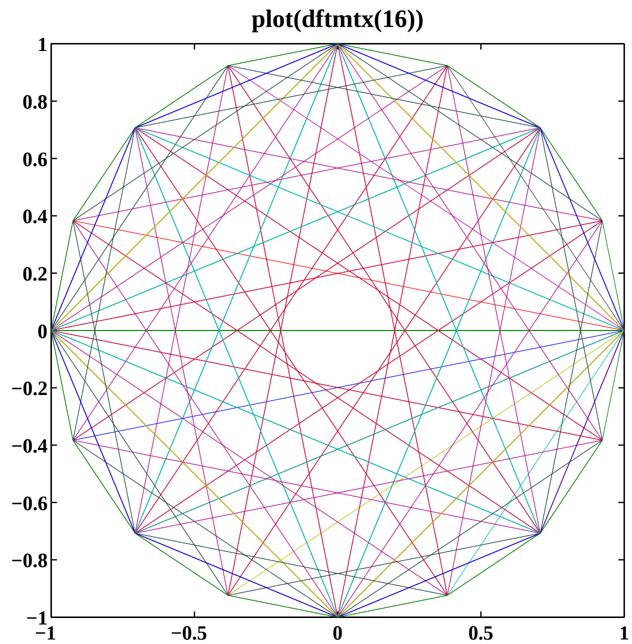


FIGURE 7.26: Plot of `plot(dftmtx(16))`.

34. (a) Solution:
The DTFT of $x[n]$ is:

$$\begin{aligned}\tilde{X}(\mathrm{e}^{\mathrm{j}\omega}) &= \sum_{n=-\infty}^{\infty} x[n]\mathrm{e}^{-\mathrm{j}n\omega} = \sum_{n=-\infty}^{\infty} (0.8)^{|n|}\mathrm{e}^{-\mathrm{j}n\omega} \\ &= \sum_{n=-\infty}^{-1} (0.8)^{-n}\mathrm{e}^{-\mathrm{j}n\omega} + 1 + \sum_{n=1}^{\infty}(0.8)^{n}\mathrm{e}^{-\mathrm{j}n\omega} \\ &= 1 + \sum_{n=1}^{\infty}(0.8)^{n}\mathrm{e}^{-\mathrm{j}n\omega} + \sum_{n=1}^{\infty}(0.8)^{n}\mathrm{e}^{\mathrm{j}n\omega} \\ &= \frac{1-0.8^2}{1-2\cdot 0.8\cos\omega + 0.8^2}\end{aligned}$$

(b) Solution:

$$\tilde{x}[n+8] = \sum_{\ell=-\infty}^{\infty} x[n-8\ell+8] = \sum_{\ell=-\infty}^{\infty} x[n-8\ell] = \tilde{x}[n]$$

We conclude that the period of $\tilde{x}[n]$ is $N=8$. Hence we can compute the DFS of $\tilde{x}[n]$ is:

$$\begin{aligned}\tilde{X}[k] &= \sum_{n=0}^{7} \tilde{x}[n]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{8}n\langle k\rangle_8} = \sum_{n=0}^{7}\sum_{\ell=-\infty}^{\infty} x[n-8\ell]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{8}n\langle k\rangle_8} \\ &= \sum_{\ell=-\infty}^{\infty}\sum_{n=0}^{7} 0.8^{|n-8\ell|}\mathrm{e}^{-\mathrm{j}\frac{2\pi}{8}n\langle k\rangle_8} \\ &= \sum_{\ell=-\infty}^{0}\sum_{n=0}^{7} 0.8^{n-8\ell}\mathrm{e}^{-\mathrm{j}\frac{2\pi}{8}n\langle k\rangle_8} + \sum_{\ell=1}^{\infty}\sum_{n=0}^{7} 0.8^{-n+8\ell}\mathrm{e}^{-\mathrm{j}\frac{2\pi}{8}n\langle k\rangle_8} \\ &= \left(1+2\sum_{\ell=1}^{\infty}0.8^{8\ell}\right)\left(\frac{1-0.8^8}{1-0.8\mathrm{e}^{\mathrm{j}\frac{2\pi}{8}\langle k\rangle_8}} + \frac{1-0.8^{-8}}{1-0.8^{-1}\mathrm{e}^{\mathrm{j}\frac{2\pi}{8}\langle k\rangle_8}}\right)\end{aligned}$$

(c) See plot below.

MATLAB script:

```
% P0734:
close all; clc
%% Specification:
```

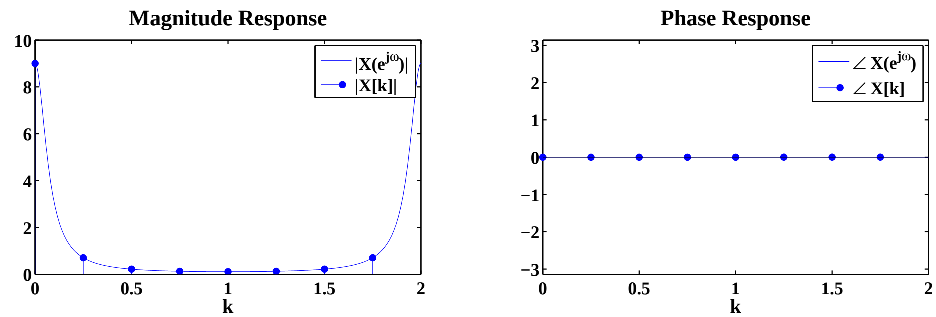


FIGURE 7.27: Plot of DTFT $\tilde{X}(\mathrm{e}^{\mathrm{j}\omega})$ and stem plot of DFS $\tilde{X}[k]$ when $x[n] = (0.8)^{|n|}$.

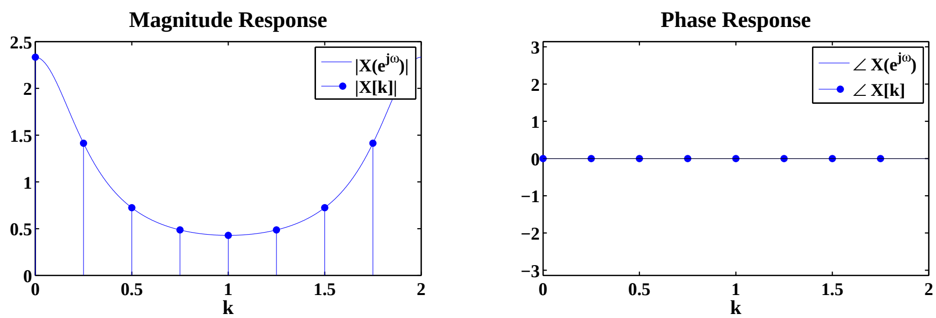


FIGURE 7.28: Plot of DTFT $\tilde{X}(\mathrm{e}^{\mathrm{j}\omega})$ and stem plot of DFS $\tilde{X}[k]$ when $x[n] = (0.4)^{|n|}$.

```
% a = 0.8;
a = 0.4;
n = -20:20;
N = 8;
k = 2*pi/N*(0:N-1);
xnp = zeros(size(n));
for jj = -10:10
    xnp = xnp + a.^abs(n-jj*8);
end
w = linspace(0,2,1000)*pi;
X = (1-a^2)./(1-2*a*cos(w)+a^2);
X_phase = zeros(size(w));
%% Part (b)
```

```
nind = n >=0 & n<=7;
Xk = fft(xnp(nind));
%% Plot:
hfb = figconfg('P0734b','long');
subplot(121)
plot(w/pi,abs(X));hold on
stem(k/pi,abs(Xk),'filled');
xlim([0 2])
xlabel('k','fontsize',LFS)
title('Magnitude Response','fontsize',TFS)
legend('|X(e^{j\omega})|','|X[k]|','location','northeast')
subplot(122)
plot(w/pi,X_phase);hold on
stem(k/pi,angle(Xk),'filled');
xlim([0 2])
ylim([-pi pi])
xlabel('k','fontsize',LFS)
title('Phase Response','fontsize',TFS)
legend('\angle X(e^{j\omega})','\angle X[k]','location','northeast')
```

35. (a) Solution:
The DTFT $\tilde{X}(e^{j\omega})$ is:

$$\tilde{X}(e^{j\omega}) = \frac{1}{1-0.8e^{-j\omega}} + \frac{1}{1+0.8e^{-j\omega}} = \frac{2}{1-0.8^2 e^{-2j\omega}}$$

(b) Solution:
$\omega = \frac{2\pi}{10}k$, $N = 10$, we can conclude $g[n]$ as

$$g[n] = \sum_{\ell=-\infty}^{\infty} x[n-10\ell]$$

36. Solution:
The ones have a real-valued 8-point DFTs are:

$$x_2[n] = \{5, 2, -9, 4, 7, 4, -9, 2\}$$

$$x_5[n] = \{10, 5, -7, -4, 5, -4, -7, 5\}$$

The ones have 8-point imaginary-valued DFTs are:

$$x_1[n] = \{0, -3, 1, -2, 0, 2, -1, 3\}$$

The ones are complex valued are:

$$x_3[n] = \{8, -3, 1, -2, 6, 2, -1, 3\}$$

$$x_4[n] = \{0, 1, 3, -2, 5, 2, -3, 1\}$$

37. (a) Solution:
The DTFT of $x[n]$ is:

$$\begin{aligned}\tilde{X}(\mathrm{e}^{\mathrm{j}\omega}) &= \sum_{n=-\infty}^{\infty} x[n]\mathrm{e}^{-\mathrm{j}\omega n} = \sum_{n=0}^{99} \cos(0.25\pi n + \pi/6)\mathrm{e}^{-\mathrm{j}\omega n} \\ &= \frac{1}{2}\sum_{n=0}^{99} \mathrm{e}^{\mathrm{j}(0.25\pi n+\pi/6)}\mathrm{e}^{-\mathrm{j}\omega n} + \frac{1}{2}\sum_{n=0}^{99} \mathrm{e}^{-\mathrm{j}(0.25\pi n+\pi/6)}\mathrm{e}^{-\mathrm{j}\omega n} \\ &= \frac{1}{2}\mathrm{e}^{\mathrm{j}\pi/6}\frac{1-\mathrm{e}^{\mathrm{j}(0.25\pi-\omega)100}}{1-\mathrm{e}^{\mathrm{j}(0.25\pi-\omega)}} + \frac{1}{2}\mathrm{e}^{-\mathrm{j}\pi/6}\frac{1-\mathrm{e}^{\mathrm{j}(0.25\pi+\omega)100}}{1-\mathrm{e}^{\mathrm{j}(0.25\pi+\omega)}}\end{aligned}$$

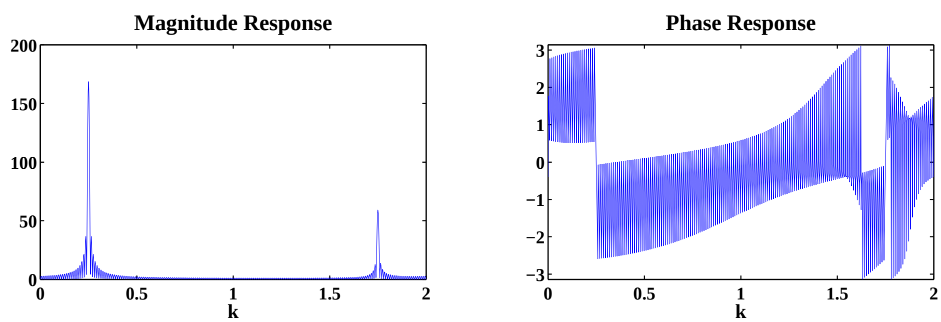


FIGURE 7.29: Plot of the DTFT $\tilde{X}(\mathrm{e}^{\mathrm{j}\omega})$ of $x[n]$.

(b) See plot below.

(c) See plot below.

(d) tba.

MATLAB script:

```
% P0737: DFT and DTFT
close all; clc
%% Specification:
n = 0:99;
% n = 0:199;
```

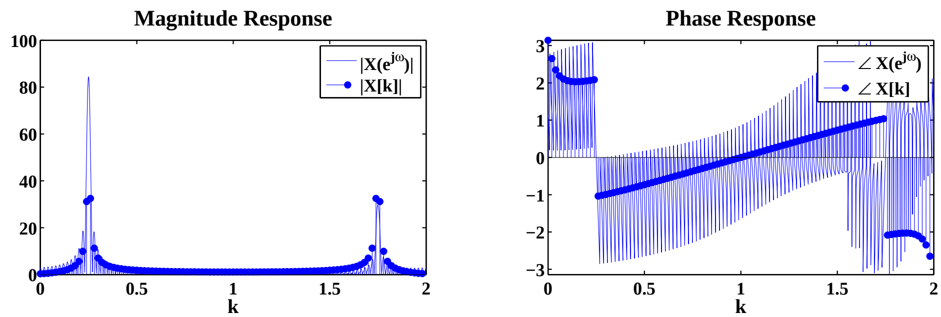


FIGURE 7.30: Plot of 100-point DFT $X[k]$ of $x[n]$ superimposed on the DTFT plot.

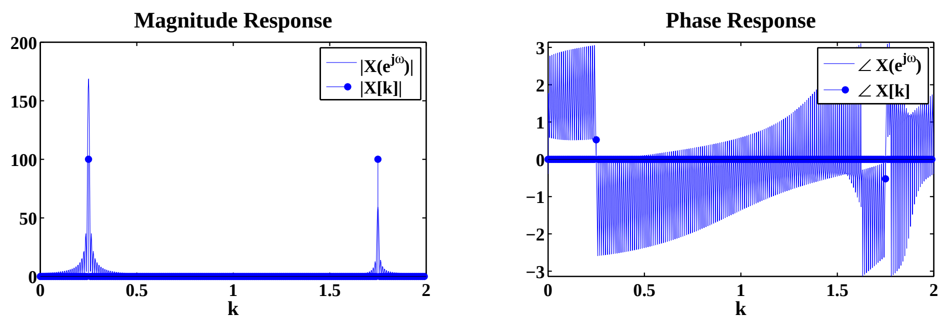


FIGURE 7.31: Plot of 200-point DFT $X[k]$ of $x[n]$ superimposed on the DTFT plot.

```
N = length(n);
k = 2*pi/N*(0:N-1);
xn = cos(0.25*pi*n+pi/6);
w = linspace(0,2,1000)*pi;
X = 1/2*exp(pi/6)*(1-exp(j*(0.25*pi-w)*N))./(1-exp(j*(0.25*pi-w)))...
    +1/2*exp(-pi/6)*(1-exp(j*(0.25*pi+w)*N))./(1-exp(j*(0.25*pi+w)));
X_phase = zeros(size(w));
%% Part (b)
Xk = fft(xn);
ind = abs(Xk) < 1e-10;
Xk(ind) = 0;
%% Plot:
hfa = figconfg('P0737a','long');
```

```
subplot(121)
plot(w/pi,abs(X));
xlim([0 2])
xlabel('k','fontsize',LFS)
title('Magnitude Response','fontsize',TFS)
subplot(122)
plot(w/pi,angle(X));
xlim([0 2])
ylim([-pi pi])
xlabel('k','fontsize',LFS)
title('Phase Response','fontsize',TFS)

hfb = figconfg('P0737b','long');
subplot(121)
plot(w/pi,abs(X));hold on
stem(k/pi,abs(Xk),'filled');
xlim([0 2])
xlabel('k','fontsize',LFS)
title('Magnitude Response','fontsize',TFS)
legend('|X(e^{j\omega})|','|X[k]|','location','northeast')
subplot(122)
plot(w/pi,angle(X));hold on
stem(k/pi,angle(Xk),'filled');
xlim([0 2])
ylim([-pi pi])
xlabel('k','fontsize',LFS)
title('Phase Response','fontsize',TFS)
legend('\angle X(e^{j\omega})','\angle X[k]','location','northeast')
```

38. Solution:
From the symmetry property of DFT of real-valued sequence, we can conclude the 9-point DFT as

$$\{4, 2 - \text{j}3, 3 + \text{j}2, -4 + \text{j}6, 8 - \text{j}7, 8 + \text{j}7, -4 - \text{j}6, 3 - \text{j}2, 2 + \text{j}3\}$$

(a) By applying the time-shifting property, the DFT of $x_1[n]$ is:

$$X_1[k] = W_9^{-2k} X[k]$$

(b) By applying the folding and time-shifting properties, the DFT of $x_2[n]$ is:

$$X_2[k] = 2W_9^{-2k} X^*[k]$$

(c) By applying the correlation property, the DFT of $x_3[n]$ is:

$$X_3[k] = X[k]X^*[k] = |X[k]|^2$$

(d) By applying the windowing property, the DFT of $x_4[n]$ is:

$$X_4[k] = \frac{1}{9}X[k] \; 9 \; X[k]$$

(e) By applying the frequency-shifting property, the DFT of $x_5[n]$ is:

$$X_5[k] = X[\langle k+2\rangle_9]$$

39. (a) Proof:

$$X[0] = \sum_{n=0}^{N-1} x[n]W_N^{n\cdot 0} = \sum_{n=0}^{N-1} x[n]$$

which is real-valued since $x[n]$ is real-valued.

(b) Proof:
If $k = 0$, since $x[0]$ is real, we have

$$X[0] = X^*[0]$$

If $1 \le k \le N-1$, we have

$$\begin{aligned} X[\langle N-k\rangle_N] &= \sum_{n=0}^{N-1} x[n]W_N^{n\langle N-k\rangle_N} = \sum_{n=0}^{N-1} x[n]W_N^{n(N-k)} \\ &= \sum_{n=0}^{N-1} x[n]W_N^{-nk} = \left(\sum_{n=0}^{N-1} x[n]W_N^{nk}\right)^* \\ &= X^*[k] \end{aligned}$$

Hence, we proved $X[\langle N-k\rangle_N] = X^*[k]$ for every k.

(c) Proof:

$$X[N/2] = \sum_{n=0}^{N-1} x[n]W_N^{n\frac{N}{2}} = \sum_{n=0}^{N-1} x[n]\mathrm{e}^{-\mathrm{j}n\pi} = \sum_{n=0}^{N-1} x[n]\cos(n\pi)$$

which is real-valued since $x[n]$ and $\cos(n\pi)$ are both real-valued.

40. Solution:

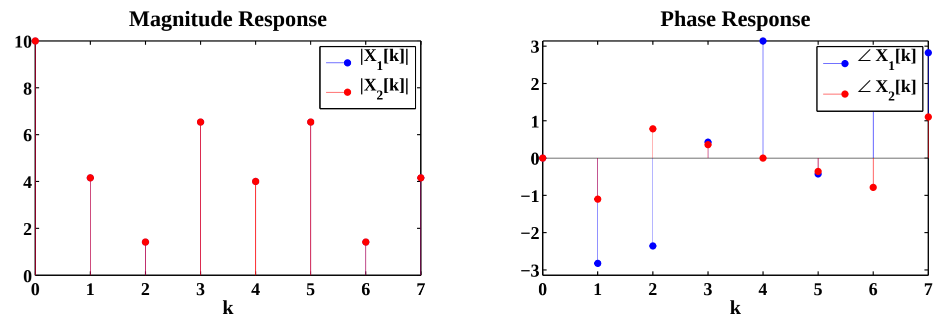


FIGURE 7.32: Verification by choosing $a = 1$, $b = 2$, $c = 3$, and $d = 4$.

MATLAB script:

```
% P0740: Investigate DFT relationship
close all; clc
a = 1; b = 2; c = 3; d = 4;
xn1 = [a, 0, b, c, 0, d, 0, 0];
xn2 = [d, 0, c, b, 0, a, 0, 0];
Xk1 = fft(xn1);
Xk2 = fft(xn2);
N = 8;
k = 0:N-1;
%% Plot:
hfa = figconfg('P0740a','long');
subplot(121)
stem(k,abs(Xk1),'filled'); hold on
stem(k,abs(Xk2),'filled','color','red');
xlim([0 N-1])
xlabel('k','fontsize',LFS)
title('Magnitude Response','fontsize',TFS)
legend('|X_1[k]|','|X_2[k]|','location','northeast')
subplot(122)
stem(k,angle(Xk1),'filled'); hold on
stem(k,angle(Xk2),'filled','color','red');
xlim([0 N-1])
ylim([-pi pi])
```

```
xlabel('k','fontsize',LFS)
title('Phase Response','fontsize',TFS)
legend('\angle X_1[k]','\angle X_2[k]','location','northeast')
```

41. (a) Solution:

Computing $x_1[n]$ 7 $x_2[n]$ using hand calculations:

$$\begin{bmatrix} -2 & 0 & 8 & 6 & -5 & -3 & 1 \\ 1 & -2 & 0 & 8 & 6 & -5 & -3 \\ -3 & 1 & -2 & 0 & 8 & 6 & -5 \\ -5 & -3 & 1 & -2 & 0 & 8 & 6 \\ 6 & -5 & -3 & 1 & -2 & 0 & 8 \\ 8 & 6 & -5 & -3 & 1 & -2 & 0 \\ 0 & 8 & 6 & -5 & -3 & 1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 46 \\ 29 \\ -7 \\ -16 \\ -9 \\ -7 \\ 14 \end{bmatrix}$$

(b) See script below.

(c) See script below.

MATLAB script:

```
% P0741: Circular convolution
close all; clc
xn1 = [-2 1 -3 -5 6 8];
xn2 = 1:4;
N = 7;
%% Part (b):
xn = circonv([xn1 zeros(1,N-length(xn1))]',...
    [xn2 zeros(1,N-length(xn2))]')';
%% Part (c):
Xk1 = fft(xn1,N);
Xk2 = fft(xn2,N);
Xk = Xk1.*Xk2;
xn_dft = ifft(Xk);
```

42. MATLAB script:

```
% P0742: Linear and Circular convolution
close all; clc
N = 10;
n = 0:N-1;
xn1 = 0.9.^n;
```

```
xn2 = n.*(0.6.^n);
%% Part (a):
x3n = conv(xn1,xn2);
%% Part (b):
Nc = 15;
x4n = circonv([xn1,zeros(1,Nc-length(xn1))]',...
    [xn2,zeros(1,Nc-length(xn2))]')';
xn3N = xn3(Nc+1:end);
en = x4n-x3n(1:Nc);
```

43. (a) MATLAB function:

```
function y = lin2circonv(x,h)
% FUNCTION 'LIN2CIRCONV' compute the circular convolution
% thru the results of linear convolution
y1 = conv(x(:)',h(:)');
N1 = length(x);
N2 = length(h);
N = max(N1,N2);
L = N1+N2-1;
nn = -L-1:L-1;
y1 = [zeros(1,L+1),y1];
ll = floor(L/N);
y = zeros(size(nn));
for ii = 0:ll
    y = y + [y1(ii*N+1:end) zeros(1,ii*N)];
end
y = y(L+2:L+N+1);
```

(b) MATLAB script:

```
% P0743: Use result of linear convolution to compute
%        circular convolution
close all; clc
xn = 1:4;
hn = [1 -1 1 -1];
y1 = lin2circonv(xn,hn);
y2 = circonv(xn',hn')';
```

44. (a) Solution:
The linear convolution $x_1[n] * x_2[n]$ is:

$$\left\{\underset{\uparrow}{0}, 2, -5, 4, -7, 7, -8, 4, -12\right\}$$

(b) Solution:

The circular convolution $x_1[n] \, 6 \, x_2[n]$ is:

$$\{\underset{\uparrow}{-8}, 6, -17, 4, -7, 7\}$$

(c) Solution:

The smallest value of N so that N-point circular convolution is equal to the linear convolution is:

$$\min N = 4 + 6 - 1 = 9$$

45. (a) Proof:

$$X_c(0) = \int_{-\infty}^{\infty} x_c(t)\mathrm{d}t$$

$$x_c(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_c(\mathrm{j}2\pi F)\mathrm{d}2\pi F$$

Hence, we can prove that

$$\Delta T_1 \Delta F_1 = \frac{\int_{-\infty}^{\infty} x_{\mathrm{c}}(t)\mathrm{d}t}{x_{\mathrm{c}}(0)} \cdot \frac{\int_{-\infty}^{\infty} X_{\mathrm{c}}(\mathrm{j}2\pi F)\,\mathrm{d}F}{X_{\mathrm{c}}(0)} = 1$$

(b) Solution:

For $x_{\mathrm{c}_1}(t) = u(t+1) - u(t-1)$,

$$X_{c1}(\mathrm{j}\Omega) = \int_{-1}^{1} \mathrm{e}^{-\mathrm{j}\Omega t}\mathrm{d}t = \frac{2\sin\Omega}{\Omega}$$

Hence, $\Delta T_1 = 2$, and $\Delta F_1 = 1/2$. Thus, $\Delta T_1 \Delta F_1 = 1$.

For $x_{\mathrm{c}_2}(t) = \cos(\pi t)[u(t+1) - u(t-1)]$,

$$X_{c2}(\mathrm{j}\Omega) = \frac{\sin(\pi - \Omega)}{\pi - \Omega} + \frac{\sin(\pi + \Omega)}{\pi + \Omega}$$

Thus, we have $X_{c2}(\mathrm{j}\Omega)|_{\Omega=0} = 0$.

We can conclude that the definition is reasonable for waveform like $x_{\mathrm{c}_1}(t)$.

(c) Proof:

$$x_c(0) = \int_{-\infty}^{\infty} x_c(\mathrm{j}2\pi F)\mathrm{d}F \implies |x_c(0)| \le \int_{-\infty}^{\infty} |x_c(\mathrm{j}2\pi F)|\mathrm{d}F$$

$$X_c(0) = \int_{-\infty}^{\infty} x_c(t)\mathrm{d}t \implies |X_c(0)| \leq \int_{-\infty}^{\infty} |x_c(t)|\mathrm{d}t$$

Hence, we proved that

$$\Delta T_2 \Delta F_2 = \frac{\int_{-\infty}^{\infty} |x_\mathrm{c}(t)|\mathrm{d}t}{|x_\mathrm{c}(0)|} \frac{\int_{-\infty}^{\infty} |X_\mathrm{c}(\mathrm{j}2\pi F)|\mathrm{d}F}{|X_\mathrm{c}(0)|} \geq 1$$

(d) tba.

46. (a) See plot below.

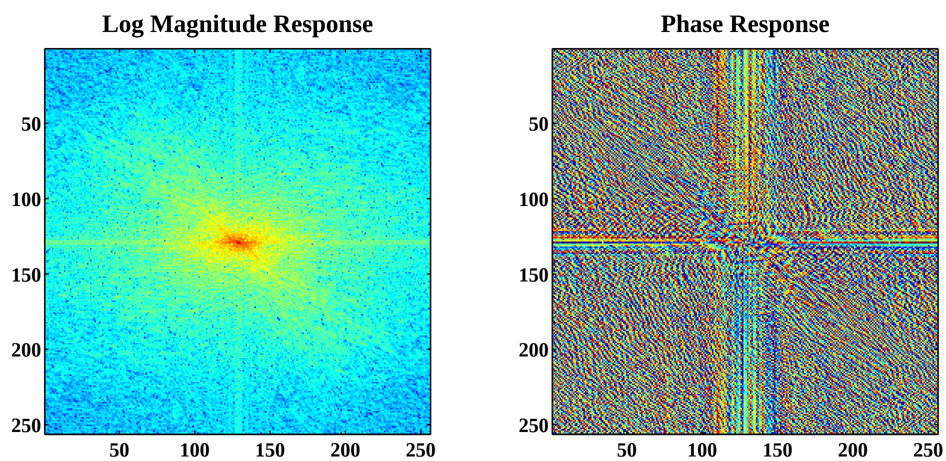


FIGURE 7.33: Plot of log-magnitude and phase as images of 2D-DFT of "Lena" image.

(b) See plot below.

(c) See plot below.

MATLAB script:

```
% P0746: 2D DFT illustration
```

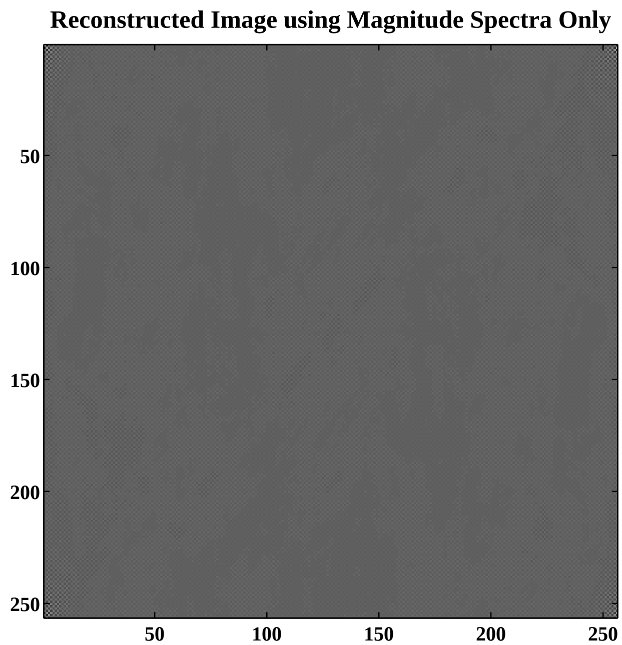


FIGURE 7.34: Plot of reconstructed image using 2D-IDFT of the magnitude array.

```
close all; clc
%% Part a:
x = imread('lena.jpg');
X = fftshift(fft2(x));
X_mag = abs(X);
X_logmag = 20*log10(X_mag);
X_phase = angle(X);
% Plot:
hfa = figure;
subplot(121)
imagesc(X_logmag); axis square
title('Log Magnitude Response','fontsize',TFS)
subplot(122)
imagesc(X_phase); axis square
title('Phase Response','fontsize',TFS)
```

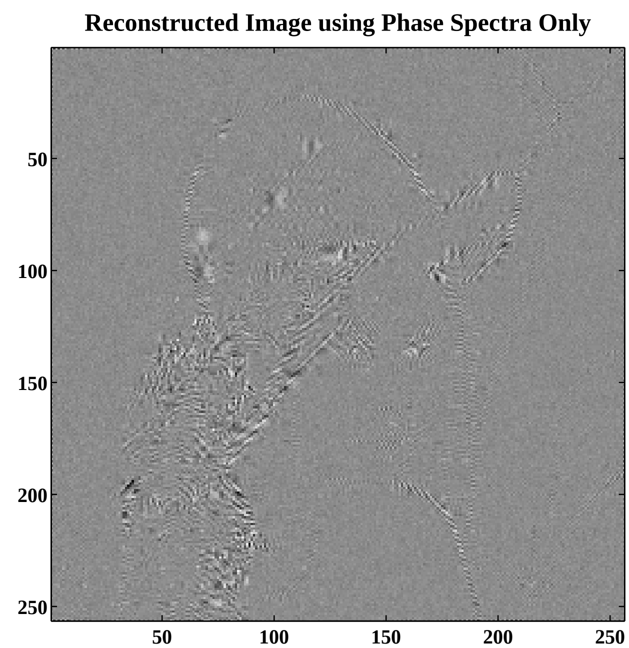


FIGURE 7.35: Plot of reconstructed image using 2D-IDFT of the phase array multiplied by constant magnitude value 128.

```
%% Part b:
x_mag = ifft2(X_mag);
% Plot:
hfb = figure;
imagesc(x_mag); axis square; colormap(gray)
title('Reconstructed Image using Magnitude Spectra Only'...
    ,'fontsize',TFS)

%% Part c:
x_phase = real(ifft2(128*exp(j*X_phase)));
% Plot:
hfc = figure;
imagesc(x_phase); axis square; colormap(gray)
title('Reconstructed Image using Phase Spectra Only'...
    ,'fontsize',TFS)
```

Assessment Problems

47. (a) Proof:

$$X[k] \triangleq \sum_{n=0}^{N-1} x[n]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}kn} \tag{7.21}$$

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\mathrm{e}^{\mathrm{j}\frac{2\pi}{N}kn} \tag{7.22}$$

Hence, we can express the (7.22) as

$$x[n] = \frac{1}{N}\left(\sum_{k=0}^{N-1} X^*[k]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}kn}\right)^*$$

(b) MATLAB function:

```
function x = myifft(X)
% Compute IDFT use FFT function only
N = length(X);
x = fft(X')'/N;
```

(c) MATLAB script:

```
% P0747: Testing Matlab function myifft
close all; clc
n = 0:9;
xn = sin(0.1*pi*n);
X = fft(xn);
xn1 = myifft(X);
xn2 = ifft(X); % Use for comparison
```

48. (a) Solution:
For sequence $\tilde{x}_2[n]$, it is possible to have all DFS values real-valued, that is

$$\tilde{x}_1[n] = \{\ldots, 0, \underset{\uparrow}{0}, 1, 1, 0, 0, 1, 1, 0, 0, \ldots\}$$

(b) Solution:
None.

(c) Solution:
For sequence $\tilde{x}_3[n]$, the DFS coefficients are zero for $k = \pm 2, \pm 4$.

49. (a) Solution:

$$\tilde{X}(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = \sum_{n=0}^{9} 0.8^n e^{-j\omega n}$$
$$= \sum_{n=0}^{9} \left(0.8e^{-j\omega}\right)^n = \frac{1 - 0.8^10 e^{-10j\omega}}{1 - 0.8e^{-j\omega}}$$

The magnitude of $\tilde{X}(e^{j\omega})$ is:

$$|\tilde{X}(e^{j\omega})| = \sqrt{\frac{1 + 0.8^{20} - 2 \times 0.8^{10} \cos 10\omega}{1 + 0.8^2 - 2 \times 0.8 \cos \omega}}$$

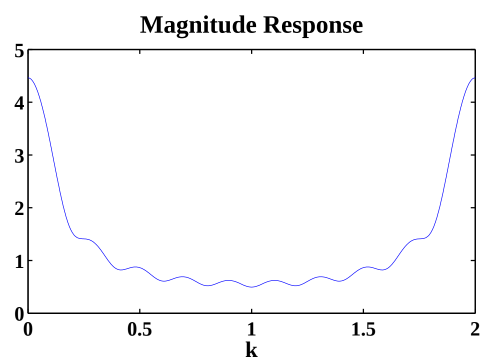


FIGURE 7.36: Magnitude plot of the DTFT $\tilde{X}(e^{j\omega})$.

(b) See plot below.

(c) See plot below.

(d) tba.

MATLAB script:

```
% P0749: DFT
close all; clc
n = 0:9;
N = 10;
k = 2*pi/N*n;
xn = 0.8.^n;
yn = xn.*exp(-j*pi*n/N);
Xk = fft(xn);
```

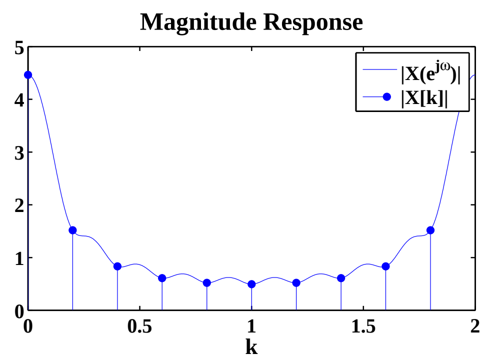


FIGURE 7.37: Magnitude plot of the 10-point DFT $X[k]$ and the DTFT $\tilde{X}(\mathrm{e}^{\mathrm{j}\omega})$.

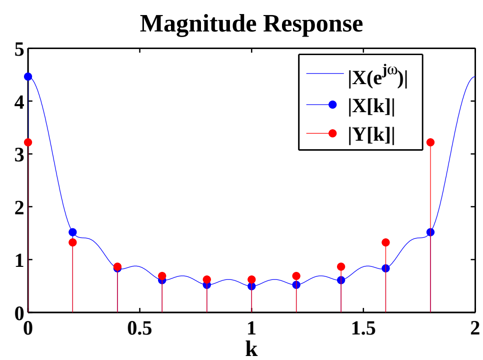


FIGURE 7.38: Magnitude plot of the 10-point DFT $X[k]$, $Y[k]$ and the DTFT $\tilde{X}(\mathrm{e}^{\mathrm{j}\omega})$.

```
Yk = fft(yn);
w = linspace(0,2,1000)*pi;
Xmag = sqrt(1+0.8^20-2*0.8^10*cos(10*w))...
    ./sqrt(1+0.8^2-2*0.8*cos(w));
%% Plot:
hfa = figconfg('P0749a','small');
plot(w/pi,Xmag);
xlim([0 2])
xlabel('k','fontsize',LFS)
title('Magnitude Response','fontsize',TFS)

hfb = figconfg('P0749b','small');
```

```
plot(w/pi,Xmag); hold on
stem(k/pi,abs(Xk),'filled')
xlim([0 2])
xlabel('k','fontsize',LFS)
title('Magnitude Response','fontsize',TFS)
legend('|X(e^{j\omega})|','|X[k]|','location','northeast')

hfc = figconfg('P0749c','small');
plot(w/pi,Xmag); hold on
stem(k/pi,abs(Xk),'filled')
stem(k/pi,abs(Yk),'filled','color','red');
xlim([0 2])
xlabel('k','fontsize',LFS)
title('Magnitude Response','fontsize',TFS)
legend('|X(e^{j\omega})|','|X[k]|','|Y[k]|','location','northeast')
```

50. (a) See plot below.

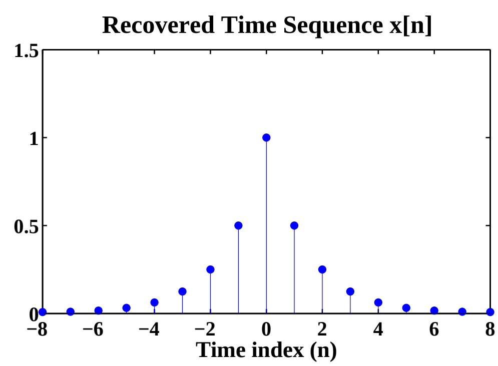


FIGURE 7.39: Stem-plot the sequence $x_1[n]$ from $-8 \leq n \leq 8$ by taking the IDFT of $X[k]$.

(b) See plot below.

(c) Solution:
From the plots, we guess the original sequence $x[n]$ is:

$$x[n] = \left(\frac{1}{2}\right)^{|n|}$$

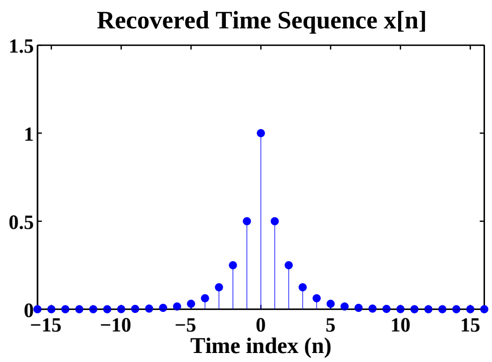


FIGURE 7.40: Stem-plot the sequence $x_2[n]$ from $-16 \le n \le 16$ by taking the IDFT of $X[k]$.

Computing the DTFT of $x[n]$ as:

$$
\begin{aligned}
X(\mathrm{e}^{\mathrm{j}\omega}) &= \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^{|n|} \mathrm{e}^{-\mathrm{j}\omega n} \\
&= 1 + \sum_{n=-\infty}^{-1} \left(\frac{1}{2}\right)^{-n} \mathrm{e}^{-\mathrm{j}\omega n} + \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n} \mathrm{e}^{-\mathrm{j}\omega n} \\
&= 1 + \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n} \mathrm{e}^{\mathrm{j}\omega n} + \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n} \mathrm{e}^{-\mathrm{j}\omega n} \\
&= 1 + \frac{\frac{1}{2}\mathrm{e}^{\mathrm{j}\omega}}{1-\frac{1}{2}\mathrm{e}^{\mathrm{j}\omega}} + \frac{\frac{1}{2}\mathrm{e}^{-\mathrm{j}\omega}}{1-\frac{1}{2}\mathrm{e}^{-\mathrm{j}\omega}} \\
&= \frac{3}{5-4\cos\omega}
\end{aligned}
$$

MATLAB script:

```
% P0750: Time sequence recover
close all; clc
N = 16; % Part (a)
% N = 32; % Part (b)
k = 2*pi/N*(0:N-1);
Xk = 3./(5-4*cos(k));
xn = ifft(Xk);
n = -N/2:N/2;
```

```
xn = [xn xn];
xn = xn(1+N/2:1+N/2+N);

%% Plot:
hfa = figconfg('P0750a','small');
stem(n,xn,'filled');
xlim([-N/2 N/2])
xlabel('Time index (n)','fontsize',LFS)
title('Recovered Time Sequence x[n]','fontsize',TFS)
```

51. (a) Solution:
The DTFT $\tilde{X}(\mathrm{e}^{\mathrm{j}\omega})$ of $x[n]$ is:

$$\begin{aligned}
\tilde{X}(\mathrm{e}^{\mathrm{j}\omega}) &= \sum_{n=0}^{99} \sin(0.6\pi n + \pi/3)\mathrm{e}^{-\mathrm{j}\omega n} \\
&= \frac{1}{2\mathrm{j}} \sum_{n=0}^{99} \left(\mathrm{e}^{\mathrm{j}0.6\pi n + \mathrm{j}\pi/3} - \mathrm{e}^{-\mathrm{j}0.6\pi n - \mathrm{j}\pi/3}\right) \mathrm{e}^{-\mathrm{j}\omega n} \\
&= \frac{1}{2\mathrm{j}} \mathrm{e}^{\mathrm{j}\frac{\pi}{3}} \sum_{n=0}^{99} \mathrm{e}^{\mathrm{j}(0.6\pi-\omega)n} - \frac{1}{2\mathrm{j}} \mathrm{e}^{-\mathrm{j}\frac{\pi}{3}} \sum_{n=0}^{99} \mathrm{e}^{-\mathrm{j}(0.6\pi+\omega)n} \\
&= \frac{1}{2\mathrm{j}} \mathrm{e}^{\mathrm{j}\frac{\pi}{3}} \frac{1-\mathrm{e}^{-\mathrm{j}(\omega-0.6\pi)100}}{1-\mathrm{e}^{-\mathrm{j}(\omega-0.6\pi)}} - \frac{1}{2\mathrm{j}} \mathrm{e}^{-\mathrm{j}\frac{\pi}{3}} \frac{1-\mathrm{e}^{-\mathrm{j}(\omega+0.6\pi)100}}{1-\mathrm{e}^{-\mathrm{j}(\omega+0.6\pi)}}
\end{aligned}$$

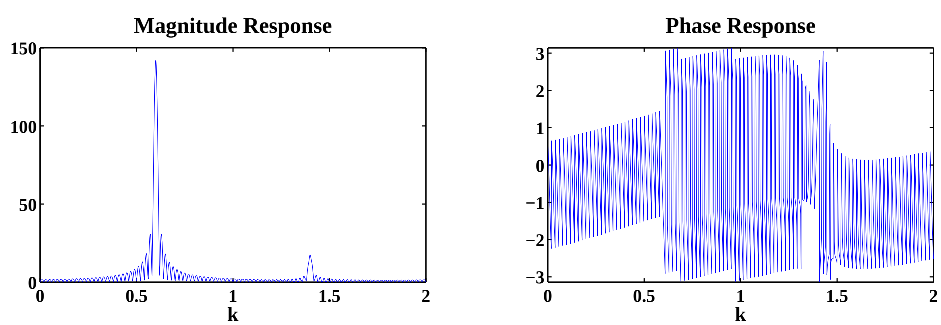


FIGURE 7.41: Plot of the DTFT $\tilde{X}(\mathrm{e}^{\mathrm{j}\omega})$.

(b) See plot below.

(c) See plot below.

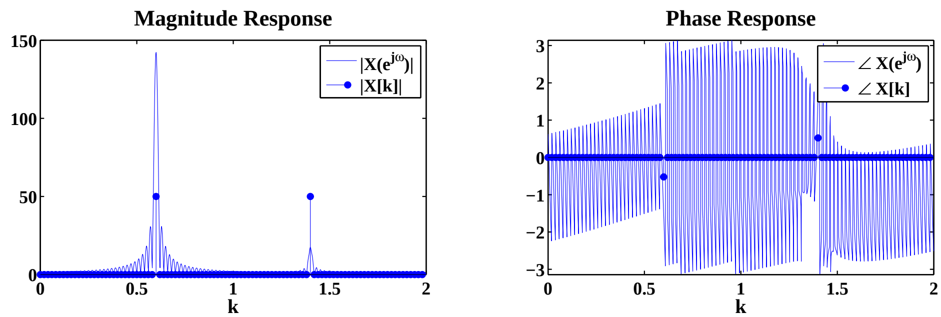


FIGURE 7.42: Plot of the 100-point DFT $X[k]$ superimposed on the DTFT $\tilde{X}(e^{j\omega})$.

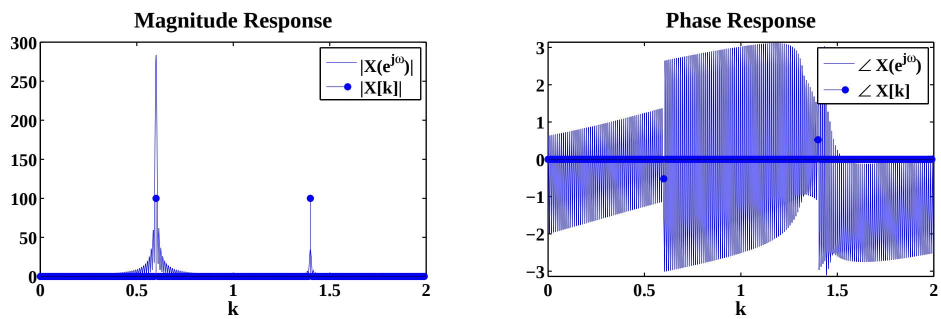


FIGURE 7.43: Plot of the 200-point DFT $X[k]$ superimposed on the DTFT $\tilde{X}(e^{j\omega})$.

(d) tba.

MATLAB script:

```
% P0751: DFT and DTFT
close all; clc
%% Specification:
n = 0:99;
% n = 0:199;
N = length(n);
k = 2*pi/N*(0:N-1);
xn = sin(0.6*pi*n+pi/3);
w = linspace(0,2,1000)*pi;
X = 1/2/j*exp(pi/3)*(1-exp(j*(0.6*pi-w)*N))./(1-exp(j*(0.6*pi-w)))...
```

```
    -1/2/j*exp(-pi/3)*(1-exp(j*(0.6*pi+w)*N))./(1-exp(j*(0.6*pi+w)));
X_phase = zeros(size(w));
%% Part (b)
Xk = fft(xn);
ind = abs(Xk) < 1e-10;
Xk(ind) = 0;
%% Plot:
hfa = figconfg('P0751a','long');
subplot(121)
plot(w/pi,abs(X));
xlim([0 2])
xlabel('k','fontsize',LFS)
title('Magnitude Response','fontsize',TFS)
subplot(122)
plot(w/pi,angle(X));
xlim([0 2])
ylim([-pi pi])
xlabel('k','fontsize',LFS)
title('Phase Response','fontsize',TFS)

hfb = figconfg('P0751b','long');
subplot(121)
plot(w/pi,abs(X));hold on
stem(k/pi,abs(Xk),'filled');
xlim([0 2])
xlabel('k','fontsize',LFS)
title('Magnitude Response','fontsize',TFS)
legend('|X(e^{j\omega})|','|X[k]|','location','northeast')
subplot(122)
plot(w/pi,angle(X));hold on
stem(k/pi,angle(Xk),'filled');
xlim([0 2])
ylim([-pi pi])
xlabel('k','fontsize',LFS)
title('Phase Response','fontsize',TFS)
legend('\angle X(e^{j\omega})','\angle X[k]','location','northeast')
```

52. Solution:
The smallest positive value ℓ is 3.

53. (a) Proof:

$$X[N/2] = \sum_{n=0}^{N-1} x[n]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}n\frac{N}{2}} = \sum_{n=0}^{N-1} x[n]\cos\pi n$$

When N is even, for any $n \in [0, N/2-1]$, there exists $x[N-1-n] = x[n]$. If we group these pairs and notice that

$$cos\pi(N-1-n) = \cos(n+1)\pi = -\cos n\pi$$

we can conclude that

$$X[N/2] = 0.$$

(b) Proof:

$$X[0] = \sum_{n=0}^{N-1} x[n]$$

When N is even, for any $n \in [0, N/2-1]$, there exists $x[N-1-n] = -x[n]$. Hence, we have

$$\begin{aligned} X[0] = \sum_{n=0}^{N-1} x[n] = X[0] &= \sum_{n=0}^{N/2-1} x[n] + \sum_{n=0}^{N/2-1} x[N-1-n] \\ &= \sum_{n=0}^{N/2-1} (x[n] - x[n]) = 0 \end{aligned}$$

(c) Proof:

$$\begin{aligned} X[2\ell+1] &= \sum_{n=0}^{N-1} x[n]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}n(2\ell+1)} \\ &= \sum_{n=0}^{N/2-1} x[n]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}n(2\ell+1)} + \sum_{n=0}^{N/2-1} x[n+N/2]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}(n+N/2)(2\ell+1)} \\ &= \sum_{n=0}^{N/2-1} x[n]e^{-\mathrm{j}\frac{2\pi}{N}n(2\ell+1)} + \sum_{n=0}^{N/2-1} x[n]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}n(2\ell+1)}\mathrm{e}^{-\mathrm{j}\pi(2\ell+1)} \\ &= \sum_{n=0}^{N/2-1} x[n]e^{-\mathrm{j}\frac{2\pi}{N}n(2\ell+1)}(1-1) \\ &= 0 \end{aligned}$$

54. (a) Proof:

$$\begin{aligned}\sum_{n=0}^{N-1} x[n]y^*[n] &= \sum_{n=0}^{N-1}\left(\frac{1}{N}\sum_{k=0}^{N-1} X[k]\mathrm{e}^{\mathrm{j}\frac{2\pi}{N}nk}\right)y^*[n]\\ &= \frac{1}{N}\sum_{k=0}^{N-1} X[k]\left(\sum_{n=0}^{N-1} y^*[n]\mathrm{e}^{\mathrm{j}\frac{2\pi}{N}nk}\right)\\ &= \frac{1}{N}\sum_{k=0}^{N-1} X[k]\left(\sum_{n=0}^{N-1} y[n]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}nk}\right)^*\\ &= \frac{1}{N}\sum_{k=0}^{N-1} X[k]Y^*[k]\end{aligned}$$

(b) Proof: According to equation (**??**), we have

$$\sum_{n=0}^{N-1} x[n]x^*[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]X^*[k]$$

which equals

$$\sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N}\sum_{k=0}^{N-1} |X[k]|^2$$

(c) Proof:

$$\boldsymbol{X} = \boldsymbol{W}_N\boldsymbol{x}$$

Hence, we have

$$\begin{aligned}\frac{1}{N}\sum_{k=0}^{N-1} |X[k]|^2 &= \frac{1}{N}\boldsymbol{X}^*\boldsymbol{X} = \frac{1}{N}(\boldsymbol{W}_N\boldsymbol{x})^H(\boldsymbol{W}_N\boldsymbol{x})\\ &= \boldsymbol{x}^H(\frac{1}{N}\boldsymbol{W}_N^H\boldsymbol{W}_N)\boldsymbol{x} = \frac{1}{N}\boldsymbol{x}^H\boldsymbol{I}_N\boldsymbol{x}\\ &= \boldsymbol{x}^H\boldsymbol{x} = \sum_{n=0}^{N-1} |x[n]|^2\end{aligned}$$

55. (a) Proof:

$$\begin{aligned}
&\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} x[\langle m-m_0\rangle_M, \langle n-n_0\rangle_N] W_M^{mk} W_N^{n\ell} \\
=&\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} x[m,n] W_M^{(m+m_0)k} W_N^{(n+n_0)\ell} \\
=&W_M^{km_0} W_N^{\ell n_0} \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} x[m,n] W_M^{mk} W_N^{n\ell} \\
=&W_M^{km_0} W_N^{\ell n_0} X[k,\ell]
\end{aligned}$$

(b) Proof:

$$\begin{aligned}
&\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} x[\langle -m\rangle_M, \langle -n\rangle_N] W_M^{mk} W_N^{n\ell} \\
=&\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} x[m,n] W_M^{-mk} W_N^{-n\ell} \\
=&\left(\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} x[m,n] W_M^{mk} W_N^{n\ell}\right)^* \\
=&X^*[k,\ell]
\end{aligned}$$

If $x[m,n]$ is real.

(c) Proof:

$$\begin{aligned}
Y[k,l] &= \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} \left(\sum_{i=0}^{M-1}\sum_{j=0}^{N-1} x[i,j] h[\langle m-i\rangle_M, \langle n-j\rangle_N]\right) W_M^{mk} W_N^{n\ell} \\
&= \sum_{i=0}^{M-1}\sum_{j=0}^{N-1} x[i,j] \left(\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} h[\langle m-i\rangle_M, \langle n-j\rangle_N] W_M^{(m-i)k} W_N^{(n-j)\ell} W_M^{ik} W_N^{n\ell}\right) \\
&= \left(\sum_{i=0}^{M-1}\sum_{j=0}^{N-1} x[i,j] W_M^{ik} W_N^{n\ell}\right)\left(\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} h[\langle m-i\rangle_M, \langle n-j\rangle_N] W_M^{(m-i)k} W_N^{(n-j)\ell}\right) \\
&= \left(\sum_{i=0}^{M-1}\sum_{j=0}^{N-1} x[i,j] W_M^{ik} W_N^{n\ell}\right)\left(\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} h[m,n] W_M^{mk} W_N^{n\ell}\right) \\
&= X[k,\ell] H[k,\ell]
\end{aligned}$$

Review Problems

56. (a) Proof:
From the equation (**??**), we can write the system function as

$$H(z) = \frac{A}{1 - \mathrm{e}^{-AT} z^{-1}}$$

that is

$$Y(z)(1 - \mathrm{e}^{-AT} z^{-1}) = AX(z)$$

Hence, we prove that the difference equation is

$$y[n] = \mathrm{e}^{-AT} y[n-1] + Ax[n]$$

(b) See plot below.

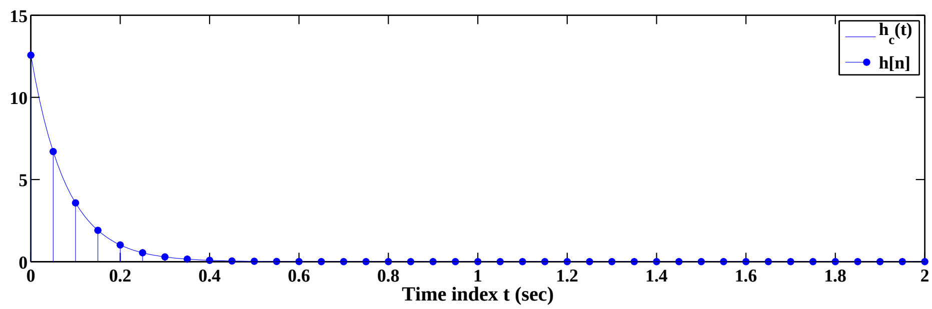


FIGURE 7.44: Graph of the impulse responses $h[n]$ and $h_{\mathrm{c}}(t)$ for $F_{\mathrm{c}} = 2$ Hz and $F_{\mathrm{s}} = 20$ Hz.

(c) See plot below.

(d) tba

(e) tba

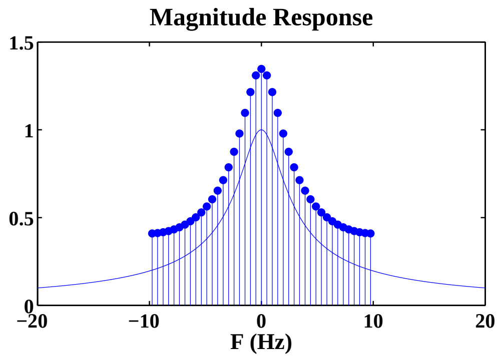


FIGURE 7.45: Graph of the magnitude responses of the analog integrator and its digital simulator for $F_c = 2$ Hz and $F_s = 20$ Hz.

57. (a) Solution:

$$x_c(t) = \sum_{k=1}^{K} \frac{1}{2}\left(\mathrm{e}^{\mathrm{j}2\pi F_k t} + \mathrm{e}^{-\mathrm{j}2\pi F_k t}\right)$$

$$X_{\mathrm{c}}(\mathrm{j}2\pi F) = \begin{cases} \frac{1}{2}, & |F| = F_k \\ 0, & \text{otherwise} \end{cases}$$

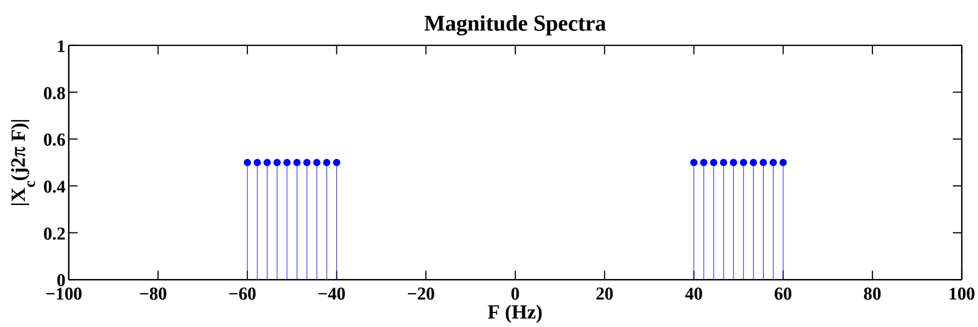


FIGURE 7.46: Magnitude plot of $X_{\mathrm{c}}(\mathrm{j}2\pi F)$ from -100 to 100 Hz.

(b) Proof:

$$\hat{x}_{\mathrm{c}}(t) = \sum_{k=1}^{K} w_c(t)\cos(2\pi F_k t) = \frac{1}{2}\sum_{k=1}^{K} w_c(t)\left(\mathrm{e}^{\mathrm{j}2\pi F_k t} + \mathrm{e}^{-\mathrm{j}2\pi F_k t}\right)$$

Hence, by applying the frequency-shift property, we have

$$\hat{X}_{\mathrm{c}}(\mathrm{j}2\pi F) = \frac{1}{2}\sum_{k=1}^{K}\left\{W_{\mathrm{c}}[\mathrm{j}2\pi(F-F_k)] + W_{\mathrm{c}}[\mathrm{j}2\pi(F+F_k)]\right\}$$

(c) Solution:
The CTFT of the window signal $w_{\mathrm{R}}(t)$ is:

$$W_R(\mathrm{j}2\pi F) = \int_{-0.1}^{0.1} \mathrm{e}^{-\mathrm{j}2\pi F t}\mathrm{d}t = \frac{\sin(0.2\pi F)}{\pi F}$$

The CTFT of the windowed signal $\hat{x}_{\mathrm{c}}(t)$ is:

$$\hat{X}_c(\mathrm{j}2\pi F) = \frac{1}{2}\sum_{k=1}^{K}\left\{\frac{\sin(0.2\pi[F-F_k])}{\pi(F-F_k)} + \frac{\sin(0.2\pi[F+F_k])}{\pi(F+F_k)}\right\}$$

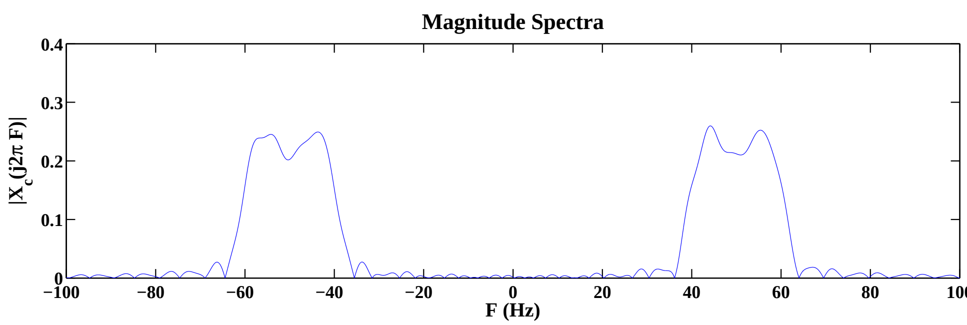


FIGURE 7.47: Magnitude plot of the CTFT of the windowed signal $\hat{x}_{\mathrm{c}}(t)$.

(d) Solution:

$$\hat{X}_c(\mathrm{j}2\pi F) = \frac{1}{2}\int_{F_L}^{F_H}\left\{W_c[\mathrm{j}2\pi(F-\theta)] + W_c[\mathrm{j}2\pi(F+\theta)]\right\}\mathrm{d}\theta$$

MATLAB script:

```
% P0757: Windowing
close all; clc
FL = 40; FH = 60;
K = 10;
dkF = (FH - FL)/(K-1);
```

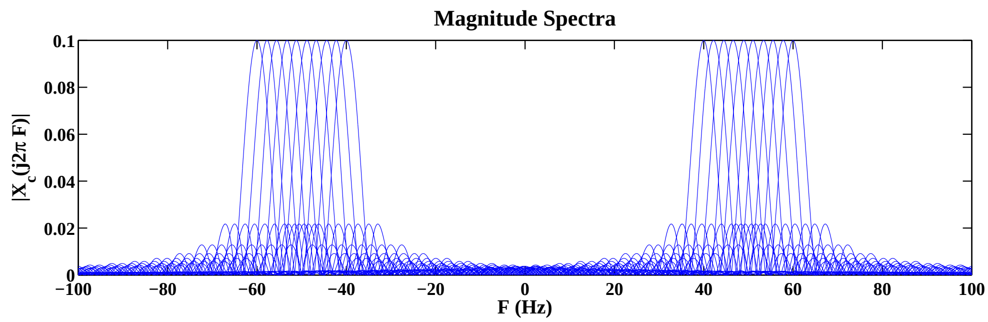


FIGURE 7.48: Magnitude plot of the CTFT of the individual windowed sinusoidal components.

```
Fk = FL:dkF:FH;
%% Part (a):
hfa = figconfg('P0757a','long');
stem([-Fk Fk],ones(1,2*K)/2,'filled')
xlabel('F (Hz)','fontsize',LFS)
ylabel('|X_c(j2\pi F)|','fontsize',LFS)
title('Magnitude Spectra','fontsize',TFS)
xlim([-100 100])
ylim([0 1])

%% Part (b)
dF = 0.001;
F = -100:dF:100;
[FF FFk] = meshgrid(F,[-Fk Fk]);
Xcpw = sinc(0.2*(FF-FFk)+sinc(0.2*(FF+FFk)))/10;
Xc = sum(Xcpw,1);
% Plot:
hfb = figconfg('P0757b','long');
plot(F,abs(Xc))
xlabel('F (Hz)','fontsize',LFS)
ylabel('|X_c(j2\pi F)|','fontsize',LFS)
title('Magnitude Spectra','fontsize',TFS)
xlim([-100 100])

hfc = figconfg('P0757c','long');
plot(F,abs(Xcpw),'color','blue')
```

```
xlabel('F (Hz)','fontsize',LFS)
ylabel('|X_c(j2\pi F)|','fontsize',LFS)
title('Magnitude Spectra','fontsize',TFS)
xlim([-100 100])
```

58. (a) See plot below.

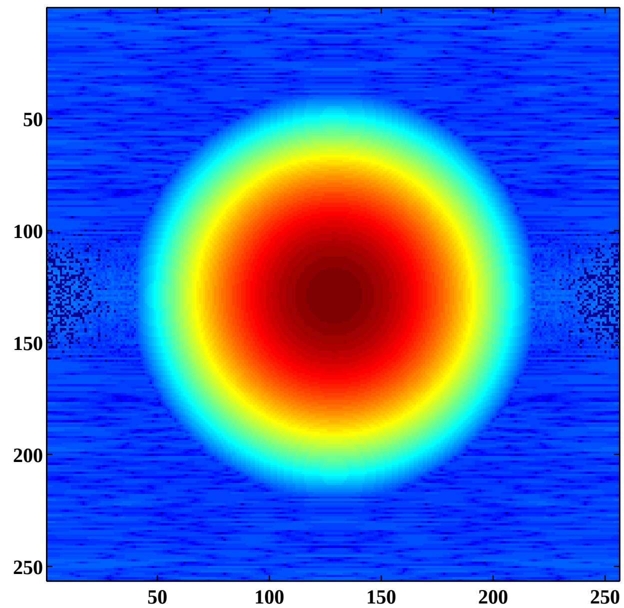


FIGURE 7.49: Plot of the log-magnitude of $H[k,\ell]$ as an image when $\sigma = 4$.

(b) See plot below.

(c) See plot below.

(d) See plot below.

(e) See plot below.

MATLAB script:

```
% P0758: 2D filtering
close all; clc
x = imread('lena.jpg');
```

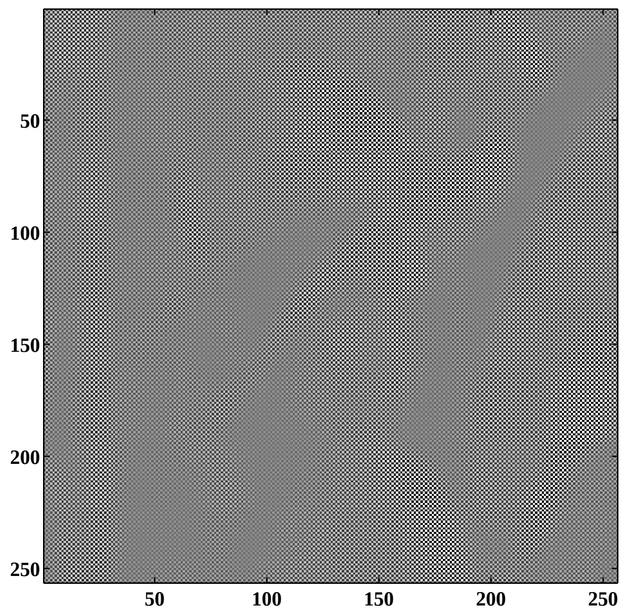


FIGURE 7.50: Reconstructed filtered image when $\sigma = 4$.

```
% Plot:
hf = figure;
imshow(x)
%% Part (a):
% sigma = 4;
sigma = 32;
M = 256; N = 256;
m = -M/2:M/2-1;
n = -N/2:N/2-1;
[NN MM] = meshgrid(n,m);
hmn = 1/2/pi/sigma^2*exp(-(MM.^2+NN.^2)/2/sigma^2);
H = real(ifftshift(fft2(hmn)));
Hmag = abs(H);
Hmaglog = 20*log10(Hmag);
% Plot:
hfa = figure;
```

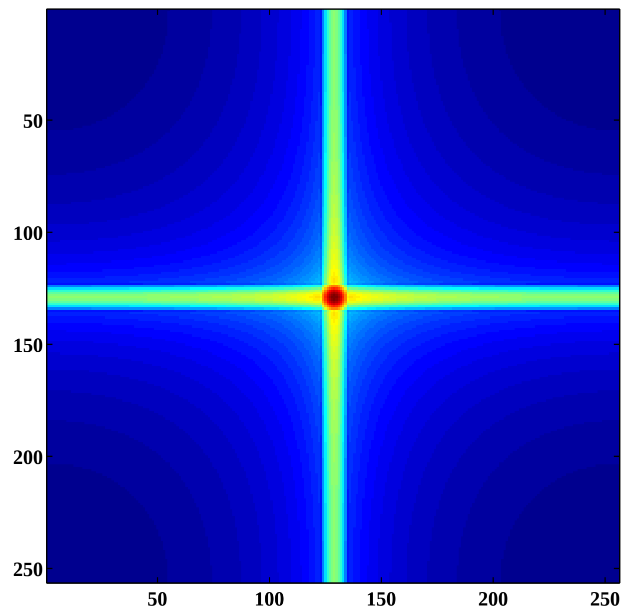


FIGURE 7.51: Plot of the log-magnitude of $H[k,\ell]$ as an image when $\sigma = 32$.

```
imagesc(Hmaglog); axis square

%% Part (b):
X = ifftshift(fft2(x));
Y = X.*H;
Y2 = X.*(1-H);
xr = fftshift(ifft2(Y));
xr2 = ifft2(Y2);
% Plot:
hfb1 = figure;
imagesc(xr); axis square; colormap(gray)
hfb2 = figure;
imagesc(xr2); axis square; colormap(gray)

%% Part(d):
xzp = zeros(2*M,2*N);
```

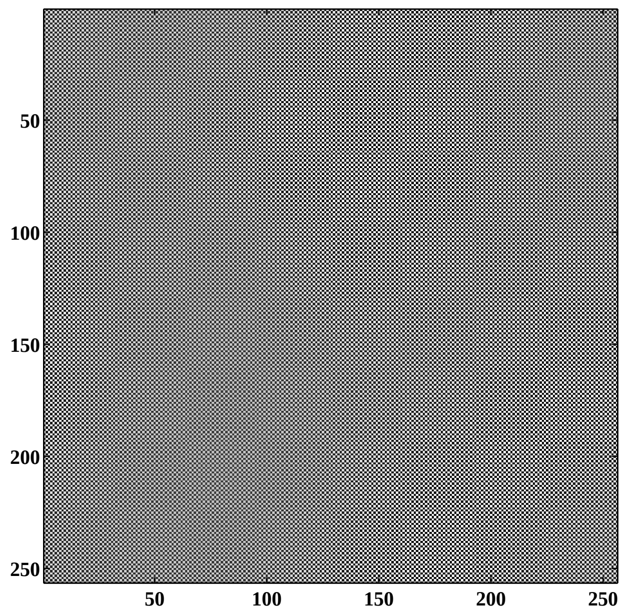


FIGURE 7.52: Reconstructed filtered image when $\sigma = 32$.

```
xzp(M/2+1:M/2+M,N/2+1:N/2+N) = x;
hmnzp = zeros(2*M,2*N);
hmnzp(M/2+1:M/2+M,N/2+1:N/2+N) = hmn;
Xzp = ifftshift(fft2(xzp));
Hzp = ifftshift(fft2(hmnzp));
xzpr = fftshift(ifft2(Xzp.*Hzp));
% Plot:
hfc = figure;
imagesc(xzpr(M/2+1:M/2+M,N/2+1:N/2+N)); axis square; colormap(gray)
```

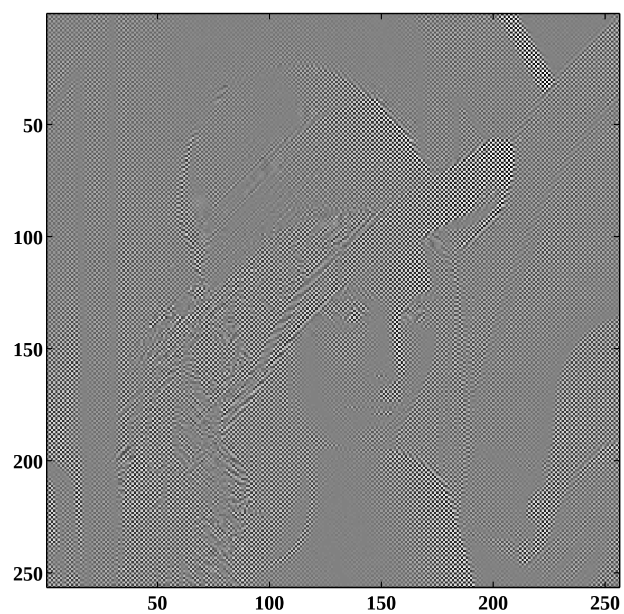


FIGURE 7.53: Reconstructed filtered image when $\sigma = 32$ and using zero padding.

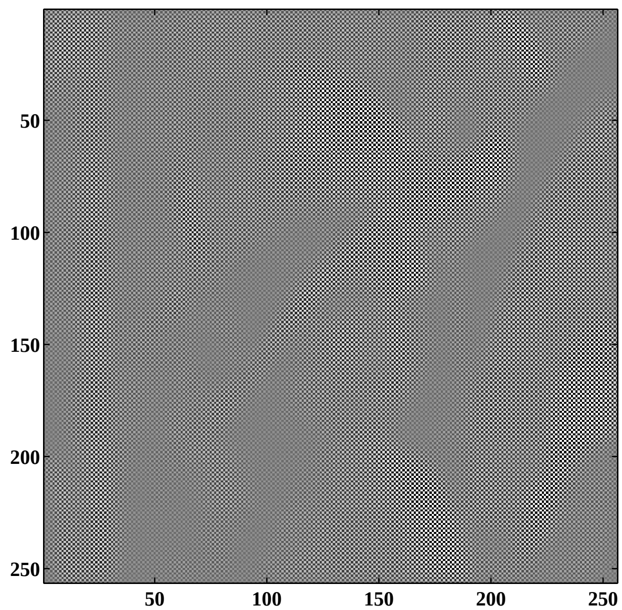


FIGURE 7.54: Reconstructed filtered image when $\sigma = 4$ but using the frequency response $1 - H[k, \ell]$ for the filtering.