CHAPTER 3

The z-Transform

Tutorial Problems

1. (a) Solution:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^n (u[n]-u[n-10])z^{-n}$$
$$= \sum_{n=0}^{9} \left(\frac{1}{2}\right)^n z^{-n} = \frac{1-\left(\frac{1}{2}z^{-1}\right)^{10}}{1-\frac{1}{2}z^{-1}} \quad \text{ROC: } |z|>0$$

(b) Solution:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^{|n|} z^{-n} = \sum_{n=-\infty}^{-1} \left(\frac{1}{2}\right)^{-n} z^{-n} + \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n} z^{-n}$$
$$= \sum_{n=1}^{\infty} \left(\frac{z}{2}\right)^n + \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^{-n} = \frac{\frac{z}{2}}{1-\frac{1}{2}z} + \frac{1}{1-\frac{1}{2}z^{-1}}$$
$$= \frac{-1}{1-2z^{-1}} + \frac{1}{1-\frac{1}{2}z^{-1}} = \frac{-\frac{3}{2}z^{-1}}{1-\frac{5}{2}z^{-1}+z^{-2}} \quad \text{ROC: } \frac{1}{2}<|z|<2$$

(c) Solution:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=-\infty}^{\infty} 5^{|n|} z^{-n} = \sum_{n=-\infty}^{-1} 5^{-n} z^{-n} + \sum_{n=0}^{\infty} 5^{n} z^{-n}$$
$$= \frac{5z}{1-5z} + \frac{1}{1-5z^{-1}} = \frac{\frac{24}{5}z^{-1}}{1-\frac{26}{5}z^{-1}+z^{-2}} \quad \text{ROC: } |z| \in \phi$$

(d) Solution:

$$x[n] = \left(\frac{1}{2}\right)^n \cos(\pi n/3)u[n] = x[n] = \left(\frac{1}{2}\right)^n (\frac{1}{2}\mathrm{e}^{j\pi n/3}+\frac{1}{2}\mathrm{e}^{-j\pi n/3})u[n]$$

$$\begin{aligned}X(z) &= \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{1}{2}\mathrm{e}^{j\pi/3}z^{-1}\right)^n + \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{1}{2}\mathrm{e}^{-j\pi/3}z^{-1}\right)^n \\ &= \frac{1}{2}\frac{1}{1-\frac{1}{2}\mathrm{e}^{j\pi/3}z^{-1}} + \frac{1}{2}\frac{1}{1-\frac{1}{2}\mathrm{e}^{-j\pi/3}z^{-1}} \\ &= \frac{1-\frac{1}{2}\cos\left(\frac{\pi}{3}\right)z^{-1}}{1-\cos\left(\frac{\pi}{3}\right)z^{-1}+\frac{1}{4}z^{-2}} = \frac{1-\frac{1}{4}z^{-1}}{1-\frac{1}{2}z^{-1}+\frac{1}{4}z^{-2}} \quad \text{ROC: } |z| > \frac{1}{2}\end{aligned}$$

2. (a) Proof:
The input in `x=filter(b,a,[1,zeros(1,N)])` is actually impulse signal $\delta[n]$. The z-transform of impulse response is 1. Hence, $Y(z) = X(z)$.

(b) Solution:

$$\begin{aligned}X(z) &= \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=-\infty}^{\infty}\left[\left(\frac{1}{2}\right)^n + \left(-\frac{1}{3}\right)^n\right]u[n]z^{-n} \\ &= \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n z^{-n} + \sum_{n=0}^{\infty}\left(-\frac{1}{3}\right)^n z^{-n} \\ &= \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{1+\frac{1}{3}z^{-1}} = \frac{2-\frac{1}{6}z^{-1}}{1-\frac{1}{6}z^{-1}-\frac{1}{6}z^{-2}} \quad \text{ROC: } |z| > \frac{1}{2}\end{aligned}$$

(c) MATLAB script:

```
% P0302: verify z-transform expression of a causal
%        sequence using function 'filter'
close all; clc
n = 0:20;
xn = (1/2).^n+(-1/3).^n;
b = [2 -1/6];
a = [1 -1/6 -1/6];
xnz = filter(b,a,[1,zeros(1,length(n)-1)]);
% Plot
hf = figconfg('P0302');
```

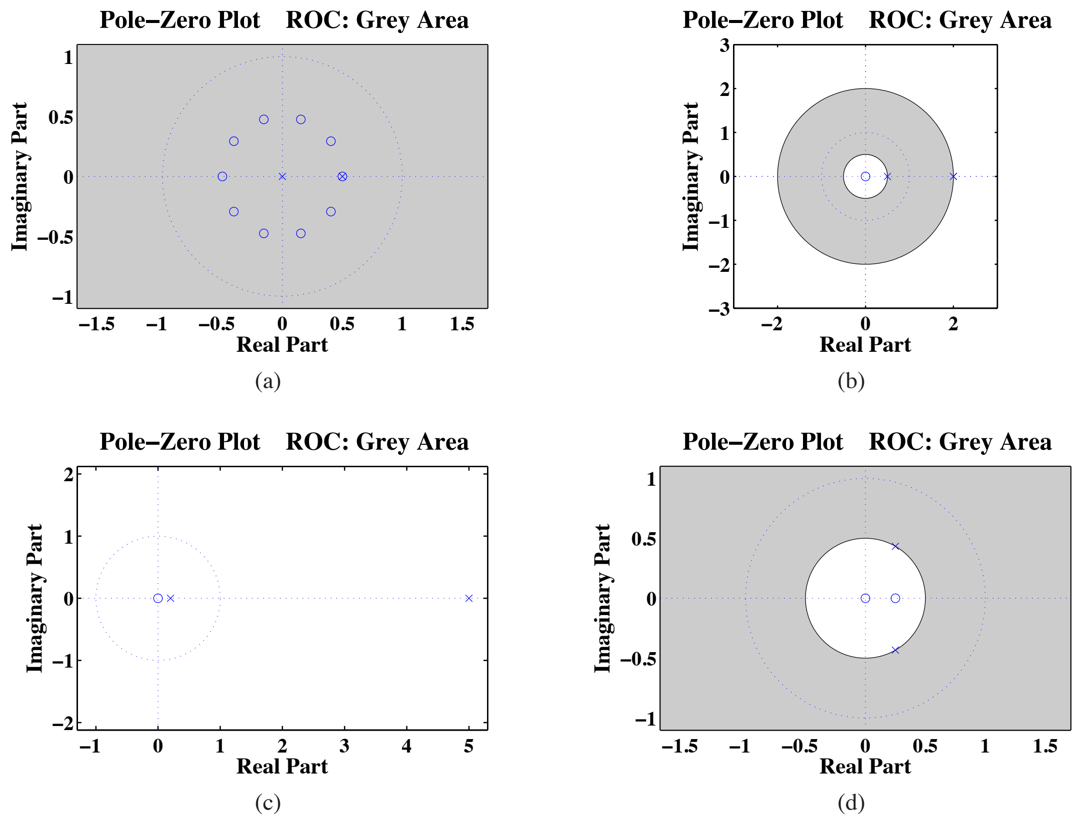


FIGURE 3.1: Pole-zero plot and ROC of (a) $x[n] = \left(\frac{1}{2}\right)^n (u[n] - u[n-10])$. (b) $x[n] = \left(\frac{1}{2}\right)^{|n|}$. (c) $x[n] = 5^{|n|}$. (d) $x[n] = \left(\frac{1}{2}\right)^n \cos(\pi n/3)u[n]$.

```
subplot(211)
stem(n,xn,'filled')
ylabel('x[n]','fontsize',LFS)
title('Original Sequence','fontsize',TFS)
subplot(212)
stem(n,xnz,'filled')
xlabel('Time Index n','fontsize',LFS)
ylabel('x[n]','fontsize',LFS)
title('Sequence Computed from z-transform','fontsize',TFS)
```

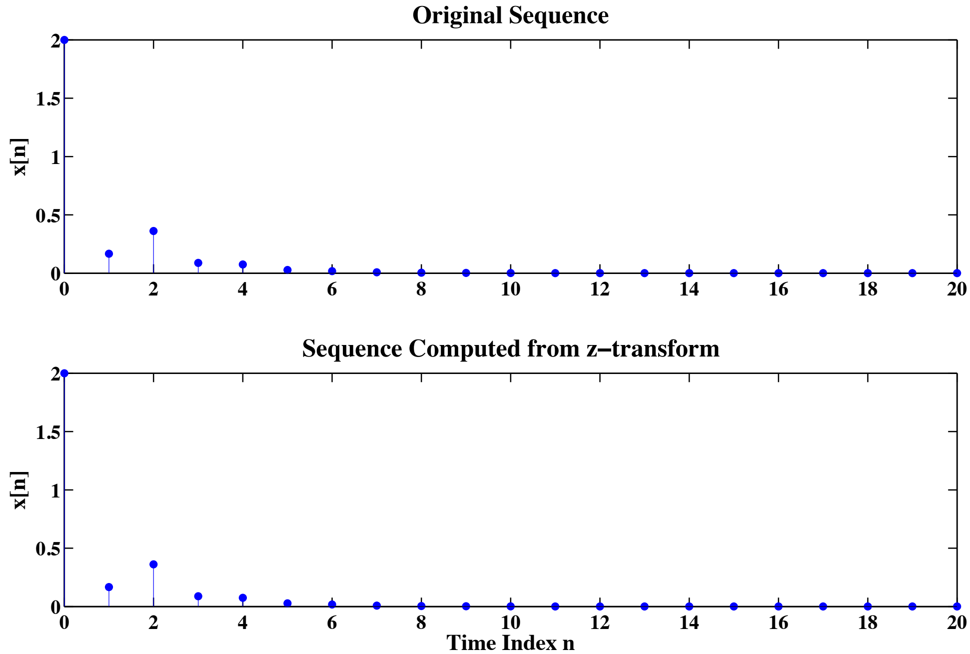


FIGURE 3.2: MATLAB verification of z-transform expression using "x=filter(b,a,[1,zeros(1,N)])".

3. Proof:

$$x[n] = (r^n \sin\omega_0 n)u[n] = r^n\left(\frac{1}{2j}\mathrm{e}^{j\omega_0 n} - \frac{1}{2j}\mathrm{e}^{-j\omega_0 n}\right)u[n]$$

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \frac{1}{2j}\sum_{n=0}^{\infty}\left(r\mathrm{e}^{j\omega_0}z^{-1}\right)^n - \frac{1}{2j}\sum_{n=0}^{\infty}\left(r\mathrm{e}^{-j\omega_0}z^{-1}\right)^n \\ &= \frac{1}{2j}\frac{1}{1-r\mathrm{e}^{j\omega_0}z^{-1}} - \frac{1}{2j}\frac{1}{1-r\mathrm{e}^{-j\omega_0}z^{-1}} \\ &= \frac{r(\sin\omega_0)z^{-1}}{1-2(r\cos\omega_0)z^{-1}+r^2z^{-2}} \quad \text{ROC: } |z| > r \end{aligned}$$

4. (a) Solution:

$$X(z) = \frac{1-\frac{1}{3}z^{-1}}{(1-z^{-1})(1+2z^{-1})} = \frac{\frac{2}{9}}{1-z^{-1}} + \frac{\frac{7}{9}}{1+2z^{-1}}$$

$$\text{ROC: } |z| > 2 \quad x[n] = \frac{2}{9}u[n] + \frac{7}{9}(-2)^n u[n]$$
$$\text{ROC: } |z| < 1 \quad x[n] = -\frac{2}{9}u[-n-1] - \frac{7}{9}(-2)^n u[-n-1]$$
$$\text{ROC: } 1 < |z| < 2 \quad x[n] = \frac{2}{9}u[n] - \frac{7}{9}(-2)^n u[-n-1]$$

(b) Solution:

$$X(z) = \frac{1 - z^{-1}}{1 - \frac{1}{4}z^{-1}} = 4 - \frac{3}{1 - \frac{1}{4}z^{-1}} \quad \text{ROC: } |z| > \frac{1}{4}$$
$$x[n] = 4\delta[n] - 3\left(\frac{1}{4}\right)^n u[n]$$

(c) Solution:

$$X(z) = \frac{2}{1 - 0.5z^{-1}} + \frac{-1}{1 - 0.25z^{-1}}, \quad \text{ROC: } |z| > 0.5$$
$$x[n] = 2(0.5)^n u[n] - (0.25)^n u[n]$$

5. Solution:

$$X(z) = z^2(1 - \frac{1}{3}z^{-1})(1 - z^{-1})(1 + 2z^{-2}) = z^2 - \frac{4}{3}z + \frac{7}{3} - \frac{8}{3}z^{-1} + \frac{2}{3}z^{-2}$$
$$x[n] = \delta[n+2] - \frac{4}{3}\delta[n+1] + \frac{7}{3}\delta[n] - \frac{8}{3}\delta[n-1] + \frac{2}{3}\delta[n-2]$$

6. (a) Solution:

$$\text{Time-shifting:} \quad Y(z) = z^{-3}X(z) = \frac{z^{-3}}{1 - 2z^{-1}}, \quad \text{ROC: } |z| < 2$$

(b) Solution:

$$\text{Scaling:} \quad Y(z) = X(3z) = \frac{1}{1 - \frac{2}{3}z^{-1}}, \quad \text{ROC: } |z| < \frac{2}{3}$$

(c) Solution:

$$\text{Folding and convolution:} \quad Y(z) = X(z)X(1/z) = \frac{1}{1 - 2z^{-1}} \cdot \frac{1}{1 - 2z}$$
$$= \frac{-\frac{1}{2}}{1 - \frac{5}{2}z^{-1} + z^{-2}}, \quad \text{ROC: } \frac{1}{2} < |z| < 2$$

(d) Solution:

Differentiation: $Y(z) = -z\dfrac{dX(z)}{dz} = \dfrac{2z^{-1}}{1-4z^{-1}+4z^{-2}}$, ROC: $|z| < 2$

(e) Solution:

Time-shifting and linearity:
$$\begin{aligned} Y(z) &= z^{-1}X(z) + z^2X(z) \\ &= \frac{z^{-1}}{1-2z^{-1}} + \frac{z^2}{1-2z^{-1}} \\ &= \frac{1+z^{-3}}{z^{-2}-2z^{-3}}, \quad \text{ROC: } 0 < |z| < 2 \end{aligned}$$

(f) Solution:

Time-shifting and convolution:
$$\begin{aligned} Y(z) &= X(z)X(z)z^{-2} \\ &= \frac{z^{-2}}{(1-2z^{-1})^{-2}}, \quad \text{ROC: } |z| < 2 \end{aligned}$$

7. Solution:
$$x[n] = \left(\frac{1}{2}\right)^n u[n]$$

(a) $Y(z) = X(1/z)$

Folding: $y[n] = x[-n] = 2^n u[-n]$

(b) $Y(z) = dX(z)/dz$

Differentiation and time-shifting:
$$\begin{aligned} y[n] &= -(n-1)x[n-1] \\ &= -(n-1)\left(\frac{1}{2}\right)^{n-1} u[n-1] \end{aligned}$$

(c) $Y(z) = X^2(z)$

Convolution: $y[n] = x[n] * x[n] = (n+1)\left(\dfrac{1}{2}\right)^n u[n]$

8. (a) Proof:
$$\begin{aligned} \sum_{n=-\infty}^{\infty} x^*[n]z^{-n} &= \sum_{n=-\infty}^{\infty} (x[n](z^*)^{-n})^* = \left(\sum_{n=-\infty}^{\infty} x[n](z^*)^{-n}\right)^* \\ &= X^*(z^*) \end{aligned}$$

(b) Proof:

$$\sum_{n=-\infty}^{\infty} x[-n]z^{-n} = \sum_{n=-\infty}^{\infty} x[n]z^{n} = \sum_{n=-\infty}^{\infty} x[n](z^{-1})^{-n} = X(1/z)$$

(c) Proof:

$$\begin{aligned}\sum_{n=-\infty}^{\infty} x_R[n]z^{-n} &= \sum_{n=-\infty}^{\infty} \frac{1}{2}(x[n]+x^*[n])z^{-n} \\ &= \frac{1}{2}\sum_{n=-\infty}^{\infty} x[n]z^{-n} + \frac{1}{2}\sum_{n=-\infty}^{\infty} x^*[n]z^{-n} = \frac{1}{2}[X(z)+X^*(z^*)]\end{aligned}$$

(d) Proof:

$$\begin{aligned}\sum_{n=-\infty}^{\infty} x_I[n]z^{-n} &= \sum_{n=-\infty}^{\infty} \frac{1}{2\mathrm{j}}(x[n]-x^*[n])z^{-n} \\ &= \frac{1}{2\mathrm{j}}\sum_{n=-\infty}^{\infty} x[n]z^{-n} - \frac{1}{2\mathrm{j}}\sum_{n=-\infty}^{\infty} x^*[n]z^{-n} = \frac{1}{2\mathrm{j}}[X(z)-X^*(z^*)]\end{aligned}$$

9. Solution:

$$\begin{aligned}X(z) &= \frac{z}{(z+\frac{3}{4})(z-\frac{1}{2}(1+\mathrm{j}))(z-\frac{1}{2}(1-\mathrm{j}))} \\ &= \frac{z}{z^3-\frac{1}{4}z^2-\frac{1}{4}z+\frac{3}{8}} \quad \text{ROC: } |z| > \frac{3}{4}\end{aligned}$$

$$Y(z) = z^{-3}X(1/z) = \frac{z^{-4}}{\frac{3}{8}-\frac{1}{4}z^{-1}-\frac{1}{4}z^{-2}+z^{-3}} \quad \text{ROC: } 0 < |z| < \frac{4}{3}$$

10. Solution:

$$H(z) = \frac{1}{1-az^{-1}}, \quad \text{ROC: } |z| > |a|$$

$$X(z) = \frac{-1}{1-z^{-1}}, \quad \text{ROC: } |z| < 1$$

$$\begin{aligned}Y(z) = H(z)X(z) &= \frac{-1}{(1-az^{-1})(1-z^{-1})} \\ &= \frac{\frac{a}{1-a}}{1-az^{-1}} + \frac{\frac{-1}{1-a}}{1-z^{-1}} \quad \text{ROC: } |a| < |z| < 1 \\ y[n] &= \left(\frac{a}{1-a}\right)a^n u[n] + \frac{1}{1-a}u[-n-1]\end{aligned}$$

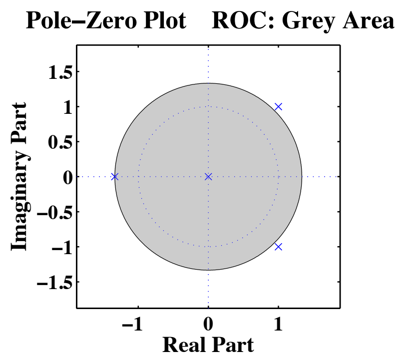


FIGURE 3.3: Pole-zero plot and ROC of $y[n]$.

11. (a) Solution:

$$H(z) = \frac{1}{1 - az^{-1}}, \quad \text{ROC: } |z| > |a|$$

$$X(z) = \frac{1}{1 - bz^{-1}}, \quad \text{ROC: } |z| > |b|$$

$$\begin{aligned} Y(z) = H(z)X(z) &= \frac{1}{1 - az^{-1}} \cdot \frac{1}{1 - bz^{-1}} \\ &= \frac{\frac{a}{a-b}}{1 - az^{-1}} + \frac{\frac{-b}{a-b}}{1 - bz^{-1}}, \quad \text{ROC: } |z| > \max\{|a|, |b|\} \end{aligned}$$

$$y[n] = \frac{a}{a-b} \cdot a^n u[n] + \frac{b}{b-a} \cdot b^n u[n]$$

(b) Solution:

$$H(z) = X(z) = \frac{1}{1 - az^{-1}}, \quad \text{ROC: } |z| > |a|$$

$$\begin{aligned} Y(z) = H(z)X(z) &= \frac{1}{(1 - az^{-1})^2} \\ &= \frac{1}{1 - az^{-1}} + \frac{az^{-1}}{1 - az^{-1}}, \quad \text{ROC: } |z| > |a| \end{aligned}$$

$$y[n] = a^n u[n] + na^n u[n]$$

(c) Solution:

$$x[n] = a^{-n}u[-n] = \delta[n] + (a^{-1})^n u[-n-1]$$

$$X(z) = 1 + \frac{-1}{1-a^{-1}z^{-1}}, \quad \text{ROC: } |z| < |a|^{-1}$$

$$Y(z) = H(z)X(z) = \left(\frac{1}{1-az^{-1}}\right) \cdot \left(1 + \frac{-1}{1-a^{-1}z^{-1}}\right)$$

$$= \frac{\frac{1}{1-a^2}}{1-az^{-1}} + \frac{\frac{-1}{1-a^2}}{1-a^{-1}z^{-1}}, \quad \text{ROC: } |a| < |z| < |a|^{-1}$$

$$y[n] = \frac{a^n}{1-a^2}u[n] + \frac{a^{-n}}{1-a^2}u[-n-1] = \frac{a^{|n|}}{1-a^2}$$

12. (a) Proof:

$$r_{xx}[\ell] \triangleq \sum_n x[n]x[n-\ell] = \sum_n x[n+\ell]x[n] = x[l] * x[-l]$$

By applying the folding property, the z-transform of sequence $x[-l]$ is $X(z^{-1})$ with ROC $\beta^{-1} < |z| < \alpha^{-1}$. Hence, we proved

$$R_{xx}(z) = X(z)X\left(z^{-1}\right) \quad \text{ROC: } \max\{\alpha, \beta^{-1}\} < |z| < \min\{\beta, \alpha^{-1}\}$$

(b) Solution:

$$X(z) = \frac{1}{1-az^{-1}}, \quad \text{ROC: } |z| > |a|$$

$$X(z^{-1}) = \frac{1}{1-az} = \frac{-az^{-1}}{1-a^{-1}z^{-1}}, \quad \text{ROC: } |z| < |a|^{-1}$$

$$R_{xx}(z) = X(z)X\left(z^{-1}\right) = \left(\frac{1}{1-az^{-1}}\right)\left(\frac{-az^{-1}}{1-a^{-1}z^{-1}}\right)$$

$$= \frac{-az^{-1}}{1-(a+a^{-1})z^{-1}+z^{-2}}, \quad \text{ROC: } |a| < |z| < |a|^{-1}$$

(c) Solution:

$$R_{xx}(z) = \frac{\frac{a^2}{1-a^2}}{1-az^{-1}} + \frac{\frac{-a^2}{1-a^2}}{1-a^{-1}z^{-1}}, \quad \text{ROC: } |a| < |z| < |a|^{-1}$$

$$r_{xx}[\ell] = \left(\frac{a^2}{1-a^2}\right)a^l u[l] + \left(\frac{a^2}{1-a^2}\right)a^{-l}u[-l-1] = \left(\frac{a^2}{1-a^2}\right)a^{|l|}$$

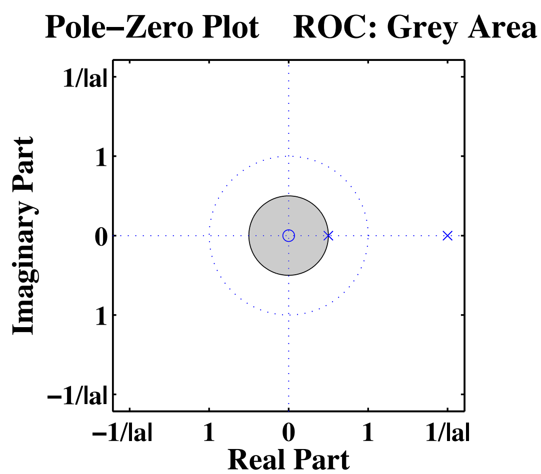


FIGURE 3.4: Pole-zero plot and ROC of $R_{xx}(z)$.

13. Solution:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-1}}{1 - \frac{5}{2}z^{-1} + z^{-2}} = \frac{\frac{2}{3}}{1 - 2z^{-1}} + \frac{-\frac{2}{3}}{1 - \frac{1}{2}z^{-1}}$$

ROC: $|z| > 2$

$$h[n] = \frac{2}{3} \cdot 2^n u[n] - \frac{2}{3} \cdot \left(\frac{1}{2}\right)^n u[n]$$

ROC: $|z| < \frac{1}{2}$

$$h[n] = -\frac{2}{3} \cdot 2^n u[-n-1] + \frac{2}{3} \cdot \left(\frac{1}{2}\right)^n u[-n-1]$$

ROC: $\frac{1}{2} < |z| < 2$

$$h[n] = -\frac{2}{3} \cdot 2^n u[-n-1] - \frac{2}{3} \cdot \left(\frac{1}{2}\right)^n u[n]$$

14. Solution:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - \frac{1}{2}z^{-1}}, \quad \text{ROC: } |z| > \frac{1}{2}$$

(a) $x[n] = e^{j(\pi/4)n}, \quad -\infty < n < \infty$

$$y[n] = \mathrm{e}^{\mathrm{j}(\pi/4)n} \cdot H(z)\mid_{z=\mathrm{e}^{\mathrm{j}(\pi/4)}} = \frac{1}{1-\frac{1}{2}e^{-\mathrm{j}(\pi/4)}} e^{\mathrm{j}(\pi/4)n}$$

(b) $x[n] = \mathrm{e}^{\mathrm{j}(\pi/4)n}u[n]$

$$X(z) = \frac{1}{1-\mathrm{e}^{\mathrm{j}(\pi/4)}z^{-1}}, \quad \text{ROC: } |z| > 1$$

$$\begin{aligned} Y(z) = X(z)H(z) &= \left(\frac{1}{1-\frac{1}{2}z^{-1}}\right)\left(\frac{1}{1-\mathrm{e}^{\mathrm{j}(\pi/4)}z^{-1}}\right) \\ &= \frac{\frac{1}{1-2\mathrm{e}^{\mathrm{j}(\pi/4)}}}{1-\frac{1}{2}z^{-1}} + \frac{\frac{1}{1-2\mathrm{e}^{-\mathrm{j}(\pi/4)}}}{1-\mathrm{e}^{\mathrm{j}(\pi/4)}z^{-1}}, \quad \text{ROC: } |z| > 1 \end{aligned}$$

$$y[n] = \frac{1}{1-2\mathrm{e}^{\mathrm{j}(\pi/4)}}\left(\frac{1}{2}\right)^n u[n] + \frac{1}{1-2\mathrm{e}^{-\mathrm{j}(\pi/4)}}\mathrm{e}^{\mathrm{j}(\pi/4)n}u[n]$$

(c) $x[n] = (-1)^n, \quad -\infty < n < \infty$

$$y[n] = (-1)^n \cdot H(z)\mid_{z=-1} = \frac{2}{3}(-1)^n$$

(d) $x[n] = (-1)^n u[n]$

$$X(z) = \frac{1}{1+z^{-1}}, \quad \text{ROC: } |z| > 1$$

$$\begin{aligned} Y(z) = X(z)H(z) &= \left(\frac{1}{1-\frac{1}{2}z^{-1}}\right)\left(\frac{1}{1+z^{-1}}\right) \\ &= \frac{\frac{1}{3}}{1-\frac{1}{2}z^{-1}} + \frac{\frac{2}{3}}{1+z^{-1}}, \quad \text{ROC: } |z| > 1 \end{aligned}$$

$$y[n] = \frac{1}{3}\left(\frac{1}{2}\right)^n u[n] + \frac{2}{3}(-1)^n u[n]$$

15. (a) Solution:

$$\begin{aligned} H(z) = \frac{Y(z)}{X(z)} &= \frac{1}{1-\frac{3}{4}z^{-1}-\frac{1}{8}z^{-1}} \\ &= \frac{1}{\left(1-\frac{1}{4}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)}, \quad \text{ROC: } |z| > \frac{1}{2} \end{aligned}$$

The system is stable if it is causal.

(b) Solution:

$$H(z) = \frac{-1}{1-\frac{1}{4}z^{-1}} + \frac{2}{1-\frac{1}{2}z^{-1}}, \quad \text{ROC: } |z| > \frac{1}{2}$$

$$h[n] = -\left(\frac{1}{4}\right)^n u[n] + 2\left(\frac{1}{2}\right)^n u[n]$$

(c) Solution:

$$U(z) = \frac{1}{1-z^{-1}}, \quad \text{ROC: } |z| > 1$$

$$\begin{aligned} S(z) = U(z)X(z) &= \frac{1}{\left(1-\frac{1}{4}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)\left(1-z^{-1}\right)}, \\ &= \frac{\frac{1}{3}}{1-\frac{1}{4}z^{-1}} + \frac{-2}{1-\frac{1}{2}z^{-1}} + \frac{\frac{8}{3}}{1-z^{-1}}, \quad \text{ROC: } |z| > \frac{1}{2} \end{aligned}$$

$$s[n] = \frac{1}{3}\left(\frac{1}{4}\right)^n u[n] - 2\left(\frac{1}{2}\right)^n u[n] + \frac{8}{3}u[n]$$

(d) MATLAB script:

```
% P0315: verify the calculated impulse and step
%        response sequences using function 'filter'
close all; clc
n = 0:10;
%% Impluse Response:
hn = -(1/4).^n + 2*(1/2).^n ;
b = 1;
a = [1 -3/4 1/8];
hnz = filter(b,a,[1,zeros(1,length(n)-1)]);
% Plot
hf1 = figconfg('P0315a');
subplot(211)
stem(n,hn,'filled')
ylabel('h[n]','fontsize',LFS)
title('Impulse Response Expression Sequence','fontsize',TFS)
subplot(212)
stem(n,hnz,'filled')
xlabel('Time Index n','fontsize',LFS)
ylabel('h[n]','fontsize',LFS)
title('Sequence Computed from z-transform','fontsize',TFS)
```

```
%% Step Response:
un = 1/3*(1/4).^n - 2*(1/2).^n +8/3;
b = 1;
a = [1 -7/4 7/8 -1/8];
unz = filter(b,a,[1,zeros(1,length(n)-1)]);
% Plot
hf2 = figconfg('P0315b');
subplot(211)
stem(n,un,'filled')
ylabel('s[n]','fontsize',LFS)
title('Step Response Expression Sequence','fontsize',TFS)
subplot(212)
stem(n,unz,'filled')
xlabel('Time Index n','fontsize',LFS)
ylabel('s[n]','fontsize',LFS)
title('Sequence Computed from z-transform','fontsize',TFS)
```

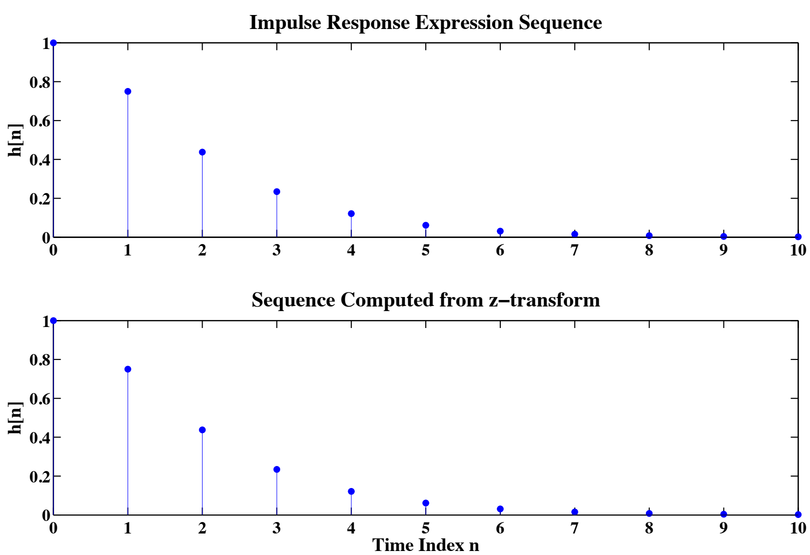


FIGURE 3.5: MATLAB verification of the impulse response expression obtained in part (b).

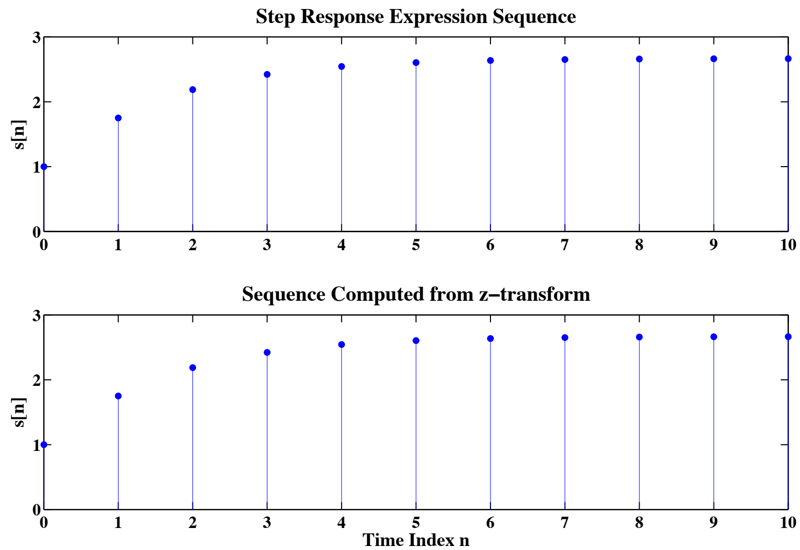


FIGURE 3.6: MATLAB verification of the step response expression obtained in part (c).

16. (a) Solution:

$$X(z) = \frac{1}{1-z^{-1}}, \quad \text{ROC: } |z| > 1$$
$$Y(z) = \frac{2}{1-\frac{1}{3}z^{-1}}, \quad \text{ROC: } |z| > \frac{1}{3}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{2(1-z^{-1})}{1-\frac{1}{3}z^{-1}} = 6 + \frac{-4}{1-\frac{1}{3}z^{-1}}, \quad \text{ROC: } |z| > \frac{1}{3}$$
$$h[n] = 6\delta[n] - 4\left(\frac{1}{3}\right)^n u[n]$$

(b) Solution:

$$X(z) = \frac{1}{1-\frac{1}{2}z^{-1}}, \quad \text{ROC: } |z| > \frac{1}{2}$$

$$\begin{aligned} Y(z) = X(z)H(z) &= \frac{2(1-z^{-1})}{\left(1-\frac{1}{3}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)} \\ &= \frac{8}{1-\frac{1}{3}z^{-1}} + \frac{-6}{1-\frac{1}{2}z^{-1}}, \quad \text{ROC: } |z| > \frac{1}{2} \end{aligned}$$

$$y[n] = 8\left(\frac{1}{3}\right)u[n] - 6\left(\frac{1}{2}\right)u[n]$$

(c) MATLAB script:

```
% P0316: verify the calculated response sequence
%        expressions using function 'filter'
close all; clc
n = 0:10;
%% Impluse Response:
hn = -4*(1/3).^n; hn(1) = hn(1) + 6;
b = [2 -2];
a = [1 -1/3];
hnz = filter(b,a,[1,zeros(1,length(n)-1)]);
% Plot
hf1 = figconfg('P0316a');
subplot(211)
stem(n,hn,'filled')
ylabel('h[n]','fontsize',LFS)
title('Impulse Response Expression Sequence','fontsize',TFS)
subplot(212)
stem(n,hnz,'filled')
xlabel('Time Index n','fontsize',LFS)
ylabel('h[n]','fontsize',LFS)
title('Sequence Computed from z-transform','fontsize',TFS)
%% Response y[n]:
yn = 8*(1/3).^n - 6*(1/2).^n;
b = [2 -2];
a = [1 -5/6 1/6];
ynz = filter(b,a,[1,zeros(1,length(n)-1)]);
% Plot
hf2 = figconfg('P0316b');
subplot(211)
stem(n,yn,'filled')
ylabel('y[n]','fontsize',LFS)
title('Response Expression Sequence','fontsize',TFS)
subplot(212)
stem(n,ynz,'filled')
xlabel('Time Index n','fontsize',LFS)
ylabel('y[n]','fontsize',LFS)
title('Sequence Computed from z-transform','fontsize',TFS)
```

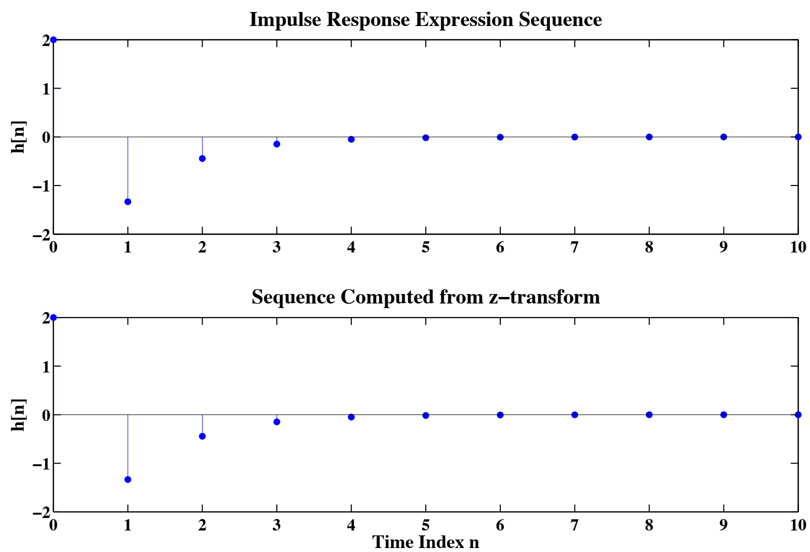


FIGURE 3.7: MATLAB verification of the impulse response expression obtained in part (a).

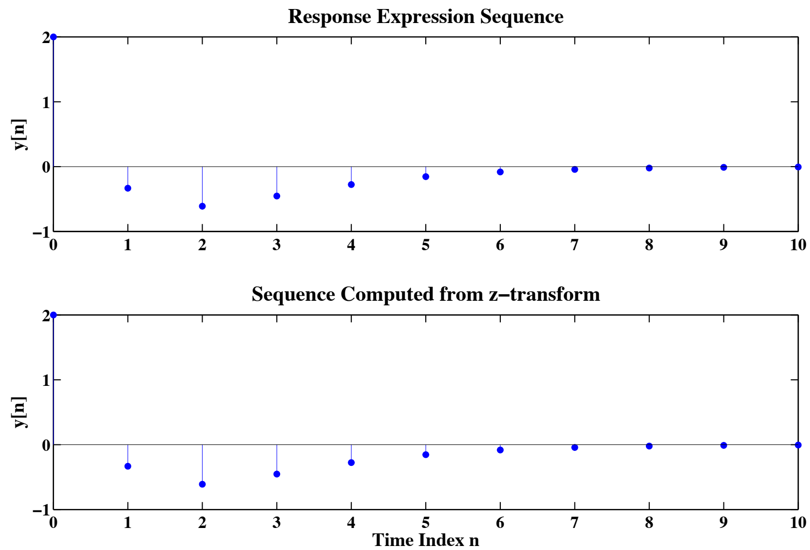


FIGURE 3.8: MATLAB verification of the expression of response $y[n]$ obtained in part (b).

17. Solution:
We repeat the formulas for the ease of comparison as follows.

$$H(z) = \frac{b_0 + b_1 z^{-1}}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

$$h[n] = a|A|r^n \cos(\omega_0 n + \theta)u[n]$$

$$(r^n \cos\omega_0 n)u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1 - (r\cos\omega_0)z^{-1}}{1 - 2(r\cos\omega_0)z^{-1} + r^2 z^{-2}}, \quad \text{ROC: } |z| > r$$

$$(r^n \sin\omega_0 n)u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{(r\sin\omega_0)z^{-1}}{1 - 2(r\cos\omega_0)z^{-1} + r^2 z^{-2}}, \quad \text{ROC: } |z| > r$$

18. Solution:
We first repeat equation (3.97)

$$H(z) = \frac{b_0 + b_1 z^{-1}}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

When the system has two real and equal poles, we have

$$a_1^2 - 4a_2 = 0, \quad p_{1,2} = -\frac{a_1}{2}$$

$$H(z) = \frac{b_0 + b_1 z^{-1}}{\left(1 + \frac{a_1}{2} z^{-1}\right)^2} = \frac{b_0}{1 + \frac{a_1}{2} z^{-1}} + \left(b_0 - \frac{2b_1}{a_1}\right) \frac{-\frac{a_1}{2} z^{-1}}{\left(1 + \frac{a_1}{2} z^{-1}\right)^2} \quad \text{ROC: } |z| > \frac{|a_1|}{2}$$

$$h[n] = b_0 \left(-\frac{a_1}{2}\right)^n u[n] + \left(b_0 - \frac{2b_1}{a_1}\right) \left(-\frac{a_1}{2}\right)^n n u[n]$$

If $|a_1| < 2$, the system $h[n]$ is stable and its shape is decaying.
If $|a_1| = 2$, the system $h[n]$ is unstable and its envelop is constant.
If $|a_1| > 2$, the system $h[n]$ is unstable and its shape is growing.

MATLAB script:

```
% P0318: Plot time sequence and its z-transform
%        plot-zero plot and ROC
close all; clc
n = 0:20;
a1 = 1;
% a1 = 2;
% a1 = 3;
a2 = a1^2/4;
b0 = 1; b1=1;
%% Impluse Response:
hn = b0*(-a1/2).^n+(b0-2*b1/a1).*n.*(-a1/2).^n;
b = [b0 b1];
a = [1 a1 a2];
hnz = filter(b,a,[1,zeros(1,length(n)-1)]);
% Plot
hf1 = figconfg('P0318a','small');
stem(n,hnz,'filled')
xlabel('Time Index n','fontsize',LFS)
ylabel('h[n]','fontsize',LFS)
title('Time Sequence','fontsize',TFS)
%% Pole-zero plot:
hf2 = figconfg('P0318b','small');
p = -a1/2;
zplane(b,a)
xlabel('Real Part','fontsize',LFS)
ylabel('Imaginary Part','fontsize',LFS)
title('Pole-Zero Plot','fontsize',TFS)
```

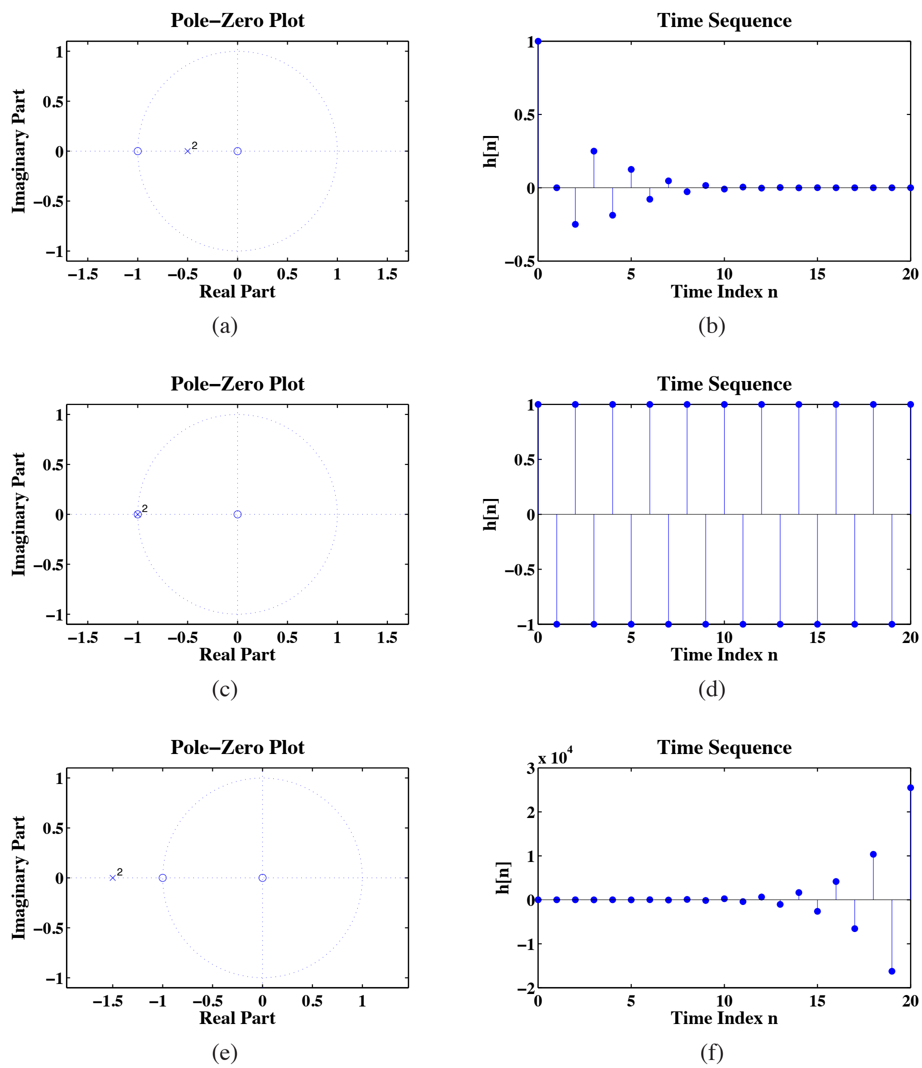


FIGURE 3.9: (a) Poles inside unit circle. (b) Decaying sequence. (c) Poles on unit circle. (b) Constant envelop sequence. (e) Poles outside unit circle. (b) Growing sequence.

19. (a) Solution:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - \frac{1}{1024}z^{-10}}{1 - \frac{1}{2}z^{-1}}$$

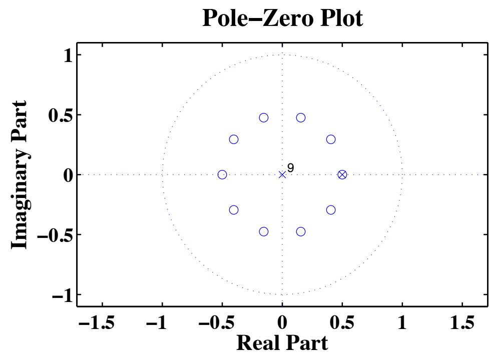


FIGURE 3.10: Pole-zero pattern of the system.

(b) Impulse response.

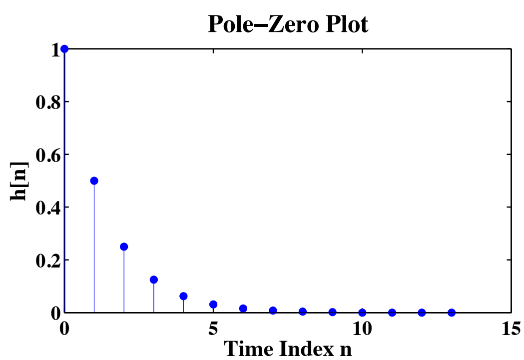


FIGURE 3.11: Impulse response $h[n]$.

(c) Comments: The pole at $z = \frac{1}{2}$ is canceled and the ROC is all the z plane. The corresponding time sequence should be of finite length.

(d) Solution:

$$H(z) = \sum_{m=0}^{9} 2^{-m} x[n-m]$$

20. (a) Solution:

$$X(z) = \frac{-\frac{1}{3}}{1-\frac{1}{2}z^{-1}} + \frac{\frac{4}{3}}{1-2z^{-1}} = \frac{1}{(1-\frac{1}{2}z^{-1})(1-2z^{-1})}, \quad \text{ROC: } \frac{1}{2} < |z| < 2$$

(b) Solution:

$$H(z) = \frac{Y(z)}{X(z)} = 1 - z^{-2}$$

$$h[n] = \delta[n] - \delta[n-2]$$

21. (a) Pole-zero plot.

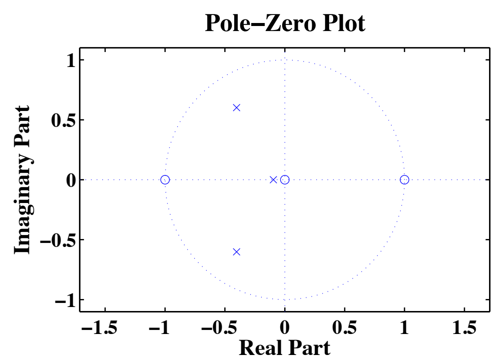


FIGURE 3.12: MATLAB verification of the impulse response expression obtained in part (a).

(b) Impulse response.

(c) Solution:

$$h[n] = -2.1760(-0.0956)^n u[n] + 2\times 1.5942\cos(2.1605n - 0.0885)0.7233^n u[n]$$

(d) See plot below.

MATLAB script:

```
% P0321: Plot plot-zero pattern and compute
%        impulse response
close all; clc
n = 0:20;
b = [1 0 -1];
a = [1 0.9 0.6 0.05];
```

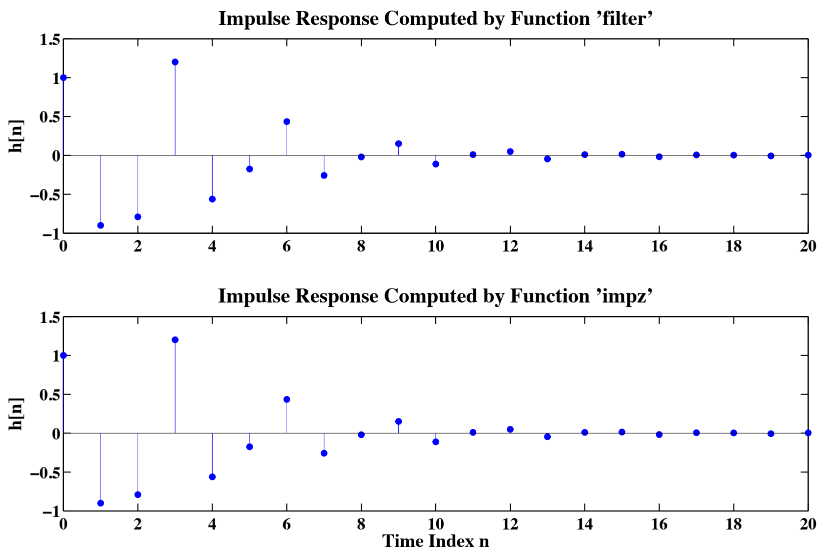


FIGURE 3.13: Impulse responses comparison using the functions `filter` and `impz`.

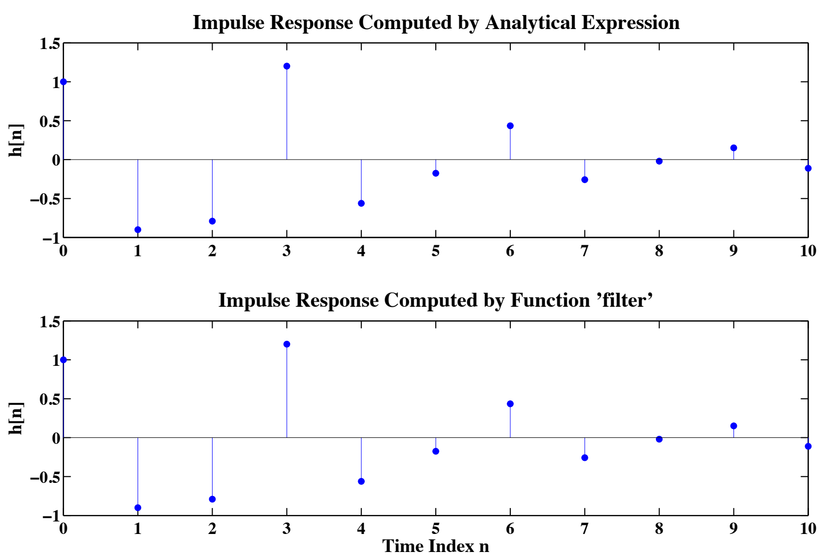


FIGURE 3.14: MATLAB verification of the analytical expression of system impulse responses obtained in part (c).

```
%% Part (a):
hf1 = figconfg('P0321a','small');
zplane(b,a)
xlabel('Real Part','fontsize',LFS)
ylabel('Imaginary Part','fontsize',LFS)
title('Pole-Zero Plot','fontsize',TFS)
%% Part (b):
hn1 = filter(b,a,[1,zeros(1,length(n)-1)]);
hn2 = impz(b,a,length(n));
% Plot
hf2 = figconfg('P0321b');
subplot(211)
stem(n,hn1,'filled')
ylabel('h[n]','fontsize',LFS)
title('Impulse Response Computed by Function ''filter''',...
    'fontsize',TFS)
subplot(212)
stem(n,hn2,'filled')
xlabel('Time Index n','fontsize',LFS)
ylabel('h[n]','fontsize',LFS)
title('Impulse Response Computed by Function ''impz''',...
    'fontsize',TFS)
%% Part (c):
[r,p,k] = residuez(b,a);
% [b2 a2] = residuez(r(1:2),p(1:2),k);
%% Part (d):
n = 0:10;
hn = r(3)*p(3).^n + 2*abs(r(1))*cos(angle(p(1)).*n...
    +angle(r(1))).*(abs(p(1)).^n);
hn_ref = filter(b,a,[1,zeros(1,length(n)-1)]);
% Plot
hf3 = figconfg('P0321c');
subplot(211)
stem(n,hn,'filled')
ylabel('h[n]','fontsize',LFS)
title('Impulse Response Computed by Analytical Expression',...
    'fontsize',TFS)
subplot(212)
stem(n,hn_ref,'filled')
xlabel('Time Index n','fontsize',LFS)
```

```
ylabel('h[n]','fontsize',LFS)
title('Impulse Response Computed by Function ''filter''',...
    'fontsize',TFS)
```

22. Solution:

$$h[n] = -\frac{2}{3}\left(-\frac{1}{2}\right)^n u[n] + \frac{5}{3}\left(\frac{1}{4}\right)^n u[n]$$

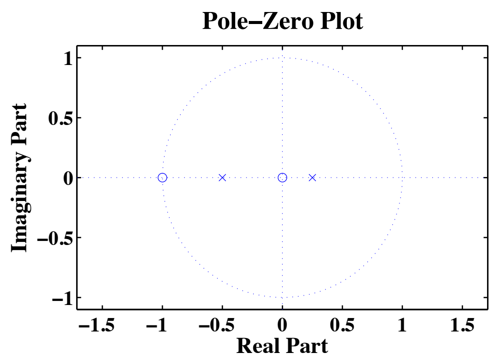


FIGURE 3.15: MATLAB verification of the impulse response expression obtained in part (a).

MATLAB script:

```
% P0322: Plot plot-zero pattern and compute
%        impulse response
close all; clc
n = 0:20;
b = [1 1];
a = [1 1/4 -1/8];
%% Part (a):
hf1 = figconfg('P0322a','small');
zplane(b,a)
xlabel('Real Part','fontsize',LFS)
ylabel('Imaginary Part','fontsize',LFS)
title('Pole-Zero Plot','fontsize',TFS)
%% Part (b):
hn1 = filter(b,a,[1,zeros(1,length(n)-1)]);
hn2 = impz(b,a,length(n));
```

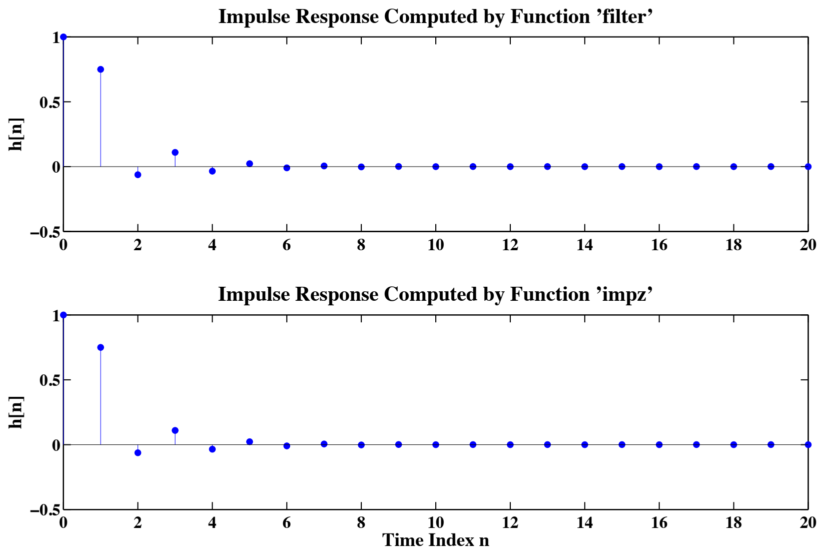


FIGURE 3.16: Impulse responses comparison using the functions `filter` and `impz`.

```
% Plot
hf2 = figconfg('P0322b');
subplot(211)
stem(n,hn1,'filled')
ylabel('h[n]','fontsize',LFS)
title('Impulse Response Computed by Function ''filter''',...
    'fontsize',TFS)
subplot(212)
stem(n,hn2,'filled')
xlabel('Time Index n','fontsize',LFS)
ylabel('h[n]','fontsize',LFS)
title('Impulse Response Computed by Function ''impz''',...
    'fontsize',TFS)
%% Part (c):
[r,p,k] = residuez(b,a);
% [b2 a2] = residuez(r(1:2),p(1:2),k);
%% Part (d):
n = 0:10;
```

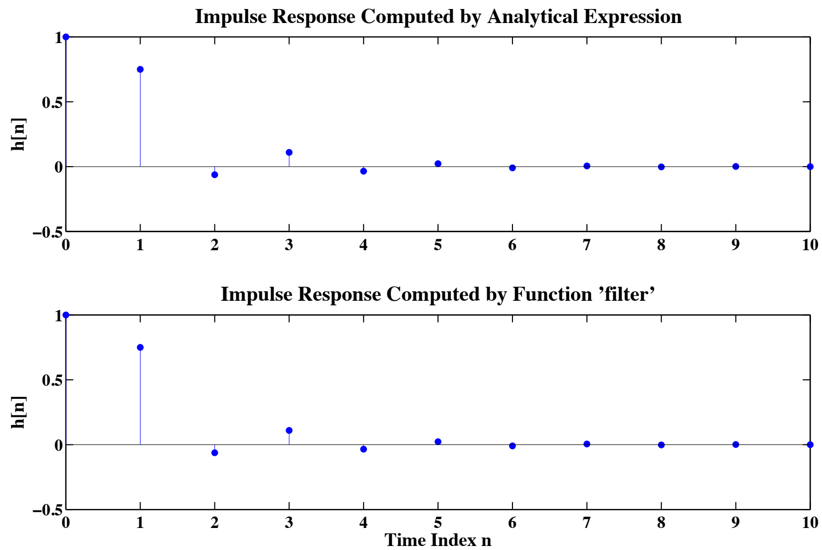
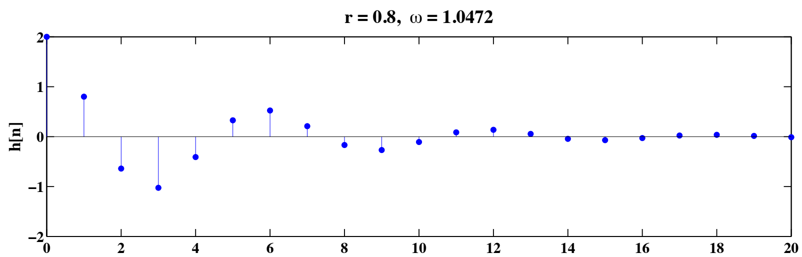


FIGURE 3.17: MATLAB verification of the analytical expression of system impulse responses obtained in part (c).

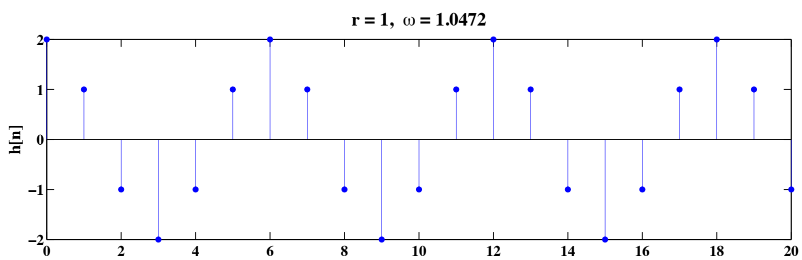
```
hn = r(1)*p(1).^n + r(2)*p(2).^n;
hn_ref = filter(b,a,[1,zeros(1,length(n)-1)]);
% Plot
hf3 = figconfg('P0322c');
subplot(211)
stem(n,hn,'filled')
ylabel('h[n]','fontsize',LFS)
title('Impulse Response Computed by Analytical Expression',...
    'fontsize',TFS)
subplot(212)
stem(n,hn_ref,'filled')
xlabel('Time Index n','fontsize',LFS)
ylabel('h[n]','fontsize',LFS)
title('Impulse Response Computed by Function ''filter''',...
    'fontsize',TFS)
```

23. MATLAB script:

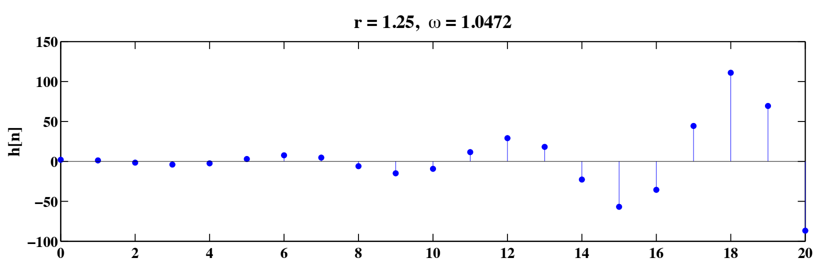
```
% P0323: Script to generate plots shown in Figure3.11
%       h[n] = 2|A|r^n*cos(w0*n+theta)*u[n]
close all; clc
w0 = pi/3;
r = [0.8 1 1.25];
n = 0:20;
A = 1; theta = 0;
rlen = length(r);
hn = zeros(rlen,length(n));
for ii = 1:rlen
hn(ii,:) = 2*A*r(ii).^n.*cos(w0.*n+theta);
end
% Plot
hf1 = figconfg('P0323a','long');
stem(n,hn(1,:),'filled')
ylabel('h[n]','fontsize',LFS)
title(['r = ',num2str(r(1)),',  \omega = ',num2str(w0)],...
    'fontsize',TFS)
hf2 = figconfg('P0323b','long');
stem(n,hn(2,:),'filled')
ylabel('h[n]','fontsize',LFS)
title(['r = ',num2str(r(2)),',  \omega = ',num2str(w0)],...
    'fontsize',TFS)
hf3 = figconfg('P0323c','long');
stem(n,hn(3,:),'filled')
ylabel('h[n]','fontsize',LFS)
title(['r = ',num2str(r(3)),',  \omega = ',num2str(w0)],...
    'fontsize',TFS)
```



(a)



(b)



(c)

FIGURE 3.18: (a) Stable system. (b) Marginally stable system. (c) Unstable system.

24. (a) Solution:

$$y[n] = b_0x[n] + b_1x[n-1] + b_2x[n-2] - a_1y[n-1] - a_2y[n-2]$$

Apply one-sided z-transform, we have

$$\begin{aligned}Y^+(z) &= \frac{b_0 + b_1z^{-1} + b_2z^{-2}}{1 + a_1z^{-1} + a_2z^{-2}}X^+(z) \\ &+ \frac{(b_2x[-1] - a_2y[-1])z^{-1} + (b_1x[-1] + b_2x[-2] - a_1y[-1] - a_2y[-2])}{1 + a_1z^{-1} + a_2z^{-2}}\end{aligned}$$

By substituting the specification in problem and using MATLAB, we have

$$\begin{aligned}y[n] = 2 \times 3.9473 \cos(1.0472n - 1.5560)u[n] \\ + 2 \times 3.7578 \cos(1.0472n + 1.5420)(0.95)^n u[n] \\ + 2 \times 1.8760 \cos(1.0472n - 1.1670)(0.95)^n u[n]\end{aligned}$$

(b) MATLAB script:

```
% P0324: Verify system response using filter
close all; clc
n = 0:20;
xn = cos(pi/3*n);
b = 1/3*ones(1,3);
a = [1 -0.95 0.9025];
bx = [1 -cos(pi/3)];
ax = [1 -2*cos(pi/3) 1];
yi = [-2 -3];
xi = [1 1];
bzi = [b(2)*xi(1)+b(3)*xi(2)-a(2)*yi(1)-a(3)*yi(2),...
    b(3)*xi(1)-a(3)*yi(1)];
bzs = conv(b,bx);
bz = bzs + conv(ax,bzi);
azs = conv(a,ax);
[rzi pzi kzi] = residuez(bzi,a);
[rzs pzs kzs] = residuez(bzs,azs);
ynzi = 2*abs(rzi(1))*cos(angle(pzi(1)).*n...
    +angle(rzi(1))).*(abs(pzi(1)).^n);
ynzs = 2*abs(rzs(1))*cos(angle(pzs(1)).*n...
    +angle(rzs(1))).*(abs(pzs(1)).^n) + ...
```

```
    2*abs(rzs(3))*cos(angle(pzs(3)).*n...
    +angle(rzs(3))).*(abs(pzs(3)).^n);
yn1 = ynzi + ynzs;
% Matlab verification:
zi = filtic(b,a,yi,xi);
yn2 = filter(b,a,xn,zi);
% Plot
hf = figconfg('P0324');
subplot(211)
stem(n,yn1,'filled')
ylabel('y[n]','fontsize',LFS)
title('System Response Computed by Analytical Expressoin',...
    'fontsize',TFS)
subplot(212)
stem(n,yn2,'filled')
xlabel('Time Index n','fontsize',LFS)
ylabel('y[n]','fontsize',LFS)
title('System Response Computed by Function ''filter''',...
    'fontsize',TFS)
```

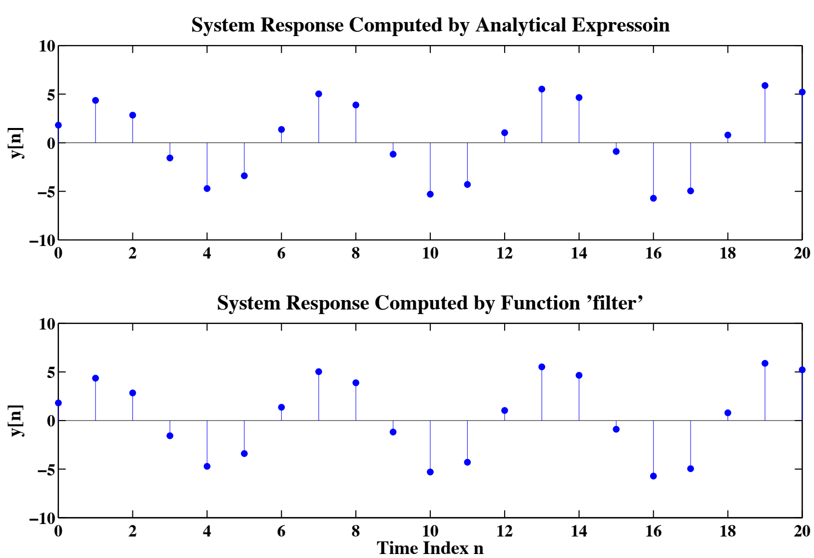


FIGURE 3.19: MATLAB verification of the analytical expression obtained in part (a).