CHAPTER 14

Random Signal Processing

Tutorial Problems

1. (a) See plot below.

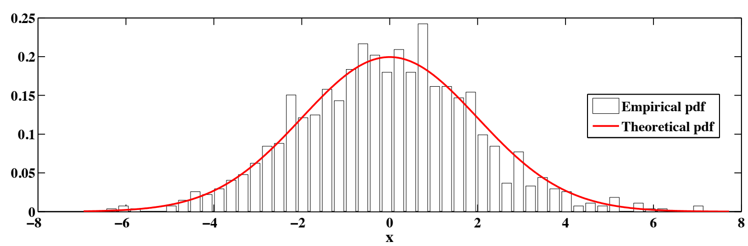

FIGURE 14.1: Empirical and theoretical pdfs of random variable $x \sim N(0, \sigma^2)$ for $\sigma = 2$.

   (b) See script below.

   (c) See plot below.

   (d) Comments:
   See script output.

   (e) See plot below.

   (f) See plots below.

   MATLAB script:

```
% P1401: Simulation for the quality of unbiased mean estimator
close all; clc
% sigma = 2; N = 40; K = 1000;
sigma = sqrt(2); N = 20; K = 1000; % part f
%% Part a
```

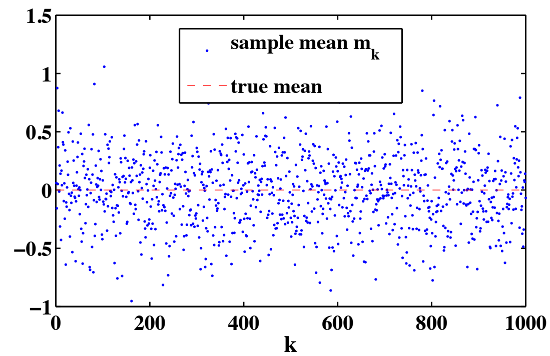

FIGURE 14.2: Plot of the true mean μ and the K sample means.

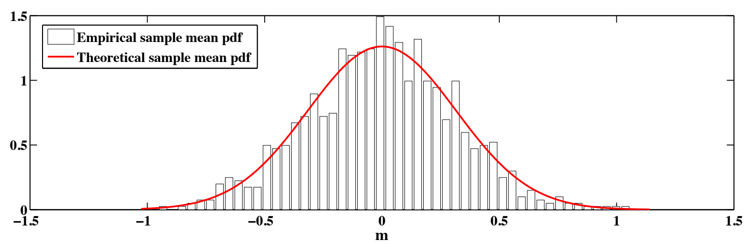

FIGURE 14.3: Empirical and theoretical pdfs of the sample mean.

```
x = randn(1,1000)*sigma;
[xo px] = epdf(x,50);
xx = linspace(1.1*min(xo),1.1*max(xo),1000);
fxx = normpdf(xx,0,sigma);
hfa = figconfg('P1401a','long');
bar(xo,px,'facecolor','w'); hold on
plot(xx,fxx,'r','linewidth',2)
xlabel('x','fontsize',LFS)
legend('Empirical pdf','Theoretical pdf','location','best')
%% Part b
X = randn(K,N)*sigma;
%% Part c
mhatk = mean(X,2);
hfb = figconfg('P1401b','small');
plot(1:K,mhatk,'.b',1:K,zeros(1,K),'--r')
xlabel('k','fontsize',LFS)
legend('sample mean m_k','true mean','location','best')
```

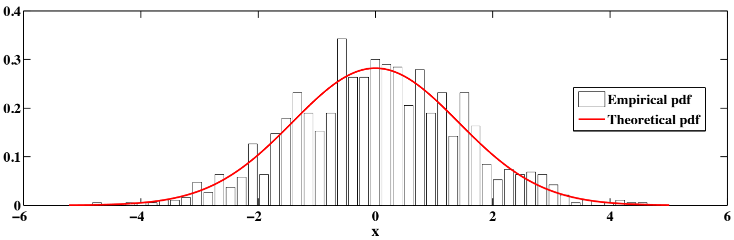

FIGURE 14.4: Empirical and theoretical pdfs of random variable $x \sim N(0, \sigma^2)$ for $\sigma = \sqrt{2}$.

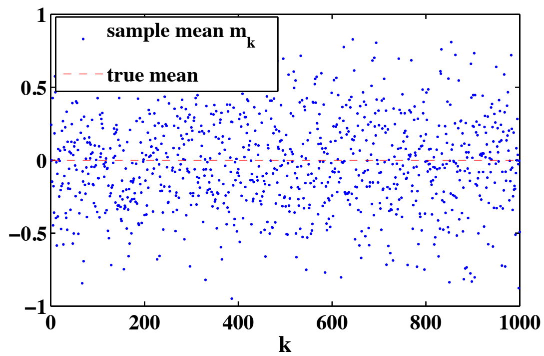

FIGURE 14.5: Plot of the true mean μ and the K sample means.

```
%% Part d
mu_m = mean(mhatk), mu_true = 0,
var_m = var(mhatk), var_true = sigma^2/N,
%% Part e
[mo pm] = epdf(mhatk,50);
mm = linspace(1.1*min(mo),1.1*max(mo),1000);
fmm = normpdf(mm,0,sigma/sqrt(N));
hfc = figconfg('P1401c','long');
bar(mo,pm,'facecolor','w'); hold on
plot(mm,fmm,'r','linewidth',2)
xlabel('m','fontsize',LFS)
legend('Empirical sample mean pdf',...
    'Theoretical sample mean pdf','location','best')
```

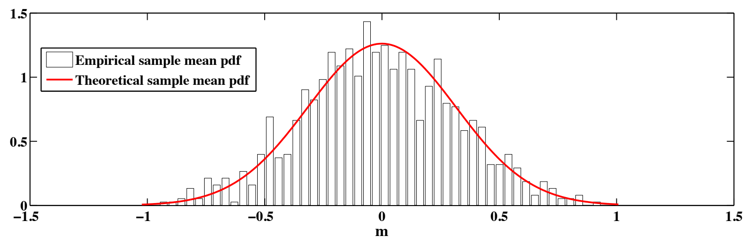

FIGURE 14.6: Empirical and theoretical pdfs of the sample mean.

2. tba.

3. (a) Solution:
We first determine the marginal pdfs of x and y.
If $-1 \le x \le 0$, the pdf of x is

$$f(x) = \int_{-2}^{0} \frac{1}{4} dy = \frac{1}{2}$$

If $0 \le x \le 2$, the pdf of x is

$$f(x) = \int_{0}^{1} \frac{1}{4} dy = \frac{1}{4}$$

Hence, the pdf of x is

$$f(x) = \begin{cases} 1/2, & -1 \le x \le 0 \\ 1/4, & 0 \le x \le 2 \\ 0, & \text{otherwise} \end{cases}$$

If $0 \le y \le 1$, the pdf of y is

$$f(y) = \int_{0}^{2} \frac{1}{4} dx = \frac{1}{2}$$

If $-2 \le y \le 0$, the pdf of y is

$$f(y) = \int_{-1}^{0} \frac{1}{4} dx = \frac{1}{4}$$

Hence, the pdf of y is

$$f(y) = \begin{cases} 1/4, & -2 \le y \le 0 \\ 1/2, & 0 \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$$

We can compute the mean of x and y as

$$\mu_x = \int_{-\infty}^{\infty} xf(x)dx = \int_{-1}^{0} \frac{x}{2}dx + \int_{0}^{2} \frac{x}{4}dx = \frac{1}{4}$$

$$\mu_y = \int_{-\infty}^{\infty} yf(y)dy = \int_{-2}^{0} \frac{y}{4}dy + \int_{0}^{1} \frac{y}{2}dy = -\frac{1}{4}$$

(b) Solution:
Given the pdfs of x and y obtained in last step, we have

$$E[x^2] = \int_{-1}^{0} \frac{x^2}{2}dx + \int_{0}^{2} \frac{x^2}{4}dx = \frac{5}{6}$$

$$\sigma_x^2 = E[x^2] - E^2[x] = \frac{5}{6} - \left(\frac{1}{4}\right)^2 = \frac{37}{48}$$

$$\sigma_x = \sqrt{\frac{37}{48}} \approx 0.8780$$

$$E[y^2] = \int_{-2}^{0} \frac{y^2}{4}dy + \int_{0}^{1} \frac{y^2}{2}dy = \frac{5}{6}$$

$$\sigma_y^2 = E[y^2] - E^2[y] = \frac{5}{6} - \left(-\frac{1}{4}\right)^2 = \frac{37}{48}$$

$$\sigma_y = \sqrt{\frac{37}{48}} \approx 0.8780$$

(c) Solution:
The correlation r_{xy} is

$$\begin{aligned} r_{xy} = E[xy] &= \iint xyf(x,y)dxdy \\ &= \frac{1}{4}\left(\int_{0}^{2} xdx\right)\left(\int_{0}^{1} ydy\right) + \frac{1}{4}\left(\int_{-1}^{0} xdx\right)\left(\int_{-2}^{0} ydy\right) = \frac{1}{2} \end{aligned}$$

$$c_{xy} = r_{xy} - \mu_x\mu_y = \frac{1}{2} - \frac{1}{4}\left(-\frac{1}{4}\right) = \frac{9}{16} = 0.5625$$

The correlation coefficient ρ_{xy} is

$$\rho_{xy} = \frac{c_{xy}}{\sigma_x\sigma_y} = \frac{9/16}{37/48} = \frac{27}{37} \approx 0.7297$$

(d) MATLAB script:

```
% P1403: Investigation of correlation between two random variables
close all; clc
% N = 1e4;
% N = 1e5;
N = 1e6;
x1 = rand(N/2,1)*2; y1 = rand(N/2,1);
x2 = rand(N/2,1)-1; y2 = (rand(N/2,1)-1)*2;
x = [x1;x2]; y = [y1;y2];
mu_x = mean(x), mu_y = mean(y),
std_x = std(x), std_y = std(y),
rxy = corr(x,y)*std_x*std_y + mu_x*mu_y,
rhoxy = corrcoef(x,y),
```

4. (a) Solution:

The mean of $x_2[n]$ is:

$$E[x_2[n]] = E[x_1[n]] + E[0.01n] = 0 + 0.01 \times \sum_{i=1}^{1000} i/1000 = 5.005$$

$$\begin{aligned} E[x_2^2[n]] &= E[(x_1[n] + 0.01n)^2] = E[x_1^2[n] + 0.02nx_1[n] + 0.0001n^2] \\ &= 1 + 0 + 10^{-4} \times \sum_{n=1}^{1000} n^2/1000 = 34.3834 \end{aligned}$$

The variance of $x_2[n]$ is:

$$\sigma_{x_2}^2 = E[x_2^2[n]] - E^2[x_2[n]] = 9.3784$$

The mean of $x_3[n]$ is:

$$E[x_3[n]] = \frac{1}{2}$$

$$E[x_3^2[n]] = \frac{1}{2} \times 1 + \frac{1}{2} \times 2 = \frac{3}{2}$$

The variance of $x_3[n]$ is:

$$\sigma_{x_3}^2 = \frac{3}{2} - \left(\frac{1}{2}\right)^2 = \frac{5}{4}$$

The mean of $x_4[n]$ is

$$E[x_4[n]] = 0$$

The variance of $x_4[n]$ is

$$\sigma_{x_4}^2 = E[x_4^2[n]] = \frac{1}{2} \times 1 + \frac{1}{2} \times 2 = \frac{3}{2}$$

(b) See plot below.

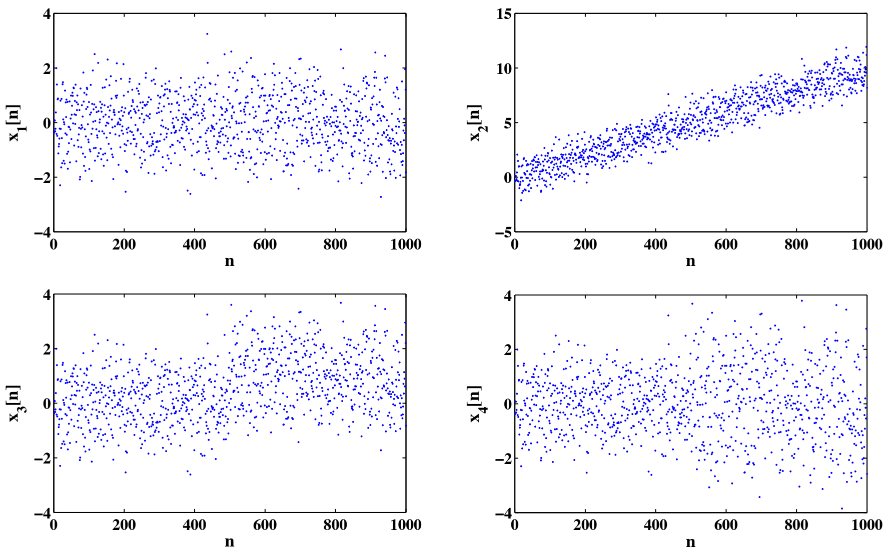

FIGURE 14.7: Plot of the data sets.

MATLAB script:

```
% P1404: Mean and Variance estimate verification
close all; clc
N = 1e3; n = (1:N)';
x1 = randn(N,1);
x2 = x1 + 0.01*n;
x3 = x1 + (n > 500);
x4 = x1; x4(501:end) = x4(501:end)*sqrt(2);
%% Plot
hfa = figconfg('P1404a','small');
plot(n,x1,'.')
xlabel('n','fontsize',LFS)
```

```
ylabel('x_1[n]','fontsize',LFS)
hfb = figconfg('P1404b','small');
plot(n,x2,'.')
xlabel('n','fontsize',LFS)
ylabel('x_2[n]','fontsize',LFS)
hfc = figconfg('P1404c','small');
plot(n,x3,'.')
xlabel('n','fontsize',LFS)
ylabel('x_3[n]','fontsize',LFS)
hfd = figconfg('P1404d','small');
plot(n,x4,'.')
xlabel('n','fontsize',LFS)
ylabel('x_4[n]','fontsize',LFS)
```

5. (a) Solution:
The mean μ_x is:

$$\mu_x = aE[x[n-1]] + E[w[n]] = 0$$

Define $\ell = m - n$, and $\ell \geq 0$ we have

$$\begin{aligned} r_{xx}[\ell] &= E[x[m]x[n]] = E[(ax[m-1] + w[m])(ax[n-1] + w[n])] \\ &= a^2 r_{xx}[\ell] + aE[x[m-1]w[n]] + aE[x[n-1]w[m]] + \sigma_w^2 \delta[\ell] \\ &= a^2 r_{xx}[\ell] + aE[x[m-1]w[n]] + \sigma_w^2 \delta[\ell] \end{aligned}$$

If $\ell = 1$,

$$r_{xx}[1] = a^2 r_{xx}[1] + a\sigma_w^2 \implies r_{xx}[1] = \frac{a\sigma_w^2}{1-a^2}$$

If $\ell = 0$,

$$r_{xx}[0] = a^2 r_{xx}[0] + \sigma_w^2 \implies r_{xx}[0] = \frac{\sigma_w^2}{1-a^2}$$

Otherwise

$$r_{xx}[\ell] = 0$$

Hence, we can conclude that

$$r_{xx}[\ell] = \frac{\sigma_w^2}{1-a^2}\delta[\ell] + \frac{a\sigma_w^2}{1-a^2}\delta[|\ell| - 1]$$

Let $\ell = 0$, the variance σ_x^2 is

$$\sigma_x^2 = r_{xx}[0] = \frac{\sigma_w^2}{1-a^2}$$

The correlation coefficient $\rho_x[\ell]$ is

$$\rho_x[\ell] = \delta[\ell] + a\,\delta[|\ell|-1]$$

(b) See plot below.

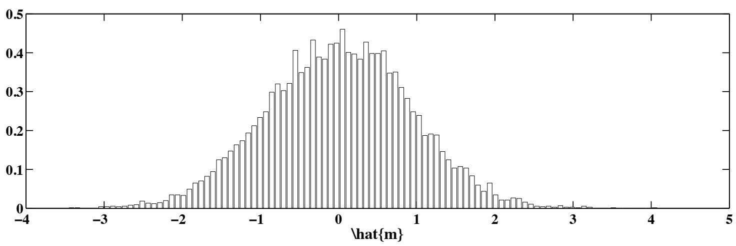

FIGURE 14.8: Plot of the histogram of the estimated mean when $a = 0.9$.

(c) See plot below.

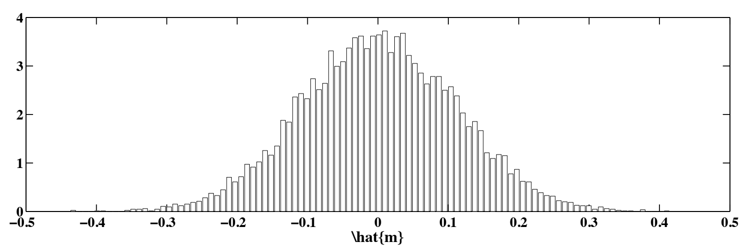

FIGURE 14.9: Plot of the histogram of the estimated mean when $a = 0.1$.

(d) tba.

MATLAB script:

```
% P1405: Sample Mean Investigation
close all; clc
N = 100; K = 1e4;
a = 0.9;
% a = 0.1; % Part c
wn = randn(N,K);
```

```
xn = filter(1,[1 -a],wn);
mhatk = mean(xn,1);
[mo pm] = epdf(mhatk,100);
%% Plot
hfa = figconfg('P1405a','long');
bar(mo,pm,'facecolor','w');
xlabel('\hat{m}','fontsize',LFS)
```

6. (a) See plot below.

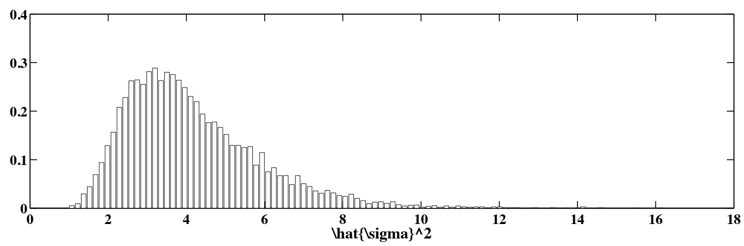

FIGURE 14.10: Plot of the histogram of the estimated mean when $a = 0.9$.

(b) See plot below.

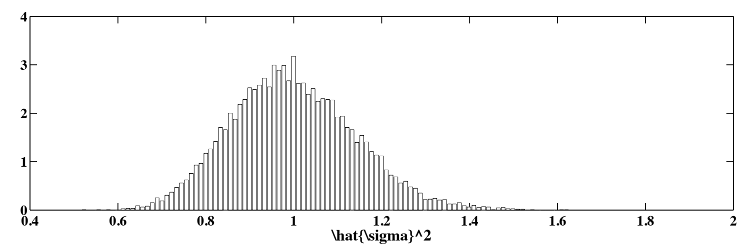

FIGURE 14.11: Plot of the histogram of the estimated mean when $a = 0.1$.

(c) tba.

MATLAB script:

```
% P1406: Sample Variance Investigation
close all; clc
N = 100; K = 1e4;
% a = 0.9;
a = 0.1; % Part c
wn = randn(N,K);
```

```
xn = filter(1,[1 -a],wn);
varhatk = var(xn,1,1);
[mo pm] = epdf(varhatk,100);
%% Plot
hfa = figconfg('P1406a','long');
bar(mo,pm,'facecolor','w');
xlabel('\hat{\sigma}^2','fontsize',LFS)
```

7. tba

8. (a) Proof:

$$E[\hat{S}_D(\omega_k)] = E\left[\frac{1}{2Q+1}\sum_{m=-Q}^{Q} I(\omega_{k-m})\right] = \frac{1}{2Q+1}\sum_{m=-Q}^{Q} E\left[I(\omega_{k-m})\right]$$

Since $I(\omega)$ is essentially an unbiased estimate, we have

$$E[\hat{S}_D(\omega_k)] = \frac{1}{2Q+1}\sum_{m=-Q}^{Q} S(\omega_{k-m}) \approx \frac{1}{2Q+1}\sum_{m=-Q}^{Q} S(\omega_k) = S(\omega_k)$$

(b) Proof:

$$\begin{aligned}
E[\hat{S}_D^2(\omega_k)] &= \frac{1}{(2Q+1)^2} E\left[\sum_{m=-Q}^{Q} I(\omega_{k-m}) \sum_{n=-Q}^{Q} I(\omega_{k-n})\right] \\
&= \frac{1}{(2Q+1)^2}\left(E\left[\sum_{m=-Q}^{Q} I(\omega_{k-m})\right] E\left[\sum_{n=-Q}^{Q} I(\omega_{k-n})\right] + \sum_{m=-Q}^{Q} \text{Var}[I(\omega_{k-m})]\right) \\
&= \frac{1}{(2Q+1)^2}\left(\sum_{m=-Q}^{Q} S(\omega_{k-m}) \sum_{n=-Q}^{Q} S(\omega_{k-n}) + \sum_{m=-Q}^{Q} \text{Var}[I(\omega_{k-m})]\right) \\
&\approx \frac{1}{(2Q+1)^2}\left[(2Q+1)^2 S^2(\omega_k) + (2Q+1)\text{Var}[I(\omega_k)]\right] \\
&= S^2(\omega_k) + \frac{\text{Var}[I(\omega_k)]}{2Q+1}
\end{aligned}$$

$$\text{Var}[\hat{S}_D(\omega_k)] = E[\hat{S}_D^2(\omega_k)] - E^2[\hat{S}_D(\omega_k)] = \frac{\text{Var}[I(\omega_k)]}{2Q+1} \approx \frac{S^2(\omega_k)}{2Q+1}$$

(c) Comments:
The variance reduction factor is $2Q+1$.

(d) See plots below.

MATLAB script:

```
% P1408: Daniell Method
close all; clc
N = 2048;
vn = randn(1,N);
xn = filter(1,[1 -0.75 0.5],vn);
Q = 4;
% Q = 8;
% Q = 16;
% Q = 32;
I = psdper(xn,N);
I_zp = [zeros(2*Q,1);I;zeros(2*Q,1)];
Im = zeros(2*Q+1,N);
for ii = 1:N
    Im(:,ii) = I_zp(ii:ii+2*Q);
end
SD = mean(Im,1);
w = linspace(0,2,N)*pi;
%% Plot
hfa = figconfg('P1408a','long');
plot(w/pi,I,'g'); hold on
plot(w/pi,SD,'linewidth',1,'color','r')
ylim([0 20])
xlabel('\omega/\pi','fontsize',LFS)
title(['Q = ',num2str(Q)],'fontsize',LFS)
legend('I(\omega_k)','\hat{S}_D(\omega_k)','location','best')
```

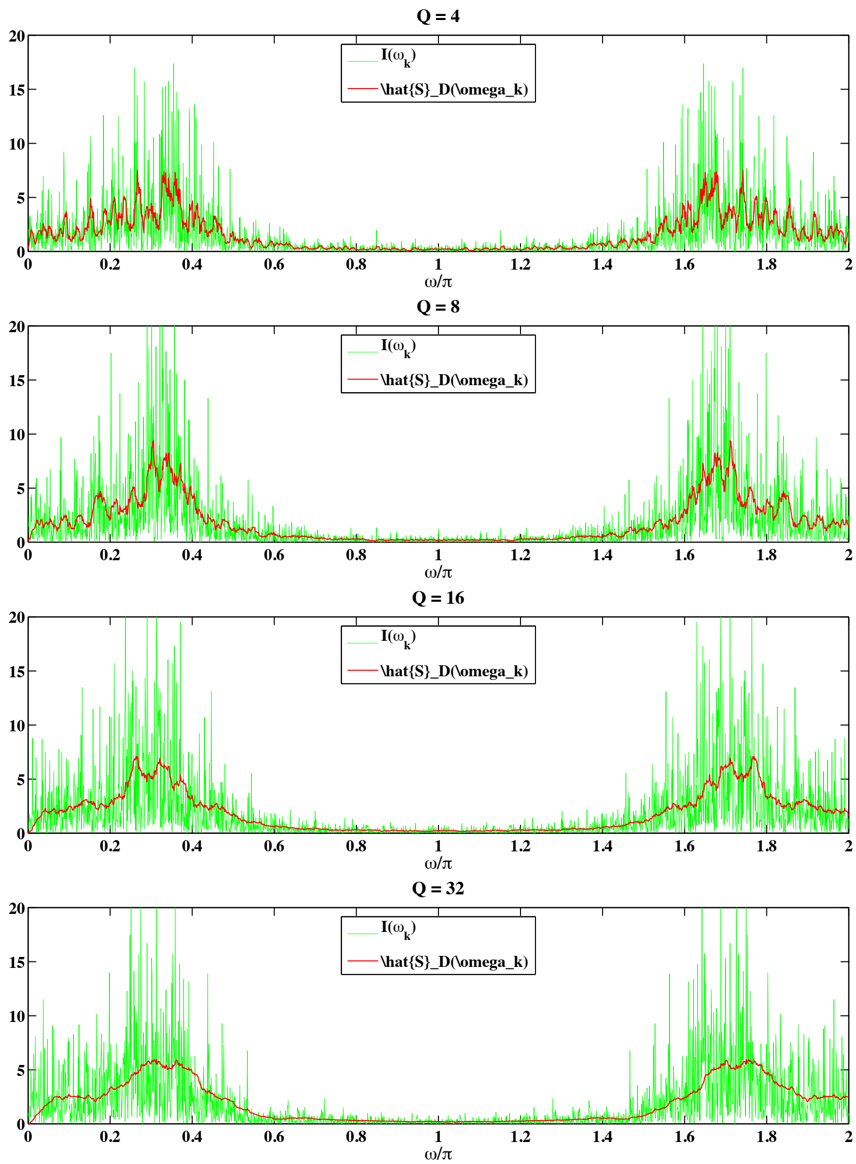

FIGURE 14.12: Plot of $I(\omega_k)$ and $\hat{S}_D(\omega_k)$ for $Q = 4$, $Q = 8$, $Q = 16$, and $Q = 32$.

9. (a) See plot below.

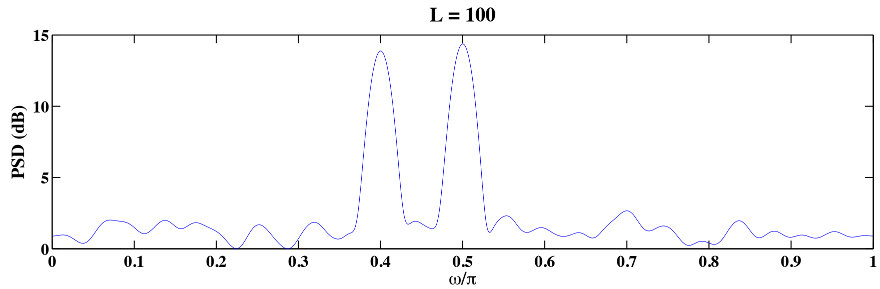

FIGURE 14.13: Plot of PSD when $L = 100$.

(b) See plot below.

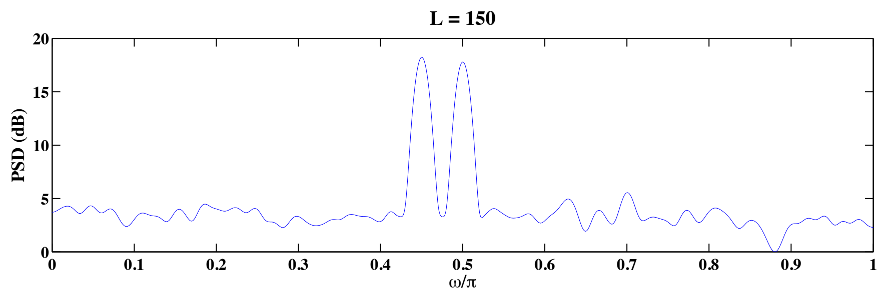

FIGURE 14.14: Plot of PSD when $L = 150$.

(c) tba.

MATLAB script:

```
% P1409: Investigate closing window aspect of Blackman-Tukey PSD estimate
close all; clc
A1 = 1; A2 = 1; A3 = 0.2; om2 = 0.5*pi; om3 = 0.7*pi; N = 5e3;
om1 = 0.4*pi; % L = 100;
% om1 = 0.45*pi; % L = 150
L = 20;
vn = randn(1,N); phi = rand(1,3)*2*pi-pi;
n = 1:N;
xn = A1*cos(om1*n+phi(1)) + A2*cos(om2*n+phi(2))...
    + A3*cos(om3*n+phi(3)) + vn;
w = linspace(0,1,2^(ceil(log2(N))-1))*pi;
```

```
hfa = figconfg('P1409a','long');
while L < N
    S = psdbt(xn',L,2^(ceil(log2(N))));
    figure(hfa)
    Sdb = 10*log10(S./min(S));
    plot(w/pi,Sdb)
    xlabel('\omega/\pi','fontsize',LFS)
    ylabel('PSD (dB)','fontsize',LFS)
    title(['L = ',num2str(L)],'fontsize',TFS)
    pause
    L = L + 10;
end
```

10. (a) See plot below.

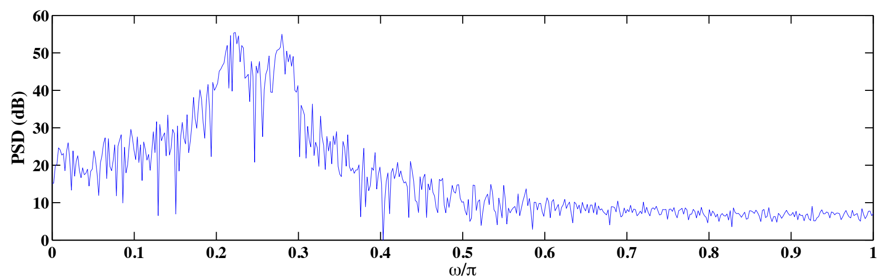

FIGURE 14.15: Periodogram of $x[n]$ using a 1024-point FFT.

(b) See plot below.

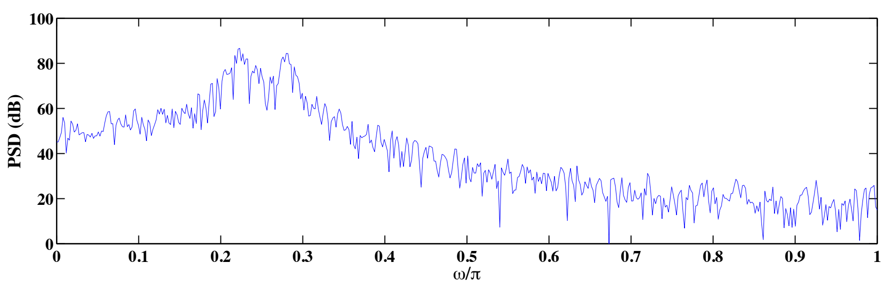

FIGURE 14.16: Modified periodogram of $x[n]$ using a 1024-point FFT based on Bartlett data window.

(c) tba.

MATLAB script:

```
% P1410: Periodogram and Modified Periodogram
close all; clc
N = 1024;
vn = randn(N,1);
xn = filter(1,[1 -2.7607 3.8106 -2.6535 0.9238],vn);
w = linspace(0,1,N/2)*pi;
%% Part a: Periodogram
I1 = psdper(xn,N);
hfa = figconfg('P1410a','long');
I1db = 10*log10(I1(1:N/2)./min(I1(1:N/2)));
plot(w/pi,I1db)
xlabel('\omega/\pi','fontsize',LFS)
ylabel('PSD (dB)','fontsize',LFS)
%% Part b: Modified Periodogram
I2 = psdmodper2(xn,N,bartlett(N));
hfb = figconfg('P1410b','long');
I2db = 10*log10(I2(1:N/2)./min(I2(1:N/2)));
plot(w/pi,I2db)
xlabel('\omega/\pi','fontsize',LFS)
ylabel('PSD (dB)','fontsize',LFS)
```

11. (a) See plot below.

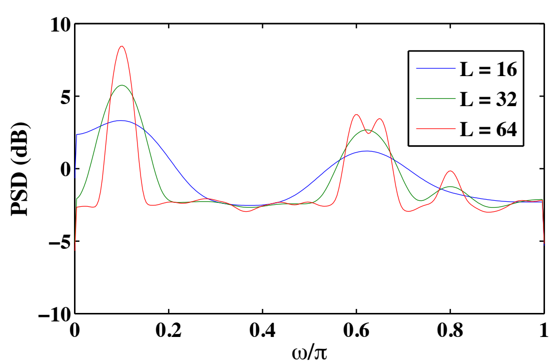

FIGURE 14.17: Plot when using 75 percent overlap, Hamming window, and $L =$ 16, 32, and 64.

(b) See plot below.

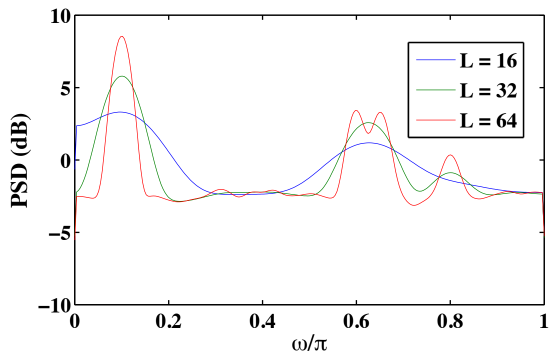

FIGURE 14.18: Plot when using 50 percent overlap, Hamming window, and $L =$ 16, 32, and 64.

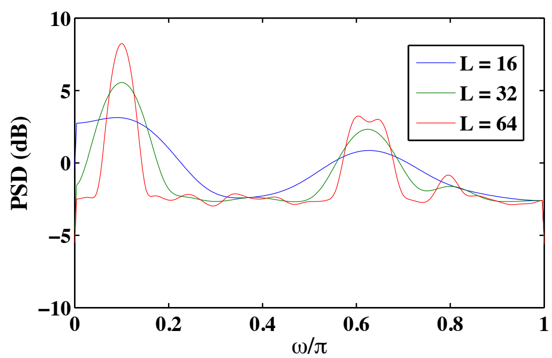

FIGURE 14.19: Plot when using 50 percent overlap, Hann window, and $L = 16$, 32, and 64.

(c) See plot below.

(d) tba.

MATLAB script:

```
% P1411: Investigate Welsh PSD estimate
close all; clc
A = [1 0.5 0.5 0.25]; om = [0.1 0.6 0.65 0.8]*pi;
phi = rand(1,4)*2*pi-pi;
N = 256; K = 50; n = (0:N-1)'; Nk = repmat(n,1,K);
V = randn(N,K);
X = V;
```

```
for jj = 1:4
    X = X + A(jj)*sin(om(jj)*Nk+phi(jj));
end
L = [16 32 64]; Nfft = 512;
Pxx = zeros(K,Nfft/2+1,length(L));
for jj = 1:length(L)
    for ii = 1:K
        [Pxx(ii,:,jj), w] = ...
            pwelch(X(:,ii),hamming(L(jj)),fix(0.75*L(jj)),Nfft); % part a
%          [Pxx(ii,:,jj), w] = ...
%              pwelch(X(:,ii),hamming(L(jj)),fix(0.5*L(jj)),Nfft); % part b
%          [Pxx(ii,:,jj), w] = ...
%              pwelch(X(:,ii),hann(L(jj)),fix(0.5*L(jj)),Nfft); % part c
    end
end
P = squeeze(mean(Pxx,1));
Pdb = bsxfun(@rdivide,P,mean(P,1));
Pdb = 10*log10(Pdb);
%% Part a: Periodogram
hfa = figconfg('P1411a','small');
plot(w/pi,Pdb)
xlabel('\omega/\pi','fontsize',LFS)
ylabel('PSD (dB)','fontsize',LFS)
legend(['L = ',num2str(L(1))],['L = ',num2str(L(2))],...
    ['L = ',num2str(L(3))],'location','best')
```

12. Solution:

$$H_1(z) = \frac{1}{1 - az^{-1}} \Leftrightarrow h_1[n] = a^n u[n] \Leftrightarrow H_1(\mathrm{e}^{\mathrm{j}\omega}) = \frac{1}{1 - a\mathrm{e}^{-\mathrm{j}\omega}}$$

$$H_2(z) = \frac{1}{1 - z^{-1}} \Leftrightarrow h_1[n] = u[n] \Leftrightarrow H_2(\mathrm{e}^{\mathrm{j}\omega}) = \frac{1}{1 - \mathrm{e}^{-\mathrm{j}\omega}}$$

$$|H_1(\mathrm{e}^{\mathrm{j}\omega})|^2 = \frac{1}{(1 - a\cos\omega)^2 + a^2\sin^2\omega}, \quad |H_2(\mathrm{e}^{\mathrm{j}\omega})|^2 = \frac{1}{2 - 2\cos\omega}$$

The variance of $x[n]$ is

$$\sigma_x^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |H_1(\mathrm{e}^{\mathrm{j}\omega})|^2 |H_2(\mathrm{e}^{\mathrm{j}\omega})|^2 \sigma_w^2 \delta(\omega) d\omega = \infty$$

Hence, the process $x[n]$ is NOT stationary.

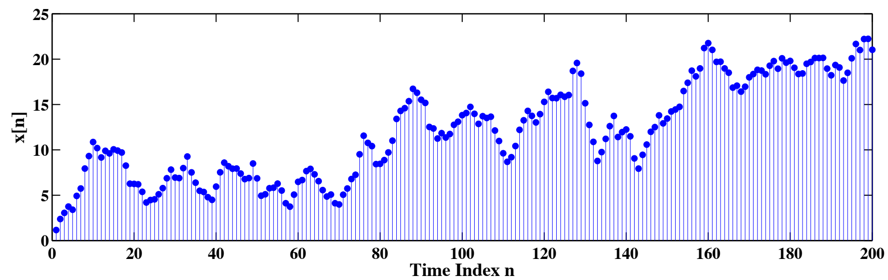

FIGURE 14.20: Plot $N = 200$ samples of $x[n]$.

(a) See plot below.

(b) Solution:

$$S(\omega) = \sigma_w^2 \times \frac{1}{(1 - a\cos\omega)^2 + a^2\sin^2\omega} \times \frac{1}{2 - 2\cos\omega}$$

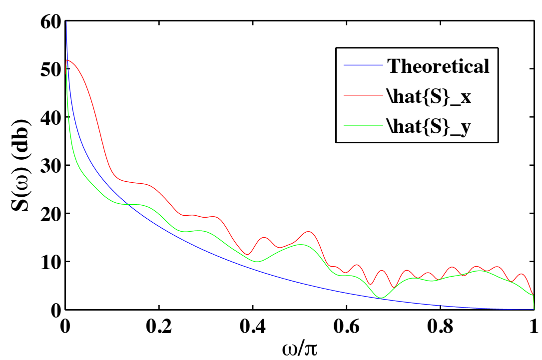

FIGURE 14.21: Plots of theoretical and empirical $S(\omega)$ and Welch PSD estimate $\tilde{S}_y(\omega)$ when $N = 200$.

(c) See plot below.

(d) See script for details.

(e) See script for details.

MATLAB script:

```
% P1412: PSD estimate
close all; clc
```

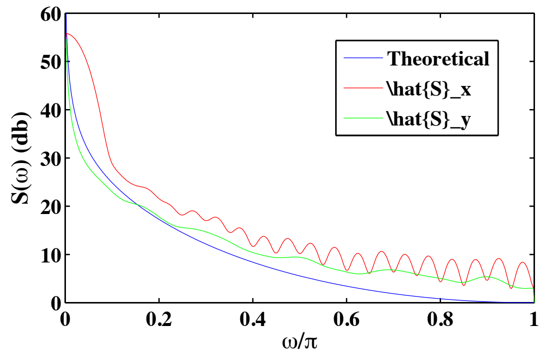

FIGURE 14.22: Plots of theoretical and empirical $S(\omega)$ and Welch PSD estimate $\tilde{S}_y(\omega)$ when $N = 1000$.

```
% N = 200;
N = 1e3;
sigmaw = 1;
a = 0.5;
randn('seed',0)
wn = randn(1,N)*sigmaw;
vn = filter(1,[1 -a],wn);
xn = filter(1,[1 -1],vn);
%% Part a:
hfa = figconfg('P1412a','long');
stem(1:N,xn,'filled')
xlabel('Time Index n','fontsize',LFS)
ylabel('x[n]','fontsize',LFS)
%% Part b:
w = linspace(0,1,1001)*pi;
Sw = sigmaw^2./(1+a^2-2*a*cos(w))./(2-2*cos(w));
Sw(1) = nan;
Swdb = 10*log10(Sw./min(Sw));
hfb = figconfg('P1412b','small');
plot(w/pi,Swdb)
ylim([0 60])
xlabel('\omega/\pi','fontsize',LFS)
ylabel('S(\omega) (db)','fontsize',LFS)
%% Part c:
L = 40; Nfft = 1024;
```

```
[Swhat, w2] = pwelch(xn,hamming(L),fix(0.5*L),Nfft);
figure(hfb)
hold on
Swhatdb = 10*log10(Swhat./min(Swhat));
plot(w2/pi,Swhatdb,'r')
%% Part d:
yn = filter([1 -1],1,xn);
[Swhat2, w3] = pwelch(yn,hamming(L),fix(0.5*L),Nfft);
Swhat2 = Swhat2./abs(1-exp(-w3)).^2;
Swhat2(1) = nan;
Swhat2db = 10*log10(Swhat2./min(Swhat2));
figure(hfb)
plot(w3/pi,Swhat2db,'g')
legend('Theoretical','\hat{S}_x','\hat{S}_y','location','best')
```

13. Solution:

$$\begin{aligned} J &= E[(y-\hat{y})^2] = E[(y-b)^2] = E[y^2 - 2by + b^2] \\ &= E[y^2] - 2bE[y] + b^2 \end{aligned}$$

Take the first derivative and assign it to zero, we have

$$\frac{dJ}{db} = -2E[y] + 2b = 0 \implies b = E[y]$$

Hence, the optimum constant linear estimator is the mean

14. (a) See script for details.
    (b) See plot below.
    (c) See plot below.
    (d) tba.

MATLAB script:

```
% P1414: Matched filtering
close all; clc
N = 200; n = 0:N-1;
sn = zeros(size(n));
%% Part b
p = 10;
ind = (n >= 0 & n <= p-1);
```

```
sn(ind) = cos(2*pi*n(ind)/10);
%% Part c
% p = 100;
% ind = (n >= 0 & n <= p-1);
% sn(ind) = cos(2*pi*n(ind)/10)/sqrt(10);

randn('seed',0)
xn = sn + randn(1,N);
yn = filter(sn(p:-1:1),1,xn);

%% Part a:
hfa = figconfg('P1414a','small');
plot(1:N,sn)
title('Signal','fontsize',LFS)

hfb = figconfg('P1414b','small');
plot(1:N,xn)
title('Signal+Noise','fontsize',LFS)

hfc = figconfg('P1414c','small');
plot(1:N,yn)
title('Matched Filter Output','fontsize',LFS)
xlabel('n','fontsize',LFS)
```

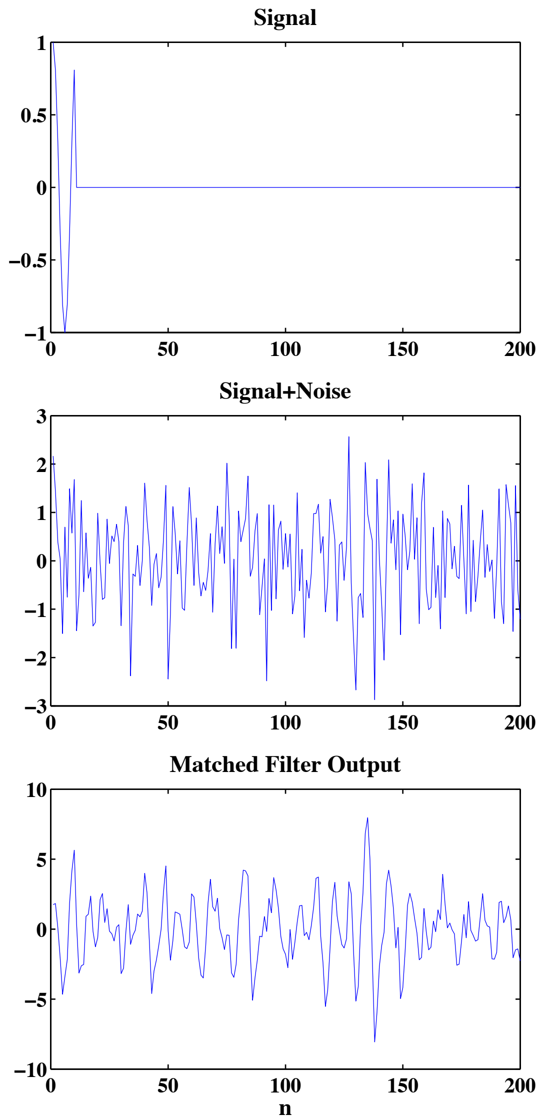

FIGURE 14.23: Plots of $s_i[n] = \cos(2\pi n/10)$, noisy signal $x[n]$ and matched filter output.

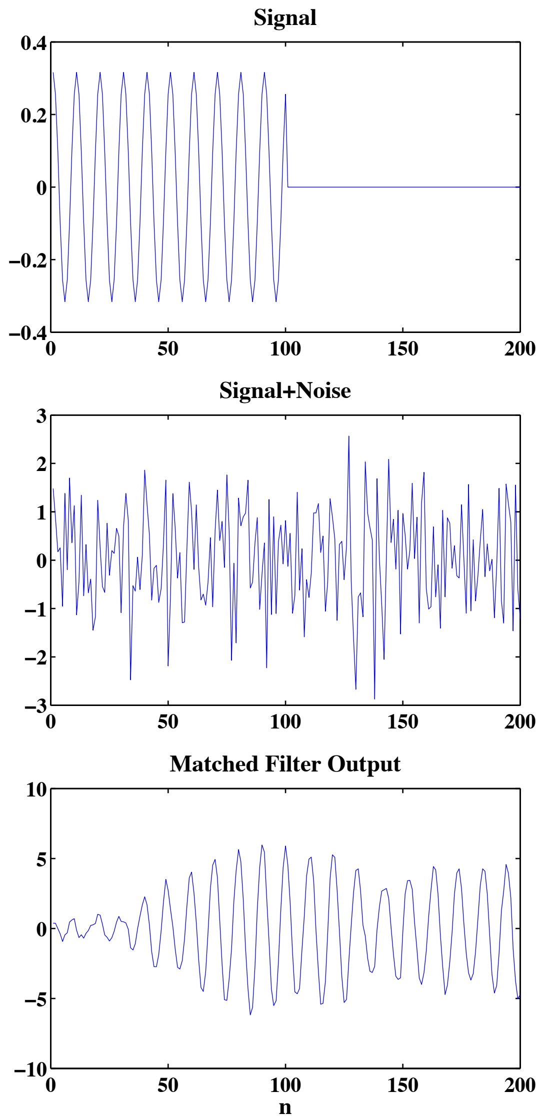

FIGURE 14.24: Plots of $s_i[n] = (1/\sqrt{10})\cos(2\pi n/10)$, noisy signal $x[n]$ and matched filter output.

15. (a) Solution:
The theoretical PSD $S_y(\omega)$ is:

$$S_y(\omega) = \left|\frac{1}{1 - 0.95\mathrm{e}^{-\mathrm{j}\omega}}\right|^2 \cdot 4 = \frac{4}{1 + 0.95^2 - 1.9\cos\omega}$$

(b) Solution:

$$H_o(\omega) = \frac{S_y(\omega)}{S_y(\omega) + 1}$$

(c) See script below.

(d) See plot below.

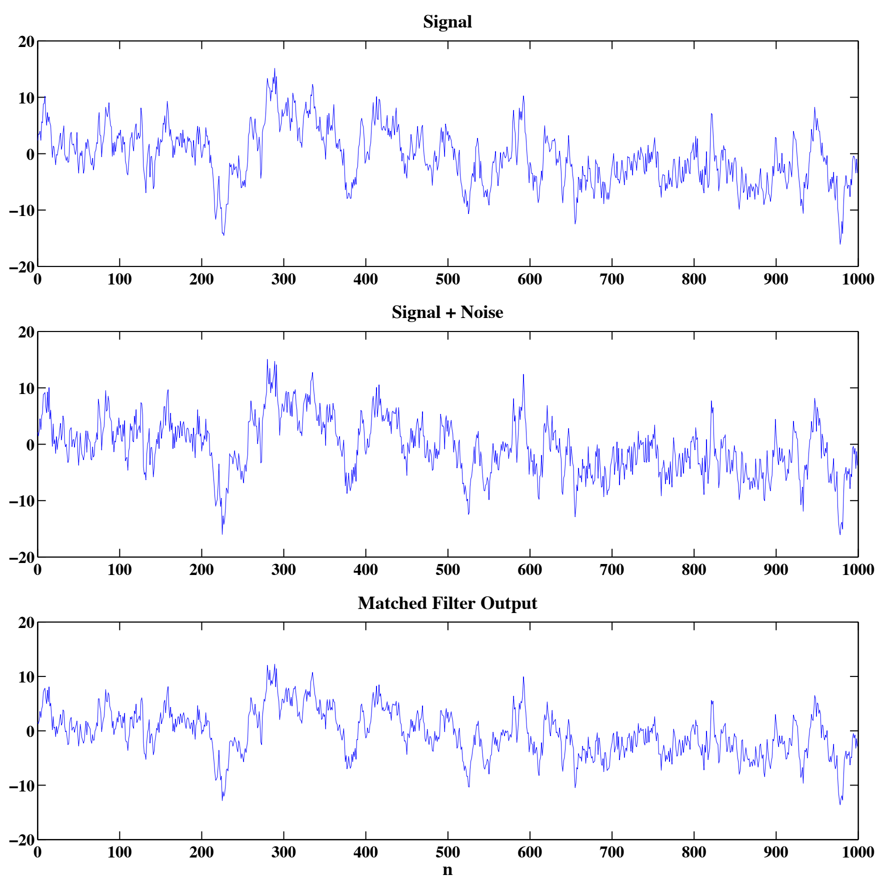

FIGURE 14.25: Plots of clean, noisy and filtered signals.

MATLAB script:

```
% P1415: Wiener Filter
close all; clc
N = 1000; n = 0:N-1;
randn('seed',0)
wn = randn(1,N)*2;
yn = filter(1,[1 -0.95],wn);
vn = randn(1,N);
xn = yn + vn;
w = linspace(0,2,1001)*pi;
Sy = 4./(1+0.95^2-1.9*cos(w));
Hw = Sy./(Sy+1);
hn = real(ifft(Hw));
ynhat = filter(hn,1,xn);
%% Plot
hfa = figconfg('P1415a','long');
plot(n,yn)
title('Signal','fontsize',LFS)

hfb = figconfg('P1415b','long');
plot(n,xn)
title('Signal + Noise','fontsize',LFS)

hfc = figconfg('P1415c','long');
plot(n,ynhat)
title('Matched Filter Output','fontsize',LFS)
xlabel('n','fontsize',LFS)
```

16. Proof:

$$J_o = r_y[0] - \sum_{k=-\infty}^{\infty} h_o[k] r_{yx}[k]$$

$$r_y[0] = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_y(\omega) d\omega$$

Applying the Parseval's theorem, we have

$$\sum_{k=-\infty}^{\infty} h_o[k] r_{yx}[k] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{S_{yx}(\omega)}{S_x(\omega)} S'_x(\omega) \frac{S'_{yx}(\omega)}{S'_x(\omega)} d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{|S_{yx}(\omega)|^2}{S_x(\omega)} d\omega$$

Hence, we have

$$J_o = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_y(\omega)d\omega - \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{|S_{yx}(\omega)|^2}{S_x(\omega)}d\omega$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[1 - \frac{|S_{yx}(\omega)|^2}{S_x(\omega)S_y(\omega)}\right]S_y(\omega)d\omega$$

17. (a) Proof:
Suppose we have

$$\begin{cases} c_1 = r\cos\alpha \\ c_2 = r\sin\alpha \end{cases} \qquad \begin{cases} x_1 = r\cos(\alpha+\theta) \\ x_2 = r\sin(\alpha+\theta) \end{cases}$$

Applying the trigonometric identities, we have

$$\begin{cases} x_1 = r\cos\alpha\cos\theta - r\sin\alpha\sin\theta \\ x_2 = r\sin\alpha\cos\theta + r\cos\alpha\sin\theta \end{cases}$$

which equals

$$\begin{cases} x_1 = c_1\cos\theta - c_2\sin\theta \\ x_2 = c_2\cos\theta + c_1\sin\theta \end{cases}$$

(b) Proof:
Suppose we have

$$\begin{cases} x_1 = r\cos\beta \\ x_2 = r\sin\beta \end{cases} \qquad \begin{cases} c_1 = r\cos(\beta-\theta) \\ c_2 = r\sin(\beta-\theta) \end{cases}$$

Applying the trigonometric identities, we have

$$\begin{cases} c_1 = r\cos\beta\cos\theta + r\sin\beta\sin\theta \\ c_2 = r\sin\beta\cos\theta - r\cos\beta\sin\theta \end{cases}$$

which equals

$$\begin{cases} c_1 = x_1\cos\theta + x_2\sin\theta \\ c_2 = x_2\cos\theta - x_1\sin\theta \end{cases}$$

(c) Proof:

$$\boldsymbol{Q}^T\boldsymbol{Q} = \begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{bmatrix} = \begin{bmatrix}1 & 0 \\ 0 & 1\end{bmatrix} = \boldsymbol{I}$$

18. (a) Proof:

$$\boldsymbol{a}^T\boldsymbol{\Gamma}\boldsymbol{a} = \sum_{i=1}^{3}\sum_{j=1}^{3} a_i a_j \gamma_{ij}$$

Hence,

$$\begin{aligned}
\frac{\partial \boldsymbol{a}^{\mathrm{T}}\boldsymbol{\Gamma}\boldsymbol{a}}{\partial \boldsymbol{a}} &= \begin{bmatrix} \frac{\sum_{i=1}^{3}\sum_{j=1}^{3} a_i a_j \gamma_{ij}}{\partial a_1} & \frac{\sum_{i=1}^{3}\sum_{j=1}^{3} a_i a_j \gamma_{ij}}{\partial a_2} & \frac{\sum_{i=1}^{3}\sum_{j=1}^{3} a_i a_j \gamma_{ij}}{\partial a_3} \end{bmatrix}^T \\
&= \begin{bmatrix} 2a_1\gamma_{11} + a_2\gamma_{12} + a_3\gamma_{13} + a_2\gamma_{21} + a_3\gamma_{31} \\ 2a_2\gamma_{22} + a_1\gamma_{12} + a_3\gamma_{32} + a_3\gamma_{23} + a_1\gamma_{21} \\ 2a_3\gamma_{33} + a_1\gamma_{13} + a_2\gamma_{23} + a_1\gamma_{31} + a_2\gamma_{32} \end{bmatrix} \\
&= \begin{bmatrix} 2a_1\gamma_{11} + 2a_2\gamma_{12} + 2a_3\gamma_{13} \\ 2a_2\gamma_{22} + 2a_1\gamma_{21} + a_3\gamma_{23} \\ 2a_3\gamma_{33} + 3a_1\gamma_{31} + 2a_2\gamma_{32} \end{bmatrix} \\
&= 2\begin{bmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} \\ \gamma_{31} & \gamma_{32} & \gamma_{33} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = 2\boldsymbol{\Gamma}\boldsymbol{a}
\end{aligned}$$

(b) Proof:

$$\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a} = \sum_{i=1}^{3} a_i^2$$

$$\begin{aligned}
\frac{\partial \boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}}{\partial \boldsymbol{a}} &= \frac{\partial \sum_{i=1}^{3} a_i^2}{\partial \boldsymbol{a}} \\
&= \begin{bmatrix} \frac{\partial \sum_{i=1}^{3} a_i^2}{\partial a_1} & \frac{\partial \sum_{i=1}^{3} a_i^2}{\partial a_2} & \frac{\partial \sum_{i=1}^{3} a_i^2}{\partial a_3} \end{bmatrix}^{\mathrm{T}} \\
&= \begin{bmatrix} 2a_1 & 2a_2 & 2a_3 \end{bmatrix}^{\mathrm{T}} = 2\boldsymbol{a}
\end{aligned}$$

19. Proof:

Suppose x_k is zero mean, that is $E[\boldsymbol{x}] = \boldsymbol{0}$, hence we have

$$E[\boldsymbol{c}] = E[\boldsymbol{A}^T\boldsymbol{x}] = \boldsymbol{A}^T E[\boldsymbol{x}] = \boldsymbol{0}$$

The variance of x_k is

$$\text{var}(x_k) = E[x_k^2]$$

The left hand of the given equation is

$$\begin{aligned}\sum_{k=1}^{p}\text{var}(x_k) &= \sum_{k=1}^{p}E[x_k^2] = E\left(\sum_{k=1}^{p}x_k^2\right) = E[\boldsymbol{x}^T\boldsymbol{x}] \\ &= E[\boldsymbol{c}^T\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{c}] = E[\boldsymbol{c}^T\boldsymbol{c}] = E\left(\sum_{k=1}^{p}c_k^2\right) \\ &= \sum_{k=1}^{p}\text{var}(c_k)\end{aligned}$$

20. (a) Proof:
The mean of random vector $\boldsymbol{y}$ is

$$E[\boldsymbol{y}] = E[\boldsymbol{\Lambda}^{1/2}\boldsymbol{z}] = \boldsymbol{\Lambda}^{1/2}E[\boldsymbol{z}] = \boldsymbol{0}$$

The covariance matrix of random vector $\boldsymbol{y}$ is

$$\begin{aligned}E[\boldsymbol{y}\boldsymbol{y}^T] &= E[\boldsymbol{\Lambda}^{1/2}\boldsymbol{z}(\boldsymbol{\Lambda}^{1/2}\boldsymbol{z})^T] \\ &= \boldsymbol{\Lambda}^{1/2}E[\boldsymbol{z}\boldsymbol{z}^T]\boldsymbol{\Lambda}^{1/2} = \boldsymbol{\Lambda}^{1/2}\boldsymbol{I}\boldsymbol{\Lambda}^{1/2} \\ &= \boldsymbol{\Lambda}\end{aligned}$$

(b) Proof:
The mean of random vector $\boldsymbol{x}$ is

$$E[\boldsymbol{x}] = E[\boldsymbol{A}\boldsymbol{y}+\boldsymbol{\mu}] = \boldsymbol{A}E[\boldsymbol{y}]+\boldsymbol{\mu} = \boldsymbol{\mu}$$

The covariance matrix of random vector $\boldsymbol{x}$ is

$$\begin{aligned}E[(\boldsymbol{x}-\boldsymbol{\mu})(\boldsymbol{x}-\boldsymbol{\mu})^T] &= E[\boldsymbol{A}\boldsymbol{y}(\boldsymbol{A}\boldsymbol{y})^T] = \boldsymbol{A}E[\boldsymbol{y}\boldsymbol{y}^T]\boldsymbol{A}^T \\ &= \boldsymbol{A}\boldsymbol{\Lambda}\boldsymbol{A}^T = \boldsymbol{\Gamma}\end{aligned}$$

(c) MATLAB function:

```
function X = Normal_ND(N,Mu,Gamma)
% P1420: Gaussian random sample vector generator
[U S A] = svd(Gamma);
Z = randn(N,length(Mu));
X = Z*sqrt(S)*A';
X = bsxfun(@plus,X,Mu(:)');
```

(d) MATLAB script:

```
% P1420: Test Gaussian Random Sample Vector Generator
close all; clc
p = 4; N = 1e6;
mu = 1:p; sig = 1:p; rho = abs(bsxfun(@minus,1:p,[1:p]'));
rho = 1-0.1*rho;
Gamma = bsxfun(@times,rho,sig);
Gamma = bsxfun(@times,Gamma,sig');
X = Normal_ND(N,mu,Gamma);
mean(X,1)
mu
cov(X,0)
Gamma
```

21. (a) See plot below.

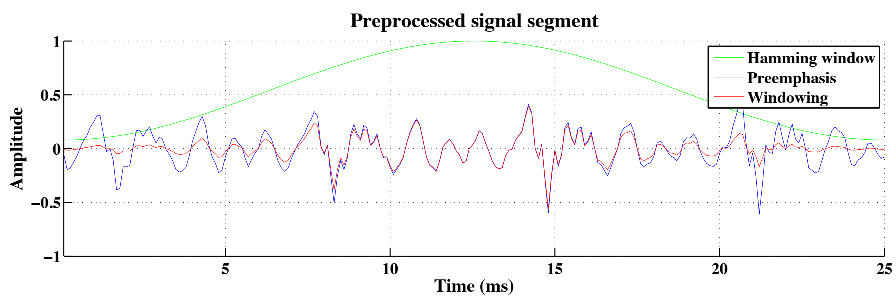

FIGURE 14.26: Replot of Figure 14.22(b).

(b) See plot below.

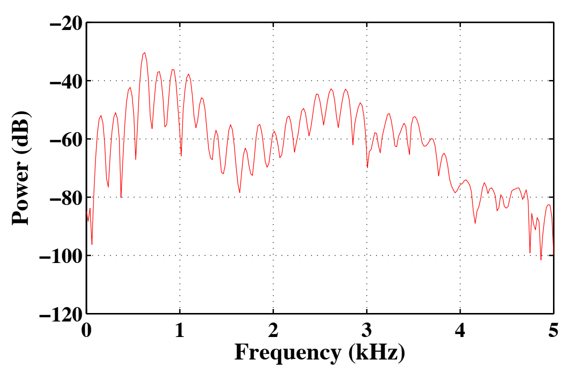

FIGURE 14.27: Plot of the periodogram estimate of the speech spectrum.

(c) See plot below.

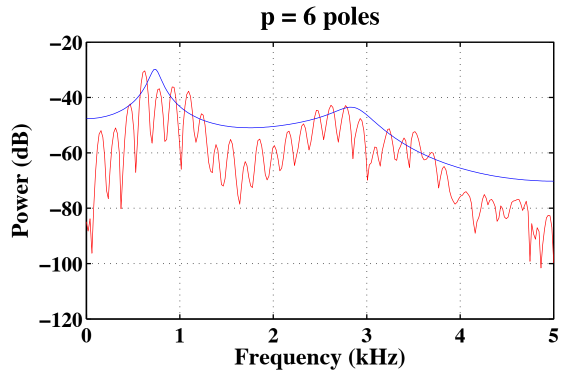

FIGURE 14.28: Plot of the spectrum using all-pole model when $p = 6$.

(d) See plot below.

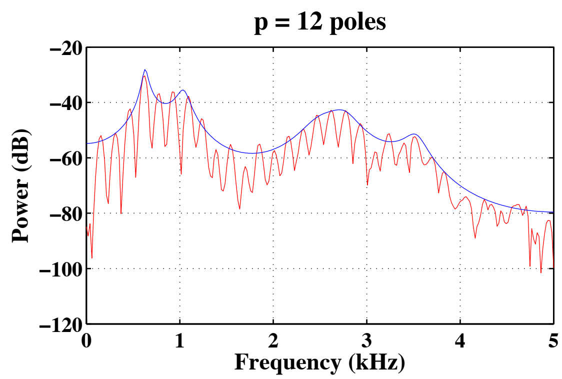

FIGURE 14.29: Plot of the spectrum using all-pole model when $p = 12$.

22. (a) See plot below.
    (b) See plot below.
    (c) Yes.
    (d) Comments:
    See script output for details.
    (e) Solution:
    Data set with $\rho = 0.9$ can be described more accurately if we retain only the first Karhunen-Loève Transform coefficient.

MATLAB script:

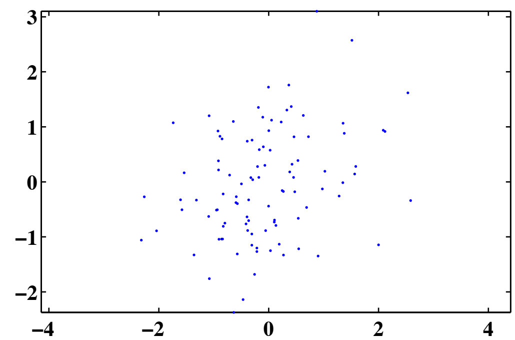

FIGURE 14.30: Scatter diagram of data set with $\sigma_1 = \sigma_2 = 1$, and $\rho = 0.4$.

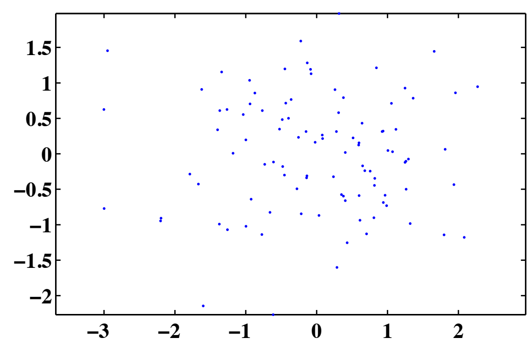

FIGURE 14.31: Scatter diagram of data set Karhunen-Loève Transform $\boldsymbol{c}[n]$.

```
% P1422: Geometrical Interpretations of KLT
close all; clc
N = 100; mu = zeros(1,2);
sig1 = 1; sig2 = 1;
rho = 0.4;
% rho = 0.9;
Gamma = [sig1^2 sig1*sig2*rho;sig1*sig2*rho sig2^2];
randn('seed',0)
X = Normal_ND(N,mu,Gamma);
%% Part b
[N,p]=size(X);
G=cov(X,1);
[U,L,A]=svd(G);
```

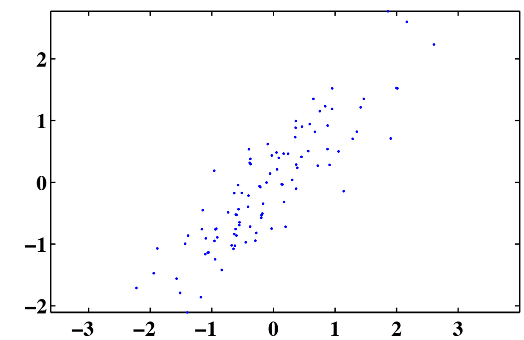

FIGURE 14.32: Scatter diagram of data set with $\sigma_1 = \sigma_2 = 1$, and $\rho = 0.9$.

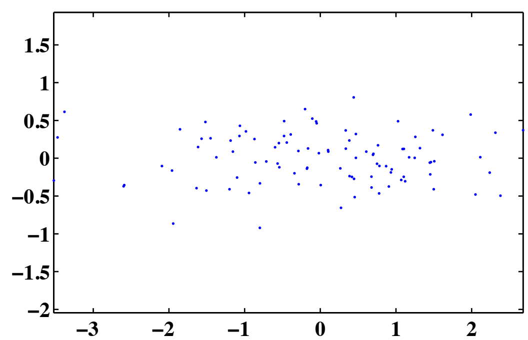

FIGURE 14.33: Scatter diagram of data set Karhunen-Loève Transform $\boldsymbol{c}[n]$.

```
lambda=diag(L);
Mx=repmat(mean(X),N,1);
C=(X-Mx)*A;
m = 1;
Xhat=C(:,1:m)*A(:,1:m)'+Mx;
E=X(:)-Xhat(:);
MSE1=sum(E.^2)/N
MSE2=sum(lambda(m+1:p))
%% Plot
hfa = figconfg('P1420a','small');
plot(X(:,1),X(:,2),'.'); axis equal
hfb = figconfg('P1420b','small');
plot(C(:,1),C(:,2),'.'); axis equal
```