CHAPTER 3:

FREQUENCY-DOMAIN REPRESENTATION OF SIGNALS

Why frequency-domain approach?

- Let us know other characteristics of signals, which are not exhibited on time-domain representation
  - E.g.: some voiced sounds are hardly distinguishable if only time-domain features are used
- Let us know other characteristics of systems, which are not exhibited on time-domain representation
  - Response of a LTI system to a sinusoid is a sinusoid with same frequency but different amplitude and phase
  - Details are presented in next chapter

Basic ideas

- Any signal can be described as a sum or integral of sinusoidal signals
- Exact form of the representation depends on whether
  - signal is continuous-time or discrete-time
  - signal is periodic or aperiodic

→ Clear understanding of sinusoidal signals are essential

Fourier representation of signals

- **Outline:**

  1. Sinusoidal signals and their properties

  2. Summary of Fourier series and Fourier transforms

Continuous-time sinusoids

Definition

$$x(t) = A\cos(2\pi F_0 t + \theta), \quad -\infty < t < \infty$$

A: amplitude

θ: phase (radians)

F_0: frequency (Hz)

angular frequency (rad/s)

$$\Omega_0 = 2\pi F_0$$

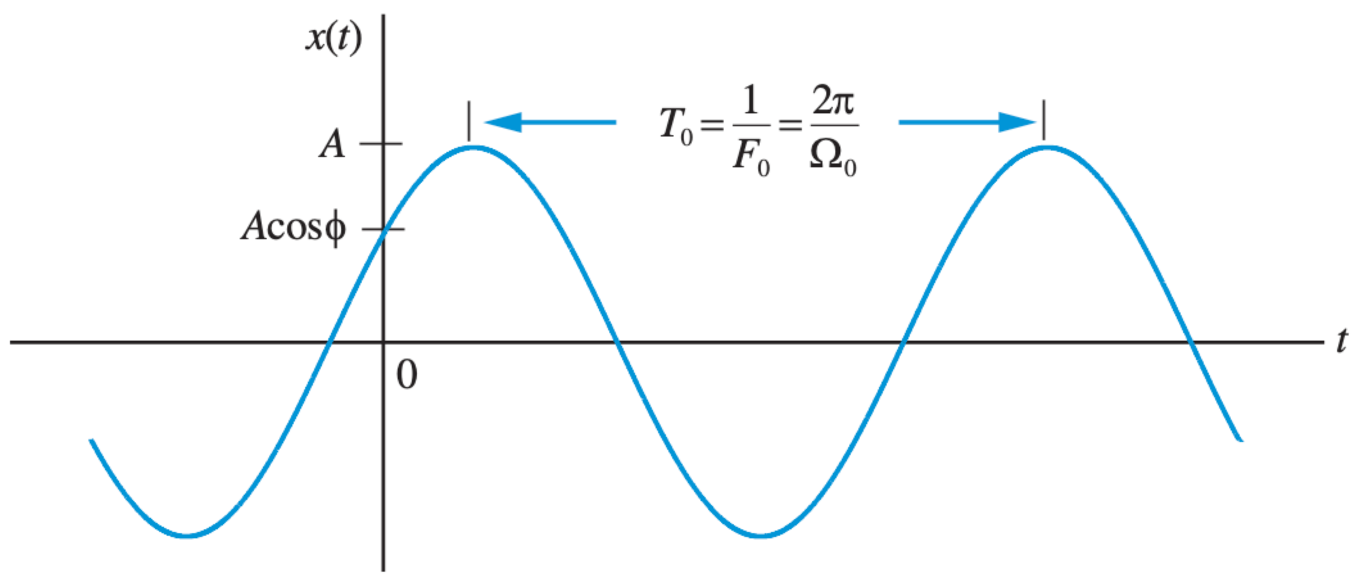


Figure 4.1 Continuous-time sinusoidal signal and its parameters.

Continuous-time sinusoids

Relation with complex exponentials

$$A\cos(\Omega_0 t + \theta) = \frac{A}{2}\mathrm{e}^{\mathrm{j}\theta}\,\mathrm{e}^{\mathrm{j}\Omega_0 t} + \frac{A}{2}\mathrm{e}^{-\mathrm{j}\theta}\,\mathrm{e}^{-\mathrm{j}\Omega_0 t}.$$

- Every sinusoidal signal is a sum of two complex exponentials with the same frequency
- Negative frequencies for mathematical convenience

Continuous-time sinusoids

Properties

- Two sinusoids with different frequencies are different
- Rate of oscillation increases indefinitely w/ increasing frequency

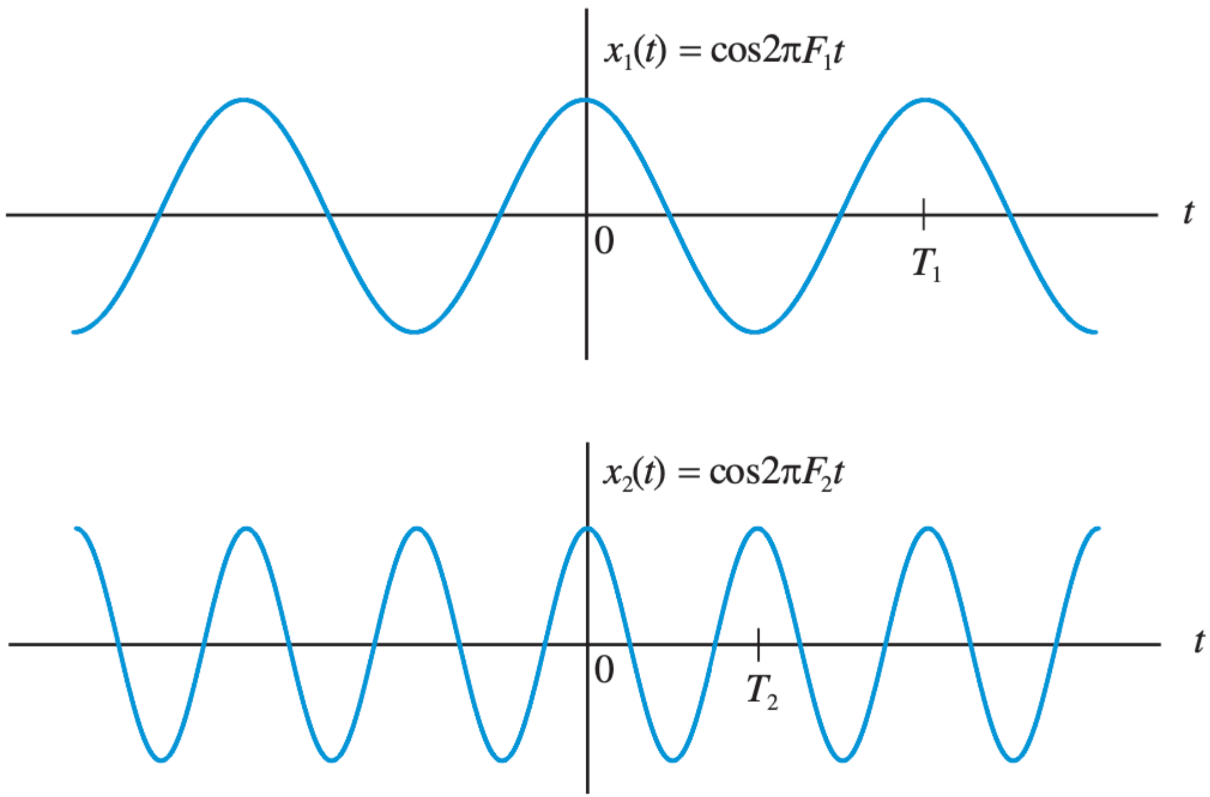


Figure 4.2 For continuous-time sinusoids, $F_1 < F_2$ always implies that $T_1 > T_2$.

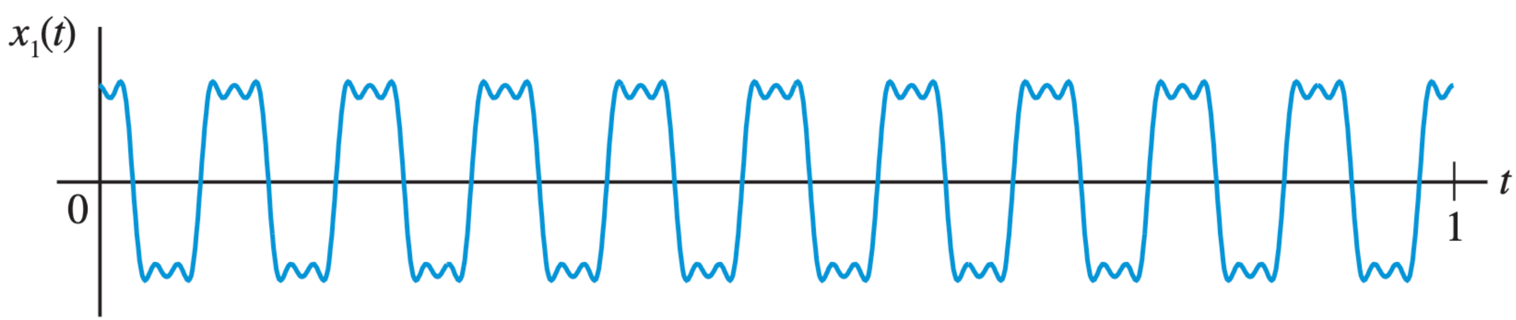
Continuous-time sinusoids

Harmonically related complex exponentials

$$s_k(t) = e^{jk\Omega_0 t} = e^{j2\pi kF_0 t}, \quad k = 0, \pm 1, \pm 2, \ldots$$

- Fundamental frequency: $\Omega_0 = 2\pi/T_0 = 2\pi F_0$
- $s_k(t)$ is the *k*th harmonic ("hài bậc k")
- Example of a periodic signal composed of three sinusoids with harmonically related frequencies

$$x_1(t) = \frac{1}{3}\cos(2\pi F_0 t) - \frac{1}{10}\cos(2\pi 3F_0 t) + \frac{1}{20}\cos(2\pi 5F_0 t)$$



Discrete-time sinusoids

Definition

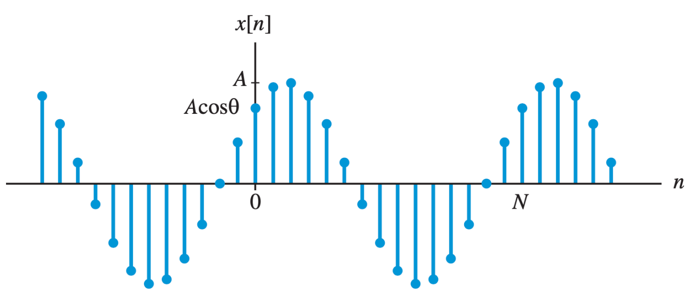
Often obtained by sampling the continuous- time sinusoid

$$x[n] = x(nT) = A\cos(2\pi F_0 nT + \theta) = A\cos\left(2\pi \frac{F_0}{F_s} n + \theta\right)$$

or $$x[n] = A\cos(2\pi f_0 n + \theta) = A\cos(\omega_0 n + \theta)$$

- Normalized frequency: $$f \triangleq \frac{F}{F_s} = FT$$

- Normalized angular frequency: $$\omega \triangleq 2\pi f = 2\pi \frac{F}{F_s} = \Omega T$$



Discrete-time sinusoids

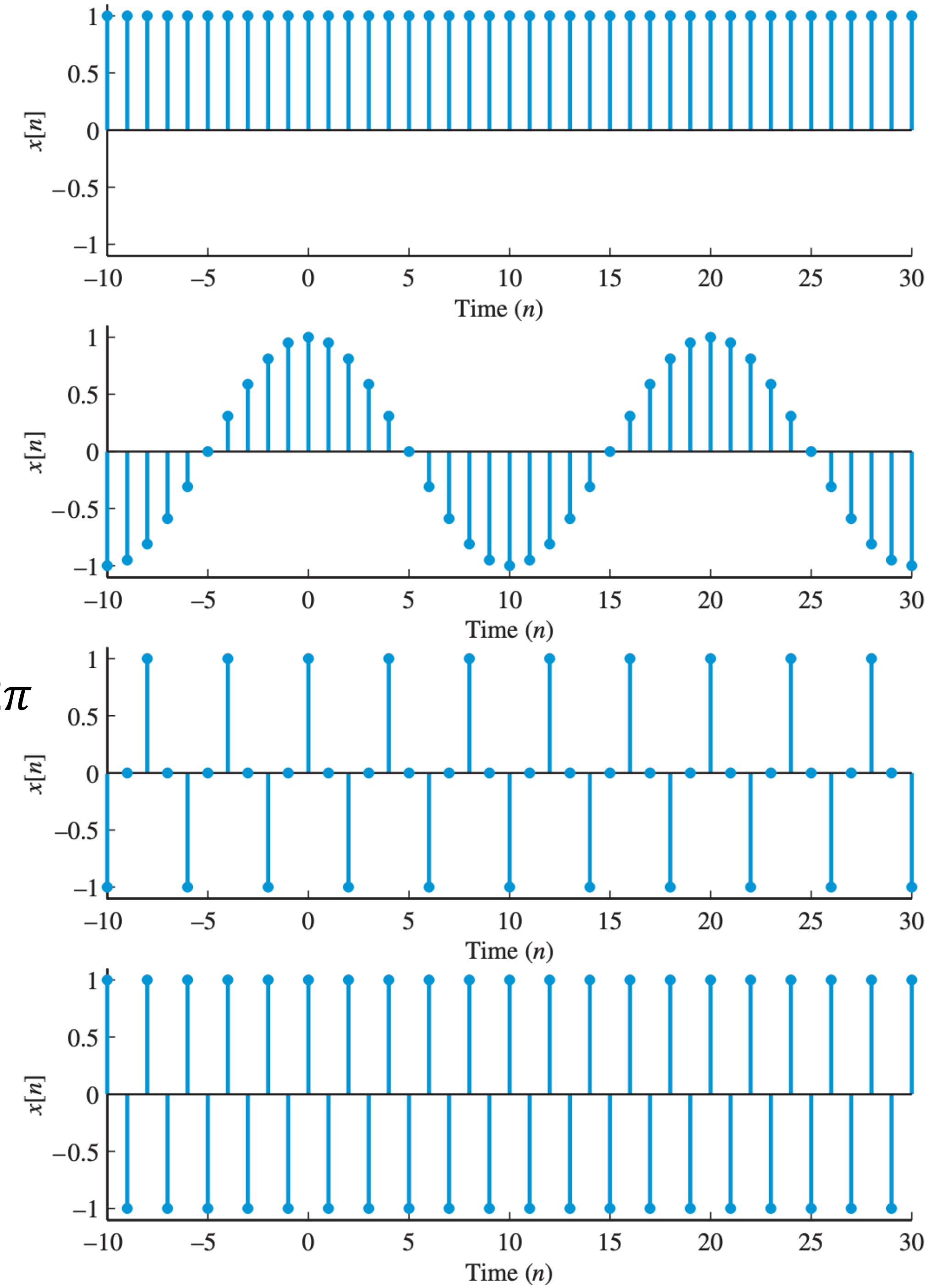
Properties

- $x[n] = A\cos(2\pi f_0 n + \theta)$ is periodic if and only if $f_0 = k/N$ (f_0 is a rational number)
  - If k and N are a pair of prime numbers, then N is the *fundamental period* of x[n]
- Sinusoidal sequences with angular frequencies separated by integer multiples of 2π are identical (*Periodicity in frequency*)
- All distinct sinusoidal sequences have frequencies within an interval of 2π radians

  → interested frequency ranges: $-\pi < \omega \le \pi$ or $0 \le \omega < 2\pi$

DT sinusoids
Properties

- Low frequencies (slow oscillations) at ~ $\omega_0 = k2\pi$
- High frequencies (rapid oscillations) at ~ $\omega_0 = \pi + k2\pi$



Discrete-time sinusoids

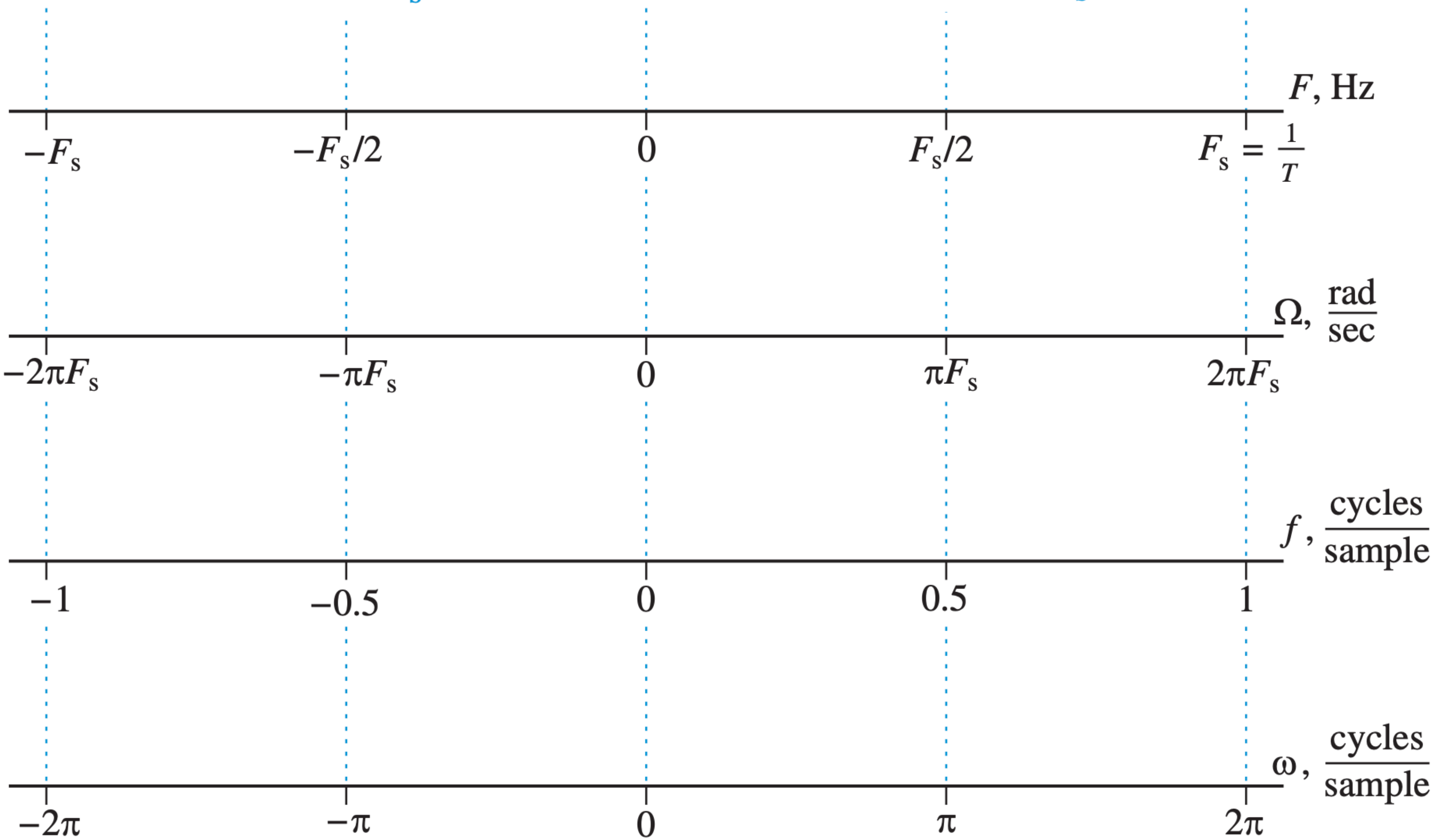
Harmonically related complex exponentials

$$s_k[n] = A_k e^{j\omega_k n}. \qquad -\infty < n < \infty$$

- For $s_k[n]$ to be periodic with fundamental period N, the frequency ω_k should be a rational multiple of 2π, that is, $\omega_k = 2\pi k/N$
- Fundamental frequency: $f_0 = 1/N$
- There are only N distinct harmonics at frequencies $f_k = k/N$ $(0 \le k \le N-1)$

Frequency variables and units

$$f \triangleq \frac{F}{F_s} = FT \qquad \omega \triangleq 2\pi f = 2\pi \frac{F}{F_s} = \Omega T$$



Fourier representation of signals

- **Outline:**

  1. Sinusoidal signals and their properties

  2. Summary of Fourier series and Fourier transforms

Basic ideas revisited

- Any signal can be described as a sum or integral of sinusoidal signals
- Exact form of the representation depends on whether
  - signal is continuous-time (CT) or discrete-time (DT)
  - signal is periodic or aperiodic

Fourier series for CT periodic signals

- Continuous-Time Fourier Series (CTFS) pair

Fourier Synthesis Equation **Fourier Analysis Equation**

$$x(t) = \sum_{k=-\infty}^{\infty} c_k \mathrm{e}^{\mathrm{j}k\Omega_0 t} \quad \stackrel{\text{CTFS}}{\longleftrightarrow} \quad c_k = \frac{1}{T_0}\int_{T_0} x(t)\mathrm{e}^{-\mathrm{j}k\Omega_0 t}\mathrm{d}t.$$

- *Analysis equation:* analyzes ("breaks-up") a periodic signal x(t) into a set of harmonic components $\{c_k \exp(jk\Omega_0 t)\}$
- *Synthesis equation:* synthesizes the signal x(t) from its harmonic components

Fourier series for CT periodic signals

- Spectrum of CT periodic signals
  - $\{c_k\}$: Fourier series coefficients ($k = 0, \pm 1, \pm 2, \ldots$)
  - *Spectrum:* the plot of c_k as a function of frequency $F = kF_0$, which constitutes a description of the signal in the frequency-domain

$$c_k = |c_k| e^{j\angle c_k}$$

$|c_k|$: magnitude spectrum of $x(t)$

$\angle c_k$: phase spectrum of $x(t)$

Fourier series for CT periodic signals

- Spectrum of CT periodic signals (cont.)
  - $\{c_k\}$: Fourier series coefficients ($k = 0, \pm 1, \pm 2, \ldots$)
  - *Parseval's relation*: $$P_{\text{av}} = \frac{1}{T_0}\int_{T_0} |x(t)|^2 \mathrm{d}t = \sum_{k=-\infty}^{\infty} |c_k|^2$$

    → $|c_k|^2$: the portion of the average power of signal x(t) that is contributed by the *k*th harmonic component
  - *Power spectrum:* the plot of $|c_k|^2$ as a function of frequency $F = kF_0$

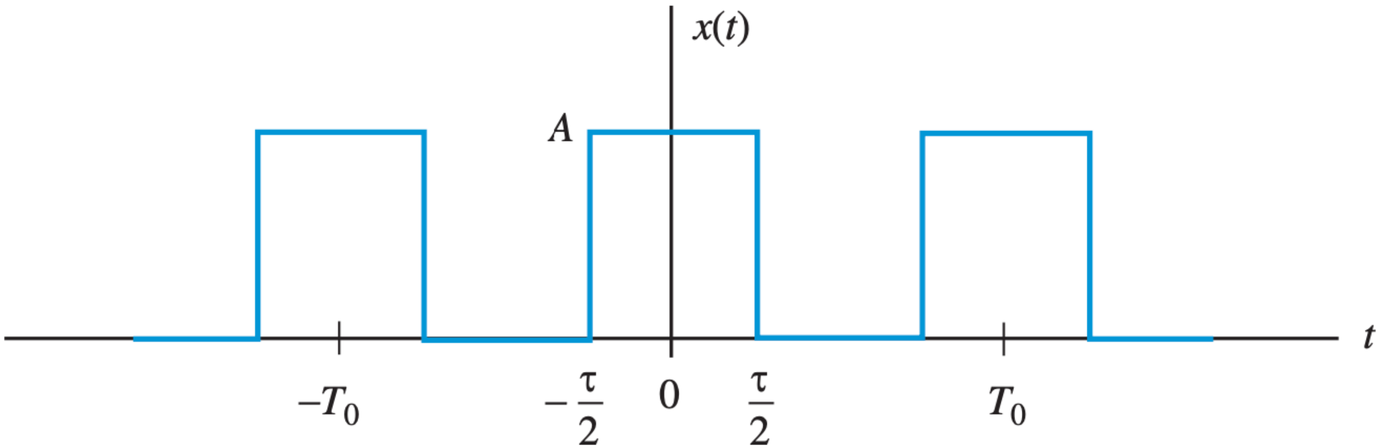
    → the power is distributed at a set of discrete frequencies

    → *discrete* or *line spectra* with uniform spacing $F_0 = 1/T_0$

    (F_0: fundamental frequency of x(t))

Fourier series for CT periodic signals

- Example: Spectrum of rectangular pulse train



$$c_k = \frac{1}{T_0}\int_{-\tau/2}^{\tau/2} A e^{-j2\pi k F_0 t}dt = \frac{A}{T_0}\left[\frac{e^{-j2\pi k F_0 t}}{-j2\pi k F_0}\right]_{-\tau/2}^{\tau/2}$$

$$= \frac{A}{\pi F_0 k T_0}\,\frac{e^{j\pi k F_0 \tau} - e^{-j\pi k F_0 \tau}}{2j}$$

$$= \frac{A\tau}{T_0}\,\frac{\sin \pi k F_0 \tau}{\pi k F_0 \tau}. \quad k = 0, \pm 1, \pm 2, \ldots$$

$(c_0 = A\tau / T_0)$

Fourier series for CT periodic signals

- Example: Spectrum of rectangular pulse train (cont.)

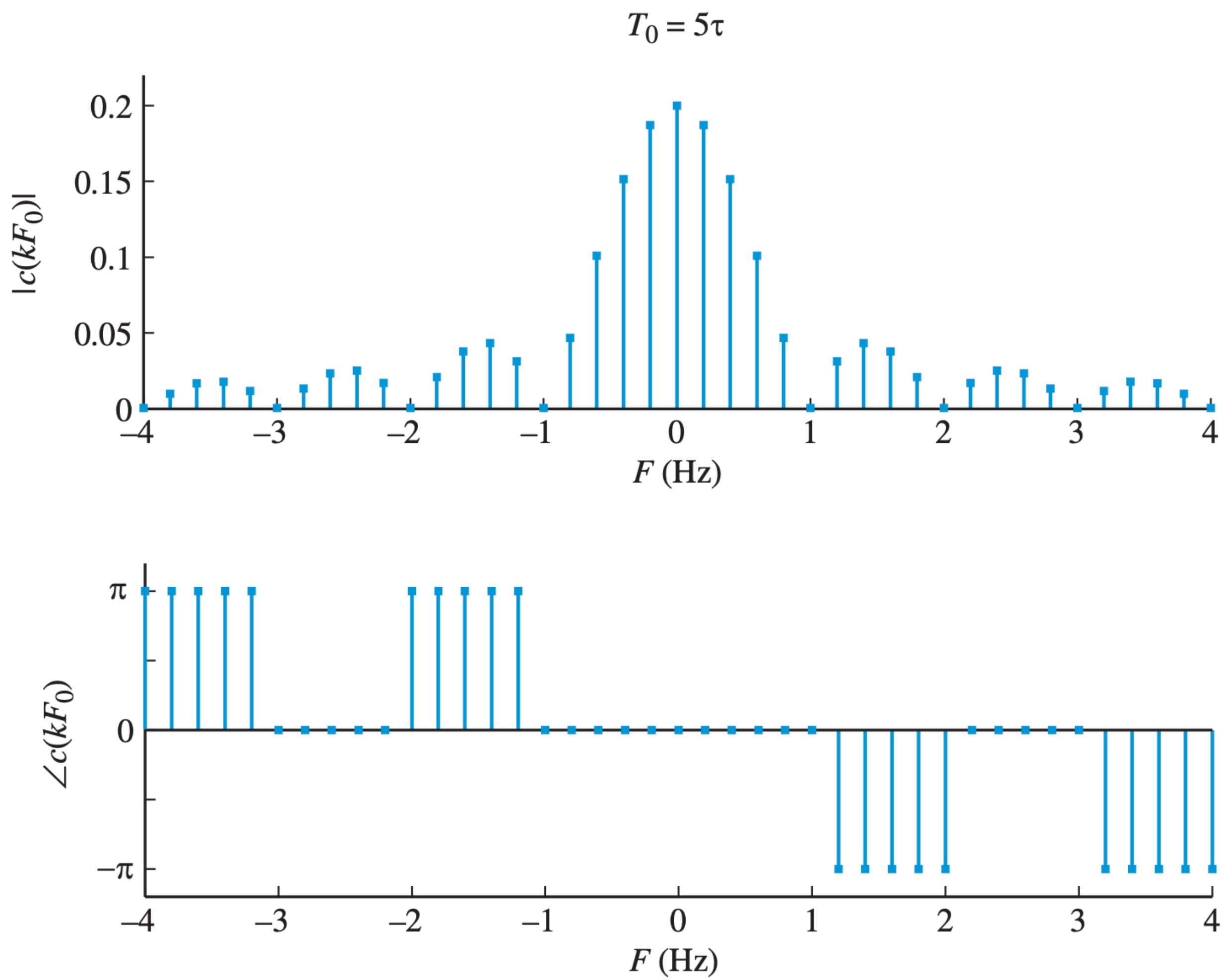


Figure 4.9 Magnitude and phase spectra of a rectangular pulse train with $A = 1$ and $T_0 = 5\tau$ =5(s)

Fourier Transform for CT aperiodic signals

- Continuous-Time Fourier Transform (CTFT) pair

Fourier Synthesis Equation **Fourier Analysis Equation**

$$x(t) = \int_{-\infty}^{\infty} X(\mathrm{j}2\pi F)\, \mathrm{e}^{\mathrm{j}2\pi F t} \mathrm{d}F \xleftrightarrow{\text{CTFT}} X(\mathrm{j}2\pi F) = \int_{-\infty}^{\infty} x(t) \mathrm{e}^{-\mathrm{j}2\pi F t} \mathrm{d}t$$

or

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\mathrm{j}\Omega)\, \mathrm{e}^{\mathrm{j}\Omega t} \mathrm{d}\Omega \xleftrightarrow{\text{CTFT}} X(\mathrm{j}\Omega) = \int_{-\infty}^{\infty} x(t) \mathrm{e}^{-\mathrm{j}\Omega t} \mathrm{d}t$$

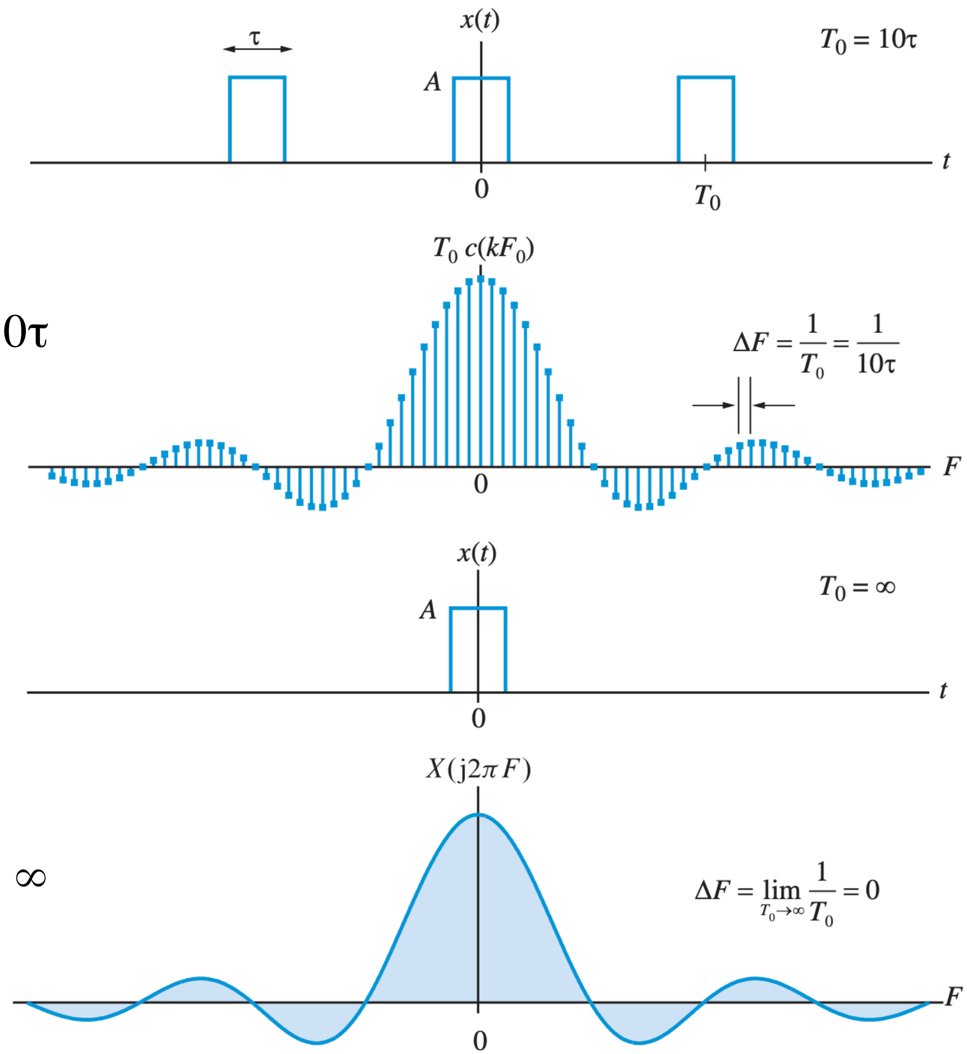
- $X(j2\pi F)$: *spectrum* of the aperiodic signal x(t) (called Fourier Transform)
- CTFT is of the same nature as CTFS with fund. frequency $F_0 = 1/T_0 \rightarrow 0$

Fourier Transform for CT aperiodic signals

- Transition from the CTFS to CTFT

CTFS of periodic signal x(t) when $T_0 = 10\tau$

CTFT of aperiodic signal x(t) when $T_0 \rightarrow \infty$



Fourier Transform for CT aperiodic signals

- Parseval's relation for aperiodic signals with finite energy

$$\int_{-\infty}^{\infty} |x(t)|^2 \mathrm{d}t = \int_{-\infty}^{\infty} |X(\mathrm{j}2\pi F)|^2 \mathrm{d}F$$

→ total energy of x(t) may be obtained either from the signal itself or from its spectrum

- ➢ $|X(j2\pi F)|^2 \Delta F$, for a small ΔF, provides the energy of the signal in a narrow frequency band of width ΔF
- ➢ *Energy-density spectrum:* the plot of $|X(j2\pi F)|^2$

Fourier Transform for CT aperiodic signals

- Example 1: Spectrum of causal exponential signal

$$x(t) = \begin{cases} e^{-at}, & t > 0 \\ 0. & t < 0 \end{cases}$$

This signal is absolutely integrable if $a > 0$.

$$X(j2\pi F) = \int_0^{\infty} e^{-at} e^{-j2\pi Ft} dt = -\frac{1}{a + j2\pi F} e^{-(a+j2\pi F)t} \Big|_0^{\infty}.$$

Hence,

$$X(j2\pi F) = \frac{1}{a + j2\pi F} \quad \text{or} \quad X(j\Omega) = \frac{1}{a + j\Omega}. \quad a > 0$$

Magnitude and phase spectra:

$$|X(j2\pi F)| = \frac{1}{\sqrt{a^2 + (2\pi F)^2}}, \quad \angle X(j2\pi F) = -\tan^{-1}\left(2\pi \frac{F}{a}\right). \quad a > 0$$

Fourier Transform for CT aperiodic signals

- Example 1: Spectrum of causal exponential signal (cont.)

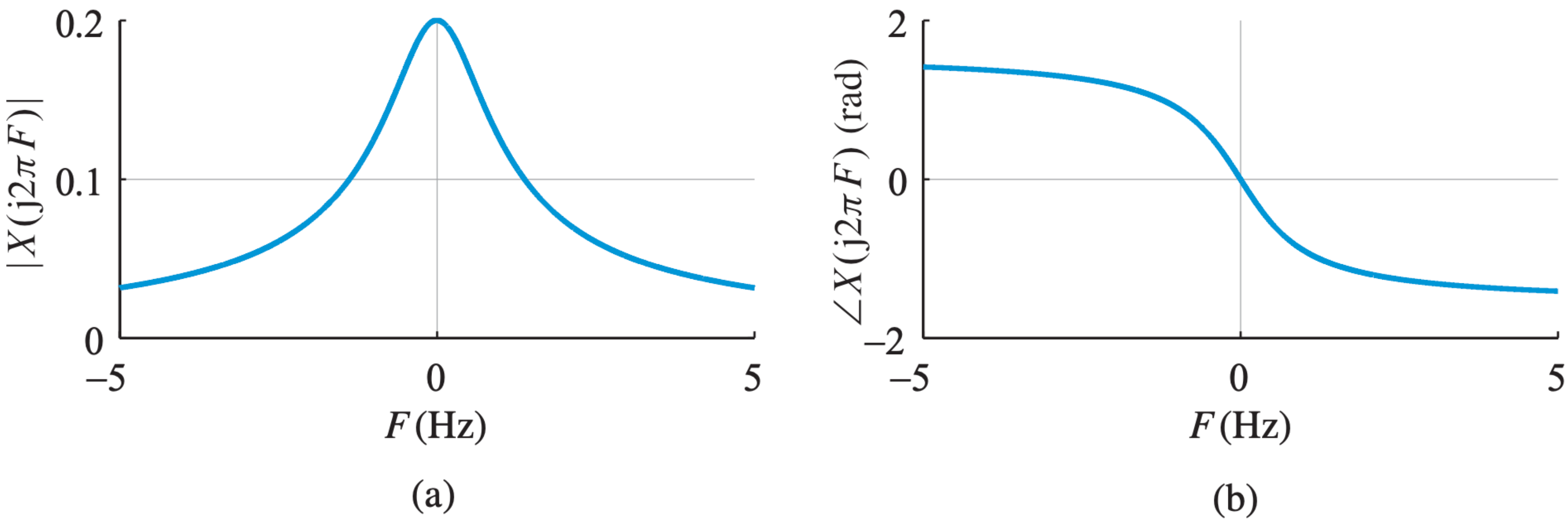


Figure 4.15 Fourier transform of the signal $x(t) = e^{-at}u(t)$ for $a = 5$. (a) Magnitude and (b) phase of $X(j2\pi F)$ in the finite interval $-5 < F < 5$ (Hz).

$x(t)$ is a real function of t → $|X(j2\pi F)|$ has even symmetry, $\angle X(j2\pi F)$ has odd symmetry

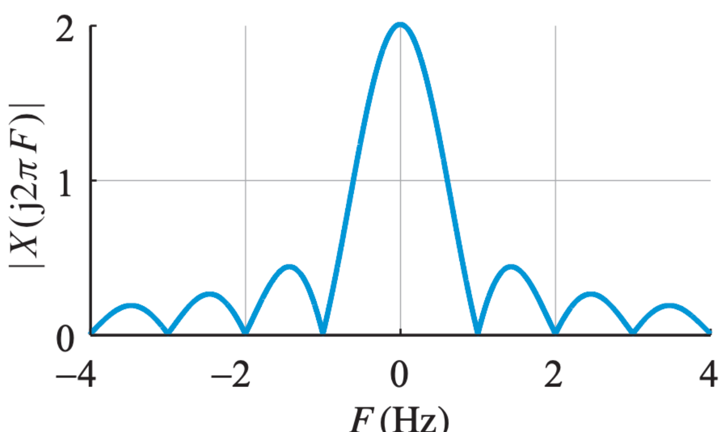
Fourier Transform for CT aperiodic signals

- Example 2: Spectrum of rectangular pulse signal

$$x(t) = \begin{cases} A, & |t| < \tau/2 \\ 0. & |t| > \tau/2 \end{cases}$$

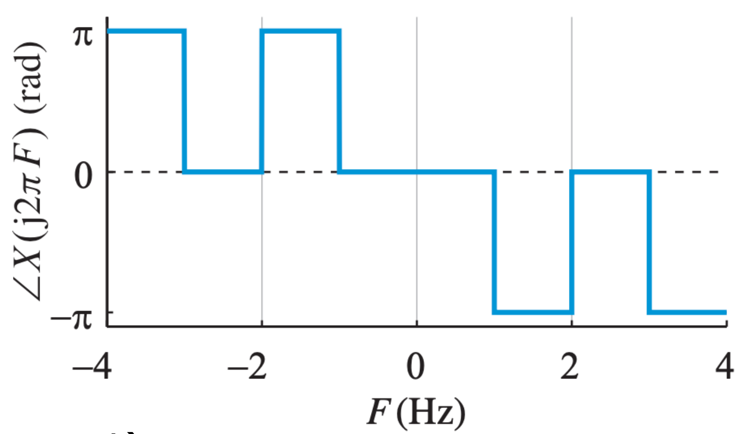
This signal is absolutely integrable for any finite τ. Its spectrum is computed as

$$X(j2\pi F) = \int_{-\tau/2}^{\tau/2} A e^{-j2\pi F t} dt = A\tau \frac{\sin(\pi F \tau)}{\pi F \tau}.$$



Magnitude spectrum

(A=2, τ =1)



Phase spectrum

Fourier series for DT periodic signals

- Discrete-Time Fourier Series (DTFS) pair

Fourier Synthesis Equation **Fourier Analysis Equation**

$$x[n] = \sum_{k=0}^{N-1} c_k e^{j\frac{2\pi}{N}kn} \quad \xleftrightarrow{\text{DTFS}} \quad c_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}kn}.$$

- *Analysis equation:* analyzes ("breaks-up") a periodic signal x[n] into a set of ***N*** harmonic components $\{c_k \exp(jk\omega_0 n)\}$, where $\omega_0 = 2\pi/N$
- *Synthesis equation:* synthesizes the signal x[n] from its *N* harmonic components
- *N*: fundamental period of periodic sequence x[n]

Fourier series for DT periodic signals

- Parseval's relation

The average power in one period of $x[n]$ can be expressed

$$P_{\mathrm{av}} = \frac{1}{N} \sum_{n=0}^{N-1} |x[n]|^2 = \sum_{k=0}^{N-1} |c_k|^2.$$

- $|c_k|^2$: portion of the average power of x[n] that is contributed by its *k*th harmonic component
- *Power spectrum:* The graph of $|c_k|^2$ as a function of $f = k/N$, $\omega = 2\pi k/N$, or simply k