CHAPTER 15

Finite Wordlength Effects

Tutorial Problems

1. MATLAB script:

```
% P1501: Binary representation conversion
clc; close all;
%% Part a: Decimal to Binary
xa_dec = [121;-48;53;-27;-347];
fprintf('Decimal\t Binary\n')
for ii = 1:length(xa_dec)
    if xa_dec(ii)<0
        xa_bin{ii} = ['1',dec2bin(abs(xa_dec(ii)))];
    else
        xa_bin{ii} = ['0',dec2bin(abs(xa_dec(ii)))];
    end
    fprintf('%d',xa_dec(ii))
    fprintf('\t%s\n',xa_bin{ii})
end
%% Part b: Binary to Decimal
xb_bin = {'1011011';'10101';'01001';'00101';'1100110'};
fprintf('Binary\t Decimal\n')
for jj = 1:length(xb_bin)
    temp = xb_bin{jj};
    if temp(1) == '0'
        xb_dec(jj) = bin2dec(temp(2:end));
    elseif temp(1) == '1'
        xb_dec(jj) = -bin2dec(temp(2:end));
    else
```

```
        error('bad input')
    end
    fprintf('%s',xb_bin{jj})
    fprintf('\t%d\n',xb_dec(jj))
end
```

2. MATLAB script:

```
% P1502: Binary representation conversion
clc; close all;
xd = [0.12345;-0.54321;0.90645;0.45388623;-0.237649];
L = 8; N = length(xd);
xd_sign = sign(xd);
%% Part a: Sign-Magnitude Representation
xb_sm = zeros(N,L);
ind = (xd < 0);
xb_sm(ind,1) = 1;
xd_abs = abs(xd);
for ii = 1:L-1
    xb_sm(:,ii+1) = floor(2*xd_abs);
    xd_abs = 2*xd_abs - xb_sm(:,ii+1);
end
disp('Sign-Magnitude Representation is:')
xb_sm
%% Part b: Two's-Complement Representation
xb_tc = zeros(N,L);
ind = (xd < 0);
xd_temp = xd; xd_temp(ind) = 2 + xd_temp(ind);
xb_tc(:,1) = floor(xd_temp);
xd_temp = xd_temp - xb_tc(:,1);
for ii = 1:L-1
    xb_tc(:,ii+1) = floor(xd_temp*2);
    xd_temp = xd_temp*2 - xb_tc(:,ii+1);
end
disp('Two''s-Complement Representation is:')
xb_tc
```

3. (a) See plot below.

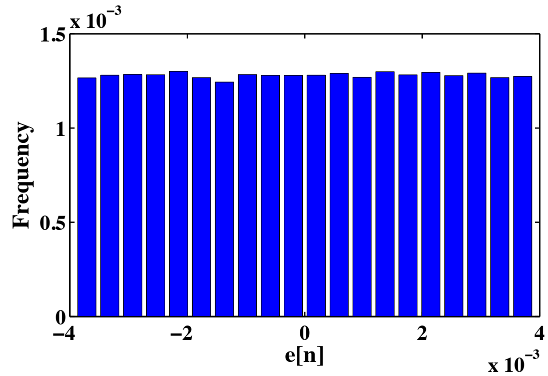

FIGURE 15.1: Plot of the histogram of $e[n]$ using 20 bins.

(b) See plot below.

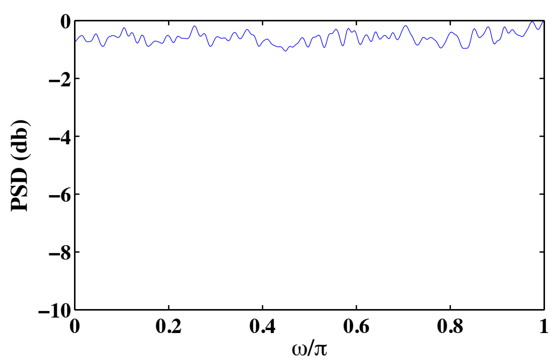

FIGURE 15.2: PSD of $e[n]$ using the psdwelch function for $B + 1 = 8$.

(c) See plot below.

MATLAB script:

```
% P1503: Quatization noise distribution analysis
clc; close all;
N = 1e5;
n = 1:N;
xn = (sin(n/11) + sin(n/31) + cos(n/67))*0.33;
%% Part a
L = 8; % L = 2:2:16;
```

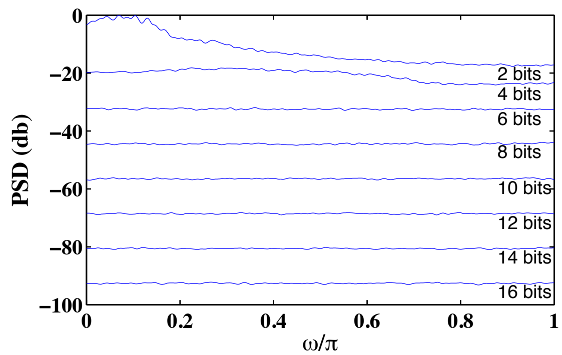

FIGURE 15.3: PSD of $e[n]$ using the `psdwelch` function for $B + 1 = 2, 4, 6, 8,$ $10, 12, 14,$ and 16.

```
[beq,E,B] = dec2beqR(xn,L);
en = beq - xn;
[eo pe] = epdf(en,20);
hfa = figconfg('P1503a','small');
bar(eo,pe/numel(en),'facecolor','b')
xlabel('e[n]','fontsize',LFS);
ylabel('Frequency','fontsize',LFS);
%% Part b
Se = psdwelch(en(:),250,1024);
w = linspace(0,1,512)*pi;
Se_mag = abs(Se); Se_db = 10*log10(Se_mag/max(Se_mag));
hfb = figconfg('P1503b','small');
plot(w/pi,Se_db); ylim([-10 0])
xlabel('\omega/\pi','fontsize',LFS);
ylabel('PSD (db)','fontsize',LFS);
%% Part c
L = 2:2:16;
hfc = figconfg('P1503c','small');
w = linspace(0,1,512)*pi;
for ii = 1:length(L)
    [beq,E,B] = dec2beqR(xn,L(ii));
    en = beq - xn;
    Se = psdwelch(en(:),250,1024);
    Se_mag = abs(Se);
    if ii == 1
```

```
        Se_max = max(Se_mag);
    end
    Se_db = 10*log10(Se_mag/max(Se_max));
    figure(hfc)
    plot(w/pi,Se_db); hold on
    text(w(450)/pi,Se_db(450)-3,[num2str(L(ii)),' bits'],'fontsize',LFS-4)
    xlabel('\omega/\pi','fontsize',LFS);
    ylabel('PSD (db)','fontsize',LFS);
end
```

4. (a) Solution:

$$f_{e_1}(x) = f_e(x) * f_e(x) = \int_{-\infty}^{\infty} f_e(y) f_e(x-y)dy$$

If $-\Delta \le x \le 0$,

$$f_{e_1}(x) = \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}+x} \frac{1}{\Delta^2} dy = \frac{1}{\Delta^2}(\Delta + x)$$

If $0 \le x \le \Delta$,

$$f_{e_1}(x) = \int_{x-\frac{\Delta}{2}}^{\frac{\Delta}{2}} \frac{1}{\Delta^2} dy = \frac{1}{\Delta^2}(\Delta - x)$$

Hence,

$$f_{e_1}(x) = \begin{cases} \frac{1}{\Delta^2}(\Delta - |x|), & |x| \le \Delta \\ 0, & \text{otherwise} \end{cases}$$

(b) See plot below.

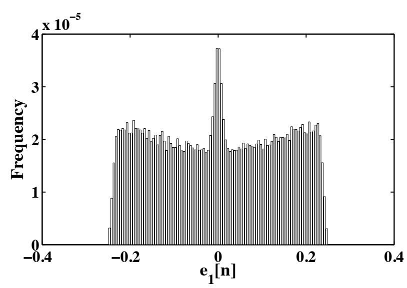

FIGURE 15.4: Plot of the histogram of $e_1[n]$ for $B+1=2$ bits.

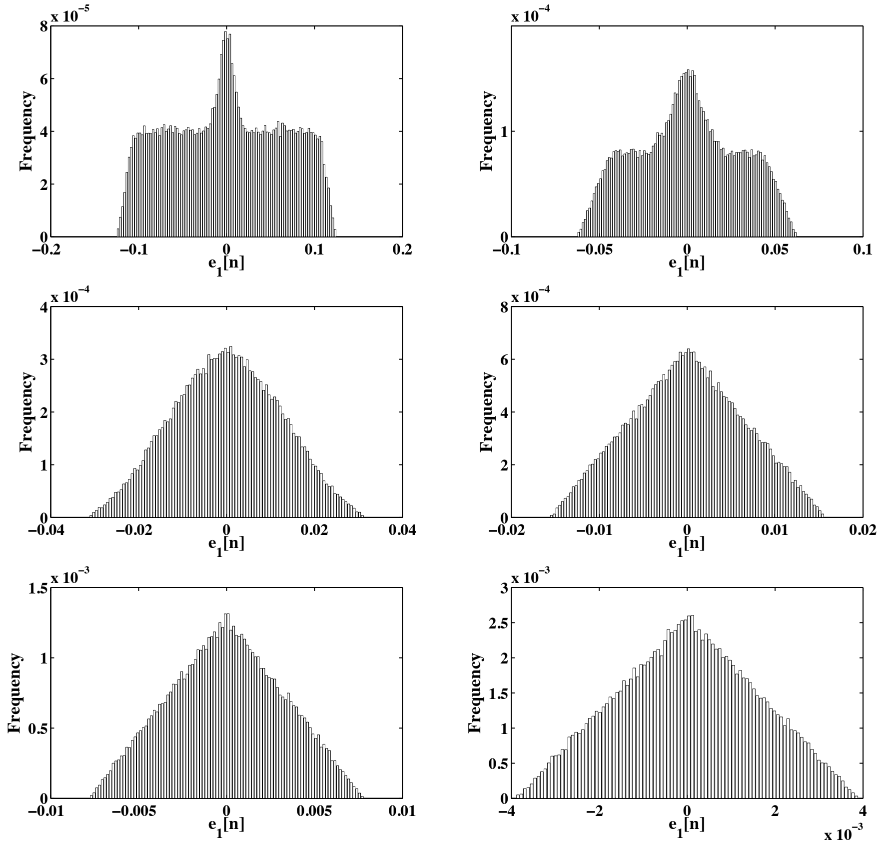

FIGURE 15.5: Plot of the histogram of $e_1[n]$ for $B + 1 = 3, 4, 5, 6, 7,$ and 8 bits.

(c) See plots below.

MATLAB script:

```
% P1504: Quatization noise distribution analysis
clc; close all;
N = 1e5;
n = 1:N;
xn = (sin(n/11) + sin(n/31) + cos(n/67))*0.33;
%% Part b
L = 8; % L = 2:8;
[beq,E,B] = dec2beqR(xn,L);
en = beq - xn;
e1n = (en + [0 en(1:end-1)])/2;
```

```
[e1o pe1] = epdf(e1n,100);
hfa = figconfg('P1503a','small');
bar(e1o,pe1/numel(e1n),'facecolor','w')
xlabel('e_1[n]','fontsize',LFS);
ylabel('Frequency','fontsize',LFS);
```

5. (a) MATLAB function:

```
function [H,bins,eavg,evar] = QNmodel(x,B)
% Compute quantization error statistics of the input sequence
xm = abs(x(:));
E = max(max(0,fix(log2(xm+eps)+1))); % Integer bits
L = B + E + 1;
[beq,~,~] = dec2beqR(x,L);
en = beq - x;
[bins H] = epdf(en,50);
eavg = mean(en);
evar = var(en);
```

(b) See plot below.

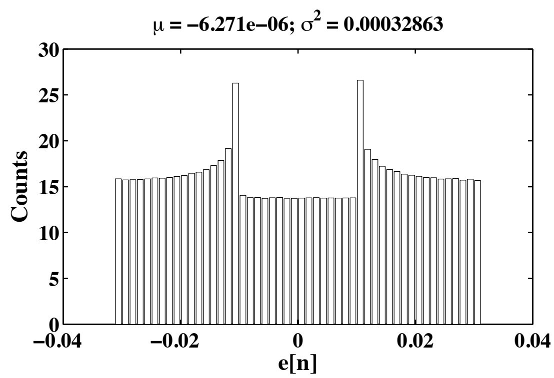

FIGURE 15.6: Plot of the histogram of the quantization error.

(c) tba.

MATLAB script:

```
% P1505: Quatization noise distribution analysis
clc; close all;
N = 1e5;
n = 1:N;
```

```
xn = 0.99*cos(n/17);
%% Part b
B = 4;
[pe,eo,eavg,evar] = QNmodel(xn,B);
hfa = figconfg('P1505a','small');
bar(eo,pe,'facecolor','w')
xlabel('e[n]','fontsize',LFS);
ylabel('Counts','fontsize',LFS);
title(['\mu = ',num2str(eavg),'; \sigma^2 = ',num2str(evar)])
```

6. MATLAB script:

```
% P1506: Variance-Gain
clc; close all;
b1 = [1 3 4]; a1 = [1 0.6 0.08];
b2 = fliplr(b1)/b1(3); a2 = fliplr(a1)/a1(3);
K = b1(3)/a1(3);
[r p k] = residuez(conv(b1,b2),conv(a1,a2));
disp('VG is')
K*(k+r(3)+r(4))
%% Verification:
hn = filter(b1,a1,[1 zeros(1,999)]);
sum(abs(hn).^2)
```

7. (a) Proof:
We first prove (15.35a). Given that

$$f(x) = \frac{1}{\sigma_x\sqrt{2\pi}}\mathrm{e}^{-\frac{x^2}{2\sigma_x^2}}, \quad \Delta = \frac{\eta\sigma_x}{2^B}$$

The granular noise variance is

$$\begin{aligned}
\sigma_{\mathrm{g}}^2 &= \frac{\Delta^2}{12}\times 2\int_0^{\eta\sigma_x} f(x)dx \overset{u=x/\sigma_x}{=} \frac{\Delta^2}{12}\times 2\int_0^{\eta} f(u)du \\
&= \frac{\Delta^2}{12}\times 2\left(\int_0^{\infty} f(u)du - \int_{\eta}^{0} f(u)du\right) \\
&= \left(\frac{\eta\sigma_x}{2^B}\right)^2 \frac{2}{12}\left[\frac{1}{2} - \Phi(\eta)\right] \\
&= 2\sigma_x^2\frac{\eta^2 2^{-2B}}{12}\left[\frac{1}{2} - \Phi(\eta)\right]
\end{aligned}$$

We now prove (15.25b). The overload noise variance is

$$\begin{aligned}
\sigma_{\mathrm{o}}^2 &= 2\left[\int_{\eta\sigma_x}^{\infty} x^2 f(x)dx - 2\eta\sigma_x \int_{\eta\sigma_x}^{\infty} x f(x)dx + \eta^2\sigma_x^2 \int_{\eta\sigma_x}^{\infty} f(x)dx\right] \\
&= 2\sigma_x^2\left[\int_{\eta}^{\infty} u^2 f(u)du - 2\eta \int_{\eta}^{\infty} u f(u)du + \eta^2 \int_{\eta}^{\infty} f(u)du\right]
\end{aligned}$$

where

$$\begin{aligned}
\int_{\eta}^{\infty} u^2 f(u)du &= \int_{\eta}^{\infty} u^2 \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{u^2}{2}}du = -\int_{\eta}^{\infty} u\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{u^2}{2}}d\left(-\frac{u^2}{2}\right) \\
&= -\left[u\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{u^2}{2}}\bigg|_{\eta}^{\infty} - \int_{\eta}^{\infty}\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{u^2}{2}}du\right] \\
&= \Phi(\eta) + \eta\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{\eta^2}{2}}
\end{aligned}$$

$$\begin{aligned}
-2\eta\int_{\eta}^{\infty} u f(u)du &= -2\eta\int_{\eta}^{\infty} u\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{u^2}{2}}du \\
&= 2\eta\int_{\eta}^{\infty}\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{u^2}{2}}d\left(-\frac{u^2}{2}\right) = 2\eta\left(\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{u^2}{2}}\bigg|_{\eta}^{\infty}\right) \\
&= -2\eta\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{\eta^2}{2}}
\end{aligned}$$

$$\eta^2\int_{\eta}^{\infty} f(u)du = \eta^2\Phi(\eta)$$

Hence,

$$\sigma_{\mathrm{o}}^2 = 2\sigma_x^2\left[(\eta^2+1)\Phi(\eta) - \eta\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{\eta^2}{2}}\right]$$

(b) See plot below.

MATLAB script:

```
% P1507: SQNR
clc; close all;
L = 4:2:16; B = L - 1; B = B(:);
eta = logspace(-1,3,101);
sig2_g = bsxfun(@times,2.^(-2*B),eta.^2/6.*(-1/2+normcdf(eta,0,1)));
sig2_o = 2*(eta.^2+1).*(1-normcdf(eta,0,1))-2*eta.*normpdf(eta,0,1);
```

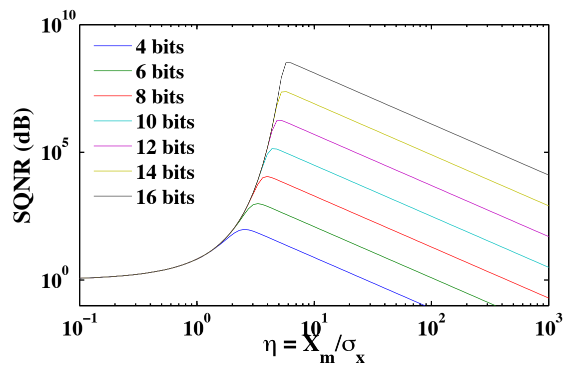

FIGURE 15.7: Plot of the noise variance as a function of η for $B+1 = 4, 6, 8, 10,$ $12, 14, 16$.

```
SQNR = 1./bsxfun(@plus,sig2_g,sig2_o);
hfa = figconfg('P1507a','small');
loglog(eta,SQNR)
xlabel('\eta = X_m/\sigma_x','fontsize',LFS);
ylabel('SQNR (dB)','fontsize',LFS);
ylim([10^(-1) 10^10])
hl = legend('4 bits','6 bits','8 bits','10 bits','12 bits','14 bits',...
    '16 bits','location','northwest');
set(hl,'box','off')
```

8. (a) Comments:
   See script output.
   (b) See plot below.
   (c) See plot below.

   MATLAB script:

```
% P1508: Quantizer performance investigation
clc; close all;
%% Part a
N = 1e5;
sigx2 = 1;
randn('seed',0)
xn = randn(N,1)*sqrt(sigx2);
L = 4;
x1 = -1; xL = 1-0.5^(L-1);
```

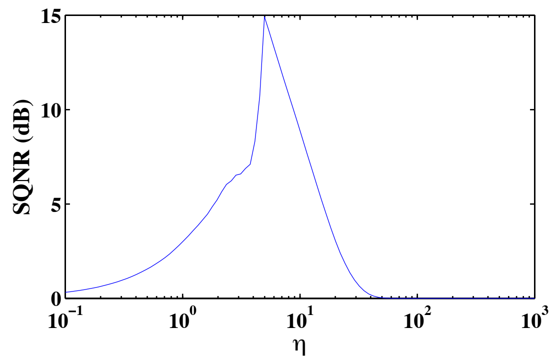

FIGURE 15.8: Plot of the noise variance as a function of η for $B + 1 = 4$.

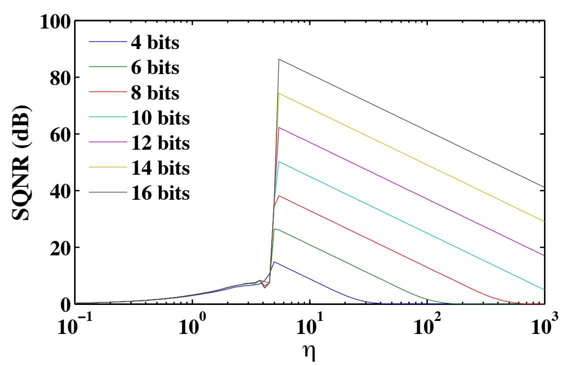

FIGURE 15.9: Plot of the noise variance as a function of η for $B + 1 = 6$, 8, 10, 12, 14, and 16.

```
ind = (xn < xL & xn > x1);
xng = xn(ind);
[xqg,E,B] = dec2beqR(xng,L);
sigg2 = mean((xng-xqg).^2);
xno = xn(~ind);
xqo = -ones(size(xno));
ind2 = (xno > x1);
xqo(ind2) = xL;
sigo2 = mean((xno-xqo).^2);
SQNRa = sigx2/(sigg2+sigo2);
%% Part b
eta = logspace(-1,3,100); N2 = length(eta);
```

```
sigx = 1./eta; sigx2 = sigx.^2;
randn('seed',0)
X = bsxfun(@times,randn(N,1),sigx);
SQNRb = zeros(1,N2);
L = 4;
x1 = -1; xL = 1-0.5^(L-1);
for ii = 1:N2
    ind = (X(:,ii) < xL & X(:,ii) > x1);
    xng = X(ind,ii);
    [xqg,E,B] = dec2beqR(xng,L);
    sigg2 = mean((xng-xqg).^2);
    xno = X(~ind,ii);
    xqo = -ones(size(xno));
    ind2 = (xno > x1);
    xqo(ind2) = xL;
    sigo2 = mean((xno-xqo).^2);
    if isnan(sigo2)
        sigo2 = 0;
    end
    SQNRb(ii) = sigx2(ii)/(sigg2+sigo2);
end
SQNRb_db = 10*log10(SQNRb/min(SQNRb));
hfb = figconfg('P1508b','small');
semilogx(eta,SQNRb_db)
xlabel('\eta','fontsize',LFS);
ylabel('SQNR (dB)','fontsize',LFS);
%% Part c
eta = logspace(-1,3,100); N2 = length(eta);
sigx = 1./eta; sigx2 = sigx.^2;
randn('seed',0)
X = bsxfun(@times,randn(N,1),sigx);
L = 4:2:16; NL = length(L);
SQNRc = zeros(NL,N2);
x1 = -1; xL = 1-0.5.^(L-1);
for jj = 1:NL
    for ii = 1:N2
        ind = (X(:,ii) < xL(jj) & X(:,ii) > x1);
        xng = X(ind,ii);
        [xqg,E,B] = dec2beqR(xng,L(jj));
        sigg2 = mean((xng-xqg).^2);
```

```
            xno = X(~ind,ii);
            xqo = -ones(size(xno));
            ind2 = (xno > x1);
            xqo(ind2) = xL(jj);
            sigo2 = mean((xno-xqo).^2);
            if isnan(sigo2)
                sigo2 = 0;
            end
            SQNRc(jj,ii) = sigx2(ii)/(sigg2+sigo2);
        end
    end
    SQNRc_db = 10*log10(SQNRc/min(SQNRc(:)));
    hfc = figconfg('P1508c','small');
    semilogx(eta,SQNRc_db)
    xlabel('\eta','fontsize',LFS);
    ylabel('SQNR (dB)','fontsize',LFS);
    hl = legend('4 bits','6 bits','8 bits','10 bits','12 bits','14 bits',...
        '16 bits','location','northwest');
    set(hl,'box','off')
```

9. (a) Proof:
Since $|z| = 1$, we have $z = \mathrm{e}^{\mathrm{j}\omega}$,

$$dz = \mathrm{j}\mathrm{e}^{\mathrm{j}\omega} d\omega \implies d\omega = \frac{1}{\mathrm{j}} z^{-1} dz$$

Hence, change the variable ω by z, we have

$$\mathrm{VG} = \frac{1}{2\pi} \oint_{\mathrm{UC}} H(z)H(z^{-1}) \frac{1}{\mathrm{j}} z^{-1} dz = \frac{1}{2\pi \mathrm{j}} \oint_{\mathrm{UC}} H(z)H(z^{-1}) z^{-1} dz$$

Compared to the inverse z-transform formula, that is

$$x[n] = Z^{-1}\left\{X(z)\right\} = \frac{1}{2\pi \mathrm{j}} \oint_{\mathrm{UC}} X(z) z^{n-1} dz$$

we can prove (15.38) that is

$$\mathrm{VG} = \frac{1}{2\pi \mathrm{j}} \oint_{\mathrm{UC}} H(z)H(z^{-1}) z^{-1} dz = Z^{-1}\left[H(z)H(z^{-1})\right]\Big|_{n=0}$$

(b) tba

10. Proof:
Consider Equation (15.59b), that is

$$S_{e_f}(\mathrm{e}^{\mathrm{j}\omega}) = |H_e(\mathrm{e}^{\mathrm{j}\omega})|^2 S_e(\mathrm{e}^{\mathrm{j}\omega}) = \sigma_e^2[2\sin(\omega/2)]^2$$

Applying inverse Fourier transform and set $n = 0$, we have the left hand side equals

$$E[e_f^2[n]]$$

The right hand side equals

$$\begin{aligned}\frac{\sigma_e^2}{2\pi}\int_{-\pi}^{\pi}[2\sin(\omega/2)]^2 d\omega &= \frac{\sigma_e^2}{\pi}\int_{-\pi}^{\pi}(1-\cos\omega)d\omega \\ &= \frac{\sigma_e^2}{\pi}(\omega - \sin\omega)|_{-\pi}^{\pi} \\ &= \frac{\sigma_e^2}{\pi}[(\pi - 0) - (-\pi - 0)] \\ &= 2\sigma_e^2 = 2E[e^2[n]]\end{aligned}$$

11. tba

12. Proof:

$$S_{e_f}(\mathrm{e}^{\mathrm{j}\omega}) = |H_e(\mathrm{e}^{\mathrm{j}\omega})|^2 S_e(\mathrm{e}^{\mathrm{j}\omega}) = \sigma_e^2[2\sin(\omega/2)]^4$$

$$\begin{aligned}E[e_o^2[n]] &= \frac{\sigma_e^2}{2\pi}\int_{-\pi/D}^{\pi/D}[2\sin(\omega/2)]^4 d\omega = \frac{\sigma_e^2}{2\pi}\int_{-\pi/D}^{\pi/D}(2-2\cos\omega)^2 d\omega \\ &= \frac{4\sigma_e^2}{2\pi}\int_{-\pi/D}^{\pi/D}(1-2\cos\omega+\cos^2\omega)d\omega = \frac{4\sigma_e^2}{2\pi}\int_{-\pi/D}^{\pi/D}(1-2\cos\omega+\frac{1+\cos 2\omega}{2})d\omega \\ &= \frac{4\sigma_e^2}{2\pi}\left[\frac{3\pi}{D} - 4\sin\frac{\pi}{D} + \frac{1}{2}\sin\frac{2\pi}{D}\right] = \frac{4\sigma_e^2}{2\pi}\left[\frac{3\pi}{D} - 4\sin\frac{\pi}{D} + \sin\frac{\pi}{D}\cos\frac{\pi}{D}\right]\end{aligned}$$

Note the approximations, that is

$$\sin\theta \approx \theta - \theta^3/6, \quad \cos\theta \approx 1 - \theta^2/2$$

Plug the above approximations into the equation, we have

$$E[e_o^2[n]] \approx \frac{4\sigma_e^2}{2\pi}\left[\frac{3\pi}{D} - 4(\frac{\pi}{D} - \frac{\pi^3}{6D^3}) + (\frac{\pi}{D} - \frac{\pi^3}{6D^3})(1 - \frac{\pi^2}{D^2})\right] = \frac{\sigma_e^2\pi^4}{6D^5}$$

Hence, we conclude

$$\mathrm{SQNR_D} = 10\ln\frac{\sigma_x^2}{\sigma_e^2} + 10\ln\frac{6}{\pi^4} + r\ln 2^5 = \mathrm{SQNR_{NR}} - 12.10 + 15.05r$$

13. (a) Proof:

$$\frac{\left.\frac{\partial D(z)}{\partial a_k}\right|_{z=p_i}}{\left.\frac{\partial D(z)}{\partial p_i}\right|_{z=p_i}} = \left.\frac{\partial D(z)}{\partial a_k}\right|_{z=p_i} \left.\frac{\partial p_i}{\partial D(z)}\right|_{z=p_i} = \frac{\partial p_i}{\partial a_k}$$

(b) Proof:

$$\left.\frac{\partial D(z)}{\partial a_k}\right|_{z=p_i} = z^{-k}|_{z=p_i} = p_i^{-k}$$

$$\left.\frac{\partial D(z)}{\partial p_i}\right|_{z=p_i} = (-z^{-1}) \prod_{j=1,j\neq i}^{N} (1-p_j z^{-1})\Bigg|_{z=p_i} = (-p_i^{-1}) \prod_{j=1,j\neq i}^{N} (1-p_j p_i^{-1})$$

$$= (-p_i^{-N}) \prod_{j=1,j\neq i}^{N} (p_i - p_j)$$

Thus,

$$\frac{\partial p_i}{\partial a_k} = \frac{p_i^{-k}}{(-p_i^{-N})\prod_{j=1,j\neq i}^{N}(p_i-p_j)} = -\frac{p_i^{N-k}}{\prod_{j=1,j\neq i}^{N}(p_i-p_j)}$$

Hence,

$$\Delta p_i = -\sum_{k=1}^{N} \frac{p_i^{N-k}}{\prod_{j=1,j\neq i}^{N}(p_i-p_j)} \Delta a_k$$

14. (a) See plots below.

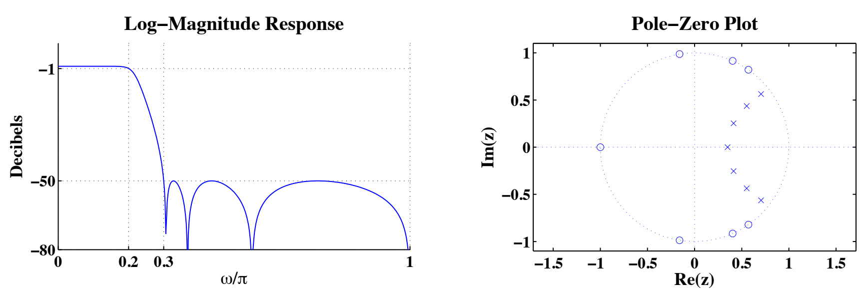

FIGURE 15.10: Plot of the magnitude response and pole-zero diagram of the filter.

(b) See plots below.

(c) See plots below.

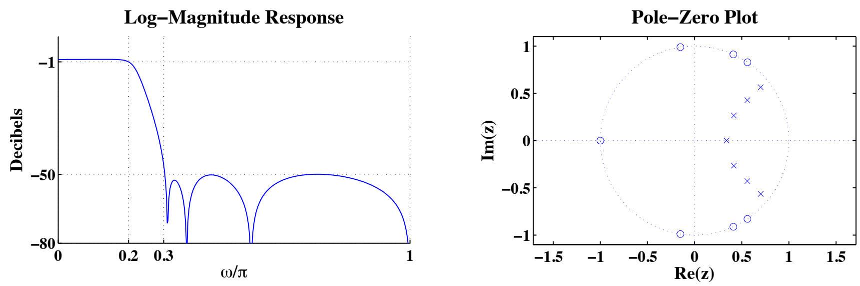

FIGURE 15.11: Plot of the magnitude response and pole-zero diagram of the filter when direct form coefficients are quantized to $L = 16$ bits.

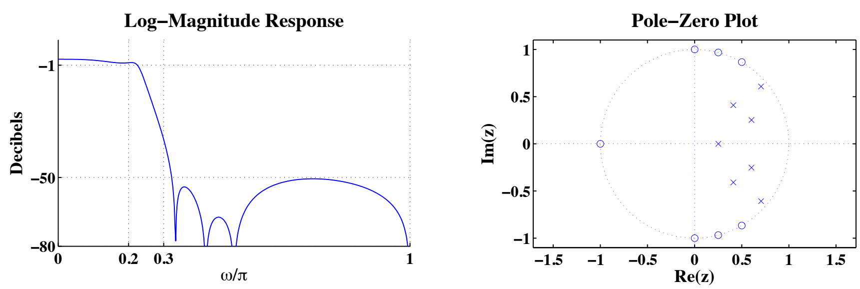

FIGURE 15.12: Plot of the magnitude response and pole-zero diagram of the filter when direct form coefficients are quantized to $L = 12$ bits.

(d) See plots below.

(e) tba.

MATLAB script:

```
% P1514: Digital lowpass filter design by Chebyshev II
close all; clc;
%% Filter Design
omegap = 0.2*pi; omegas = 0.3*pi; As = 50; Ap = 1; % Specification
[N,omegac] = cheb2ord(omegap/pi,omegas/pi,Ap,As); % Order define
[B,A] = cheby2(N,As,omegac); % coefficients
% [sos G] = tf2sos(B,A); % cascade form
%% Filter Coefficient Quantization of Direct Form
L = 16; % Part b
% L = 12; % Part c
% L = 8; % Part d L = 8
[BAhat,E1,B1] = dec2beqR([B;A],L);
```

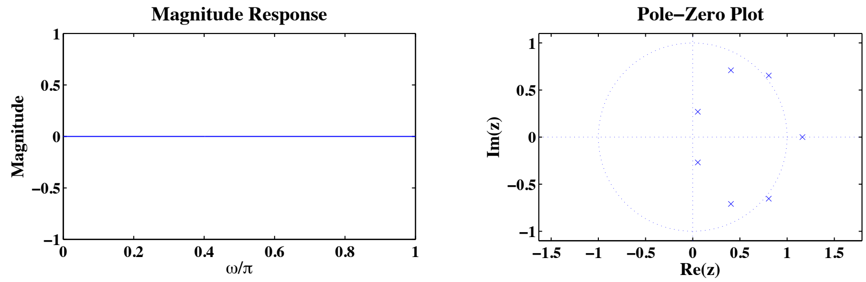

FIGURE 15.13: Plot of the magnitude response and pole-zero diagram of the filter when direct form coefficients are quantized to $L = 8$ bits.

```
Bhat = BAhat(1,:); Ahat = BAhat(2,:);
%% Plotting Parameters and Filter Responses
BP = B; AP = A; % Plot infinite precision
% BP = Bhat; AP = Ahat; % Plot finite precision
om = linspace(0,1,501)*pi;
H = freqz(BP,AP,om); Hmag = abs(H); Hdb = 20*log10(Hmag);
%% Design Plots
hfa = figconfg('P1514a','small'); % Log-Magnitude Response in dB
plot(om/pi,Hdb,'b','linewidth',1); axis([0,1,-80,10]);
xlabel('\omega/\pi','fontsize',LFS);
ylabel('Decibels','fontsize',LFS);
title('Log-Magnitude Response','fontsize',TFS);
set(gca,'xtick',[0,omegap,omegas,pi]/pi);
set(gca,'ytick',[-80,-As,-Ap]); grid; box off;

% hfa = figconfg('P1514a','small'); % Magnitude Response
% plot(om/pi,Hmag,'b','linewidth',1); axis([0,1,-1,1]);
% xlabel('\omega/\pi','fontsize',LFS);
% ylabel('Magnitude','fontsize',LFS);
% title('Magnitude Response','fontsize',TFS);

hfb = figconfg('P1514b','small'); % Pole-Zero Plot
zplane(BP,AP);
xlabel('Re(z)','fontsize',LFS);
ylabel('Im(z)','fontsize',LFS);
title('Pole-Zero Plot','fontsize',TFS);
```

15. (a) See plot below.

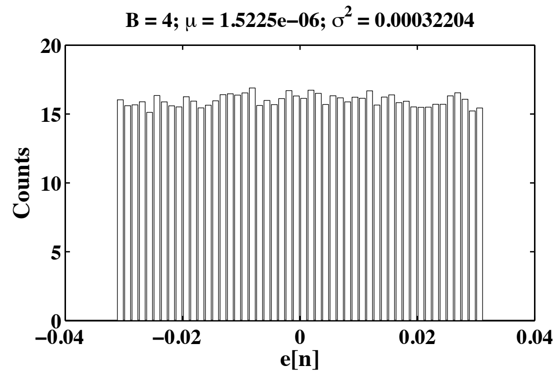

FIGURE 15.14: Plot of the histogram of the resulting error sequence when $ax[n]$ is quantized to $B = 4$.

(b) See plot below.

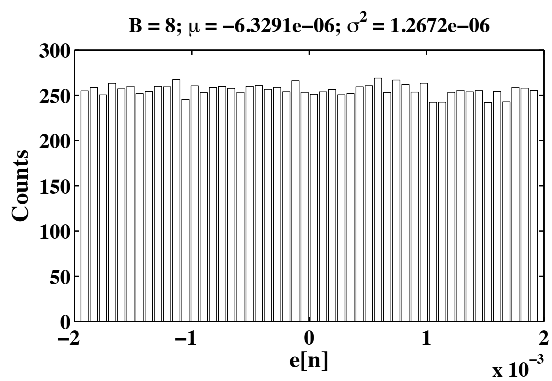

FIGURE 15.15: Plot of the histogram of the resulting error sequence when $ax[n]$ is quantized to $B = 8$.

(c) See plot below.

(d) tba.

MATLAB script:

```
% P1515: Quatization noise distribution analysis
clc; close all;
N = 1e5;
n = 1:N;
xn = (cos(n/11) + sin(n/17) + cos(n/31))/3;
```

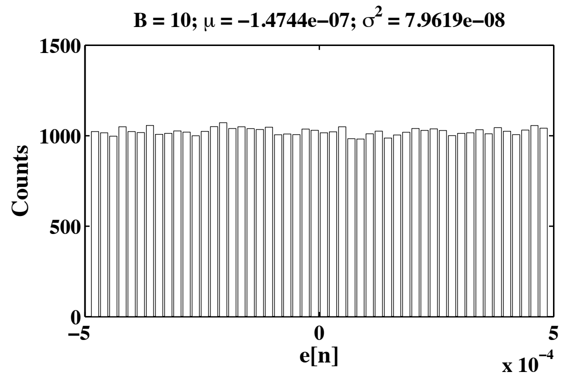

FIGURE 15.16: Plot of the histogram of the resulting error sequence when $ax[n]$ is quantized to $B = 10$.

```
a = 0.9375;
yn = a*xn;
B = 4; % Part a
% B = 8; % Part b
% B = 10; % Part c
[pe,eo,eavg,evar] = QNmodel(yn,B);
hfa = figconfg('P1515a','small');
bar(eo,pe,'facecolor','w')
xlabel('e[n]','fontsize',LFS);
ylabel('Counts','fontsize',LFS);
title(['B = ',num2str(B),'; \mu = ',num2str(eavg),...
    '; \sigma^2 = ',num2str(evar)])
```

16. (a) Comments:
    See script for detail.
    (b) See plot below.
    (c) See plot below.
    (d) Comments:
    See the script output.

MATLAB script:

```
% P1516:
clc; close all;
a = -0.375; N = 1e5; n = 1:N; S = 1 - abs(a);
```

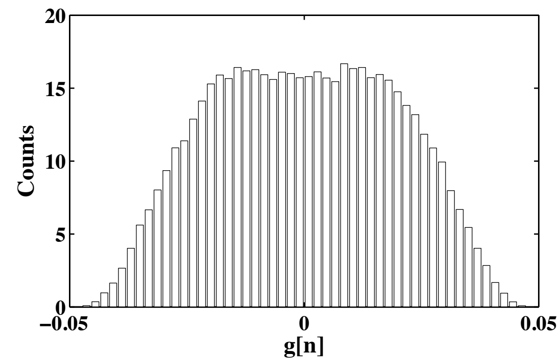

FIGURE 15.17: Plot of the histogram of the resulting error sequence $g[n]$ for $B = 4$.

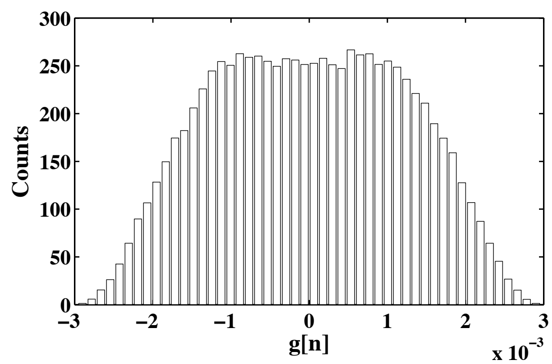

FIGURE 15.18: Plot of the histogram of the resulting error sequence $g[n]$ for $B = 8$.

```
B = 4; % Part a & b
% B = 8; % Part c
xn = cos(n/7);
[xnq,E1,B1] = dec2beqR(S*xn,B+1);
yn = filter(1,[1 a],xnq); % Part a
delta = 2^(-B);
rand('seed',0);
en = rand(1,N)*delta-delta/2;
ynhat = filter(1,[1 a],xnq+en);
gn = ynhat - yn;
[eo pe] = epdf(gn,50);
```

```
SNR = var(yn)/var(gn);
SNR_ref = 2^(2*B+2)*(1-abs(a))^2; % Part d
%% Plot
hfa = figconfg('P1516a','small');
bar(eo,pe,'facecolor','w')
xlabel('g[n]','fontsize',LFS);
ylabel('Counts','fontsize',LFS);
```

17. (a) Solution:
The output display oscillation and the amplitude is 0.125 and frequency is one sample.

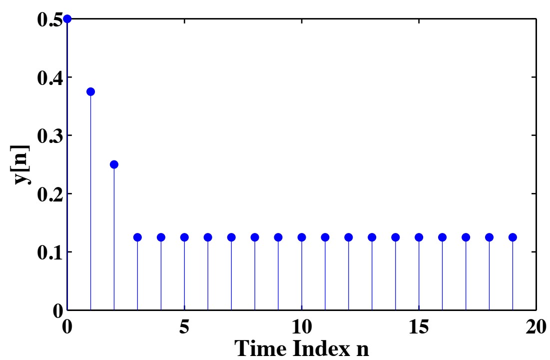

FIGURE 15.19: Plot of the first 20 samples of $y[n]$ when two's-complement overflow is used in the addition.

(b) Solution:
The output display oscillation and the amplitude is 0.125 and frequency is one sample.

MATLAB script:

```
% P1517:
clc; close all;
a0 = 0.5; a1 = 0.625; N = 20; B = 3;
yn = zeros(1,N);
yn(1) = a0;
%% Part a:
for ii = 2:20
    [yn(ii) E1 B1] = dec2beqR(yn(ii-1)*a1,B+1);
    if yn(ii) < -1 || yn(ii) >= 1
```

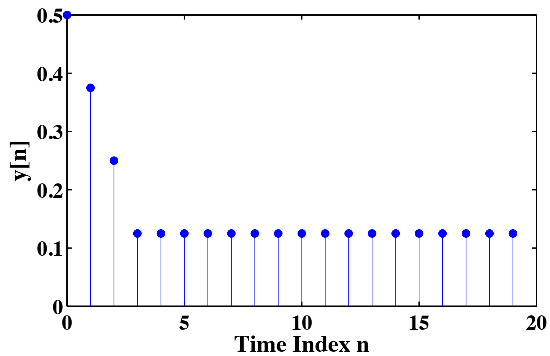

FIGURE 15.20: Plot of the first 20 samples of $y[n]$ when saturation characteristics is used in the addition.

```
            yn(ii) = -sign(yn(ii))*(abs(yn(ii)-1));
        end
    end
    %% Part b:
    % for ii = 2:20
    %     [yn(ii) E1 B1] = dec2beqR(yn(ii-1)*a1,B+1);
    %     if yn(ii) < -1 || yn(ii) >= 1
    %         yn(ii) = sign(yn(ii));
    %     end
    % end
    %% Plot
    hfa = figconfg('P1517a','small');
    stem(0:N-1,yn,'filled')
    xlabel('Time Index n','fontsize',LFS);
    ylabel('y[n]','fontsize',LFS);
```

18. (a) Proof:

We first prove the sufficiency. Given

$$|x[n]| < \frac{1}{N}$$

we have

$$|X(k)| = \left|\sum_{n=0}^{N-1} x[n]W_N^{kn}\right| \leq \sum_{n=0}^{N-1} \left|x[n]W_N^{kn}\right| = \sum_{n=0}^{N-1} |x[n]| < \sum_{n=0}^{N-1} \frac{1}{N} = 1$$

We next prove the necessity. Given

$$|X(k)| < 1$$

If $|x[n]| = a \geq \frac{1}{N}$, we can assign that

$$x[n] = aW_N^{-kn}$$

Hence,

$$X(k) = Na \geq 1$$

which proves the necessity.

(b) Proof:

$$E[|X(k)|^2] = E[\frac{1}{N^2}\left|\sum_{n=0}^{N-1} x[n]W_N^{kn}\right|^2] = \frac{1}{N^2}E[\left|\sum_{n=0}^{N-1} x[n]W_N^{kn}\right|^2]$$

Hence, the signal power decreases by $1/N^2$, while the noise power remains the same, that proves that SQNR decreases by $1/N^2$.

19. Proof:

$$E[x[n]] = 0$$

$$\sigma_x^2 = E[x^2[n]] = \int_{-1/N}^{1/N} x^2 \frac{N}{2} dx = \frac{1}{3N^2}$$

$$E[|X(k)|^2] = E(\sum_{n=0}^{N-1} x[n]W_N^{nk} \sum_{m=0}^{N-1} x[m]W_N^{-mk})$$

If $m \neq n$, we have

$$E[x[n]x[m]] = E[x[n]]E[x[m]] = 0$$

Hence,

$$E[|X(k)|^2] = E(\sum_{n=0}^{N-1} x^2[n]) = \sum_{n=0}^{N-1} E[x^2[n]] = N\frac{1}{3N^2} = \frac{1}{3N}$$