CHAPTER 12

Multirate Signal Processing

Tutorial Problems

1. (a) See script below.

   (b) See plot below.

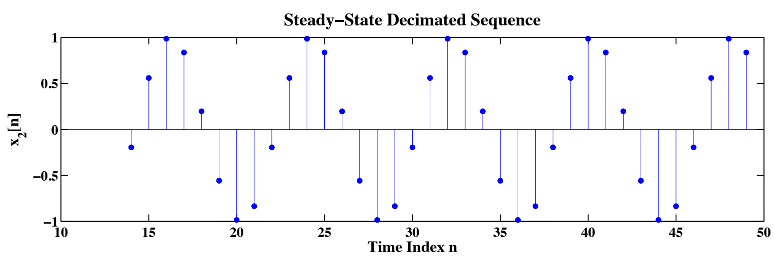

FIGURE 12.1: Steady-state values of $x_{\mathrm{D}}[n]$ computed by `src` function.

   (c) See plot below.

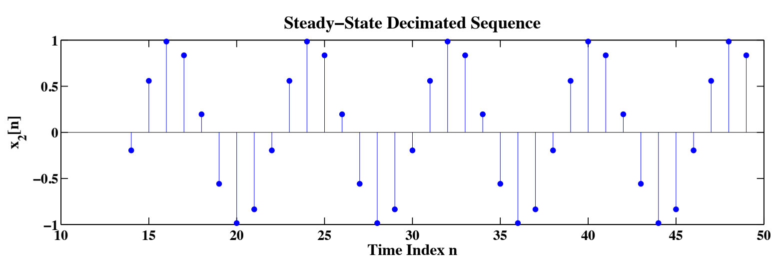

FIGURE 12.2: Steady-state values of $x_{\mathrm{D}}[n]$ computed by `firdec` function.

   (d) See plot below.

   (e) tba.

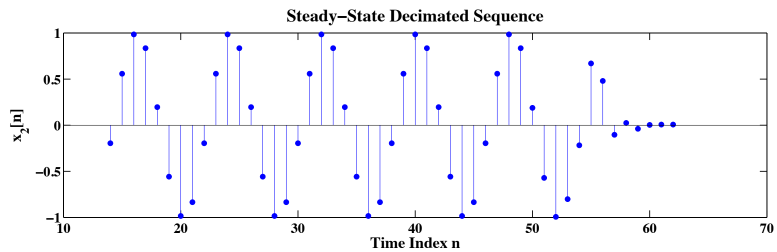

FIGURE 12.3: Steady-state values of $x_{\mathrm{D}}[n]$ computed by upfirdn function.

MATLAB script:

```
% P1201: Decimation functions comparison
close all; clc
N = 100;
n = 0:N-1;
om0 = 0.125*pi;
xn = sin(om0*n);
D = 2;
%% Part a: Lowpass filter
M = 25;
hn = firpm(M,[0 0.5 0.6 1],[1 1 0 0],[1 1]);
vn = filter(hn,1,xn);
%% Part b:
ynb = src(vn,D);
%% Part c:
ync = firdec(hn,xn,D);
%% Part d:
ynd = upfirdn(xn,hn,1,D);
%% Plot
yn = ynb; % part b
% yn = ync; % part c
% yn = ynd; % part d
hfa = figconfg('P1201a','long');
Lp = length(yn); Ls = 15;
stem(Ls-1:Lp-1,yn(Ls:end),'filled')
xlabel('Time Index n','fontsize',LFS);
ylabel(['x_',num2str(D),'[n]'],'fontsize',LFS);
title('Steady-State Decimated Sequence','fontsize',TFS);
```

2. (a) Comments: tba.
   (b) Comments: tba.

3. (a) See script below.
   (b) See plot below.

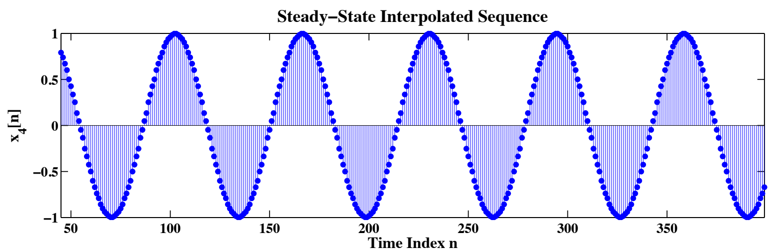

FIGURE 12.4: Steady-state values of $x_{\mathrm{I}}[n]$ computed by `sre` function.

   (c) See plot below.

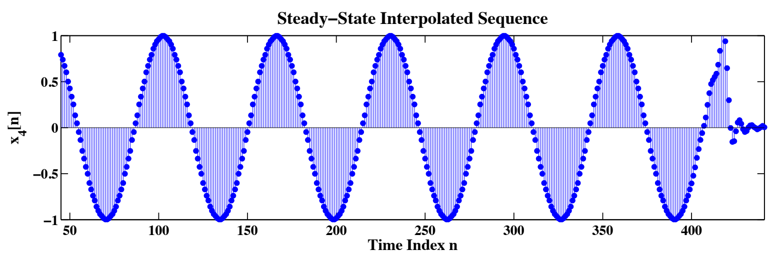

FIGURE 12.5: Steady-state values of $x_{\mathrm{I}}[n]$ computed by `upfirdn` function.

   (d) See plot below.

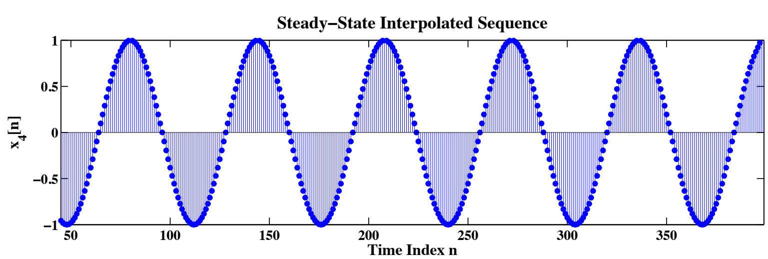

FIGURE 12.6: Steady-state values of $x_{\mathrm{I}}[n]$ computed by `interp` function.

   (e) tba.

MATLAB script:

```
% P1203: Interpolation functions comparison
close all; clc
N = 100;
n = 0:N-1;
om0 = 0.125*pi;
xn = sin(om0*n);
I = 4;
%% Part a: Lowpass filter
M = 45;
hn = firpm(M,[0 0.25 0.35 1],[1 1 0 0],[1 1]);
hn = hn*I;
w = linspace(0,1,501)*pi;
H = freqz(hn,1,w); Hmag = abs(H);
figure, plot(w/pi,Hmag)
vn = sre(xn,I);
%% Part b:
ynb = filter(hn,1,vn);
%% Part c:
ync = upfirdn(xn,hn,I,1);
%% Part d:
ynd = interp(xn,I);
%% Plot
% yn = ynb; % part b
% yn = ync; % part c
yn = ynd; % part d
hfa = figconfg('P1203a','long');
Lp = length(yn); Ls = length(hn);
stem(Ls-1:Lp-1,yn(Ls:end),'filled')
ylim([-1 1]); xlim([Ls-1 Lp-1])
xlabel('Time Index n','fontsize',LFS);
ylabel(['x_',num2str(I),'[n]'],'fontsize',LFS);
title('Steady-State Interpolated Sequence','fontsize',TFS);
```

4. (a) See plot below.
   (b) See plot below.

   MATLAB script:

```
% P1204: Interpolation function "interp"
close all; clc
```

```
n = 0:50;
xn = cos(0.9*pi*n);
I = 2;
% I = 4;
% I = 8;
[yn b] = interp(xn,I);
%% Plot
hfa = figconfg('P1204a','long');
stem(n,xn,'filled')
ylim([-1 1]);
xlabel('Time Index n','fontsize',LFS);
ylabel('x[n]','fontsize',LFS);
title('Original Sequence','fontsize',TFS);

hfb = figconfg('P1204b','long');
stem(0:length(yn)-1,yn,'filled')
ylim([-1 1]); xlim([0 50*I])
xlabel('Time Index n','fontsize',LFS);
ylabel(['x_',num2str(I),'[n]'],'fontsize',LFS);
title('Interpolated Sequence','fontsize',TFS);

w = linspace(0,1,501)*pi;
H = freqz(b,1,w); Hmag = abs(H);
hfc = figconfg('P1204c','small');
plot(w/pi,Hmag)
xlabel('\omega/\pi','fontsize',LFS);
ylabel('Magnitude','fontsize',LFS);
title('Magnitude Response','fontsize',TFS);
```

5. (a) See plot below.
   (b) See plot below.
   (c) See plot below.
   (d) See plot below.
   (e) tba.

   MATLAB script:

```
% P1205: Interpolation; Frequency Investigation
close all; clc
L = 101; M = L - 1;
```

```
xn = fir2(M,[0,0.1,0.2,0.5,0.55,0.6,1],[2,2,1.5,1,0.5,0,0]);
w = linspace(0,1,501)*pi;
Hx = freqz(xn,1,w); Hxmag = abs(Hx);
I = 2;
% I = 3;
% I = 4;
yn = interp(xn,I);
Hy = freqz(yn,1,w); Hymag = abs(Hy);
%% Plot
hfa = figconfg('P1205a','small');
plot(w/pi,Hxmag)
xlabel('\omega/\pi','fontsize',LFS);
ylabel('Magnitude','fontsize',LFS);
title('Magnitude Response','fontsize',TFS);
hfb = figconfg('P1205b','small');
plot(w/pi,Hymag)
xlabel('\omega/\pi','fontsize',LFS);
ylabel('Magnitude','fontsize',LFS);
title('Magnitude Response','fontsize',TFS);
```

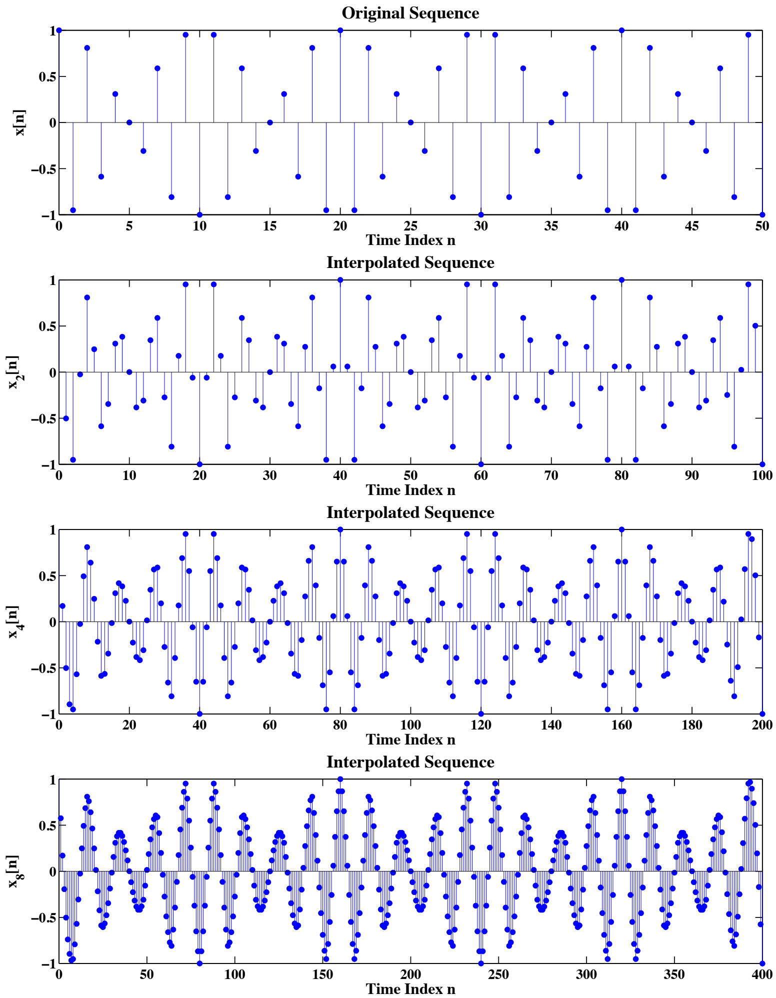

FIGURE 12.7: Original signal $x[n]$ and interpolated signal using $I = 2$, $I = 4$ and $I = 8$.

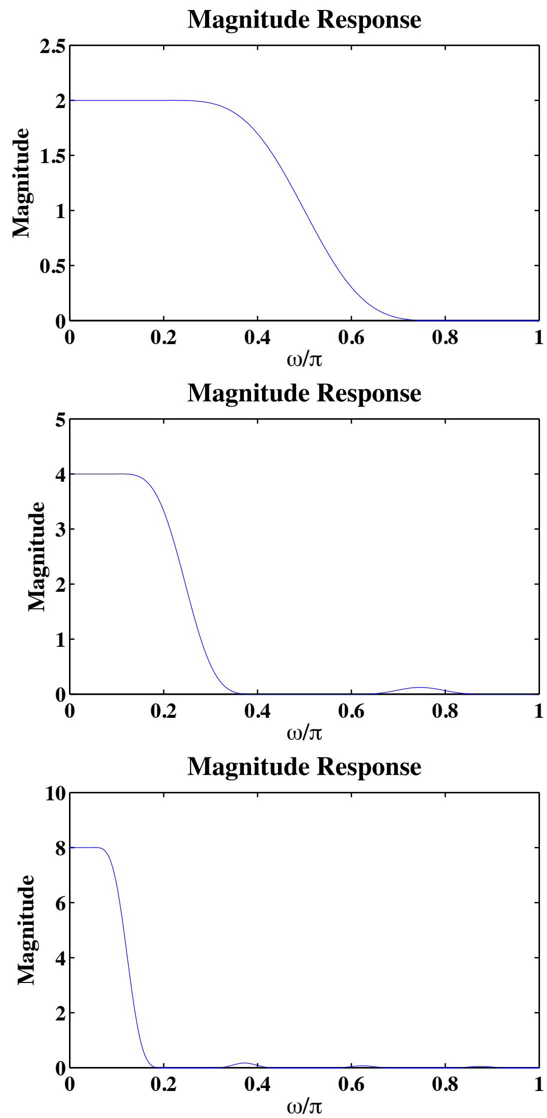

FIGURE 12.8: Magnitude response of the lowpass filter when $I = 2$, $I = 4$ and $I = 8$.

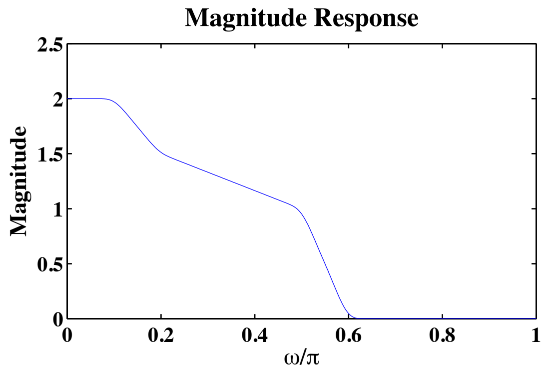

FIGURE 12.9: Magnitude spectra of $x[n]$.

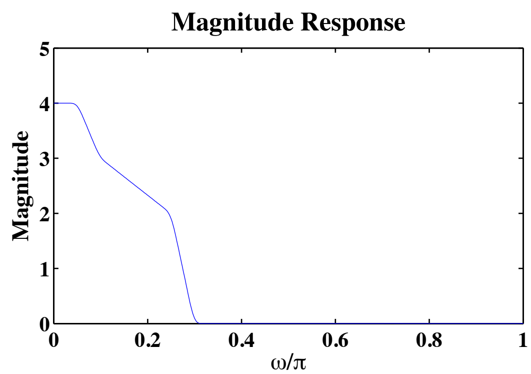

FIGURE 12.10: Magnitude spectra of the decimated signal by $I = 2$.

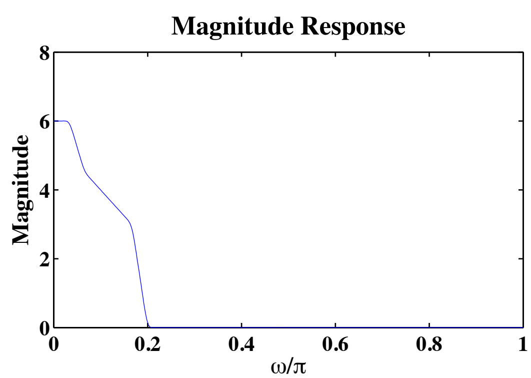

FIGURE 12.11: Magnitude spectra of the decimated signal by $I = 3$.

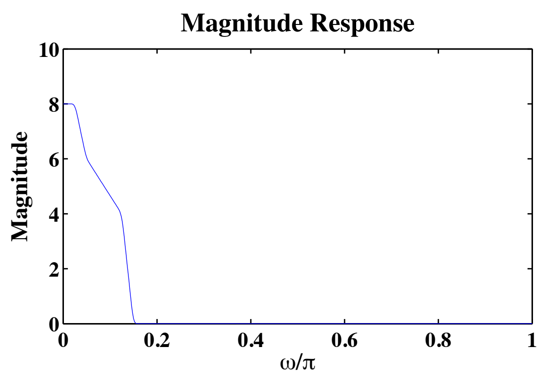

FIGURE 12.12: Magnitude spectra of the decimated signal by $I = 4$.

6. Solution:
   We can express a similar convolution summation given in (12.44) as:

$$x_{\rm I}[n] = \sum_{k=-\infty}^{\infty} x_{\rm u}[k] g_{\rm lin}[n-k]$$

A brief verification is as follows:

$$\begin{aligned}
x_{\rm I}[(m-1)I+k] &= \sum_{p=-\infty}^{\infty} x_{\rm u}[pI] g_{\rm lin}[(m-1)I+k-pI] \\
&= x_{\rm u}[(m-1)I] g_{\rm lin}[k] + x_{\rm u}[mI] g_{\rm lin}[-I+k] \\
&= x_{\rm u}[(m-1)I]\left(1-\frac{|k|}{I}\right) + x_{\rm u}[mI]\left(1-\frac{|-I+k|}{I}\right) \\
&= x_{\rm u}[(m-1)I]\left(1-\frac{k}{I}\right) + x_{\rm u}[mI]\left(\frac{k}{I}\right) \qquad k=0,1,\ldots,I-1
\end{aligned}$$

7. (a) See plot below.

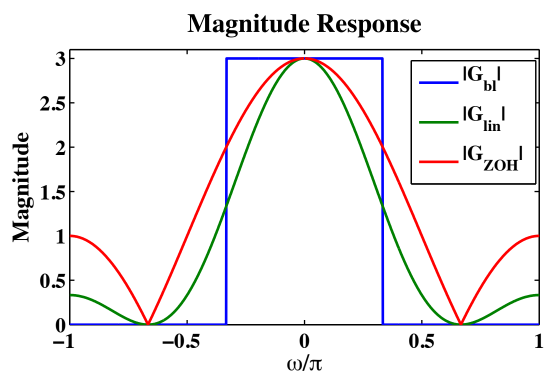

FIGURE 12.13: Magnitude responses of the ideal, ZOH, and FOH interpolators for $I = 3$.

(b) See plot below.

MATLAB script:

```
% P1207: Zero-order-hold (ZOH) Interpretor
close all; clc
w = linspace(-1,1,1001)*pi;
I = 3; % Part a
```

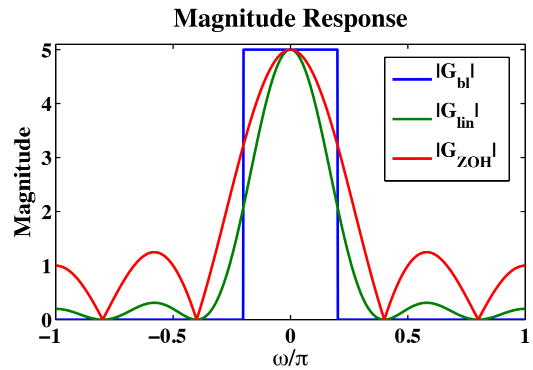

FIGURE 12.14: Magnitude responses of the ideal, ZOH, and FOH interpolators for $I = 5$.

```
% I = 5; % Part b
G_bl = zeros(size(w));
ind = (abs(w) <= pi/I);
G_bl(ind) = I; magG_bl = abs(G_bl);
G_lin = (sin(w*I/2)./sin(w/2)).^2/I;
magG_lin = abs(G_lin);
G_zoh = (1-exp(-1j*I*w))./(1-exp(-1j*w));
magG_zoh = abs(G_zoh);
%% Plot:
hfa = figconfg('P1207a','small');
plot(w/pi,[magG_bl;magG_lin;magG_zoh],'linewidth',2)
ylim([0 I+0.1])
xlabel('\omega/\pi','fontsize',LFS);
ylabel('Magnitude','fontsize',LFS);
title('Magnitude Response','fontsize',TFS);
legend('|G_{bl}|','|G_{lin}|','|G_{ZOH}|','location','best')
```

8. (a) See plot below.

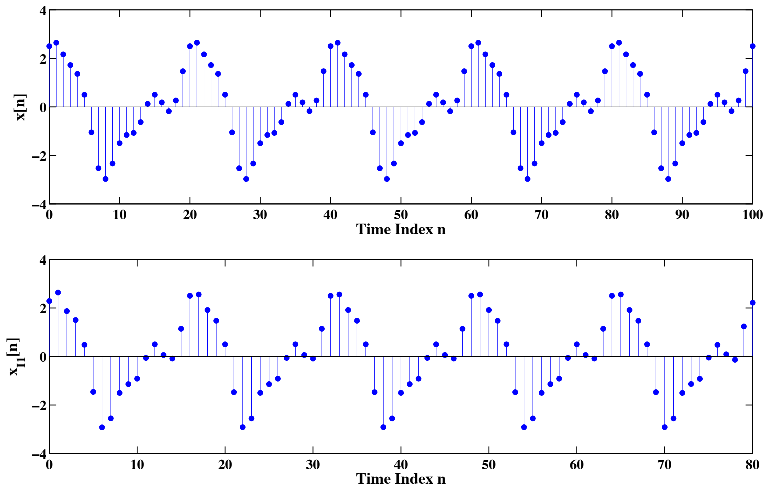

FIGURE 12.15: Stem plots of sequence $x[n]$ and resampled sequence $x_{\mathrm{I}_1}[m]$.

(b) See plot below.

(c) See plot below.

(d) Comments:
Upsampled sequence $x_{\mathrm{I}_2}[m]$ retains the "shape" of the original sequence $x[n]$.

MATLAB script:

```
% P1208: Illustrating function "resample"
close all; clc
n = 0:100;
xn = 2*cos(0.1*pi*n) + sin(0.2*pi*n) + 0.5*cos(0.4*pi*n);
xI1 = resample(xn,4,5); % Part a
xI2 = resample(xn,5,4); % Part b
xI3 = resample(xn,2,3); % Part c
%% Plot
hfa = figconfg('P1208a','long');
stem(n,xn,'filled')
xlabel('Time Index n','fontsize',LFS);
ylabel('x[n]','fontsize',LFS);

hfb = figconfg('P1208b','long');
```

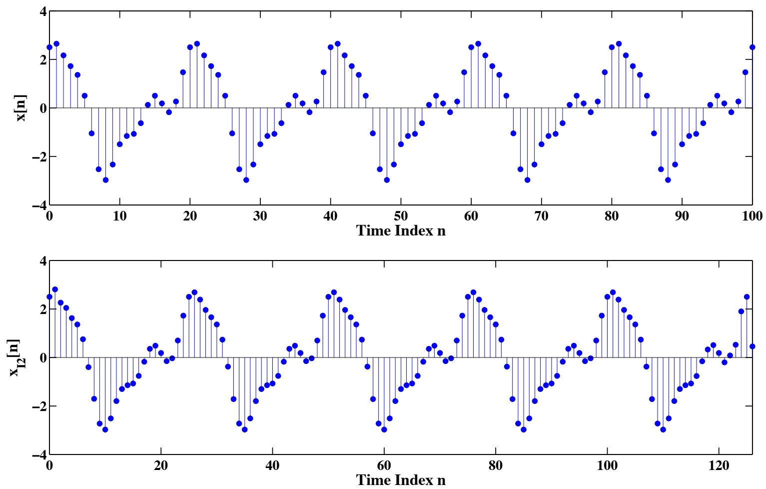

FIGURE 12.16: Stem plots of sequence $x[n]$ and resampled sequence $x_{\mathrm{I}_2}[m]$.

```
stem(0:length(xI1)-1,xI1,'filled')
xlabel('Time Index n','fontsize',LFS);
ylabel('x_{I1}[n]','fontsize',LFS);
xlim([0 length(xI1)-1])

hfc = figconfg('P1208c','long');
stem(0:length(xI2)-1,xI2,'filled')
xlabel('Time Index n','fontsize',LFS);
ylabel('x_{I2}[n]','fontsize',LFS);
xlim([0 length(xI2)-1])

hfd = figconfg('P1208d','long');
stem(0:length(xI3)-1,xI3,'filled')
xlabel('Time Index n','fontsize',LFS);
ylabel('x_{I3}[n]','fontsize',LFS);
xlim([0 length(xI3)-1])
```

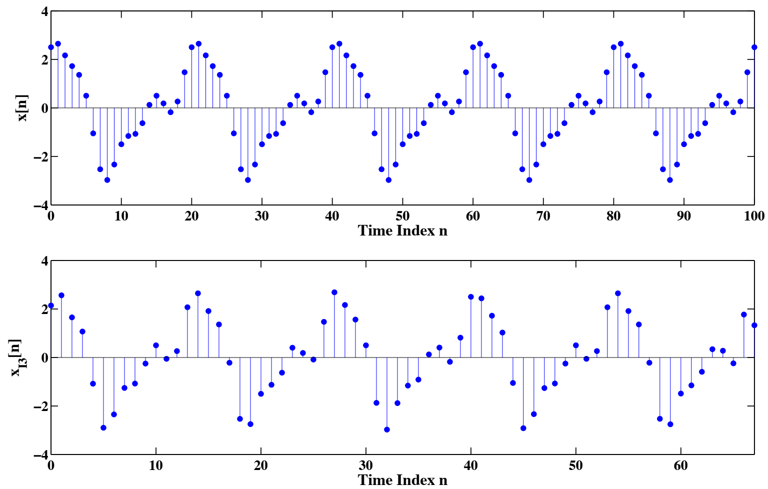

FIGURE 12.17: Stem plots of sequence $x[n]$ and resampled sequence $x_{\mathrm{I}_3}[m]$.

9. (a) See plot below.
   (b) See plot below.

   MATLAB script:

```
% P1209: Decimation recovered by Interpolation
close all; clc
n = 0:80;
xn = cos(0.04*pi*n) + 3*sin(0.0072*pi*n);
% D = 3; I = 3;
% D = 5; I = 5;
D = 10; I = 10;
xd = downsample(xn,D);
xu = upsample(xd,I);
glin = 1:I-1; glin = [glin I fliplr(glin)]/I;
xi = filter(glin,1,xu);
%% Plot:
hfa = figconfg('P1209a','long');
stem(0:length(xn)-1,xn,'filled');
xlim([0 length(xn)-1])
xlabel('Time Index n','fontsize',LFS);
ylabel('x[n]','fontsize',LFS);
```

```
hfb = figconfg('P1209b','long');
stem(0:length(xd)-1,xd,'filled');
xlim([0 length(xd)-1])
xlabel('Time Index n','fontsize',LFS);
ylabel('x_d[n]','fontsize',LFS);

hfc = figconfg('P1209c','long');
stem(0:length(xu)-1,xu,'filled');
xlim([0 length(xu)-1])
xlabel('Time Index n','fontsize',LFS);
ylabel('x_u[n]','fontsize',LFS);

hfd = figconfg('P1209d','long');
stem(0:length(xi)-1,xi,'filled');
xlim([0 length(xi)-1])
xlabel('Time Index n','fontsize',LFS);
ylabel('x_i[n]','fontsize',LFS);
```

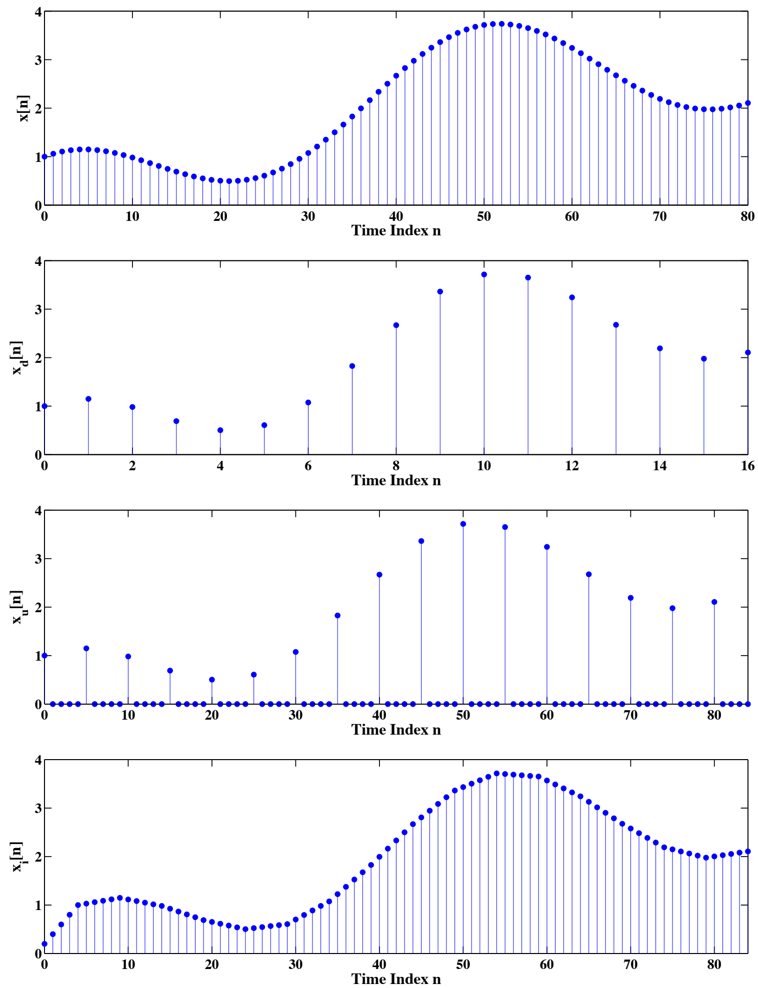

FIGURE 12.18: Stem plots of sequence $x[n]$, $x_d[n]$, $x_u[n]$ and $x_i[n]$ for $D = I = 5$.

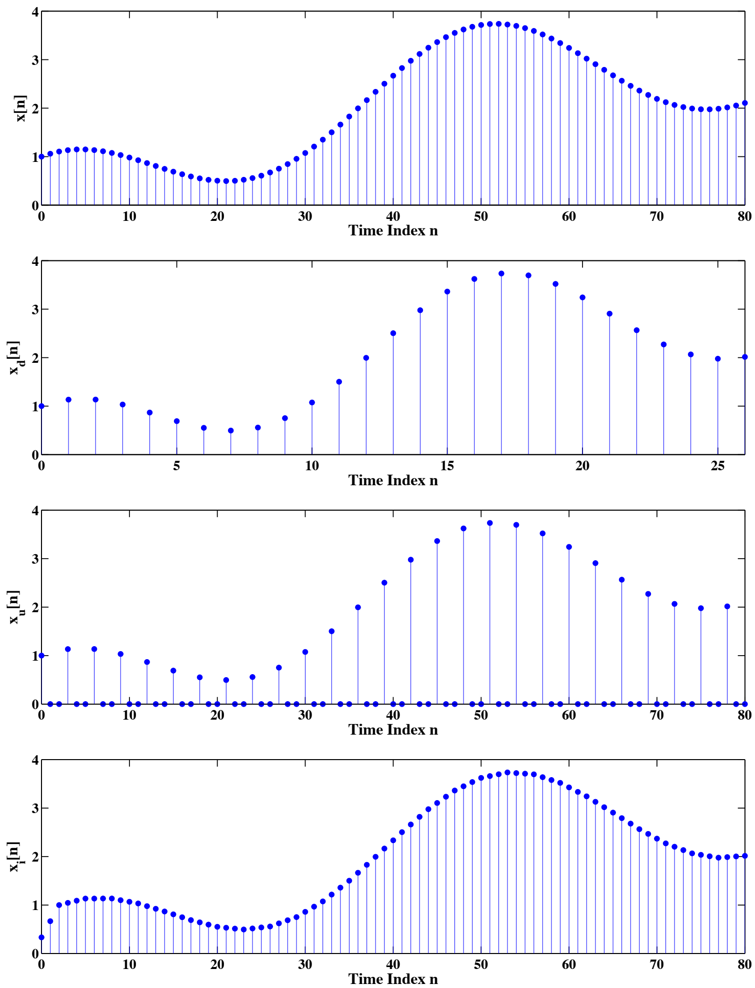

FIGURE 12.19: Stem plots of sequence $x[n]$, $x_d[n]$, $x_u[n]$ and $x_i[n]$ for $D = I = 3$.

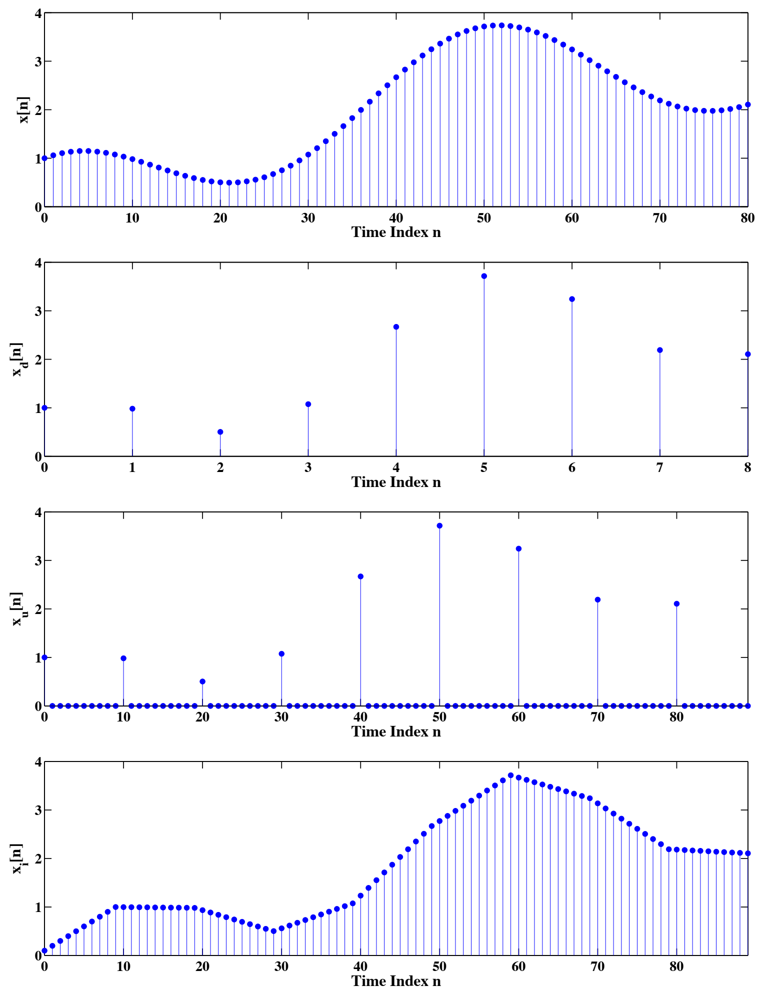

FIGURE 12.20: Stem plots of sequence $x[n]$, $x_d[n]$, $x_u[n]$ and $x_i[n]$ for $D = I = 5$.

10. Proof:
Given (12.89), that is

$$H(z) = \frac{1}{K} + \sum_{k=1}^{K-1} z^{-k} P_k\left(z^K\right)$$

we have

$$\begin{aligned}
H\left(zW_K^p\right) &= \frac{1}{K} + \sum_{k=1}^{K-1} \left(zW_K^p\right)^{-k} P_k\left((zW_K^p)^K\right) \\
&= \frac{1}{K} + \sum_{k=1}^{K-1} z^{-k} W_K^{p-k} P_k(z^K)
\end{aligned}$$

Hence,

$$\begin{aligned}
\sum_{p=0}^{K-1} H\left(zW_K^p\right) &= \sum_{p=0}^{K-1} \left(\frac{1}{K} + \sum_{k=1}^{K-1} z^{-k} W_K^{p-k} P_k(z^K)\right) \\
&= \sum_{p=0}^{K-1} \frac{1}{K} + \sum_{p=0}^{K-1} \sum_{k=1}^{K-1} z^{-k} W_K^{p-k} P_k(z^K) \\
&= 1 + \sum_{k=1}^{K-1} z^{-k} W_K^{-k} P_k(z^K) \left(\sum_{p=0}^{K-1} W_K^p\right) \\
&= 1
\end{aligned}$$

Substitute $z = \mathrm{e}^{\mathrm{j}\omega}$, $W_K = \mathrm{e}^{-\mathrm{j}2\pi/K}$ into (12.90), we have

$$\sum_{k=0}^{K-1} H\left(\mathrm{e}^{\mathrm{j}\omega} \cdot \mathrm{e}^{-\mathrm{j}\frac{2\pi}{K}k}\right) = \sum_{k=0}^{K-1} H\left(\mathrm{e}^{\mathrm{j}(\omega - \frac{2k\pi}{K})}\right) = 1$$

11. Solution:
Step I:
Suppose $I = I_1 I_2$. A single interpolation of I can be implemented by a two stage process of I_1 and I_2, that is

$$x[n] \longrightarrow \boxed{\uparrow I_1} \longrightarrow \boxed{H_1(z)} \longrightarrow \boxed{\uparrow I_2} \longrightarrow \boxed{H_2(z)} \longrightarrow x_{\mathrm{I}}[n]$$

where $\omega_c^{(1)} = \pi/I_1$ of $H_1(z)$, and $\omega_c^{(2)} = \pi/I_2$ of $H_2(z)$.
Step II:

Interchange the order of lowpass filters $H_1(z)$ and upsampler $\uparrow I_2$ by the multirate identity for the sampling. We obtain that

$$x[n] \longrightarrow \boxed{\uparrow I_1} \longrightarrow \boxed{\uparrow I_2} \longrightarrow \boxed{H_1(z^{I_2})} \longrightarrow \boxed{H_2(z)} \longrightarrow x_{\mathrm{I}}[n]$$

Step III:
Combining the two upsamplers and two filters yields the equivalent single-stage interpolator as follows. The equivalent single-stage interpolator has a factor of $I = I_1 I_2$ and a lowpass filter with system function $H(z) = H_1(z^{I_2})H_2(z)$.

$$x[n] \longrightarrow \boxed{\uparrow (I_1 I_2)} \longrightarrow \boxed{H(z)} \longrightarrow x_{\mathrm{I}}[n]$$

12. See plot below

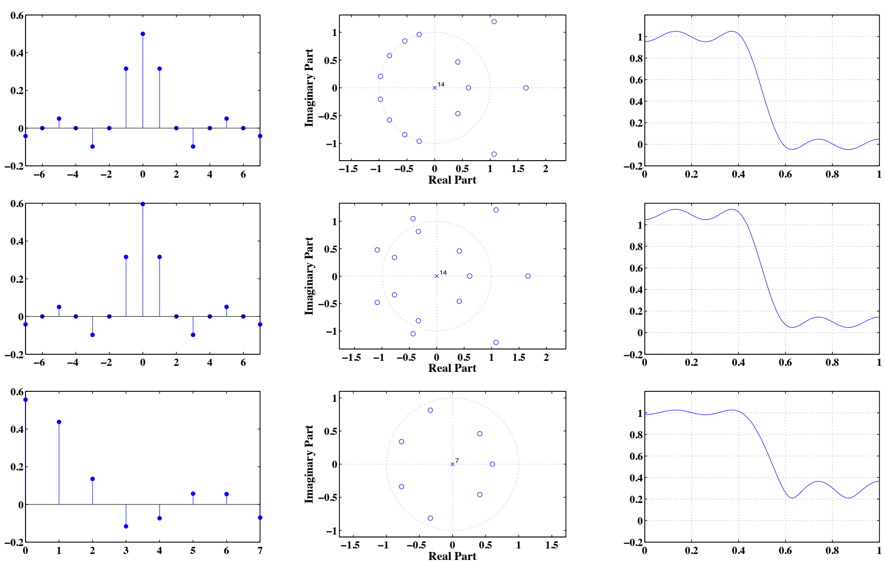

FIGURE 12.21: Illustration of the modified design process for conjugate quadrature filter banks using the Parks-McClellan algorithm.

13. (a) $I = 2$, and $D = 3$.
    (b) tba
    (c) tba
    (d) tba

14. Solution:
    Complexity:
    (1) The upsamplers of the two structure are of the same complexity.
    (2) $H(z)$ and $H(z^I)$ has same number of nonzero coefficients if we omit the multiplications by zero which is trivial.
    Hence, the two structures have the same complexity.

    Rate:
    (1) Figure 12.25(a) has I time higher rate after upsampler before subband filter and adders.
    (2) Figure 12.25(b) only takes higher rate before adders.

15. tba

16. tba

17. Solution:

$$H(z) = \sum_{m=0}^{3} z^{-m} P_m(z^4)$$

$$P_m(z) = \sum_{n=0}^{\infty} p_k[4n+m] z^{-n}$$

$$H_k(z) = \sum_{m=0}^{3} z^{-m} W_4^{-km} P_m(z^4 W_4^{4k}) = \sum_{m=0}^{3} z^{-m} W_4^{-km} P_m(z^4)$$

$$\begin{bmatrix} H_0(z) \\ H_1(z) \\ H_2(z) \\ H_3(z) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_4^1 & W_4^2 & W_4^3 \\ 1 & W_4^2 & W_4^4 & W_4^6 \\ 1 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} \begin{bmatrix} P_0(z^4) \\ z^{-1}P_1(z^4) \\ z^{-2}P_2(z^4) \\ z^{-3}P_3(z^4) \end{bmatrix}$$

    The plot will be available shortly.

18. tba