CHAPTER 13

Random Signals

Tutorial Problems

1. (a) Solution:
   $P\{R = 1, B = 1, G = 1\} = \frac{1}{6\times6\times6} = \frac{1}{216}$

   (b) Solution:
   We can suppose the observed one is green, that is $P\{R = 1, B = 1|G = 1\} = \frac{1}{6\times6} = \frac{1}{36}$

   (c) Solution:
   $P\{G = 1, B = 1|R = 1\} = \frac{1}{6\times6} = \frac{1}{36}$

   (d) Solution: tba

2. MATLAB script:

```
% P1302: Figure 13.4 reproduction
clc; close all
load f16.mat
N = 20000; Fs = 19.98e3;
x = f16(1:N);
clear f16
[xo px] = epdf(x,50);
%% Plot
hfa = figconfg('P1302a','long');
plot((1:N)/Fs,x)
xlim([1 N]/Fs)
xlabel('Time (sec)','fontsize',LFS)
ylabel('Amplitude','fontsize',LFS)
title(['N = ',num2str(N),' samples'],'fontsize',TFS)
hfb = figconfg('P1302b','long');
```

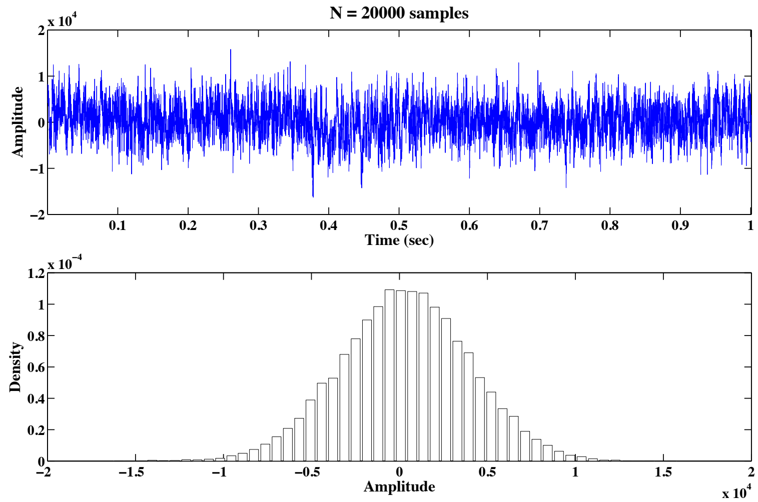

FIGURE 13.1: Waveform of F-16 noise recorded at co-pilot's seat and its empirical pdf using function `epdf` with 50 bins.

```
bar(xo,px,'w')
ylim([0 1.2e-4])
xlabel('Amplitude','fontsize',LFS)
ylabel('Density','fontsize',LFS)
```

3. (a) See plot below.

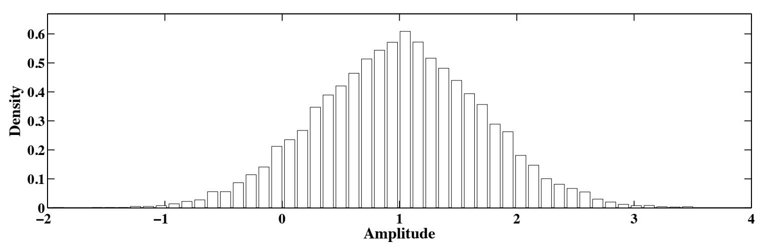

FIGURE 13.2: Plot of empirical pdf of X in part (a) using function `epdf` with 50 bins.

(b) See plot below.

(c) See plot below.

(d) tba.

MATLAB script:

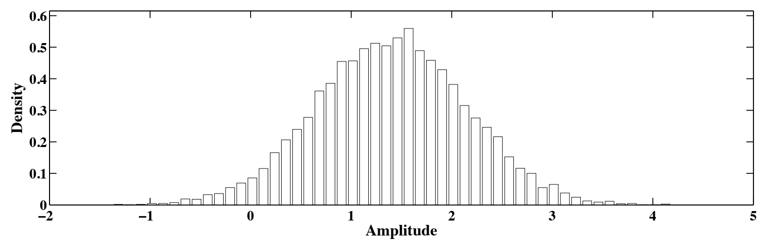

FIGURE 13.3: Plot of empirical pdf of X in part (b) using function `epdf` with 50 bins.

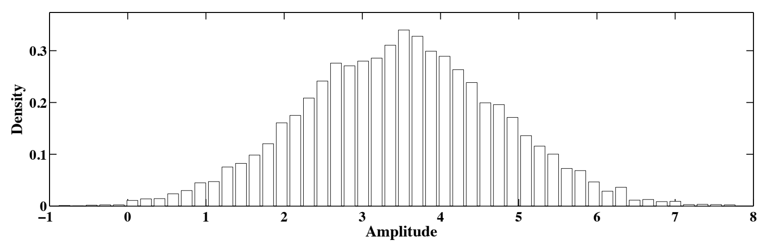

FIGURE 13.4: Plot of empirical pdf of X in part (c) using function `epdf` with 50 bins.

```
% P1303: Mixture of two Gaussians
clc; close all
N = 10000;
% a1 = 0.5; a2 = 0.5; % part a
a1 = 0.3; a2 = 0.7; % part b & c
mu1 = 0; sigma1 = 1;
% mu2 = 2; sigma2 = 1; % part a & b
mu2 = 5; sigma2 = sqrt(3); % part c
fx = a1*(sigma1*randn(1,N)+mu1) + a2*(sigma2*randn(1,N)+mu2);
[xo px] = epdf(fx,50);
disp('The mean is:')
disp(mean(fx))
disp('The standard deviation is:')
disp(std(fx))
%% Plot
hfa = figconfg('P1303a','long');
bar(xo,px,'w')
ylim([0 1.1*max(px)])
```

```
xlabel('Amplitude','fontsize',LFS)
ylabel('Density','fontsize',LFS)
```

4. (a) Proof: If we suppose $a > 0$, we have

$$F_Y(y) = P\{Y \le y\} = P\{aX+b \le y\} = P\{x \le \frac{y-b}{a}\} = F_X\left(\frac{y-b}{a}\right)$$

Take the derivative of cdf of Y, we have

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \frac{d}{dy}F_X\left(\frac{y-b}{a}\right) = \frac{1}{a}f_X\left(\frac{y-b}{a}\right)$$

We can similar prove when $a < 0$,

$$f_Y(y) = -\frac{1}{a}f_X\left(\frac{y-b}{a}\right)$$

Combination of the two cases above yields,

$$f_Y(y) = \frac{1}{|a|}f_X\left(\frac{y-b}{a}\right)$$

(b) Proof:
The affine transformation of a Gaussian distributed rv is still Gaussian. We will prove the mean and variance rules under such affine transformation.

$$E[Y] = E[aX+b] = aE[X] + b = a\mu + b$$

$$\begin{aligned} E[Y^2] &= E[(aX+b)^2] = a^2E[X^2] + 2abE[X] + b^2 \\ &= a^2(\mu^2+\sigma^2) + 2ab\mu + b^2 = a^2\sigma^2 + (a\mu+b)^2 \end{aligned}$$

$$\sigma_y^2 = E[Y^2] - E^2[Y] = a^2\sigma^2$$

(c) See script below.

(d) See plots below.

(e) Comments: The numerical computation of mean and variance of Y can be verified by theoretical results. See the script below for details.

MATLAB script:

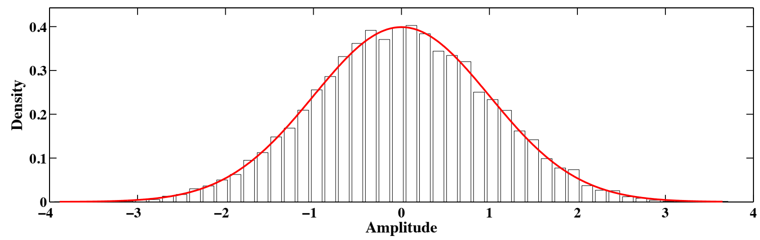

FIGURE 13.5: Plots of empirical and theoretical pdf of X.

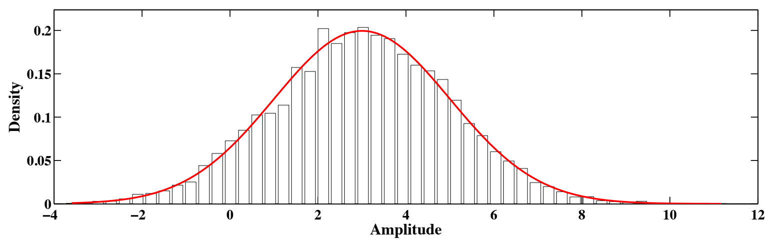

FIGURE 13.6: Plots of empirical and theoretical pdf of Y.

```
% P1304: Affine transformation of Gaussian
clc; close all
N = 10000; a = 2; b = 3;
x = randn(1,N);
y = a*x + b;
[xo px] = epdf(x,50);
[yo py] = epdf(y,50);
disp('The mean is:')
disp(mean(y))
disp('The variance is:')
disp(var(y))
xp = linspace(min(xo),max(xo),1000);
px_ref = pdf('normal',xp,0,1);
yp = linspace(min(yo),max(yo),1000);
py_ref = pdf('normal',yp,b,a);
%% Plot
hfa = figconfg('P1304a','long');
bar(xo,px,'w'); hold on
plot(xp,px_ref,'r','linewidth',2)
ylim([0 1.1*max(px)])
```

```
xlabel('Amplitude','fontsize',LFS)
ylabel('Density','fontsize',LFS)

hfb = figconfg('P1304b','long');
bar(yo,py,'w'); hold on
plot(yp,py_ref,'r','linewidth',2)
ylim([0 1.1*max(py)])
xlabel('Amplitude','fontsize',LFS)
ylabel('Density','fontsize',LFS)
```

5. (a) See plot below.

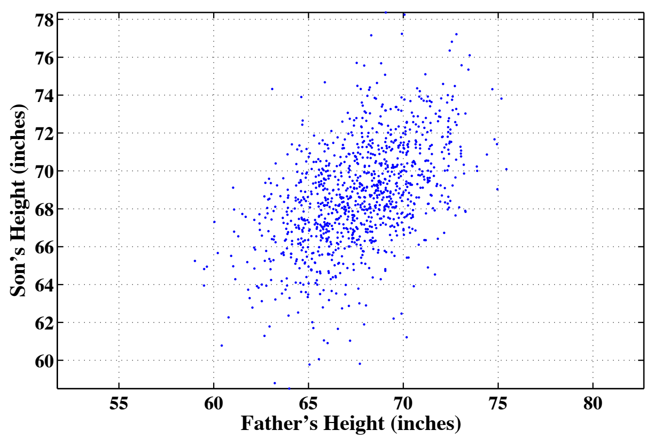

FIGURE 13.7: Scatter plot of the data between father and son heights.

(b) See plot below.

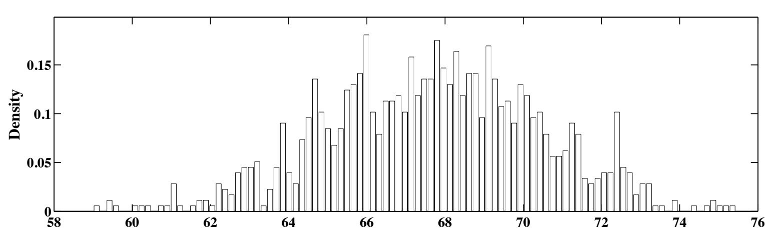

FIGURE 13.8: Normalized bar-graph for the father-height data.

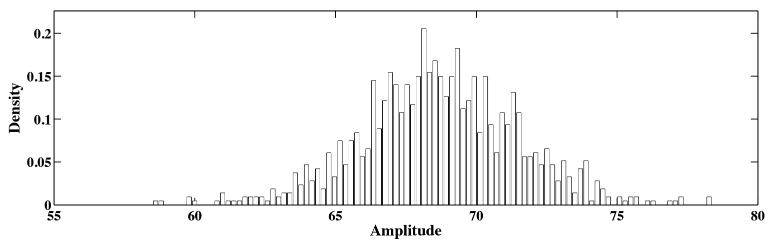

FIGURE 13.9: Normalized bar-graph for the son-height data.

(c) See plot below.

(d) See plot below.

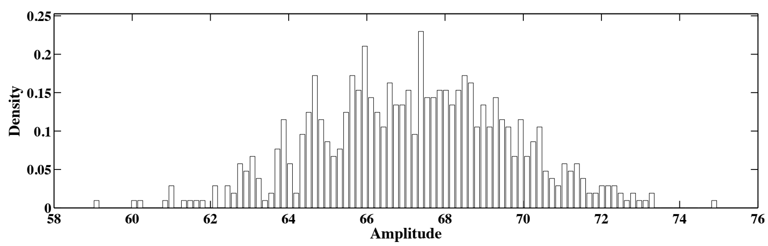

FIGURE 13.10: Normalized bar-graph for the conditional father-height data when son's heights are between 65 inches and 70 inches.

(e) See plot below.

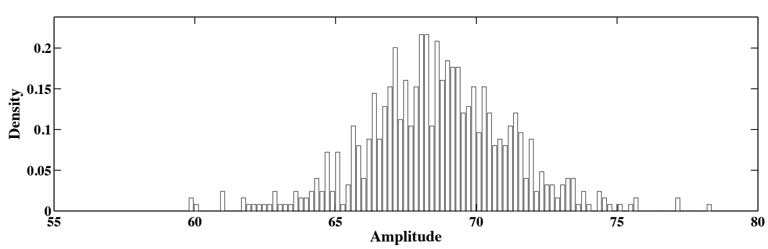

FIGURE 13.11: Normalized bar-graph for the conditional father-height data when father's heights are between 65 inches and 70 inches.

MATLAB script:

```
% P1305: fatherson
clc; close all
```

```
xx = load('fatherson.txt');
f = xx(:,1); s = xx(:,2);
%% Part a
hfa = figconfg('P1305a');
plot(f,s,'.'); axis equal; grid
xlabel('Father''s Height (inches)','fontsize',LFS)
ylabel('Son''s Height (inches)','fontsize',LFS)
%% Part b
[fo pf] = epdf(f,100);
hfb = figconfg('P1305b','long');
bar(fo,pf,'w');
ylim([0 1.1*max(pf)])
xlabel('Amplitude','fontsize',LFS)
ylabel('Density','fontsize',LFS)
%% Part c
[so ps] = epdf(s,100);
hfbc = figconfg('P1305c','long');
bar(so,ps,'w');
ylim([0 1.1*max(ps)])
xlabel('Amplitude','fontsize',LFS)
ylabel('Density','fontsize',LFS)
%% Part d
ind = (s >= 65 & s <= 70);
fc = f(ind);
[fco pfc] = epdf(fc,100);
hfd = figconfg('P1305d','long');
bar(fco,pfc,'w');
ylim([0 1.1*max(pfc)])
xlabel('Amplitude','fontsize',LFS)
ylabel('Density','fontsize',LFS)
%% Part e
ind = (f >= 65 & f <= 70);
sc = s(ind);
[sco psc] = epdf(sc,100);
hfe = figconfg('P1305e','long');
bar(sco,psc,'w');
ylim([0 1.1*max(psc)])
xlabel('Amplitude','fontsize',LFS)
ylabel('Density','fontsize',LFS)
```

6. (a) Solution:

$$f(y) = \int_{-\infty}^{\infty} f(x,y)dx$$

if $0 \leq y \leq 1$, the general integral is

$$f(y) = \int_{1-y}^{1+y} \frac{1}{2} dy = y$$

if $1 \leq y \leq 2$, the general integral is

$$f(y) = \int_{y-1}^{3-y} \frac{1}{2} dy = 2 - y$$

Combination of the two cases above yields that

$$f(y) = \begin{cases} y, & 0 \leq y \leq 1 \\ 2-y, & 1 \leq y \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

Since, the symmetric behavior of the two random variables, we can conclude that

$$f(x) = \begin{cases} x, & 0 \leq x \leq 1 \\ 2-x, & 1 \leq x \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

(b) Proof:
The expectations of X and Y are equal, and we will calculate one of them,

$$E[x] = \int_{-\infty}^{\infty} xf(x)dx = \int_0^1 x^2 dx + \int_1^2 (2x - x^2)dx = 1$$

We then calculate the expectation of XY,

$$\begin{aligned} E[XY] &= \int \int xyf(x,y)dxdy = \int_0^1 y \left(\int_{1-y}^{1+y} \frac{x}{2} dx \right) dy + \int_1^2 y \left(\int_{y-1}^{3-y} \frac{x}{2} dx \right) dy \\ &= \int_0^1 y^2 dy + \int_1^2 (2y - y^2) dy = 1 \end{aligned}$$

Hence, we can make the conclusion that X and Y are uncorrelated since $E[XY] = E[X]E[Y]$.

(c) Proof:
Since we have $f(x,y) \neq f(x)f(y)$, that implies X and Y are not independent.

7. (a) See plot below.

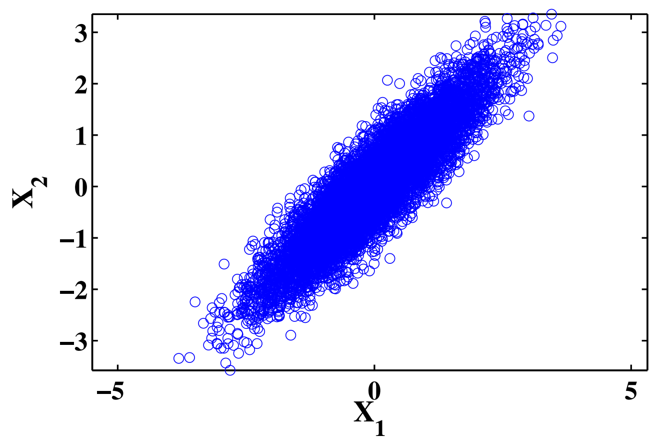

FIGURE 13.12: Scatter plot for $\rho = 0.9$.

(b) See plot below.

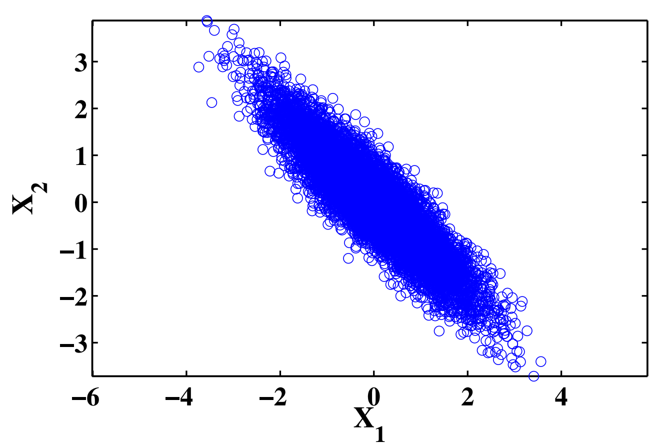

FIGURE 13.13: Scatter plot for $\rho = -0.9$.

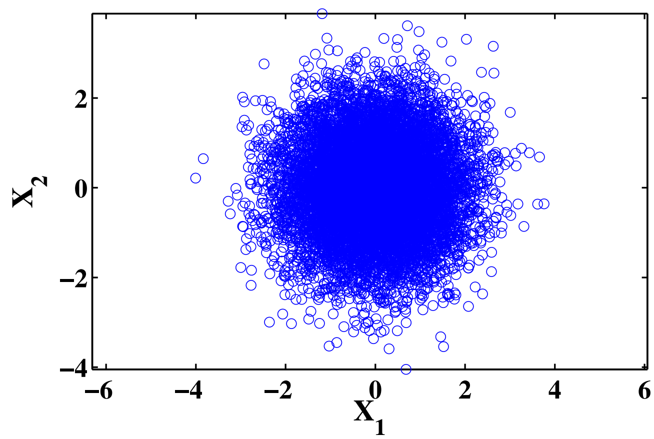

FIGURE 13.14: Scatter plot for $\rho = 0$.

(c) See plot below.

(d) tba.

MATLAB script:

```
% P1307: Investigation of correlation coefficient
clc; close all
% rho = 0.9; % part a
% rho = -0.9; % part b
rho = 0; % part c
N = 10000;
C = [1 rho;rho 1];
L = chol(C)';
x = L*randn(2,N);
%% Plot
hfa = figconfg('P1307a','small');
scatter(x(1,:),x(2,:)); axis equal; box on
xlabel('X_1','fontsize',LFS)
ylabel('X_2','fontsize',LFS)
```

8. (a) Solution:
Integrate the pdf function with respect to x_1, x_2, and x_3, we have

$$\iiint f(x_1, x_2, x_3) dx_1 dx_2 dx_3 = \int_0^1 x_1 \left[\int_0^{x_1} x_2 \left(\int_0^{x_2} x_3 dx_3 \right) dx_2 \right] dx_1 = \frac{1}{48}$$

Since the integral of a valid pdf equals one, we have $K = 48$.
The mean of each random variable x_i, $i = 1, 2, 3$, can be computed as

$$E[x_i] = \iiint Kx_i f(x_1, x_2, x_3) dx_1 dx_2 dx_3$$

Using the codes at the end of this problem, we compute that

$$\boldsymbol{\mu} = \begin{bmatrix} \frac{6}{7} & \frac{24}{35} & \frac{16}{35} \end{bmatrix}^T$$

(b) Solution:
The i, jth element r_{ij} of autocorrelation matrix $\boldsymbol{R}$ can be computed as

$$E[x_i x_j] = \iiint Kx_i x_j f(x_1, x_2, x_3) dx_1 dx_2 dx_3$$

Using the codes at the end of this problem, we compute that

$$\boldsymbol{R} = \begin{bmatrix} \frac{3}{4} & \frac{3}{5} & \frac{2}{5} \\ \frac{3}{5} & \frac{1}{2} & \frac{1}{3} \\ \frac{2}{5} & \frac{1}{3} & \frac{1}{4} \end{bmatrix}$$

(c) Solution:
The i, jth element c_{ij} of autocorrelation matrix $\boldsymbol{C}$ can be computed as

$$E[(x_i-\mu_i)(x_j-\mu_j)] = \iiint K(x_i-\mu_i)(x_j-\mu_j) f(x_1, x_2, x_3) dx_1 dx_2 dx_3$$

Using the codes at the end of this problem, we compute that

$$\boldsymbol{C} = \begin{bmatrix} \frac{3}{196} & \frac{3}{245} & \frac{2}{245} \\ \frac{3}{245} & \frac{73}{2450} & \frac{73}{3675} \\ \frac{2}{245} & \frac{73}{3675} & \frac{21}{4900} \end{bmatrix}$$

MATLAB script:

```
% P1308: Compute mean vector and autocorrelation matrix and autocovariance
% matirx
clc; close all
syms x1 x2 x3
disp(int(int(int(x1*x2*x3,x3,0,x2),x2,0,x1),x1,0,1))
K = 48;
%% Mean
disp('X_1 mean is: ')
disp(int(int(int(K*x1^2*x2*x3,x3,0,x2),x2,0,x1),x1,0,1))
disp('X_2 mean is: ')
disp(int(int(int(K*x1*x2^2*x3,x3,0,x2),x2,0,x1),x1,0,1))
disp('X_3 mean is: ')
disp(int(int(int(K*x1*x2*x3^2,x3,0,x2),x2,0,x1),x1,0,1))
%% Autocorrelation
disp('R_11 mean is: ')
disp(int(int(int(K*x1^3*x2*x3,x3,0,x2),x2,0,x1),x1,0,1))
disp('R_12 mean is: ')
disp(int(int(int(K*x1^2*x2^2*x3,x3,0,x2),x2,0,x1),x1,0,1))
disp('R_13 mean is: ')
disp(int(int(int(K*x1^2*x2*x3^2,x3,0,x2),x2,0,x1),x1,0,1))
disp('R_21 mean is: ')
disp(int(int(int(K*x1^2*x2^2*x3,x3,0,x2),x2,0,x1),x1,0,1))
disp('R_22 mean is: ')
disp(int(int(int(K*x1*x2^3*x3,x3,0,x2),x2,0,x1),x1,0,1))
disp('R_23 mean is: ')
disp(int(int(int(K*x1*x2^2*x3^2,x3,0,x2),x2,0,x1),x1,0,1))
disp('R_31 mean is: ')
disp(int(int(int(K*x1^2*x2*x3^2,x3,0,x2),x2,0,x1),x1,0,1))
disp('R_32 mean is: ')
disp(int(int(int(K*x1*x2^2*x3^2,x3,0,x2),x2,0,x1),x1,0,1))
disp('R_33 mean is: ')
disp(int(int(int(K*x1*x2*x3^3,x3,0,x2),x2,0,x1),x1,0,1))
%% Autocovariance
disp('C_11 mean is: ')
disp(int(int(int(K*x1*x2*x3*(x1-6/7)^2,x3,0,x2),x2,0,x1),x1,0,1))
disp('C_12 mean is: ')
disp(int(int(int(K*x1*x2*x3*(x1-6/7)*(x2-24/35),x3,0,x2),x2,0,x1),x1,0,1))
disp('C_13 mean is: ')
disp(int(int(int(K*x1*x2*x3*(x1-6/7)*(x3-16/35),x3,0,x2),x2,0,x1),x1,0,1))
disp('C_21 mean is: ')
```

```
disp(int(int(int(K*x1*x2*x3*(x1-6/7)*(x2-24/35),x3,0,x2),x2,0,x1),x1,0,1))
disp('C_22 mean is: ')
disp(int(int(int(K*x1*x2*x3*(x2-24/35)^2,x3,0,x2),x2,0,x1),x1,0,1))
disp('C_23 mean is: ')
disp(int(int(int(K*x1*x2*x3*(x2-24/35)*(x3-16/35),x3,0,x2),x2,0,x1),x1,0,1))
disp('C_31 mean is: ')
disp(int(int(int(K*x1*x2*x3*(x1-6/7)*(x3-16/35),x3,0,x2),x2,0,x1),x1,0,1))
disp('C_32 mean is: ')
disp(int(int(int(K*x1*x2*x3*(x2-24/35)*(x3-16/35),x3,0,x2),x2,0,x1),x1,0,1))
disp('C_33 mean is: ')
disp(int(int(int(K*x1*x2*x3*(x3-16/35)^2,x3,0,x2),x2,0,x1),x1,0,1))
```

9. (a) Solution:

$$\begin{aligned}\mu_x[n] &= E\left[x[n]\right] = E[A\cos(\Omega n + \Theta)] \\ &= E[A\cos(\Omega n)\cos\Theta - A\sin(\Omega n)\sin\Theta] \\ &= E[A]E[\cos(\Omega n)]E[\cos\Theta] - E[A]E[\sin(\Omega n)]E[\sin\Theta] \\ &= 0\end{aligned}$$

(b) Solution:

$$\begin{aligned}c_X[m,n] &= r_X[m,n] = E\left[A\cos(\Omega m + \Theta)A\cos(\Omega n + \Theta)\right] \\ &= E\left[\frac{A^2}{2}\left\{\cos[\Omega(m+n)+2\Theta] + \cos[\Omega(m-n)]\right\}\right] \\ &= \frac{1}{2}E[A^2]\left\{E[\cos\Omega(m+n)]E[\cos 2\Theta] - E[\sin\Omega(m+n)]E[\sin 2\Theta]\right. \\ &\quad \left. + E[\cos\Omega(m-n)]\right\} \\ &= \frac{1}{2}E[A^2]E[\cos\Omega(m-n)]\end{aligned}$$

$$E[A^2] = \int_0^1 a^2 da = \frac{1}{3}$$

$$E[\cos(m-n)\Omega] = \frac{1}{2}\cos 10(m-n) + \frac{1}{2}\cos 20(m-n)$$

Hence, the ACVS $c_X[m,n]$ is

$$c_X[m,n] = \frac{1}{12}\cos 10(m-n) + \frac{1}{12}\cos 20(m-n)$$

(c) Comment:
$x[n]$ is wide-sense stationary, since its mean is constant and its second order statistic is only dependent on the lag.

10. (a) Solution:
We first note that for an exponential distributed random variable with parameter λ, that is

$$f(x) = \lambda e^{-\lambda x},\ x \geq 0$$

Its first order and second order statistics are

$$E[x] = \frac{1}{\lambda}, \quad Var[x] = \frac{1}{\lambda^2}$$

$$\begin{aligned}\mu_y[n] &= E\left[y[n]\right] = E\left[x[n] + x[n-1] + v[n]\right] \\ &= E\left[x[n]\right] + E\left[x[n-1]\right] + E\left[v[n]\right] = 1 + 1 + \frac{1}{2} = \frac{5}{2}\end{aligned}$$

(b) Solution:
Without loss of generality, we can first suppose $m \geq n$. The ACRS $r_y[m,n]$ is

$$\begin{aligned}r_y[m,n] &= E[y[m]y[n]] = E[(x[m] + x[m-1] + v[m])(x[n] + x[n-1] + v[n])] \\ &= E[x[m]x[n]] + E[x[m-1]x[n]] + E[v[m]]E[x[n]] + E[x[m]x[n-1]] \\ &\quad + E[x[m-1]x[n-1]] + E[v[m]]E[x[n-1]] + E[x[m]]E[v[n]] \\ &\quad + E[x[m-1]]E[v[n]] + E[v[m]v[n]]\end{aligned}$$

If $m - 1 > n$, we have

$$r_y[m,n] = 1 + 1 + \frac{1}{2} + 1 + 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{4} = \frac{25}{4}$$

If $m - 1 = n$, we have

$$r_y[m,n] = 1 + (1 + 1) + \frac{1}{2} + 1 + 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{4} = \frac{29}{4}$$

If $m = n$, we have

$$r_y[m,n] = (1+1) + 1 + \frac{1}{2} + 1 + (1+1) + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + (\frac{1}{4} + \frac{1}{4}) = \frac{34}{4}$$

Combine all the cases above and also include $m < n$, we conclude that

$$r_y[m,n] = \frac{25}{4} + \delta[|m-n| - 1] + \frac{9}{4}\delta[m-n]$$

(c) Solution:
We first note a theorem that ”The density of the sum of two independent random variables is the convolution of the two pdfs.” Hence, the marginal density of $f(y)$ can be obtained by

$$f_y(x) = f_x(x) * f_x(x) * f_v(x)$$

Define $z[n] = x[n] + x[n-1]$, hence the density $f_z(x)$ is

$$f_z(z) = f_x(z)*f_x(z) = \int_0^z \mathrm{e}^{-(z-x)} \cdot \mathrm{e}^{-x} dx = \int_0^z \mathrm{e}^{-z} dx = z\mathrm{e}^{-z}, \quad z \geq 0$$

The density of $f_y(x)$ is

$$\begin{aligned} f_y(y) = f_z(y) * f_v(y) &= \int f_z(y-z) f_v(z) dz = \int_0^y (y-z)\mathrm{e}^{-(y-z)} \cdot 2\mathrm{e}^{-2z} dz \\ &= \int_0^y 2(y-z)\mathrm{e}^{-(y+z)} dz = 2y\mathrm{e}^{-y} - 2\mathrm{e}^{-y} + 2\mathrm{e}^{-2y}, \quad y \geq 0 \end{aligned}$$

That is

$$f_y(y) = \begin{cases} 2y\mathrm{e}^{-y} - 2\mathrm{e}^{-y} + 2\mathrm{e}^{-2y}, & y \geq 0 \\ 0, & y < 0 \end{cases}$$

11. Proof:
The output $y[n]$ is defined as

$$y[n] = h[n] * x[n] = \sum_{m=0}^{M} h[m] x[n-m]$$

Hence, the average power is

$$\begin{aligned} E[Y^2[n]] &= E\left[\sum_{m=0}^{M} h[m]x[n-m] \sum_{k=0}^{M} h[k]x[n-k]\right] \\ &= E\left[\sum_{m=0}^{M}\sum_{k=0}^{M} h[k]h[m]x[n-k]x[n-m]\right] \\ &= \sum_{m=0}^{M}\sum_{k=0}^{M} h[k]h[m]E[x[n-k]x[n-m]] \\ &= \sum_{m=0}^{M}\sum_{k=0}^{M} h[k]h[m]r_x[m-k] \\ &= \boldsymbol{h}^T \boldsymbol{R}_x \boldsymbol{h} \quad \text{(in matrix form)} \end{aligned}$$

The ACRS of $y[n]$ is

$$r_y[\ell] = h[\ell] * h[-\ell] * r_x[\ell]$$

and

$$S_{yy}(\omega) = |H(\mathrm{e}^{\mathrm{j}\omega})|^2 S_{xx}(\omega)$$

Use the inverse DTFT relation,

$$\begin{aligned} E[Y^2[n]] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} S_{yy}(\omega)\cdot \mathrm{e}^{\mathrm{j}\omega\cdot 0} d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_{yy}(\omega) d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|H\left(\mathrm{e}^{\mathrm{j}\omega}\right)\right|^2 S_{xx}(\omega) d\omega \end{aligned}$$

12. Proof:
We firstly prove the covariance between $x[m]$ and $y[n]$, $\ell = m - n$, that is

$$c_{xy}[\ell] = \sum_{k=-\infty}^{\infty} h[k] c_{xx}[\ell + k] = h[-\ell] * c_{xx}[\ell]$$

$$\begin{aligned} c_{xy}[\ell] &= E[(x[m] - E(x[m]))(y[n] - E(y[n]))] \\ &= E\left[(x[m] - E(x[m])) \sum_{k=-\infty}^{\infty} h[k](x[n-k] - E(x[n-k]))\right] \\ &= \sum_{k=-\infty}^{\infty} h[k] \cdot E[(x[m] - E(x[m]))(x[n-k] - E(x[n-k]))] \\ &= \sum_{k=-\infty}^{\infty} h[k] c_{xx}[(m-n) + k] \\ &= \sum_{k=-\infty}^{\infty} h[k] c_{xx}[\ell + k] = h[-\ell] * c_{xx}[\ell] \end{aligned}$$

We secondly prove the covariance between $y[m]$ and $x[n]$, $\ell = m - n$, that is

$$c_{yx}[\ell] = \sum_{k=-\infty}^{\infty} h[k] c_{xx}[\ell - k] = h[\ell] * c_{xx}[\ell]$$

$$
\begin{aligned}
c_{yx}[\ell] &= E\left[(y[m]-E(y[m]))(x[n]-E(x[n]))\right] \\
&= E\left[\sum_{k=-\infty}^{\infty} h[k](x[m-k]-E(x[m-k]))(x[n]-E(x[n]))\right] \\
&= \sum_{k=-\infty}^{\infty} h[k]\cdot E\left[(x[m-k]-E(x[m-k]))(x[n]-E(x[n]))\right] \\
&= \sum_{k=-\infty}^{\infty} h[k]\cdot c_{xx}[m-k-n] \\
&= \sum_{k=-\infty}^{\infty} h[k]c_{xx}[\ell-k] = h[\ell]*c_{xx}[\ell]
\end{aligned}
$$

We thirdly prove the covariance between $y[m]$ and $y[n]$, $\ell = m-n$, that is

$$
c_{yy}[\ell] = \sum_{k=-\infty}^{\infty} h[k]c_{xy}[\ell-k] = h[\ell]*c_{xy}[\ell]
$$

$$
\begin{aligned}
c_{yy}[\ell] &= E\left[(y[m]-E(y[m]))(y[n]-E(y[n]))\right] \\
&= \sum_{k=-\infty}^{\infty} h[k]\cdot E\left[(x[m-k]-E(x[m-k]))(y[n]-E(y[n]))\right] \\
&= \sum_{k=-\infty}^{\infty} h[k]\cdot c_{xy}[m-k-n] \\
&= \sum_{k=-\infty}^{\infty} h[k]c_{xy}[\ell-k] = h[\ell]*c_{xy}[\ell]
\end{aligned}
$$

Finally, the fourth expression can be easily proved by previous results, that is

$$
\begin{aligned}
c_{yy}[\ell] &= h[\ell]*c_{xy}[\ell] = h[\ell]*(h[-\ell]*c_{xx}[\ell]) = (h[\ell]*(h[-\ell)]*c_{xx}[\ell] \\
&= r_{hh}[\ell]*c_{xx}[\ell] = \sum_{m=-\infty}^{\infty} r_{hh}[m]c_{xx}[\ell-m]
\end{aligned}
$$

13. (a) Solution:

$$
\begin{aligned}
J(a,b) &= E[(Y-aX-b)(Y-aX-b)] \\
&= E[Y^2] + a^2E[X^2] + b^2 - 2aE[XY] - 2bE[Y] + 2abE[X]
\end{aligned}
$$

(b) Solution:

$$
\frac{\partial J(a,b)}{\partial a} = 2aE[X^2] - 2E[XY] + 2bE[X] = 0 \qquad \text{(P13A)}
$$

$$
\frac{\partial J(a,b)}{\partial b} = 2b - 2E[Y] + 2aE[X] = 0 \qquad \text{(P13B)}
$$

Solving Eq. (P13B) for b, we result in Eq. (13.58), that is

$$
b = E[Y] - aE[X]
$$

Plug the above equation into Eq. (P13A), and solve for a, we have

$$
a = \frac{E[XY] - E[X]E[Y]}{\sigma_x^2} = \frac{c_{xy}}{\sigma_x^2} = \rho_{xy}\frac{\sigma_y}{\sigma_x}
$$

which is exactly Eq. (13.62).

14. Proof:
Using the results from Problem 13-12, we have

$$
c_{xy}[\ell] = h[-\ell] * c_{xx}[\ell]
$$

Apply z-transform to both sides of the equation above will result in

$$
C_{xy}(z) = H(1/z)C_{xx}(z)
$$

$$
c_{yx}[\ell] = h[\ell] * c_{xx}[\ell]
$$

Apply z-transform to both sides of the equation above will result in

$$
C_{yx}(z) = H(z)C_{xx}(z)
$$

$$
c_{yy}[\ell] = h[\ell] * h[-\ell] * c_{xx}[\ell]
$$

Apply z-transform to both sides of the equation above will result in

$$
C_{yy}(z) = H(z)H(1/z)C_{xx}(z)
$$

15. Solution:

$$R_{yx}(z) = H(z)R_{xx}(z)$$

which implies

$$H(z) = \frac{R_{xx}(z)}{R_{yx}(z)}$$

16. Solution:

$$r_{yy}[0]a_1 + r_{yy}[1]a_2 = -r_{yy}[1] \tag{13.147}$$

$$r_{yy}[1]a_1 + r_{yy}[0]a_2 = -r_{yy}[2] \tag{13.148}$$

$$\sigma_x^2 = r_{yy}[0] + a_1 r_{yy}[1] + a_2 r_{yy}[2] \tag{13.150}$$

Plug $a_1 = -5/6$, $a_2 = 1/6$, and $\sigma_x^2 = 2$ into the three equations above and solve for $r_{yy}[0]$, $r_{yy}[1]$, and $r_{yy}[2]$, we have

$$r_{yy}[0] = \frac{21}{5}, \quad r_{yy}[1] = 3, \quad r_{yy}[2] = \frac{9}{5}$$

We can also conclude that $r_{yy}[\ell] = 0$, for $\ell > 2$.

17. (a) Proof:

$$\begin{aligned} E[x[n]] &= E\left[\sum_{k=1}^{p} A_k \cos(\omega_k n + \phi_k)\right] \\ &= \sum_{k=1}^{p} A_k E[\cos(\omega_k n + \phi_k)] \end{aligned}$$

Since ϕ_k is uniformly distributed in the interval $(0, 2\pi)$, we have

$$E[\cos(\phi_k)] = E[\sin\phi_k] = 0$$

which implies

$$E[\cos(\omega_k n + \phi_k)] = E[\cos(\omega_k n)\cos(\phi_k) - \sin(\omega_k n)\sin(\phi_k)] = 0$$

Hence, we conclude that

$$E[x[n]] = 0$$

(b) Proof:
Since the random process has zero mean, its autocorrelation sequence equals its autocovariance sequence, that is $r_{xx}[\ell] = c_{xx}[\ell]$. For the rest of this proof, we only verify the expression for $r_{xx}[\ell]$.

$$\begin{aligned}
r_{xx}[\ell] = E[x[m]x[n]] &= E\left[\sum_{k=1}^{p} A_k \cos(\omega_k m + \phi_k) \sum_{q=1}^{p} A_q \cos(\omega_q n + \phi_q)\right] \\
&= \sum_{k=1}^{p}\sum_{q=1}^{p} A_q A_k E[\cos(\omega_k m + \phi_k)\cos(\omega_q n + \phi_q)] \\
&= \frac{1}{2}\sum_{k=1}^{p}\sum_{q=1}^{p} A_q A_k E[\cos(\omega_k m + \phi_k + \omega_q n + \phi_q) + \cos(\omega_k m + \phi_k - \omega_q n - \phi_q)]
\end{aligned}$$

Since we have

$$E[\cos(\omega_k m + \phi_k + \omega_q n + \phi_q)] = 0$$

$$E[\cos(\omega_k m + \phi_k - \omega_q n - \phi_q)] = 0, \quad k \neq q$$

The double summation can be simplified by removing all zero terms and written as

$$\begin{aligned}
r_{xx}[\ell] &= \frac{1}{2}\sum_{k=1}^{p} A_k^2 E[\cos(\omega_k m - \omega_k n)] \\
&= \frac{1}{2}\sum_{k=1}^{p} A_k^2 \cos\omega_k \ell, \quad \text{where} \quad \ell = m - n
\end{aligned}$$