CHAPTER 5

Transform Analysis of LTI Systems

Tutorial Problems

1. (a) Solution:

$$y[n] = a\,y[n-1] + b\,x[n], \quad -1 < a < 1$$

$$H\big(e^{j\omega}\big) = \frac{b}{1 - a e^{-j\omega}}$$

$$x[n] = 3\cos(\pi n/2) = \frac{3}{2}\left(e^{j\frac{\pi}{2}n} + e^{-j\frac{\pi}{2}n}\right)$$

$$y[n] = \frac{3}{2}\left(e^{j\frac{\pi}{2}n} H\big(e^{j\frac{\pi}{2}}\big) + e^{-j\frac{\pi}{2}n} H\big(e^{j-\frac{\pi}{2}}\big)\right) = \frac{6}{5}\cos\frac{\pi}{2}n + \frac{3}{5}\sin\frac{\pi}{2}n$$

$$x[n] = 3\sin(\pi n/4) = \frac{3}{2j}\left(e^{j\frac{\pi}{4}n} - e^{-j\frac{\pi}{4}n}\right)$$

$$\begin{aligned} y[n] &= \frac{3}{2j}\left(e^{j\frac{\pi}{4}n} H\big(e^{j\frac{\pi}{4}}\big) - e^{-j\frac{\pi}{4}n} H\big(e^{j-\frac{\pi}{4}}\big)\right) \\ &= \frac{3\left[2\sin\frac{\pi}{4}n - \sin\frac{\pi}{4}(\sin\frac{\pi}{4}n + \cos\frac{\pi}{4}n)\right]}{1.25 - \cos\frac{\pi}{4}} \end{aligned}$$

(b) See plot below.

(c) See plot below.

MATLAB script:

```
% P0501: Plot magnitude and phase response
close all;clc
%% Frequencey Domain:
```

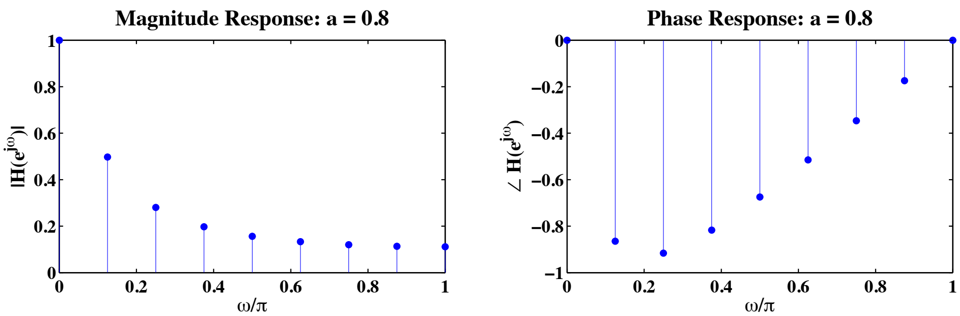


FIGURE 5.1: Magnitude and phase response of the system for $0 \leq \omega \leq \pi$ at increments of $\pi/8$.

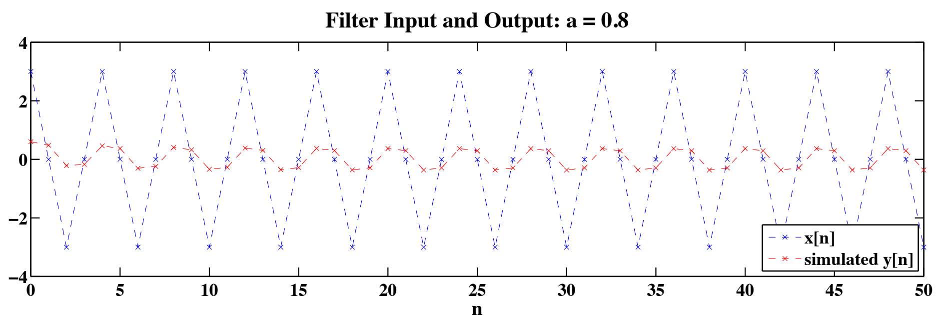


FIGURE 5.2: Input $x[n]$ and the output $y[n]$ where $x[n] = 3\cos(\pi n/2)$.

```
% X = [0 0 0 0 3/2 0 0 0 0]; % (i) x[n]
X = [0 0 3/2/j 0 0 0 0 0 0]; % (ii) x[n]
w = (0:1/8:1)*pi;
% a = 0.5; % part a
a = 0.8; % part b
% a = -0.8; % part c
b = 1-abs(a);
H = freqz(b,[1 -a],w);
H_mag = abs(H);
H_phase = angle(H);
Y = H.*X;
%% Time Domain:
n = 0:50;
xn = 3*cos(pi*n/2);
```

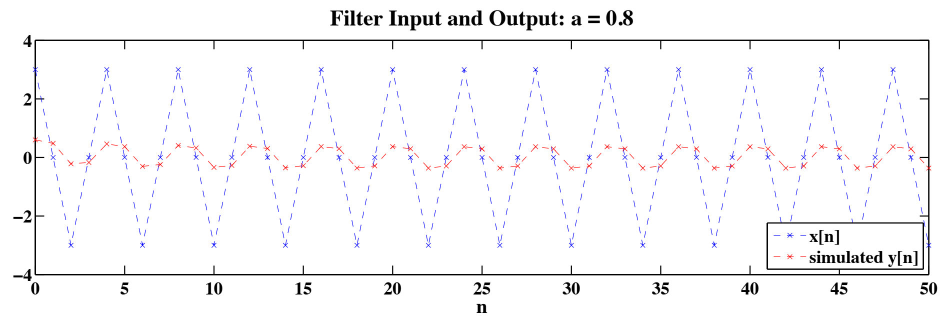


FIGURE 5.3: Input $x[n]$ and the output $y[n]$ where $x[n] = 3\sin(\pi n/4)$.

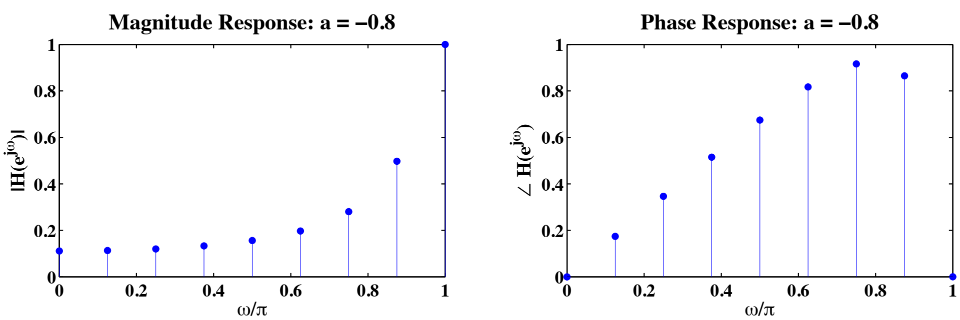


FIGURE 5.4: Magnitude and phase response of the system for $0 \leq \omega \leq \pi$ at increments of $\pi/8$.

```
yn = filter(b,[1 -a],xn);
yn_ref = 6/5*cos(pi*n/2)+3/5*sin(pi*n/2);
%% Plot:
hfa = figconfg('P0501a','long');
subplot(121)
stem(w/pi,H_mag,'filled')
xlabel('\omega/\pi','fontsize',LFS)
ylabel('|H(e^{j\omega})|','fontsize',LFS)
title(['Magnitude Response: a = ',num2str(a)],'fontsize',TFS)
subplot(122)
stem(w/pi,H_phase,'filled')
xlabel('\omega/\pi','fontsize',LFS)
ylabel('\angle H(e^{j\omega})','fontsize',LFS)
title(['Phase Response: a = ',num2str(a)],'fontsize',TFS)
```

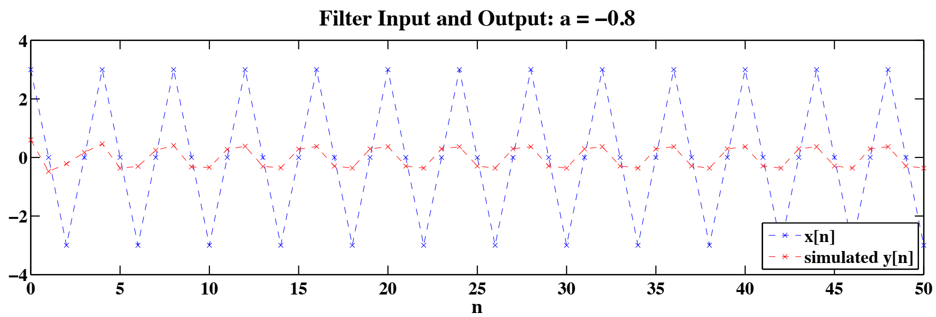


FIGURE 5.5: Input $x[n]$ and the output $y[n]$ where $x[n] = 3\cos(\pi n/2)$.

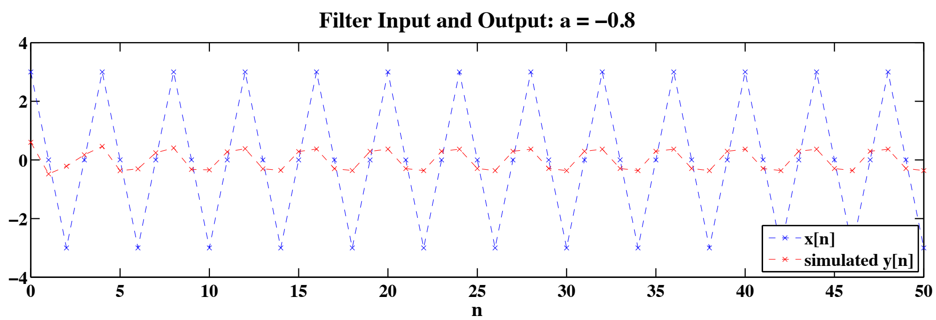


FIGURE 5.6: Input $x[n]$ and the output $y[n]$ where $x[n] = 3\sin(\pi n/4)$.

```
hfb = figconfg('P0501b','long');
plot(n,xn,'--xb',n,yn,'--xr')
legend('x[n]','simulated y[n]','theoretical y[n]'...
    ,'Location','Southeast')
xlabel('n','fontsize',LFS)
title(['Filter Input and Output: a = ',num2str(a)],'fontsize',TFS)
```

2. (a) Solution:

$$H(e^{j\omega}) = \frac{b}{1 + 0.81e^{-2j\omega}}$$

(b) Solution:

$$H(e^{j\omega}) = \frac{b}{1 + 0.81\cos 2\omega - j0.81\sin 2\omega}$$

$$\begin{aligned}|H(e^{j\omega})| &= \frac{|b|}{\sqrt{(1 + 0.81\cos 2\omega)^2 + (0.81\sin 2\omega)^2}} \\ &= \frac{|b|}{\sqrt{1 + 0.81^2 + 2 * 0.81\cos 2\omega}}\end{aligned}$$

$$\begin{aligned}\max|H(e^{j\omega})| &= \frac{|b|}{\sqrt{1 + 0.81^2 - 2 * 0.81}} = 1 \\ |b| &= 0.19\end{aligned}$$

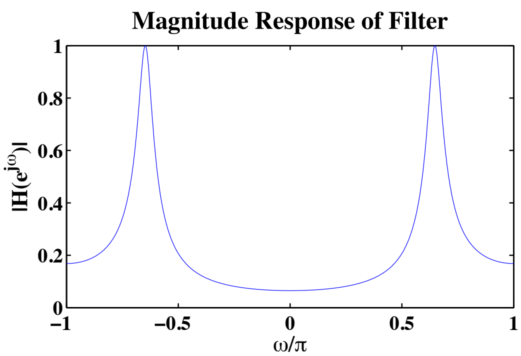


FIGURE 5.7: Magnitude response of the system.

(c) See plot below.

(d) Solution:

$$x[n] = 2\cos(0.5\pi n + 60^\circ) = e^{j\frac{\pi}{3}}e^{j\frac{\pi}{2}n} + e^{-j\frac{\pi}{3}}e^{-j\frac{\pi}{2}n}$$

$$\begin{aligned}y[n] &= e^{j\frac{\pi}{3}}e^{j\frac{\pi}{2}n}H(e^{j\frac{\pi}{2}}) + e^{-j\frac{\pi}{3}}e^{-j\frac{\pi}{2}n}H(e^{j-\frac{\pi}{2}}) \\ &= 2\cos(\frac{\pi}{2}n + \frac{\pi}{3})\end{aligned}$$

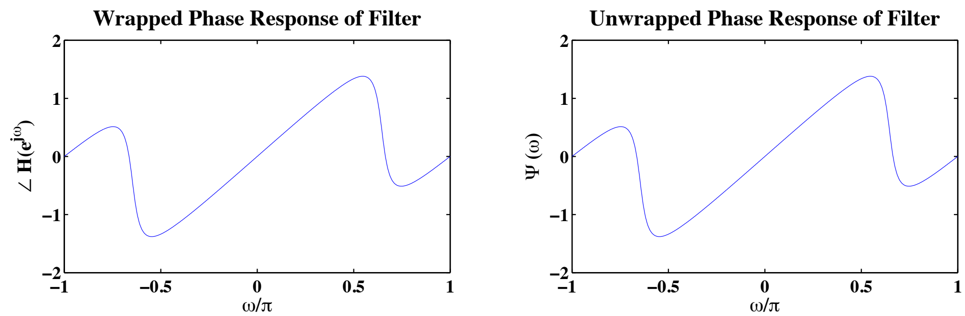


FIGURE 5.8: Wrapped and the unwrapped phase responses of the system.

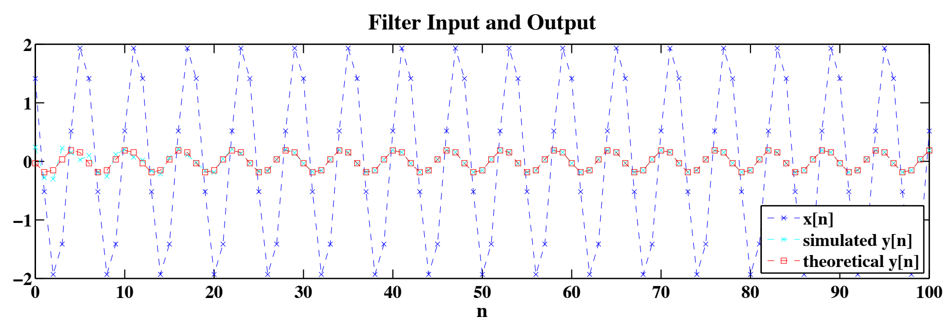


FIGURE 5.9: MATLAB verification of the steady-state response to $x[n]$.

(e) See plot below.
MATLAB script:

```
% P0502: Plot magnitude and phase response
close all;clc
b = sqrt(1+0.8^2+0.81^2-2*0.81-0.8^2*1.81^2/2/2/0.81);
a = [1 0.8 0.81];
w = linspace(-pi,pi,1000);
H = freqz(b,a,w);
H_mag = abs(H);
H_phase = angle(H);
H_phase_unwrap = unwrap(H_phase);
n = 0:100;
xn = 2*cos(pi*n/3+pi/4);
yn = filter(b,a,xn);
yn_ref = 2*0.0577*cos(pi*n/3+pi/4)-2*0.0809*sin(pi*n/3+pi/4);
```

```
%% Plot:
hfa = figconfg('P0502a','small');
plot(w/pi,H_mag)
xlabel('\omega/\pi','fontsize',LFS)
ylabel('|H(e^{j\omega})|','fontsize',LFS)
title('Magnitude Response of Filter','fontsize',TFS)
hfb = figconfg('P0502b','long');
subplot(121)
plot(w/pi,H_phase)
xlabel('\omega/\pi','fontsize',LFS)
ylabel('\angle H(e^{j\omega})','fontsize',LFS)
title('Wrapped Phase Response of Filter','fontsize',TFS)
subplot(122)
plot(w/pi,H_phase_unwrap)
xlabel('\omega/\pi','fontsize',LFS)
ylabel('\Psi (\omega)','fontsize',LFS)
title('Unwrapped Phase Response of Filter','fontsize',TFS)
hfc = figconfg('P0502c','long');
plot(n,xn,'--xb',n,yn,'--xc',n,yn_ref,'--sr')
legend('x[n]','simulated y[n]','theoretical y[n]','Location','Southeast')
xlabel('n','fontsize',LFS)
title('Filter Input and Output','fontsize',TFS)
```

3. (a) Solution:

$$
\begin{aligned}
|H(\mathrm{e}^{\mathrm{j}2\pi F})| &= \frac{0.2}{\sqrt{(1-0.8\cos\omega)^2+(0.8\sin\omega)^2}} \\
&= \frac{0.2}{\sqrt{1.64-1.6\cos\omega}}, \quad \omega = 2\pi F/F_s
\end{aligned}
$$

(c) See plot below.

MATLAB script:

```
% P0503: Linear FM signal
close all; clc
%% Specification:
B = 10;
Fs = 100;
tau = 10;
N = tau*Fs;
```

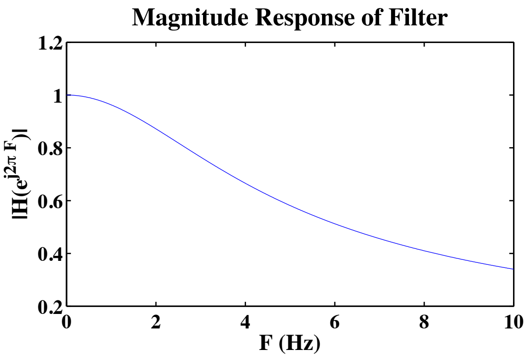


FIGURE 5.10: Plot of $|H(e^{j2\pi F})|$ over $0 \le F \le B$ Hz.

(b)

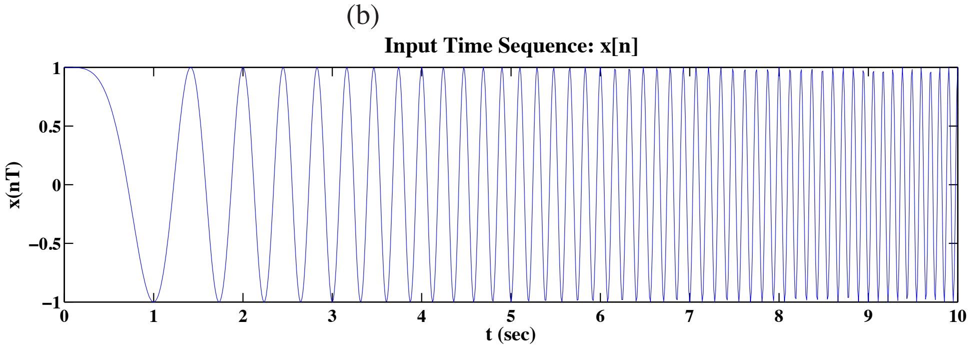


FIGURE 5.11: Plot of $x[n] = x(nT)$ over $0 \le t \le \tau$ sec.

```
n = 0:N;
F = linspace(0,B,1001);
w = 2*pi*F/Fs;
%% Part a:
H = 0.2./(1-0.8*exp(-j*w));
% H_mag = abs(H);
H_mag = 0.2./sqrt(1.64-1.6*cos(w));
%% Part b:
xn = cos(pi*B/Fs/N*n.^2);
%% Part c:
yn = filter(0.2,[1 -0.8],xn);
%% Plot:
hfa = figconfg('P0503a','small');
plot(F,H_mag)
```

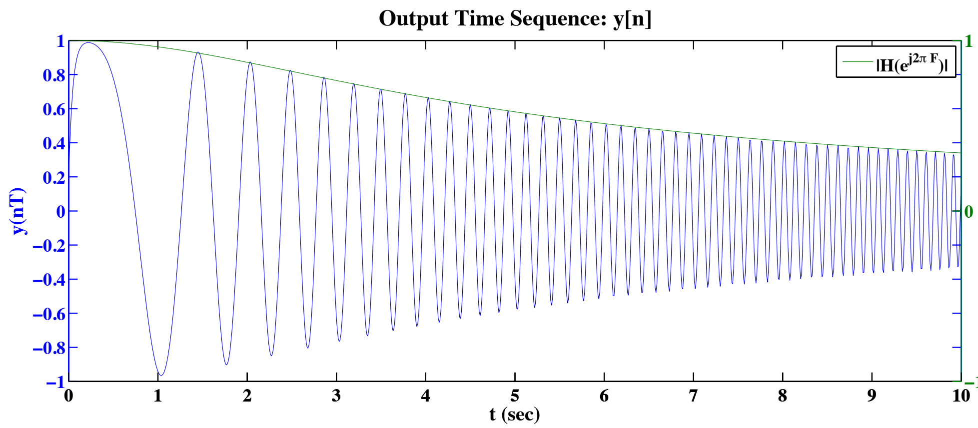


FIGURE 5.12: Plot of $y[n] = y(nT)$ over $0 \le t \le \tau$ sec.

```
xlabel('F (Hz)','fontsize',LFS)
ylabel('|H(e^{j2\pi F})|','fontsize',LFS)
title('Magnitude Response of Filter','fontsize',TFS)
hfb = figconfg('P0503b','long');
plot(n/Fs,xn)
xlabel('t (sec)','fontsize',LFS)
ylabel('x(nT)','fontsize',LFS)
title('Input Time Sequence: x[n]','fontsize',TFS)
hfc = figconfg('P0503c','long');
[AX H1 H2] = plotyy(n/Fs,yn,n/Fs,H_mag);
set(AX(2),'ylim',[-1 1],'YTick',-1:1)
xlabel('t (sec)','fontsize',LFS)
ylabel('y(nT)','fontsize',LFS)
title('Output Time Sequence: y[n]','fontsize',TFS)
legend('|H(e^{j2\pi F})|')
```

4. (a) Solution:

$$X(z) = \frac{1}{1 - \frac{1}{2}z^{-1}}, \quad |z| > \frac{1}{2}$$

$$Y(z) = \frac{1}{1 - \frac{1}{3}z^{-1}}, \quad |z| > \frac{1}{3}$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1 - \frac{1}{2}e^{-j\omega}}{1 - \frac{1}{3}e^{-j\omega}}$$

which is unique.

(b) Solution:

$$H(e^{j\frac{\pi}{3}}) = 2$$

Hence, the frequency response function exists but not unique.

(c) $x[n] = \frac{\sin \pi n/4}{\pi n} \xrightarrow{\mathcal{H}} y[n] = \frac{\sin \pi n/2}{\pi n}$
Solution:

$$y[n] = 2x[2n]$$

which is time-varying.

(d) Solution:

$$y[n] = x[n] - x[n-1]$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = 1 - e^{-j\omega}$$

which is unique.

5. (a) See plot below.

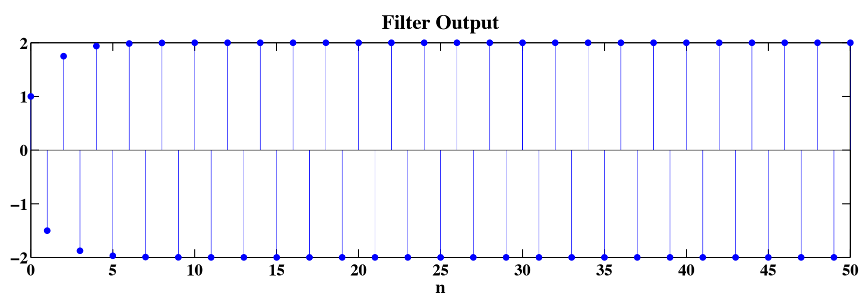


FIGURE 5.13: System response to the input $x[n] = (-1)^n u[n]$.

(b) See plot below.

(c) See plot below.

MATLAB script:

```
% P0505: Checking whether a system is LTI
%        Compute impulse response and frequence response
close all; clc
%% Part a:
n = 0:50;
N = length(n);
xn1 = (-1).^n;
```

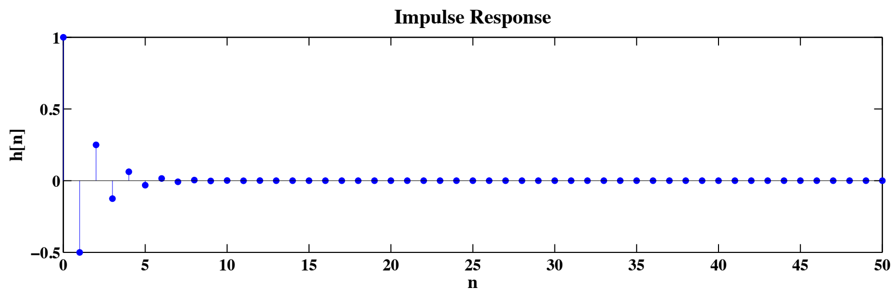


FIGURE 5.14: System frequency response.

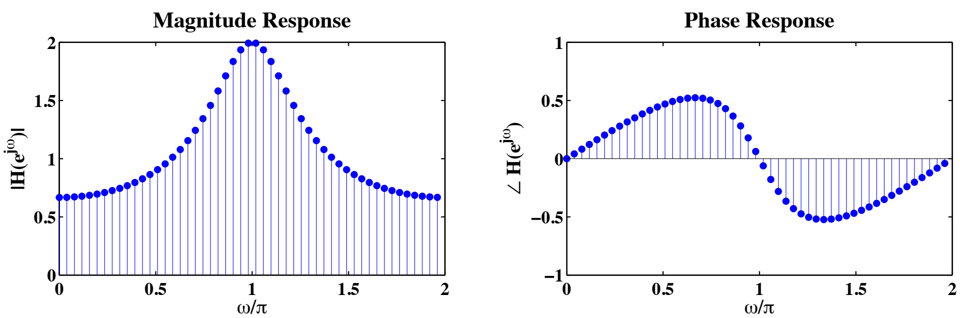


FIGURE 5.15: System frequency response.

```
yn1 = ltiwhich(xn1);
%% Part b:
delta = zeros(size(n));
delta(1) = 1;
hn = ltiwhich(delta);
H = fft(hn);
H_mag = abs(H);
H_phase = angle(H);
%% Part c:
w = (0:10)/10*pi;
H_ref_mag = zeros(size(w));
H_ref_phase = zeros(size(w));
for ii = 1:length(w)
    xn = cos(w(ii)*n);
    yn = ltiwhich(xn);
    [H_ref_mag(ii) ind] = max(yn(10:end));
```

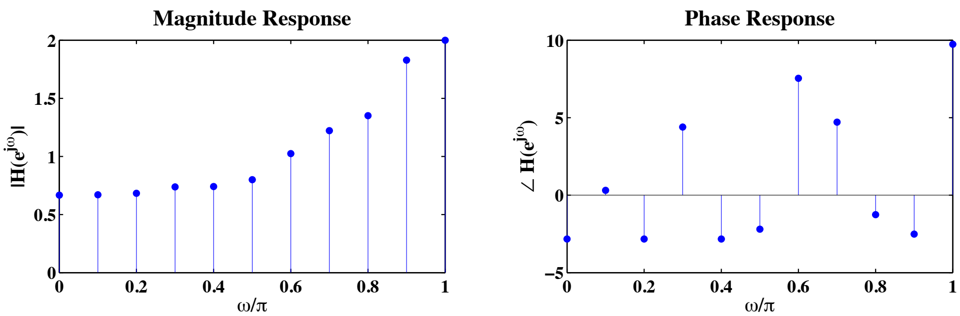


FIGURE 5.16: System frequency response for comparison.

```
    H_ref_phase(ii) = (ind(1)+9-20)/20*2*pi;
%     H_ref_phase(ii) = atan(tan(H_ref_phase(ii)));
end
%% Plot
hfa = figconfg('P0505a','long');
stem(n,yn1,'filled')
xlabel('n','fontsize',LFS)
title('Filter Output','fontsize',TFS)
hfb = figconfg('P0505b','long');
stem(n,hn,'filled')
xlabel('n','fontsize',LFS)
ylabel('h[n]','fontsize',LFS)
title('Impulse Response','fontsize',TFS)
hfc = figconfg('P0505c','long');
subplot(121)
stem(2*(0:N-1)/N,H_mag,'filled')
xlabel('\omega/\pi','fontsize',LFS)
ylabel('|H(e^{j\omega})|','fontsize',LFS)
title('Magnitude Response','fontsize',TFS)
subplot(122)
stem(2*(0:N-1)/N,H_phase,'filled')
xlabel('\omega/\pi','fontsize',LFS)
ylabel('\angle H(e^{j\omega})','fontsize',LFS)
title('Phase Response','fontsize',TFS)
hfd = figconfg('P0505d','long');
subplot(121)
stem((0:10)/10,H_ref_mag,'filled')
```

```
xlabel('\omega/\pi','fontsize',LFS)
ylabel('|H(e^{j\omega})|','fontsize',LFS)
title('Magnitude Response','fontsize',TFS)
subplot(122)
stem((0:10)/10,H_ref_phase,'filled')
xlabel('\omega/\pi','fontsize',LFS)
ylabel('\angle H(e^{j\omega})','fontsize',LFS)
title('Phase Response','fontsize',TFS)
```

6. (a) Solution:

$$H(e^{j\omega}) = \frac{(1+e^{-j\omega})^6}{2^6} = \frac{(1+\cos\omega - j\sin\omega)^6}{2^6}$$

$$\angle H(e^{j\omega}) = 6\tan^{-1}\frac{-\sin\omega}{1+\cos\omega} = 6\tan^{-1}\frac{-2\sin\frac{\omega}{2}\cos\frac{\omega}{2}}{2\cos^2\frac{\omega}{2}} = -3\omega$$

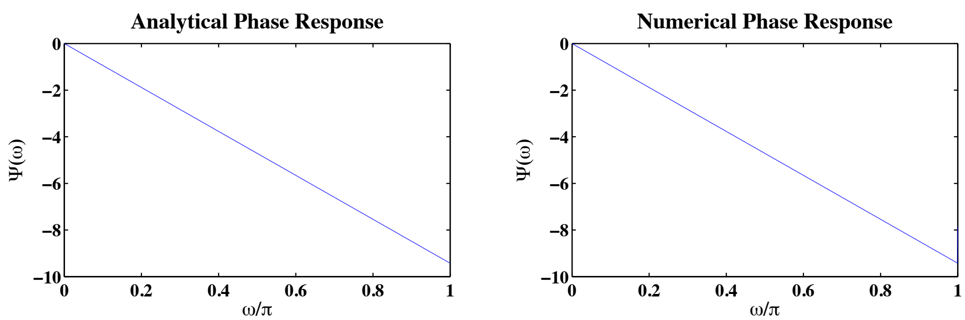


FIGURE 5.17: Phase response comparison between analytical and numerical results.

(b) MATLAB script:

```
  % P0506: Compute and plot phase response
close all; clc
%% Part a: Analytic Solution
w = linspace(0,1,1000)*pi;
H_phase_ana = -3*w;
%% Part b: Numerical Solution
b = poly(-ones(1,6));
H_ref = freqz(b,1,w);
H_phase_ref = unwrap(angle(H_ref));
```

```
%% Plot
hfa = figconfg('P0506','long');
subplot(121)
plot(w/pi,H_phase_ana)
xlabel('\omega/\pi','fontsize',LFS)
ylabel('\Psi(\omega)','fontsize',LFS)
title('Analytical Phase Response','fontsize',TFS)
subplot(122)
plot(w/pi,H_phase_ref)
xlabel('\omega/\pi','fontsize',LFS)
ylabel('\Psi(\omega)','fontsize',LFS)
title('Numerical Phase Response','fontsize',TFS)
```

7. (a) Solution:

$$\text{System function:} \quad H(z) = \frac{Y(z)}{X(z)} = \frac{1}{2} - \frac{1}{2}z^{-1}$$

$$\text{Frequency response:} \quad H(\mathrm{e}^{\mathrm{j}\omega}) = \frac{1}{2} - \frac{1}{2}\mathrm{e}^{-\mathrm{j}\omega} = \frac{1}{2}(1 - \cos\omega + \mathrm{j}\sin\omega)$$

$$\text{Magnitude response:} \quad |H(\mathrm{e}^{\mathrm{j}\omega})| = \frac{\sqrt{(1-\cos\omega)^2 + \sin^2\omega}}{2} = |\sin\frac{\omega}{2}|$$

$$\text{Phase response:} \quad \angle H(\mathrm{e}^{\mathrm{j}\omega}) = \tan^{-1}\frac{\sin\omega}{1-\cos\omega} = \frac{\pi}{2} - \frac{\omega}{2}$$

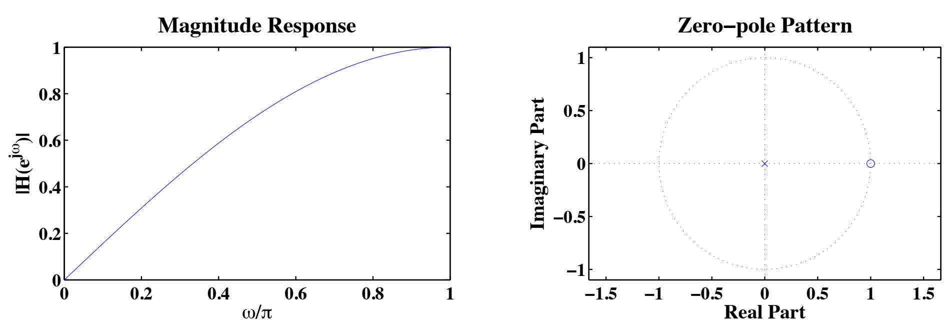


FIGURE 5.18: Magnitude response and pole-zero plot of $y[n] = \frac{1}{2}(x[n] - x[n-1])$.

(b) Solution:

$$\text{System function:}\quad H(z) = \frac{Y(z)}{X(z)} = \frac{1}{2} - \frac{1}{2}z^{-2}$$

$$\text{Frequency response:}\quad H\big(e^{j\omega}\big) = \frac{1}{2} - \frac{1}{2}e^{-2j\omega} = \frac{1}{2}(1-\cos 2\omega + j\sin 2\omega)$$

$$\text{Magnitude response:}\quad |H\big(e^{j\omega}\big)| = \frac{\sqrt{(1-\cos 2\omega)^2 + \sin^2 2\omega}}{2} = |\sin\omega|$$

$$\text{Phase response:}\quad \angle H\big(e^{j\omega}\big) = \tan^{-1}\frac{\sin 2\omega}{1-\cos 2\omega} = \frac{\pi}{2} - \omega$$

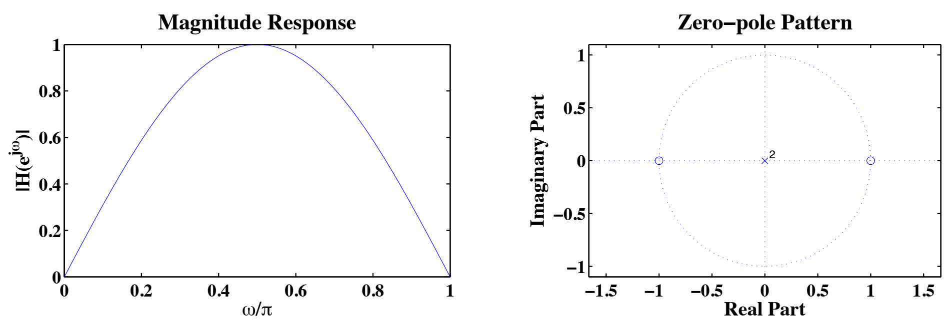


FIGURE 5.19: Magnitude response and pole-zero plot of $y[n] = \frac{1}{2}(x[n] - x[n-2])$.

(c) Solution:

$$\text{System function:}\quad H(z) = \frac{Y(z)}{X(z)} = \frac{1}{4}(1 + z^{-1} - z^{-2} - z^{-3})$$

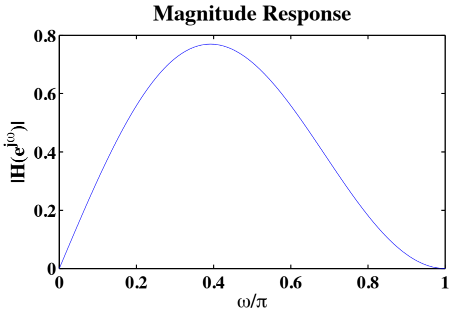
Frequency response:

$$H\big(e^{j\omega}\big) = \frac{1}{4}[(1+\cos\omega - \cos 2\omega - \cos 3\omega) + j(-\sin\omega + \sin 2\omega + \sin 3\omega)]$$

Magnitude response:

$$|H\big(e^{j\omega}\big)| = \frac{1}{4}\sqrt{(1+\cos\omega - \cos 2\omega - \cos 3\omega)^2 + (-\sin\omega + \sin 2\omega + \sin 3\omega)^2}$$

$$\text{Phase response:}\quad \angle H\big(e^{j\omega}\big) = \frac{-\sin\omega + \sin 2\omega + \sin 3\omega}{1 + \cos\omega - \cos 2\omega - \cos 3\omega}$$



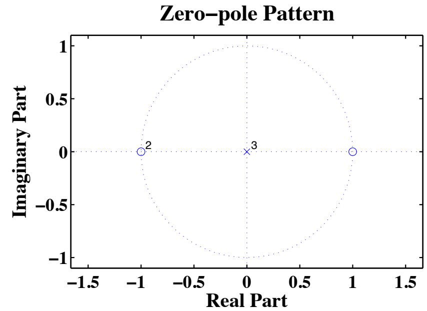


FIGURE 5.20: Magnitude response and pole-zero plot of $y[n] = \frac{1}{4}(x[n] + x[n-1]) - \frac{1}{4}(x[n-2] + x[n-3])$.

(d) Solution:

$$\text{System function:} \quad H(z) = \frac{Y(z)}{X(z)} = \frac{1}{4}(1 + z^{-1} - z^{-3} - z^{-4})$$

Frequency response:

$$H(e^{j\omega}) = \frac{1}{4}[(1+\cos\omega-\cos 3\omega-\cos 4\omega)+j(-\sin\omega+\sin 3\omega+\sin 4\omega)]$$

Magnitude response:

$$|H(e^{j\omega})| = \frac{1}{4}\sqrt{(1 + \cos\omega - \cos 3\omega - \cos 4\omega)^2 + (-\sin\omega + \sin 3\omega + \sin 4\omega)^2}$$

$$\text{Phase response:} \quad \angle H(e^{j\omega}) = \frac{-\sin\omega + \sin 3\omega + \sin 4\omega}{1 + \cos\omega - \cos 3\omega - \cos 4\omega}$$

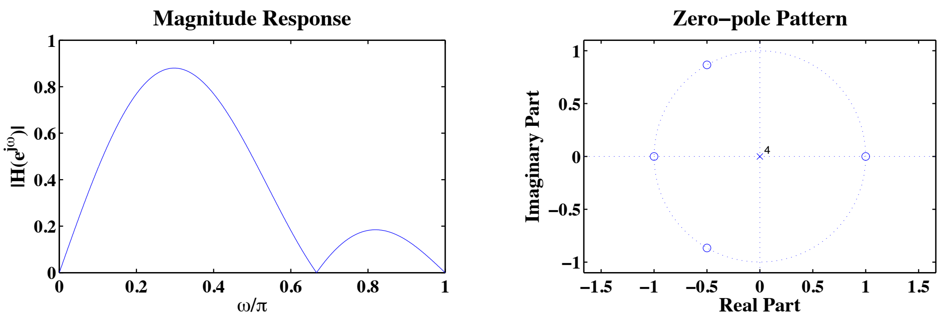


FIGURE 5.21: Magnitude response and pole-zero plot of $y[n] = \frac{1}{4}(x[n] + x[n-1]) - \frac{1}{4}(x[n-3] + x[n-4])$.

8. (a) Solution:

$$h_{bp}[n] = 2\frac{\sin\omega_c(n-n_d)}{\pi(n-n_d)}\cos\omega_0 n \quad (5.72)$$

$$h_{lp}[n] = \frac{\sin\omega_c(n-n_d)}{\pi(n-n_d)} \quad (5.70)$$

Modulation Property:

$$x[n]\cos\omega_0 n = \frac{1}{2}X\big(e^{j(\omega+\omega_0)}\big) + \frac{1}{2}X\big(e^{j(\omega-\omega_0)}\big)$$

$$H_{bp}\big(e^{j\omega}\big) = \begin{cases} e^{-j\omega n_d}, & \omega_\ell \le |\omega| \le \omega_h \\ 0, & \text{otherwise} \end{cases}$$

$$H_{lp}\big(e^{j\omega}\big) = \begin{cases} e^{-j\omega n_d}, & |\omega| \le \omega_c \\ 0, & \omega_c < |\omega| \le \pi \end{cases}$$

$$H_{bp}\big(e^{j\omega}\big) = H_{lp}\big(e^{j(\omega-\omega_0)}\big) + H_{lp}\big(e^{j(\omega+\omega_0)}\big)$$

where $\omega_0 = \frac{\omega_\ell+\omega_h}{2}$ and $\omega_c = \frac{\omega_h-\omega_\ell}{2}$.
Hence,

$$h_{bp}[n] = 2h_{lp}\cos\omega_0 n = 2\frac{\sin\omega_c(n-n_d)}{\pi(n-n_d)}\cos\omega_0 n$$

(b) Solution:

$$H_{bp}\big(e^{j\omega}\big) = H_{lp1}\big(e^{j\omega}\big) - H_{lp2}\big(e^{j\omega}\big)$$

where

$$H_{lp1}(e^{j\omega}) = \begin{cases} e^{-j\omega n_d}, & |\omega| \le \omega_h \\ 0, & \omega_h < |\omega| \le \pi \end{cases}$$

$$H_{lp2}(e^{j\omega}) = \begin{cases} e^{-j\omega n_d}, & |\omega| \le \omega_\ell \\ 0, & \omega_\ell < |\omega| \le \pi \end{cases}$$

9. MATLAB function:

```
function [grp,omega] = mygrpdelay(b,a)
% Implement equation (5.89) to compute group delay
p = roots(a);
p = p(:);
N = size(p,1);
r = abs(p);
phi = angle(p);

z = roots(b);
z = z(:);
M = size(z,1);
q = abs(z);
theta = angle(z);

K = 1024;
omega = 2*pi*(0:K-1)/K;

r_epd = repmat(r,1,K);
phi_epd = repmat(phi,1,K);
q_epd = repmat(q,1,K);
theta_epd = repmat(theta,1,K);
temp1 = cos(repmat(omega,N,1)-phi_epd);
temp2 = cos(repmat(omega,M,1)-theta_epd);

grp = sum((r_epd.^2-r_epd.*temp1)./(1+r_epd.^2-2*r_epd.*temp1),1);
grp = -grp + sum((q_epd.^2-q_epd.*temp2)./(1+q_epd.^2-2*q_epd.*temp2),1);
```

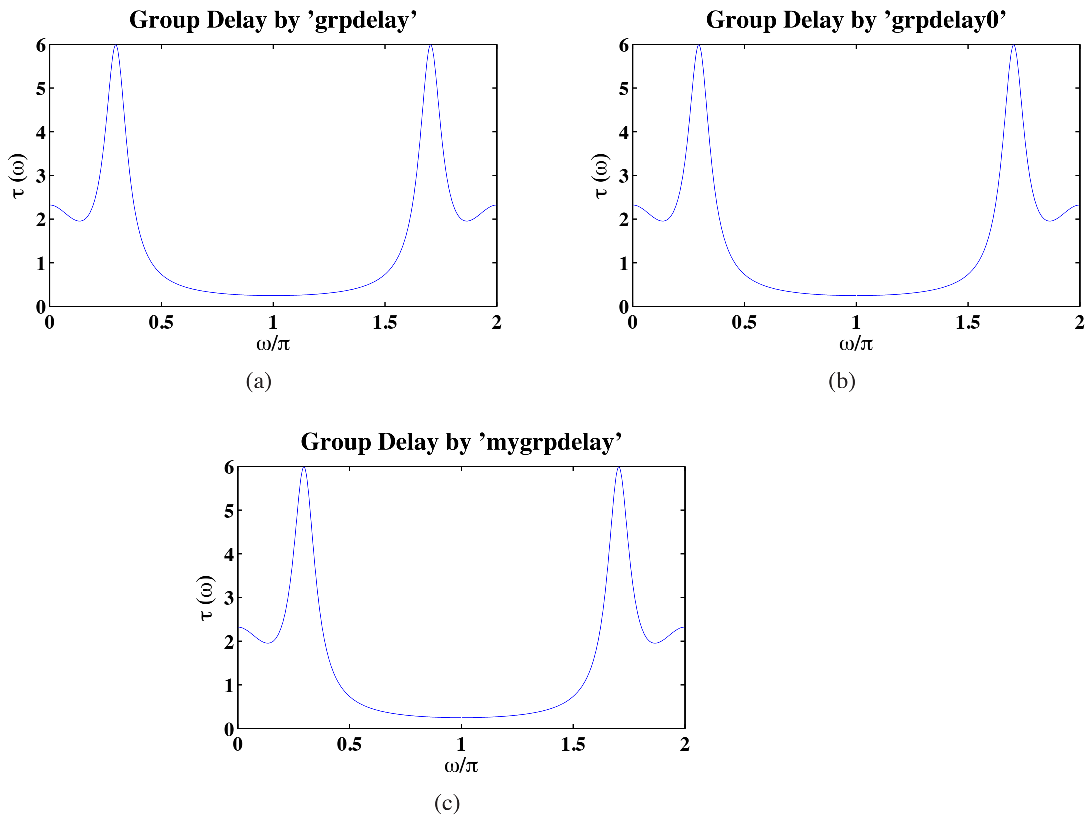


FIGURE 5.22: (a) Group delay computed by MATLAB function `grpdelay`. (b) Group delay computed by MATLAB function `grpdelay0`. (c) Group delay computed by MATLAB function `mygrpdelay`.

MATLAB script:

```
% P0509: Compute group delay using different functions
%        and compare with results for system (5.99)
close all; clc
%% Specification:
b = [1 1.655 1.655 1];
a = [1 -1.57 1.264 -0.4];
w = linspace(0,2,1000)*pi;
gd1 = grpdelay(b,a,w); % method I
[gd2 w2] = grpdelay0(b,a); % method II
```

```
[gd3,w3] = mygrpdelay(b,a); % method III
%% Plot
hfa = figconfg('P0509a','small');
plot(w/pi,gd1)
xlabel('\omega/\pi','fontsize',LFS)
ylabel('\tau (\omega)','fontsize',LFS)
title('Group Delay by ''grpdelay''','fontsize',TFS)
hfb = figconfg('P0509b','small');
plot(w2/pi,gd2)
xlabel('\omega/\pi','fontsize',LFS)
ylabel('\tau (\omega)','fontsize',LFS)
title('Group Delay by ''grpdelay0''','fontsize',TFS)
hfc = figconfg('P0509c','small');
plot(w3/pi,gd3)
xlabel('\omega/\pi','fontsize',LFS)
ylabel('\tau (\omega)','fontsize',LFS)
title('Group Delay by ''mygrpdelay''','fontsize',TFS)
```

10. Proof:

$$\tau_{gd}(\omega) = -\frac{d\Psi(\omega)}{d\omega}$$

$$\Psi(\omega) = \tan^{-1}\frac{H_I(\omega)}{H_R(\omega)} + 2k\pi, \quad \Psi(\omega) \text{ is continuous}$$

$$-\frac{d\Psi(\omega)}{d\omega} = -\frac{d}{d\omega}\left(\tan^{-1}\frac{H_I(\omega)}{H_R(\omega)} + 2k\pi\right) = \frac{\left(\frac{dH_I(\omega)}{d\omega}\right)H_R(\omega) - \left(\frac{dH_R(\omega)}{d\omega}\right)H_I(\omega)}{H_R^2(\omega) + H_I^2(\omega)}$$

$$nh[n] \overset{\text{DTFT}}{\longrightarrow} \mathrm{j}\cdot\frac{d\left[H_R(\omega) + \mathrm{j}H_I(\omega)\right]}{d\omega} = \mathrm{j}\frac{dH_R(\omega)}{d\omega} - \frac{dH_I(\omega)}{d\omega}$$

Hence, we have

$$\frac{dH_R(\omega)}{d\omega} = G_I(\omega); \quad \frac{dH_I(\omega)}{d\omega} = -G_R(\omega)$$

$$H_R^2(\omega) + H_I^2(\omega) = |H\left(\mathrm{e}^{\mathrm{j}\omega}\right)|^2$$

Thus, we proved that

$$\tau_{gd}(\omega) = -\frac{d\Psi(\omega)}{d\omega} = \frac{H_{\mathrm{R}}(\omega)G_{\mathrm{R}}(\omega) + H_{\mathrm{I}}(\omega)G_{\mathrm{I}}(\omega)}{|H(\mathrm{e}^{\mathrm{j}\omega})|^2}$$

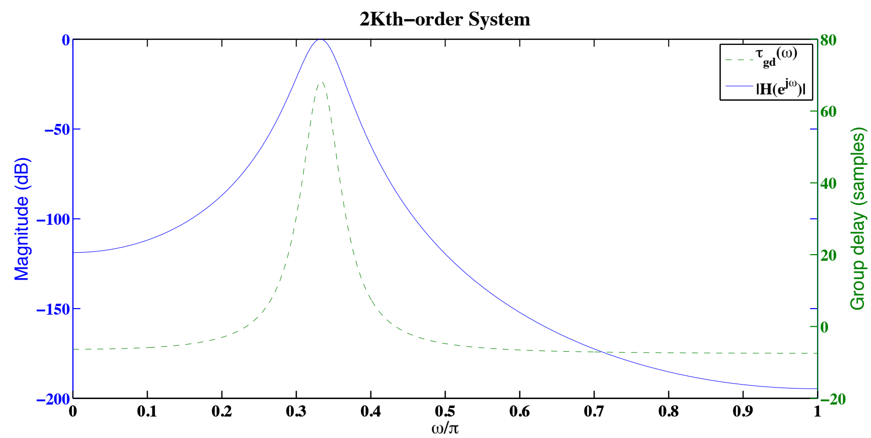


FIGURE 5.23: Frequency response of a single $2K$th-order system.

11. Solution:

    MATLAB script:

```
% P0511: Generate Figures 5.7 and 5.8 in Example5.6
close all; clc
%% Speicification:
r = 0.9;
w0 = pi/3;
K = 8;
b0 = 0.0271^4;
w = linspace(0,1,1000)*pi;

N = 100;
mu = 0;
sigma = 2;
n = linspace(-5,5,N);
sn = normpdf(n,mu,sigma);
n = 0:4*N-1;
w1 = 0.34*pi;
w2 = 0.6*pi;
xn = zeros(1,length(n));
xn(1:N) = sn.*cos(w1*(0:N-1));
xn(N+1:2*N) = sn.*cos(w2*(0:N-1));
```

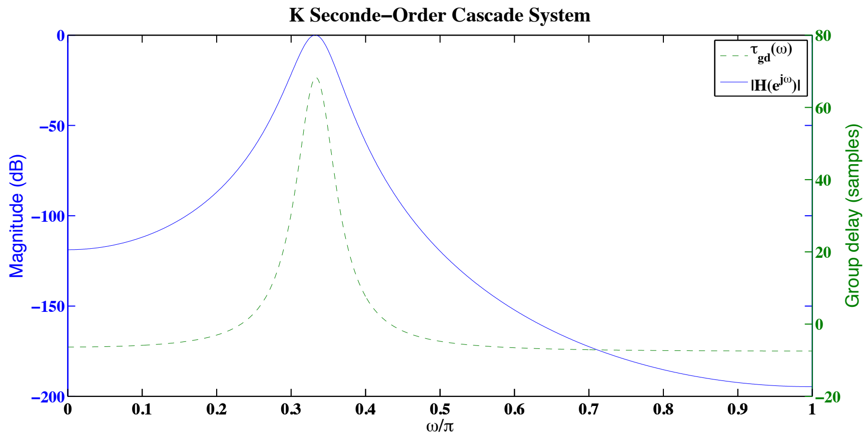


FIGURE 5.24: Frequency response of a cascade connection of K second-order systems.

```
X = dtft12(xn,0,w);

%% Part a:
bcas = b0^(1/K);
acas = [1 -2*r*cos(w0) r^2];

Hcas = freqz(bcas,acas,w);
H_mag_db_cas = 10*log10(abs(Hcas).^2)*K;
gd_cas = K*grpdelay(bcas,acas,w);
```

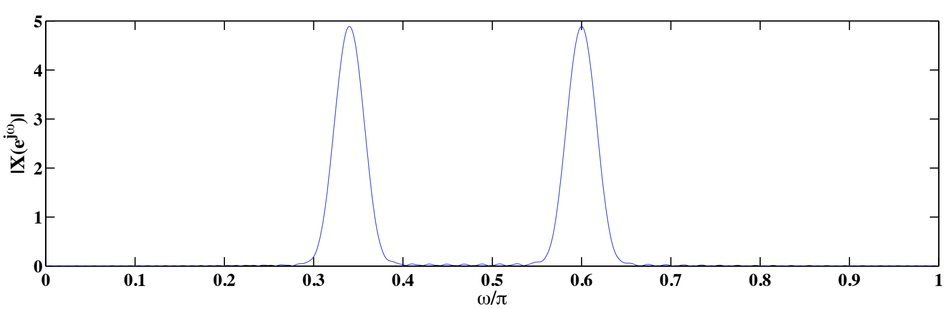


FIGURE 5.25: Frequency response input sequence $x[n]$.

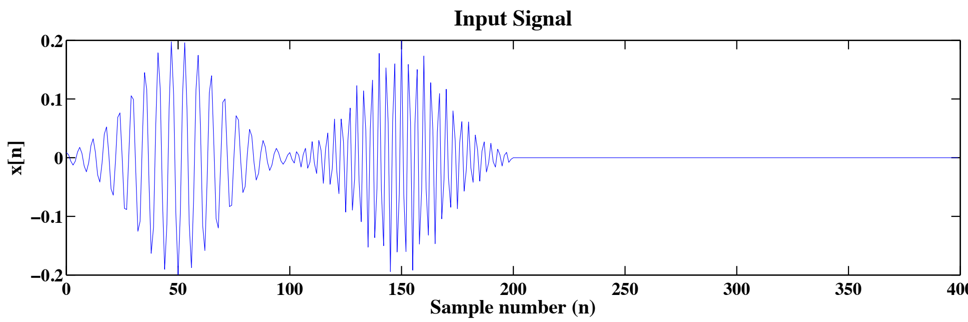


FIGURE 5.26: Input sequence $x[n]$.

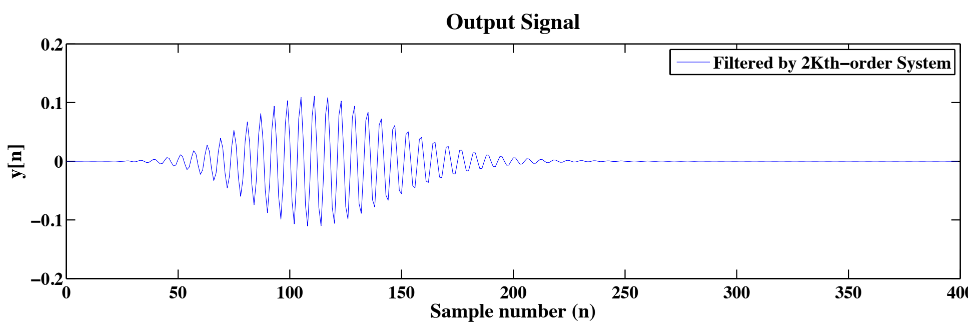


FIGURE 5.27: Output sequence $y[n]$ of a single $2K$th-order system.

```
yn_cas = xn;
for ii=1:K
yn_cas = filter(bcas,acas,yn_cas);
end

%% Part b:
a = acas;
for ii = 1:K-1
    a = conv(a,acas);
end
H = freqz(b0,a,w);
H_mag_db = 10*log10(abs(H).^2);
gd = grpdelay(b0,a,w);
yn = filter(b0,a,xn);
```

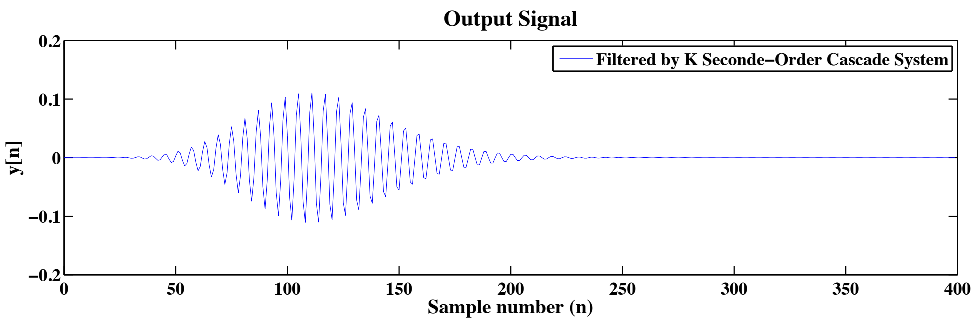


FIGURE 5.28: Output sequence $y[n]$ of a cascade connection of K second-order systems.

```
%% Plot
hfa = figconfg('P0511a','long');
[AX H1 H2] = plotyy(w/pi,H_mag_db,w/pi,gd);
xlabel('\omega/\pi','fontsize',LFS)
set(get(AX(1),'Ylabel'),'String','Magnitude (dB)','fontsize',LFS)
set(get(AX(2),'Ylabel'),'String','Group delay (samples)','fontsize',LFS)
set(AX(1),'ylim',[-200 0])
set(H2,'LineStyle','--')
legend('\tau_{gd}(\omega)','|H(e^{j\omega})|','Location','Northeast')
title('2Kth-order System','fontsize',TFS)

hfb = figconfg('P0511b','long');
[AX H1 H2] = plotyy(w/pi,H_mag_db_cas,w/pi,gd_cas);
xlabel('\omega/\pi','fontsize',LFS)
set(get(AX(1),'Ylabel'),'String','Magnitude (dB)','fontsize',LFS)
set(get(AX(2),'Ylabel'),'String','Group delay (samples)','fontsize',LFS)
set(AX(1),'ylim',[-200 0])
set(H2,'LineStyle','--')
legend('\tau_{gd}(\omega)','|H(e^{j\omega})|','Location','Northeast')
title('K Seconde-Order Cascade System','fontsize',TFS)

hfc = figconfg('P0511c','long');
plot(w/pi,abs(X))
xlabel('\omega/\pi','fontsize',LFS)
ylabel('|X(e^{j\omega})|','fontsize',LFS)
```

```
hfd = figconfg('P0511d','long');
plot(n,xn)
xlabel('Sample number (n)','fontsize',LFS)
ylabel('x[n]','fontsize',LFS)
title('Input Signal','fontsize',TFS)

hfe = figconfg('P0511e','long');
plot(n,yn)
xlabel('Sample number (n)','fontsize',LFS)
ylabel('y[n]','fontsize',LFS)
title('Output Signal','fontsize',TFS)
legend('Filtered by 2Kth-order System','Location','Northeast')

hff = figconfg('P0511f','long');
plot(n,yn_cas)
xlabel('Sample number (n)','fontsize',LFS)
ylabel('y[n]','fontsize',LFS)
title('Output Signal','fontsize',TFS)
legend('Filtered by K Seconde-Order Cascade System','Location','Northeast')
```

12. Proof:

$$x[n] = s[n]\cos\omega_c n \quad (5.62)$$

$$\Psi(\omega) \approx \Psi(\omega_c) + \left.\frac{\Phi(\omega)}{d\omega}\right|_{\omega=\omega_c}(\omega-\omega_c) = -\tau_{pd}\omega_c - \tau_{gd}(\omega_c)(\omega-\omega_c) \quad (5.63)$$

$$y[n] \approx |H(\mathrm{e}^{\mathrm{j}\omega_c})|s[n-\tau_{gd}(\omega_c)]\cos\{\omega_c[n-\tau_{pd}(\omega_c)]\} \quad (5.64)$$

Suppose the system magnitude response is constant near ω_c,

$$\begin{aligned}
y[n] &\approx |H(\mathrm{e}^{\mathrm{j}\omega_c})|\left(\frac{1}{2\pi}\int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} s(\mathrm{e}^{\mathrm{j}\omega})\mathrm{e}^{\mathrm{j}n\omega}d\omega\right)\cos\omega_c n\mathrm{e}^{-\mathrm{j}\tau_{pd}(\omega_c)\omega_c}\mathrm{e}^{-\mathrm{j}\tau_{gd}(\omega_c)(\omega-\omega_c)} \\
&= |H(\mathrm{e}^{\mathrm{j}\omega_c})|\left(\frac{1}{2\pi}\int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} s(\mathrm{e}^{\mathrm{j}\omega})\mathrm{e}^{\mathrm{j}n\omega}\mathrm{e}^{-\mathrm{j}\tau_{gd}(\omega_c)\omega}d\omega\right)\cos\omega_c n\mathrm{e}^{-\mathrm{j}\tau_{pd}(\omega_c)\omega_c} \\
&= |H(\mathrm{e}^{\mathrm{j}\omega_c})|\left(\frac{1}{2\pi}\int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} s(\mathrm{e}^{\mathrm{j}\omega})\mathrm{e}^{\mathrm{j}\omega(n-\tau_{gd}(\omega_c))}d\omega\right)\left(\cos\omega_c n\mathrm{e}^{-\mathrm{j}\tau_{pd}(\omega_c)\omega_c}\right) \\
&= |H(\mathrm{e}^{\mathrm{j}\omega_c})|s[n-\tau_{gd}(\omega_c)]\cos\{\omega_c[n-\tau_{pd}(\omega_c)]\}
\end{aligned}$$

13. (a) Solution:

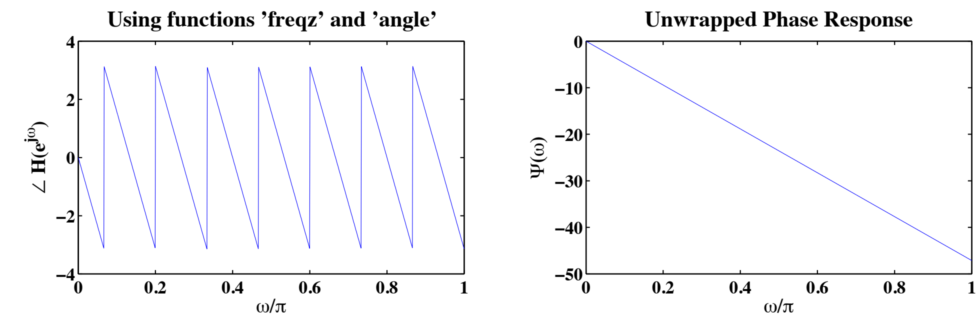


FIGURE 5.29: Phase response of pure delay $y[n] = x[n-15]$ using the functions freqz, angle and unwrap.

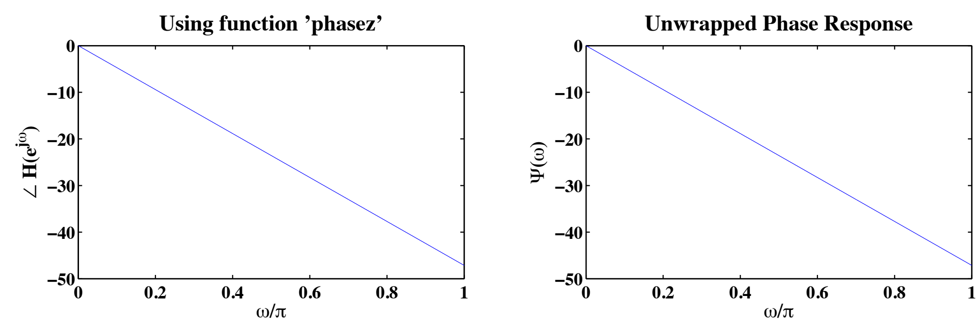


FIGURE 5.30: Phase response of pure delay $y[n] = x[n-15]$ using the function phasez.

(b) Solution:

MATLAB script:

```
% P0513: Plot phase response
close all; clc
%% Specification:
%% Part a:
% b = zeros(1,16);
% b(end) = 1;
% a = 1;

%% Part a:
```

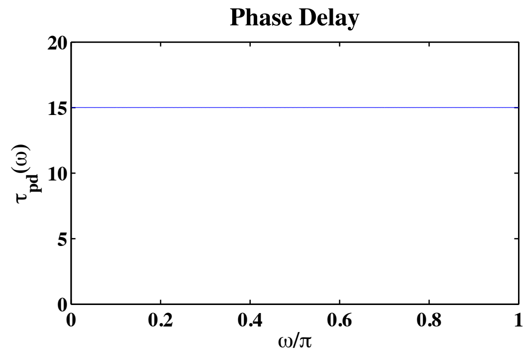


FIGURE 5.31: Phase delay of pure delay $y[n] = x[n - 15]$ using the function phasedelay.

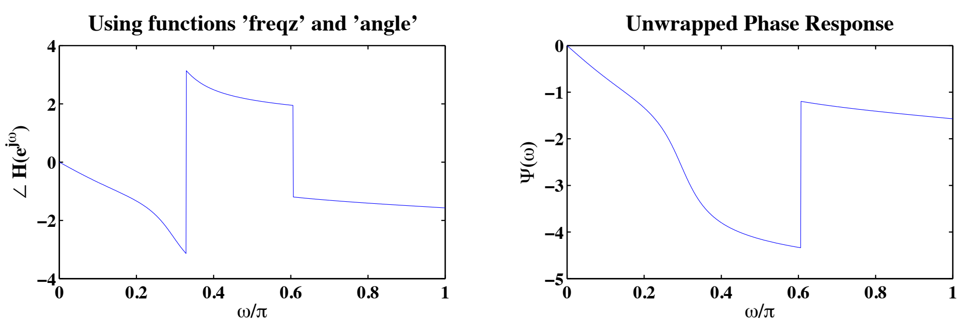


FIGURE 5.32: Phase response of the system defined by (5.99) using the functions freqz, angle and unwrap.

```
b = [1 1.655 1.655 1];
a = [1 -1.57 1.264 -0.4];

%% Computation:
w = linspace(0,1,1000)*pi;
H = freqz(b,a,w);
H_phase = angle(H);
H_phase2 = phasez(b,a,w);
Pd = phasedelay(b,a,w);
%% Plot:
hfa = figconfg('P0513a','long');
subplot(121)
```

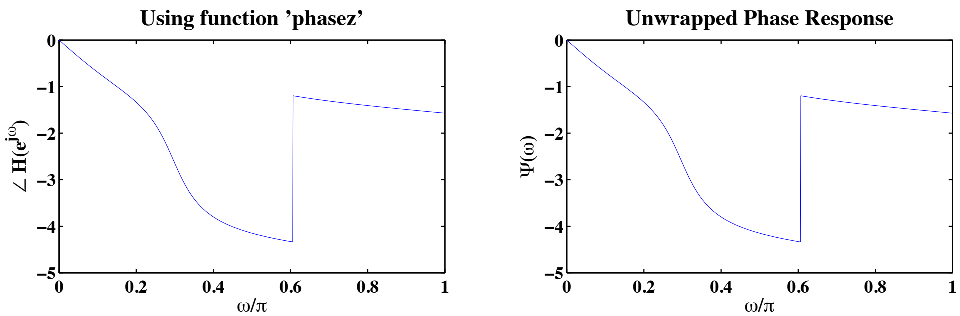


FIGURE 5.33: Phase response of the system defined by (5.99) using the function phasez.

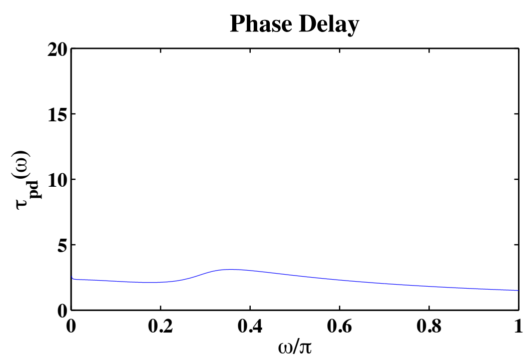


FIGURE 5.34: Phase delay of the system defined by (5.99) using the function phasedelay.

```
plot(w/pi,H_phase)
xlabel('\omega/\pi','fontsize',LFS)
ylabel('\angle H(e^{j\omega})','fontsize',LFS)
title('Using functions ''freqz'' and ''angle''','fontsize',TFS)
subplot(122)
plot(w/pi,unwrap(H_phase))
xlabel('\omega/\pi','fontsize',LFS)
ylabel('\Psi(\omega)','fontsize',LFS)
title('Unwrapped Phase Response','fontsize',TFS)

hfb = figconfg('P0513b','long');
subplot(121)
```

```
plot(w/pi,H_phase2)
xlabel('\omega/\pi','fontsize',LFS)
ylabel('\angle H(e^{j\omega})','fontsize',LFS)
title('Using function ''phasez''','fontsize',TFS)
subplot(122)
plot(w/pi,unwrap(H_phase2))
xlabel('\omega/\pi','fontsize',LFS)
ylabel('\Psi(\omega)','fontsize',LFS)
title('Unwrapped Phase Response','fontsize',TFS)

hfc = figconfg('P0513c','small');
plot(w/pi,Pd)
xlabel('\omega/\pi','fontsize',LFS)
ylabel('\tau_{pd}(\omega)','fontsize',LFS)
ylim([0 20])
title('Phase Delay','fontsize',TFS)
```

14. (a) Solution:

$$H(e^{j\omega}) = b_0 \frac{1 - e^{-2j\omega}}{1 - (2r\cos(\phi)e^{-j\omega}) + r^2 e^{-2j\omega}}$$

Solve for $|H(e^{j\phi})| = 1$, we have

$$b_0 = 0.095$$

(b) Solution:

$$h[n] = b_0 r^n \frac{\sin[(n+1)\phi]}{\sin\phi} u[n] - b_0 r^{n-2} \frac{\sin[(n-1)\phi]}{\sin\phi} u[n-2]$$

(c) Solution:
The approximate 3-dB bandwidth of the resonator is 0.2086.

(d) Solution:
The exact 3-dB bandwidth is 0.2094.

(e) tba

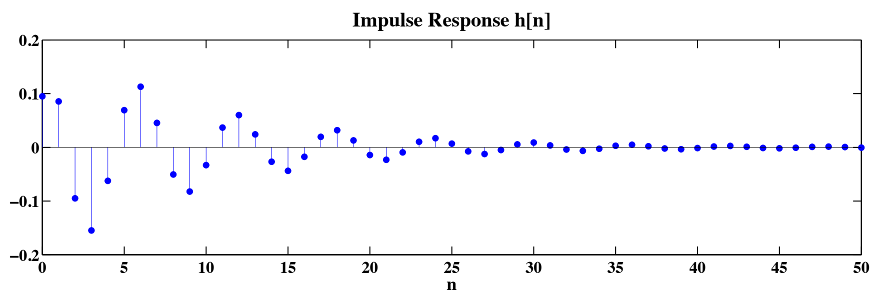


FIGURE 5.35: Impulse response $h[n]$ of the above resonator.

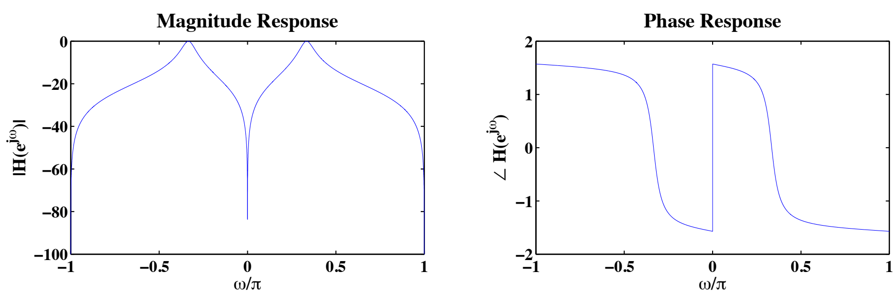


FIGURE 5.36: Magnitude response and phase response of the above resonator.

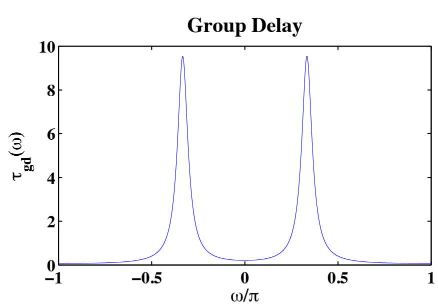


FIGURE 5.37: Group-delay of the above resonator.

15. (a) tba

    (b) See plot below.

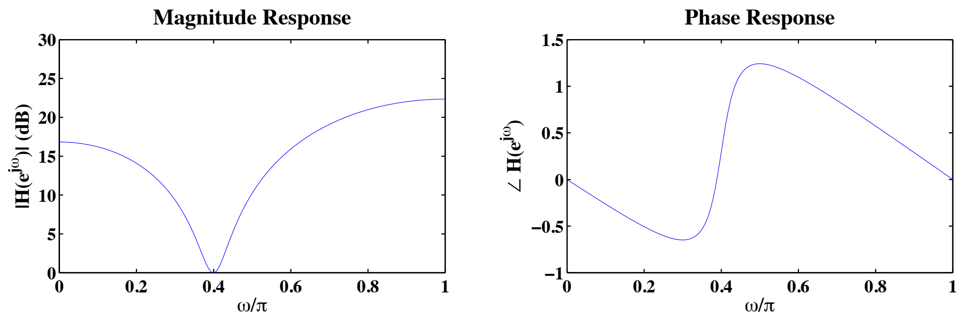


FIGURE 5.38: Magnitude response and phase response of the above resonator.

   (c) Solution: The analytic 3-dB bandwidth is 0.2118 while the approximate bandwidth is 0.2107.

16. (a) Solution:

$$H_{lp}\big(\mathrm{e}^{\mathrm{j}\omega}\big) = \frac{\sum_{k=0}^{M} b_k \mathrm{e}^{-\mathrm{j}k\omega}}{1 + \sum_{k=0}^{N} a_k \mathrm{e}^{-\mathrm{j}k\omega}}$$

$$H_{hp}\big(\mathrm{e}^{\mathrm{j}\omega}\big) = H_{lp}\big(\mathrm{e}^{\mathrm{j}(\omega-\pi)}\big) = \frac{\sum_{k=0}^{M} b_k \mathrm{e}^{-\mathrm{j}k(\omega-\pi)}}{1 + \sum_{k=0}^{N} a_k \mathrm{e}^{-\mathrm{j}k(\omega-\pi)}}$$

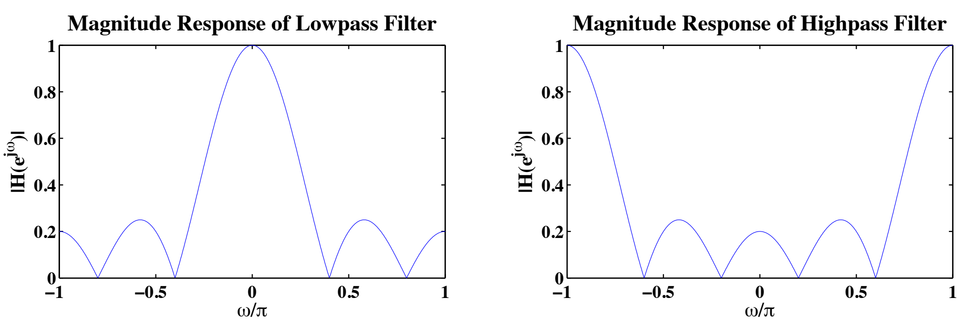


FIGURE 5.39: Magnitude response of lowpass and highpass filter.

   (b) Solution:

$$H_{lp}\big(\mathrm{e}^{\mathrm{j}\omega}\big) = \frac{1 + 1.655\mathrm{e}^{-\mathrm{j}\omega} + 1.655\mathrm{e}^{-2\mathrm{j}\omega} + \mathrm{e}^{-3\mathrm{j}\omega}}{1 - 1.57\mathrm{e}^{-\mathrm{j}\omega} + 1.264\mathrm{e}^{-2\mathrm{j}\omega} - 0.4\mathrm{e}^{-3\mathrm{j}\omega}}$$

$$H_{hp}(e^{j\omega}) = \frac{1 - 1.655e^{-j\omega} + 1.655e^{-2j\omega} - e^{-3j\omega}}{1 + 1.57e^{-j\omega} + 1.264e^{-2j\omega} + 0.4e^{-3j\omega}}$$

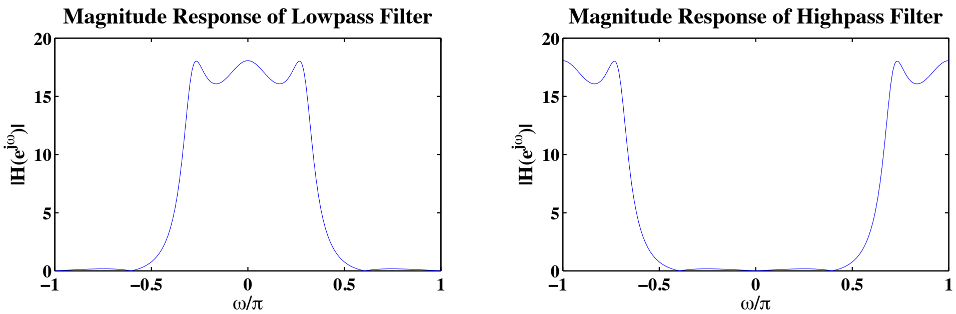


FIGURE 5.40: Magnitude response of lowpass and highpass filter.

17. Proof:

$$H_k(z) = z^{-1}\frac{1 - p_k^* z}{1 - p_k z^{-1}} = \frac{z^{-1} - p_k^*}{1 - p_k z^{-1}} \quad (5.157)$$

$$|H_k(e^{j\omega})| = 1, \quad (5.160)$$

$$\angle H_k(e^{j\omega}) = -\omega - 2\tan^{-1}\frac{r_k \sin(\omega - \phi_k)}{1 - r_k\cos(\omega - \phi_k)} \quad (5.161)$$

$$\tau_k(\omega) = \frac{1 - r_k^2}{1 + r_k^2 - 2r_k\cos(\omega - \phi_k)} \quad (5.162)$$

The frequency response is

$$H_k(e^{j\omega}) = e^{-j\omega}\frac{1 - p_k^* e^{j\omega}}{1 - p_k e^{-j\omega}} = \frac{e^{-j\omega} - p_k^*}{1 - p_k e{-}j\omega}$$

also we have

$$p_k = r_k e^{j\phi_k}$$

First, we prove (5.160)

$$|H_k(e^{j\omega})| = |e^{-j\omega}|\frac{|1 - p_k^* e^{j\omega}|}{|1 - p_k e^{-j\omega}|} = 1$$

Then, we prove (5.161)

$$
\begin{aligned}
\angle H_k\big(\mathrm{e}^{\mathrm{j}\omega}\big) &= \angle \mathrm{e}^{-\mathrm{j}\omega} + \angle(1 - p_k^* \mathrm{e}^{\mathrm{j}\omega}) - \angle(1 - p_k \mathrm{e}^{-\mathrm{j}\omega}) \\
&= -\omega + \angle(1 - r_k \mathrm{e}^{-\mathrm{j}\phi_k} \mathrm{e}^{\mathrm{j}\omega}) - \angle(1 - r_k \mathrm{e}^{\mathrm{j}\phi_k} \mathrm{e}^{-\mathrm{j}\omega}) \\
&= -\omega + \angle(1 - r_k \cos(\omega - \phi_k) - \mathrm{j} r_k \sin(\omega - \phi_k)) \\
&\quad - \angle(1 - r_k \cos(\omega - \phi_k) + \mathrm{j} r_k \sin(\omega - \phi_k)) \\
&= -\omega - 2\tan^{-1} \frac{r_k \sin(\omega - \phi_k)}{1 - r_k \cos(\omega - \phi_k)}
\end{aligned}
$$

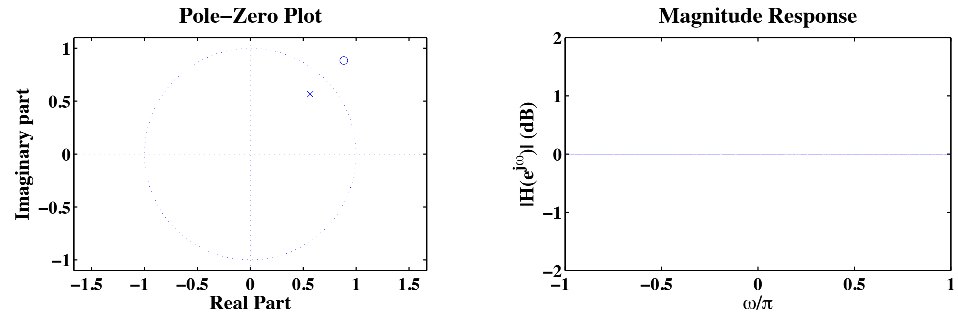
Finally, we prove (5.162)

$$
\begin{aligned}
\tau_{gd}(\omega) &= -\frac{d\angle H\big(\mathrm{e}^{\mathrm{j}\omega}\big)}{d\omega} = 1 + 2 \times \frac{1}{1 + \left[\frac{r_k \sin(\omega-\phi_k)}{1 - r_k \cos(\omega-\phi_k)}\right]^2} \\
&\quad \times \frac{r_k \cos(\omega - \phi_k)[1 - r_k \cos(\omega - \phi_k)] - r_k \sin(\omega - \phi_k) r_k \sin(\omega - \phi_k)}{[1 - r_k \cos(\omega - \phi_k)]^2} \\
&= 1 + 2 \times \frac{r_k \cos(\omega - \phi_k) - r_k^2 \cos^2(\omega - \phi_k) - r_k^2 \sin^2(\omega - \phi_k)}{[1 - r_k \cos(\omega - \phi_k)]^2 + [r_k \sin(\omega - \phi_k)]^2} \\
&= \frac{1 - r_k^2}{1 + r_k^2 - 2 r_k \cos(\omega - \phi_k)}
\end{aligned}
$$

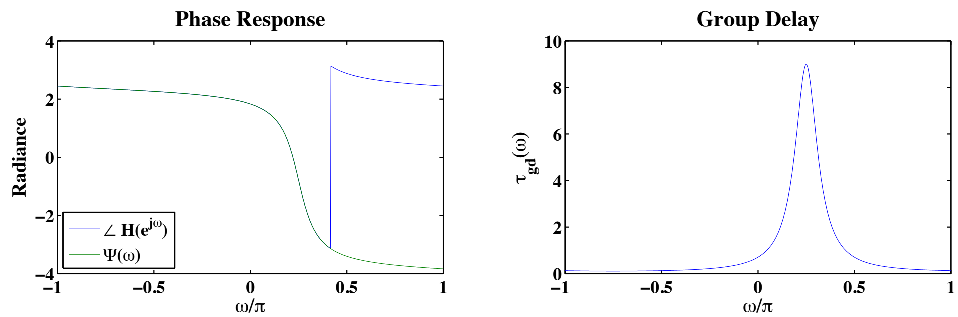
MATLAB scripts:

```
% P0517: Illustrate a first order allpass filter
close all; clc
%% Specification:
p = 0.8*exp(j*pi/4);
b = [-p',1];
a = [1 -p];
w = linspace(-1,1,1000)*pi;
H = freqz(b,a,w);
H_mag = abs(H);
H_mag_db = 10*log10(H_mag.^2);
H_phase = angle(H);
H_phase_unwrap = unwrap(H_phase);
gd = grpdelay(b,a,w);

%% Plot:
hfa = figconfg('P0517a','long');
```



(a)



(b)

FIGURE 5.41: Figure 5.29 regeneration.

```
subplot(121)
zplane(b,a)
xlabel('Real Part','fontsize',LFS)
ylabel('Imaginary part','fontsize',LFS)
title('Pole-Zero Plot','fontsize',TFS)
subplot(122)
plot(w/pi,H_mag_db)
ylim([-2 2])
xlabel('\omega/\pi','fontsize',LFS)
ylabel('|H(e^{j\omega})| (dB)','fontsize',LFS)
title('Magnitude Response','fontsize',TFS)

hfb = figconfg('P0517b','long');
subplot(121)
plot(w/pi,H_phase,w/pi,H_phase_unwrap)
xlabel('\omega/\pi','fontsize',LFS)
```

```
ylabel('Radiance','fontsize',LFS)
title('Phase Response','fontsize',TFS)
legend('\angle H(e^{j\omega})','\Psi(\omega)','location','southwest')
subplot(122)
plot(w/pi,gd)
xlabel('\omega/\pi','fontsize',LFS)
ylabel('\tau_{gd}(\omega)','fontsize',TFS)
title('Group Delay','fontsize',TFS)
```

18. (a) Solution:
The frequency response is:

$$\begin{aligned} H(e^{j\omega}) &= \frac{2 + 3.125e^{-2j\omega}}{1 - 0.9e^{-j\omega} + 0.81e^{-2j\omega}} \\ &= \frac{2 + 3.125\cos 2\omega - 3.125j\sin 2\omega}{(1 - 0.9\cos\omega + 0.81\cos 2\omega) + j(0.9\sin\omega - 0.81\sin 2\omega)} \end{aligned}$$

The magnitude response is:

$$\begin{aligned} |H(e^{j\omega})| &= \frac{\sqrt{(2 + 3.125\cos 2\omega)^2 + 3.125^2\sin^2 2\omega}}{\sqrt{(1 - 0.9\cos\omega + 0.81\cos 2\omega)^2 + (0.9\sin\omega - 0.81\sin 2\omega)^2}} \\ &= \frac{\sqrt{4 + 3.125^2 + 12.5\cos 2\omega}}{\sqrt{1 + 0.9^2 + 0.81^2 - 2\times 0.9\times 1.81\cos\omega + 2\times 0.81\cos 2\omega}} \end{aligned}$$

The phase response is:

$$\angle H(e^{j\omega}) = -\tan^{-1}\frac{3.125\sin 2\omega}{2 + 3.125\cos 2\omega} - \tan^{-1}\frac{0.9\sin\omega - 0.81\sin 2\omega}{1 - 0.9\cos\omega + 0.81\cos 2\omega}$$

(b) Solution:

$$H(z) = \frac{2(1 + \frac{25}{16}z^{-2})}{1 - 0.9z^{-1} + 0.81z^{-2}} = \frac{2(1 + \frac{5}{4}jz^{-1})(1 - \frac{5}{4}jz^{-1})}{1 - 0.9z^{-1} + 0.81z^{-2}}$$

$$\begin{aligned} H_{\min}(z) &= \frac{2\times\frac{25}{8}(1 - \frac{4}{5}jz^{-1})(1 + \frac{4}{5}jz^{-1})}{1 - 0.9z^{-1} + 0.81z^{-2}} \\ &= \frac{3.125 + 2z^{-2}}{1 - 0.9z^{-1} + 0.81z^{-2}} \end{aligned}$$

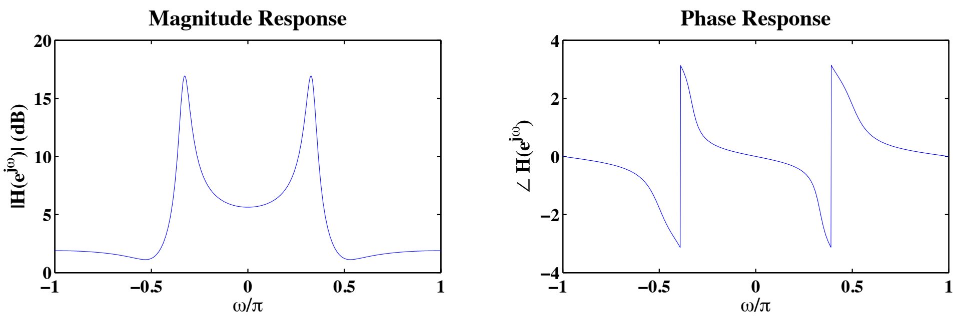


FIGURE 5.42: Magnitude response and phase responses of the system $H(z)$.

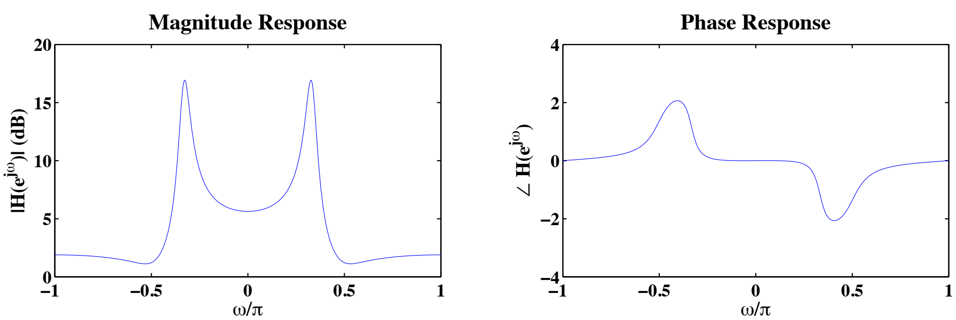


FIGURE 5.43: Magnitude response and phase responses of the minimum-phase system $H_{\text{min}}(z)$.

(c) Solution:

$$H_{\text{eq}}(z) = \frac{Gz^{-n_d}}{H_{\text{min}}(z)}$$

Choose $G = 1$ and $n_d = 0$, we have the equalizer system:

$$H_{\text{eq}}(z) = \frac{1}{H_{\text{min}}(z)} = \frac{1 - 0.9z^{-1} + 0.81z^{-2}}{3.125 + 2z^{-2}}$$

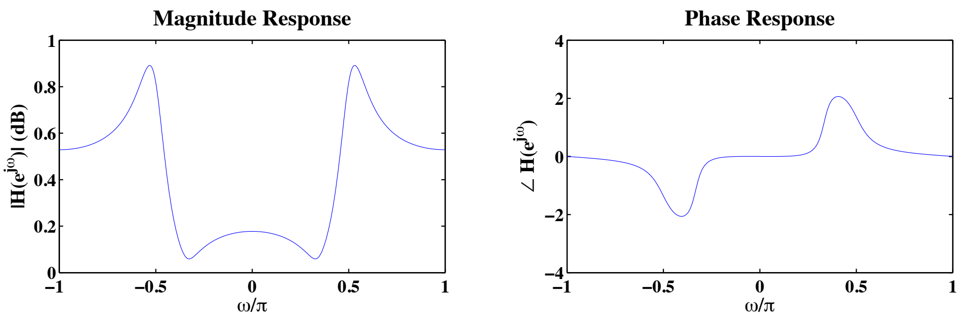


FIGURE 5.44: Magnitude response and phase responses of the equalizer system $H_{\rm eq}(z)$.

19. (a) Proof:
Substitute $z = r\mathrm{e}^{\mathrm{j}\theta}$ into $H(z)$, we have

$$H(z) = \frac{r^{-1}\mathrm{e}^{-\mathrm{j}\theta} - a}{1 - ar^{-1}\mathrm{e}^{-\mathrm{j}\theta}} = \frac{(-a + r^{-1}\cos\theta) - \mathrm{j}r^{-1}\sin\theta}{(1 - ar^{-1}\cos\theta) + \mathrm{j}ar^{-1}\sin\theta}$$

$$|H(z)| = \sqrt{\frac{(-a + r^{-1}\cos\theta)^2 + r^{-2}\sin^2\theta}{(1 - ar^{-1}\cos\theta)^2 + a^2r^{-2}\sin^2\theta}} = \sqrt{\frac{a^2r^2 + 1 - 2ar\cos\theta}{r^2 + a^2 - 2ar\cos\theta}}$$

where $a^2r^2 + 1 - 2ar\cos\theta > 0$ and $r^2 + a^2 - 2ar\cos\theta > 0$.

$$a^2r^2 + 1 - (r^2 + a^2) = (1 - r^2)(1 - a^2)$$

where $1 - a^2 > 0$. Hence, we proved that

$$|H(z)| \begin{cases} < 1 & \text{for } r > 1, \\ = 1 & \text{for } r = 1, \\ > 1 & \text{for } r < 1. \end{cases}$$

(b) Proof:
The frequency response of the system is

$$H\big(\mathrm{e}^{\mathrm{j}\omega}\big) = \frac{\mathrm{e}^{-\mathrm{j}\omega} - a}{1 - a\mathrm{e}{-}\mathrm{j}\omega} = \frac{(\cos\omega - a) - \mathrm{j}\sin\omega}{(1 - a\cos\omega) + \mathrm{j}a\sin\omega}$$

The group-delay $\tau(\omega)$ is

$$\tau(\omega) = \frac{1 - a^2}{1 + a^2 - 2a\cos\omega}$$

Substitute $\tau(\omega)$ into the integral, we have

$$\int_0^\pi \tau(\omega)\mathrm{d}\omega = \left(\omega + 2\tan^{-1}\frac{a\sin\omega}{1-a\cos\omega}\right)\Bigg|_0^\pi = \pi$$

20. (a) Solution:
Using time-expansion property,

$$x_{(k)}[n] = \begin{cases} x[r], & n = rk, \\ 0, & n \neq rk. \end{cases} \xrightarrow{z} X(z^k)$$

The time sequence corresponding to z-transform

$$A(z) = \frac{z^{-1}}{1-az^{-1}}$$

is

$$x[n] = a^{n-1}u[n-1]$$

Hence, the impulse response $h[n]$ can be defined as

$$h[n] = \begin{cases} x[r], & n = r \cdot D \\ 0, & n \neq r \cdot D \end{cases}$$

(b) Solution:
The frequency response can be written as

$$H(\mathrm{e}^{\mathrm{j}\omega}) = \frac{\mathrm{e}^{-\mathrm{j}D\omega}}{1-a\mathrm{e}^{-\mathrm{j}D\omega}} = \frac{\mathrm{e}^{-\mathrm{j}D\omega}}{1-a\cos D\omega + \mathrm{j}a\sin D\omega}$$

The magnitude response is

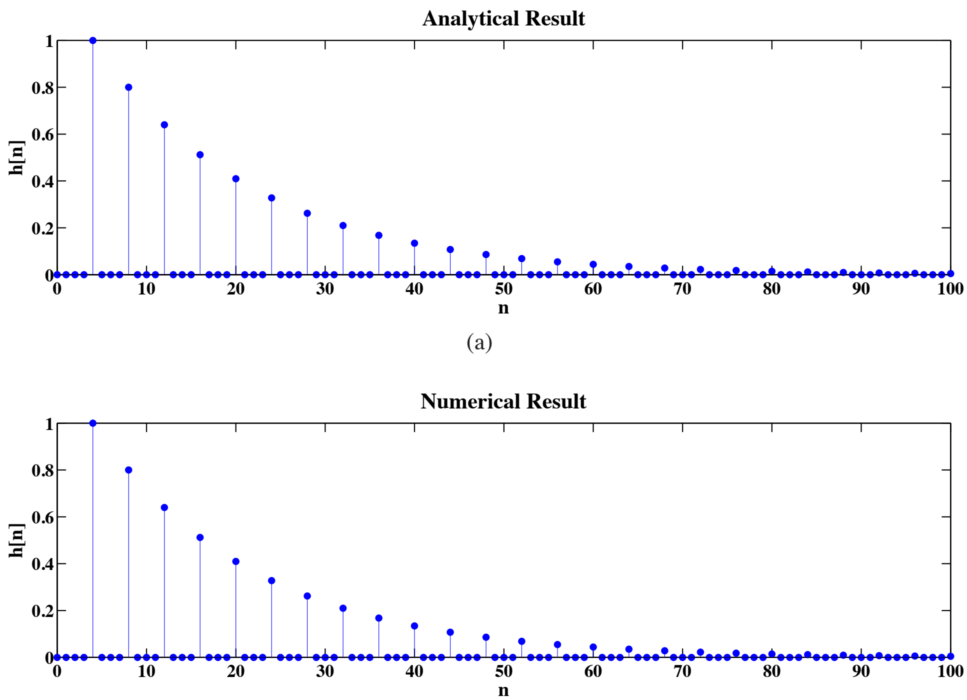
$$|H(\mathrm{e}^{\mathrm{j}\omega})| = \frac{1}{\sqrt{(1-a\cos D\omega)^2 + a^2\sin^2 D\omega}} = \frac{1}{\sqrt{1+a^2-2a\cos D\omega}}$$

which exhibits D peaks and D dips over $0 \leq \omega \leq 2\pi$.
When $a > 0$,

$$\begin{cases} \text{Dip locations:} & \omega = \frac{(2k+1)\pi}{D}, 0 \leq k \leq D-1 \\ \text{Peak locations:} & \omega = \frac{2k\pi}{D}, 0 \leq k \leq D-1 \end{cases}$$

When $a < 0$,

$$\begin{cases} \text{Peak locations:} & \omega = \frac{(2k+1)\pi}{D}, 0 \leq k \leq D-1 \\ \text{Dip locations:} & \omega = \frac{2k\pi}{D}, 0 \leq k \leq D-1 \end{cases}$$



(a)

(b)

FIGURE 5.45: MATLAB verification of analytical expression of the impulse response $h[n]$ for $D = 4$ and $a = 0.8$.

(c) MATLAB script:

```
% P0520: Illustrate Comb filter
close all; clc
%% Specification:
% D = 4;
% a = 0.8;
% D = 5;
% a = 0.9;
D = 8;
a = -0.8;
bh = zeros(1,D+1);
bh(end) = 1;
ah = zeros(1,D+1);
ah(1) = 1; ah(end) = -a;
```

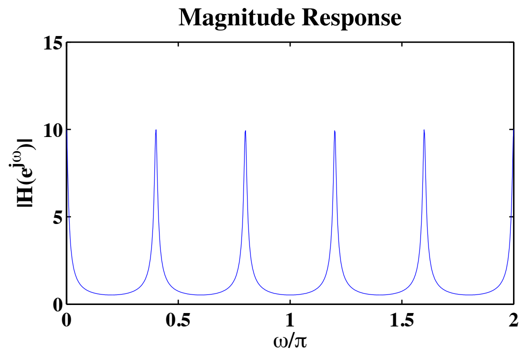


FIGURE 5.46: Magnitude response for $D = 4$ and $a = 0.8$.

```
n = 0:100;
xn = a.^(n-1);
hn_ana = zeros(size(n));
ind = (mod(n,D) == 0);
hn_ana(ind) = xn(1:sum(ind));
hn_ana(1) = 0;

[hn nn] = impz(bh,ah,n(end)+1);

w = linspace(0,2,1000)*pi;
H = freqz(bh,ah,w);
H_mag = abs(H);
%% Plot:
hfa = figconfg('P0520a','long');
stem(n,hn_ana,'filled')
xlabel('n','fontsize',LFS)
ylabel('h[n]','fontsize',LFS)
title('Analytical Result','fontsize',TFS)

hfb = figconfg('P0520b','long');
stem(n,hn,'filled')
xlabel('n','fontsize',LFS)
ylabel('h[n]','fontsize',LFS)
title('Numerical Result','fontsize',TFS)

hfc = figconfg('P0520c','small');
```

```
plot(w/pi,H_mag)
xlabel('\omega/\pi','fontsize',LFS)
ylabel('|H(e^{j\omega})|','fontsize',LFS)
title('Magnitude Response','fontsize',TFS)
```

21. (a) Proof:

$$H(s) = \frac{(s-3)(s-2-\mathrm{j})(s-2+\mathrm{j})(s+1)}{(s+5)(s+3-3\mathrm{j})(s+3+3\mathrm{j})(s+2-2\mathrm{j})(s+2+2\mathrm{j})}$$

Hence, there are three zero on the right-hand plane which proved that the system $H(s)$ is NOT minimum phase system.

(b) Solution:

$$H(s) = H_{\min}(s) \cdot H_{\mathrm{ap}}(s)$$

$$H_{\min}(s) = \frac{(s+3)(s+2-\mathrm{j})(s+2+\mathrm{j})(s+1)}{(s+5)(s+3-3\mathrm{j})(s+3+3\mathrm{j})(s+2-2\mathrm{j})(s+2+2\mathrm{j})}$$

$$H_{\mathrm{ap}}(s) = \frac{(s-3)(s-2-\mathrm{j})(s-2+\mathrm{j})}{(s+3)(s+2-\mathrm{j})(s+2+\mathrm{j})}$$

(c) See plot below.

(d) See plot below.

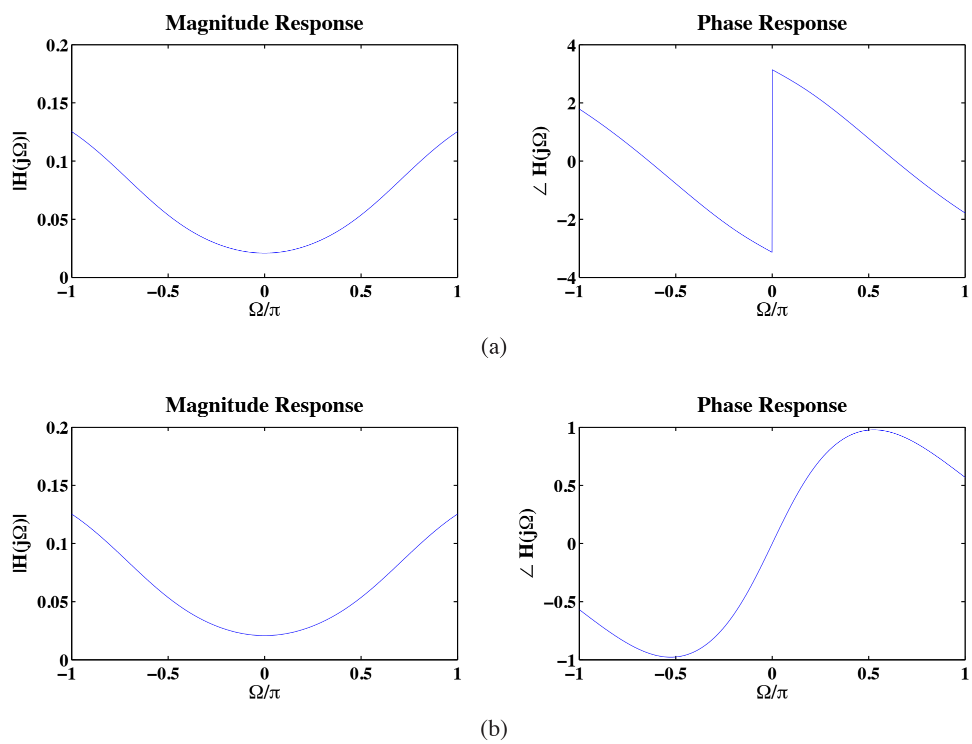


FIGURE 5.47: Magnitude and phase responses of (a) $H(s)$ and (b) $H_{\min}(s)$.

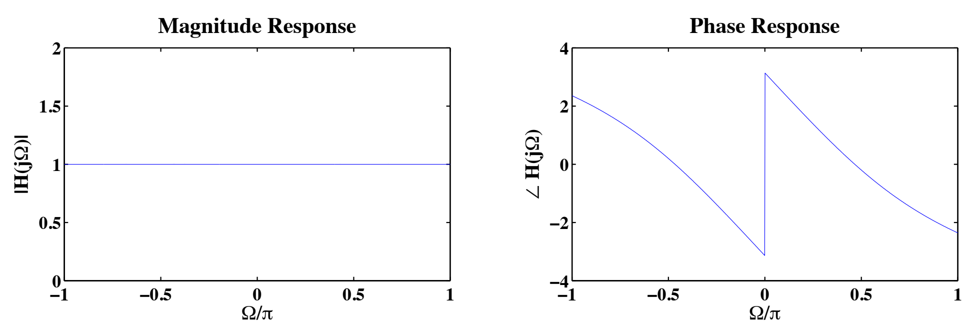


FIGURE 5.48: Magnitude and phase responses of $H_{\text{ap}}(s)$.

22. (a) See plot below.

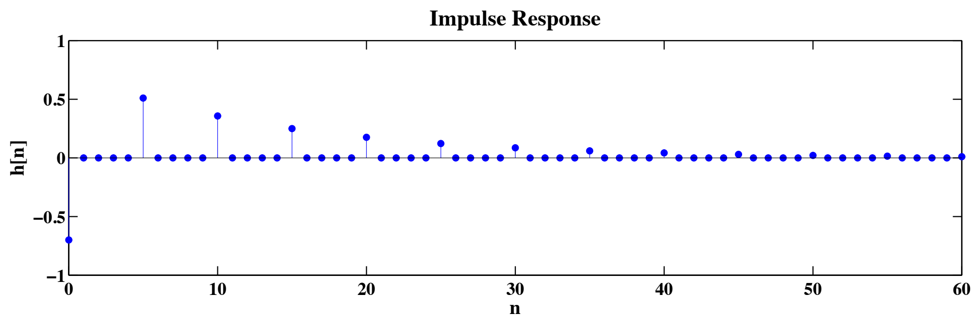


FIGURE 5.49: Impulse response $h[n]$.

(b) See plot below.

(c) See plot below.

(d) tba.

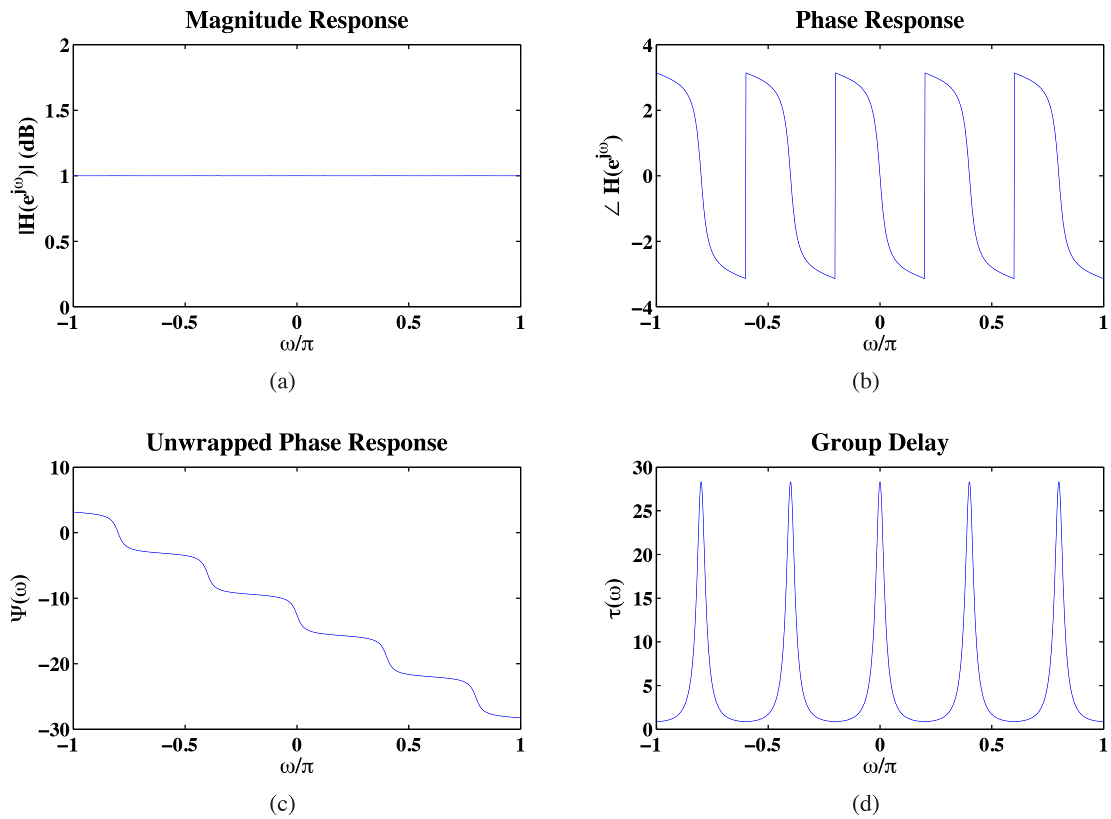


FIGURE 5.50: Magnitude, wrapped-phase, unwrapped-phase, and group-delay responses for $D = 5$ and $a = 0.7$.

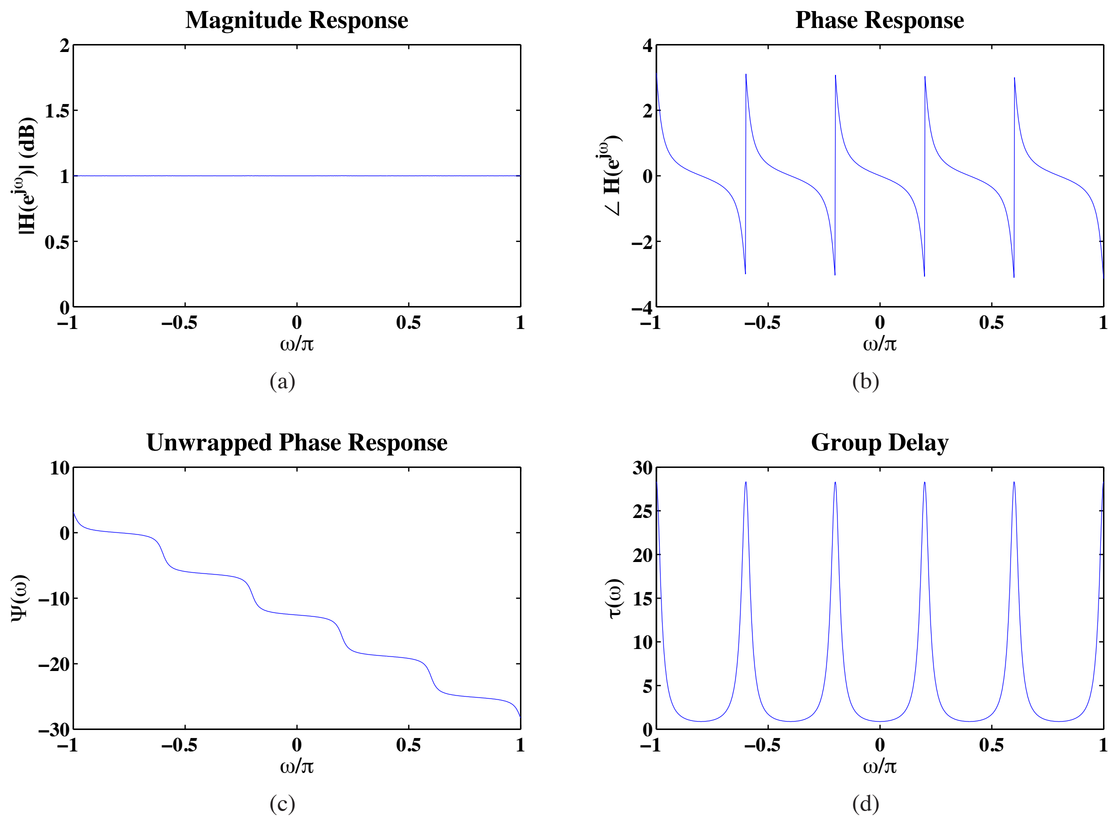


FIGURE 5.51: Magnitude, wrapped-phase, unwrapped-phase, and group-delay responses for $D = 5$ and $a = -0.7$.