CHAPTER 4

Fourier Representation of Signals

Tutorial Problems

1. Solution:
   If there exists a fundamental period T, we have

$$\begin{aligned}x(t+T) &= x_1(t+T)+x_2(t+T) = x_1(t+mT_1)+x_2(t+nT_2)\\ &= x_1(t)+x_2(t) = x(t), \quad m,n = 1,2,3,\ldots\end{aligned}$$

   The condition is a finite T exists that

$$T = mT_1 = nT_2, \quad m,n = 1,2,3,\ldots$$

2. (a) Solution:
   $x_1(t)$ is periodic and its fundamental period is $T = 24$.
   (b) Solution:
   $x_2(t)$ is aperiodic.
   (c) Solution:
   $x_3[n]$ is aperiodic.
   (d) Solution:
   $x_4[n]$ is periodic and its fundamental period is $N = 24$.
   (e) Solution:
   $x_5(t)$ is periodic and its fundamental period is $T = 6$.

3. (a) $x_1(t) = 2\cos(10\pi t) \times 3\cos(20\pi t)$, $-0.2 \le t \le 0.2$.
   (b) $x_2(t) = 3\sin(0.2\pi t) \times 5\cos(2\pi t)$, $0 \le t \le 20$.
   (c) $x_3(t) = 5\cos(5\pi t) \times 4\sin(10\pi t)$, $0 \le t \le 2$.
   (d) $x_4(t) = 4\sin(100\pi t) \times 2\cos(400\pi t)$, $0 \le t \le 0.01$.
   MATLAB script:

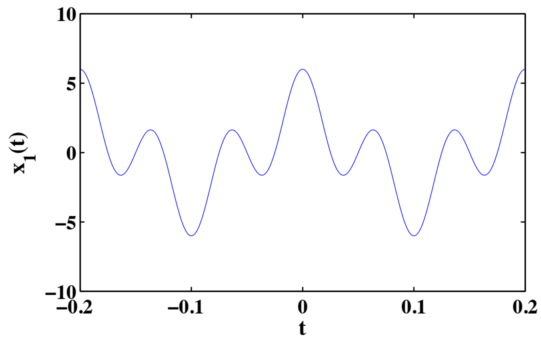


FIGURE 4.1: $x_1(t) = 2\cos(10\pi t) \times 3\cos(20\pi t), -0.2 \leq t \leq 0.2.$

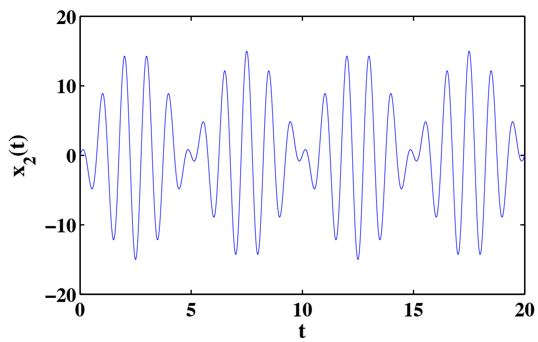


FIGURE 4.2: $x_2(t) = 3\sin(0.2\pi t) \times 5\cos(2\pi t), 0 \leq t \leq 20.$

```
% P0403: Verify the area under the function is zero
close all; clc
N = 100000;
%% Part (a):
% t = linspace(-0.2,0.2,N);
% x1 = 2*cos(10*pi*t).*3.*cos(20*pi*t);
% hf = figconfg('P0403','small');
% % hf = figconfg('P0403');
% plot(t,x1)
% xlabel('t','fontsize',LFS)
% ylabel('x_1(t)','fontsize',LFS)
% sum(x1.*0.4/N)
```

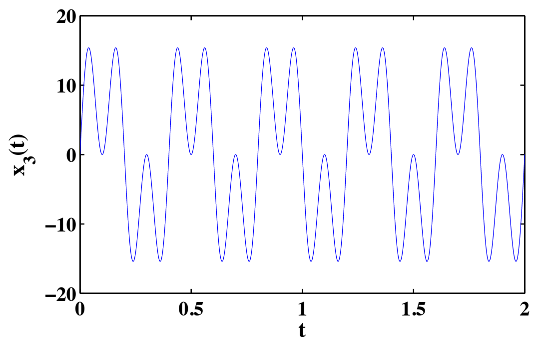


FIGURE 4.3: $x_3(t) = 5\cos(5\pi t) \times 4\sin(10\pi t), 0 \leq t \leq 2.$

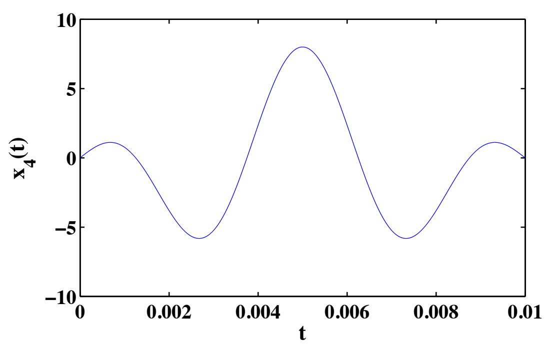


FIGURE 4.4: $x_4(t) = 4\sin(100\pi t) \times 2\cos(400\pi t), 0 \leq t \leq 0.01.$

```
%% Part (b):
% t = linspace(0,20,N);
% x2 = 3*sin(0.2*pi*t).*5.*cos(2*pi*t);
% hf = figconfg('P0403','small');
% plot(t,x2)
% xlabel('t','fontsize',LFS)
% ylabel('x_2(t)','fontsize',LFS)
% sum(x2.*20/N)

%% Part (c):
% t = linspace(0,2,N);
% x3 = 5*cos(5*pi*t).*4.*sin(10*pi*t);
% hf = figconfg('P0403','small');
```

```
% plot(t,x3)
% xlabel('t','fontsize',LFS)
% ylabel('x_3(t)','fontsize',LFS)
% sum(x3.*2/N)

%% Part (d):
t = linspace(0,0.01,N);
x4 = 4*sin(100*pi*t).*2.*cos(400*pi*t);
hf = figconfg('P0403','small');
plot(t,x4)
xlabel('t','fontsize',LFS)
ylabel('x_4(t)','fontsize',LFS)
sum(x4.*0.01/N)
```

4. (a) Solution:

The fundamental period of $x(t)$ is $T = 2$.

$$\int_0^2 \sin(3\pi t)dt = \int_0^2 \cos(8\pi t+\pi/3)dt = \int_0^2 \sin(3\pi t)\cos(8\pi t+\pi/3)dt = 0$$

$$\begin{aligned} P_{\text{av}} &= \frac{1}{T}\int_0^2 |x(t)|^2 dt \\ &= \frac{1}{2}\int_0^2 4dt + \frac{1}{2}\int_0^2 16\cos^2(3\pi t - \pi/2)dt + \frac{1}{2}\int_0^2 36\cos^2(8\pi t+\pi/3)dt \\ &= 4 + 8\int_0^2 \frac{1-\cos(6\pi t-\pi)}{2}dt + 18\int_0^2 \frac{1-\cos(16\pi t + 2\pi/3)}{2}dt \\ &= 30 \end{aligned}$$

(b) Solution:

$$\Omega_0 = 2\pi \cdot \frac{1}{T} = \pi$$

(c) Solution:

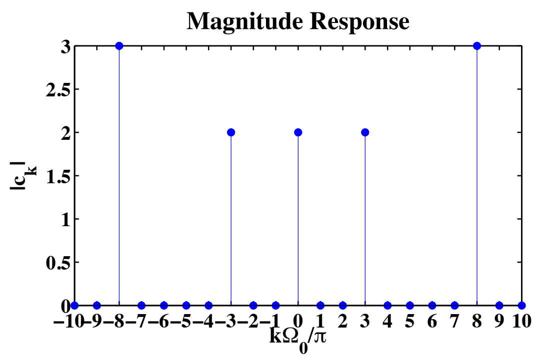
$$x(t) = 2\mathrm{e}^{\mathrm{j}0\pi t} + 2\mathrm{e}^{-\mathrm{j}\frac{\pi}{2}}\mathrm{e}^{j3\pi t} + 2\mathrm{e}^{\mathrm{j}\frac{\pi}{2}}\mathrm{e}^{-j3\pi t} + 3\mathrm{e}^{\mathrm{j}\frac{\pi}{3}}\mathrm{e}^{j8\pi t} + 3\mathrm{e}^{-\mathrm{j}\frac{\pi}{3}}\mathrm{e}^{-j8\pi t}$$

$$c_0 = 2, c_3 = 2\mathrm{e}^{-\mathrm{j}\frac{\pi}{2}}, c_{-3} = 2\mathrm{e}^{\mathrm{j}\frac{\pi}{2}}, c_8 = 3\mathrm{e}^{\mathrm{j}\frac{\pi}{3}}, c_{-8} = 3\mathrm{e}^{-\mathrm{j}\frac{\pi}{3}}.$$

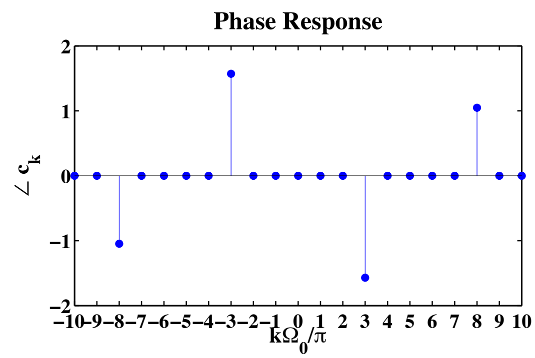
(d) Solution:

$$P_{\text{av}} = \sum_{k=-\infty}^{\infty} |c_k|^2 = 30$$

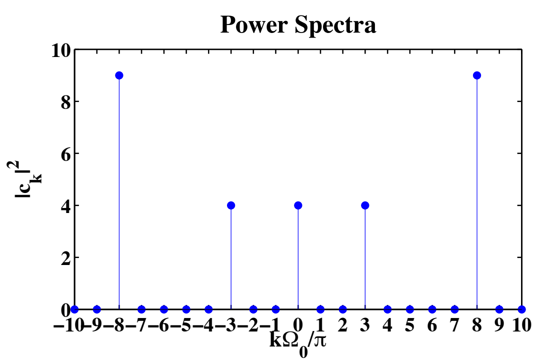
which verifies our computation in part (a).



(a)



(b)



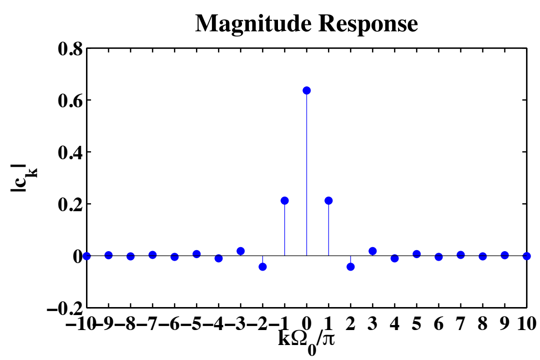
(c)

FIGURE 4.5: (a) Magnitude response of $x(t)$. (b) Phase response of $x(t)$. (c) Power spectra of $x(t)$.

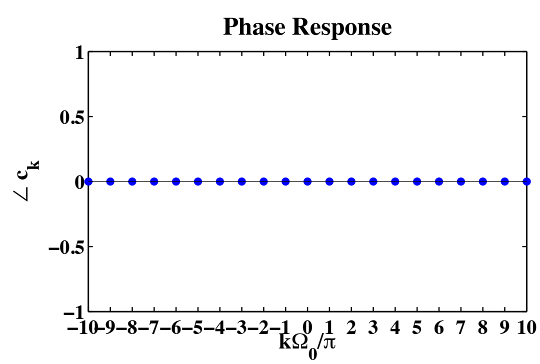
5. Solution:

$$T = \frac{2\pi}{10\pi} \cdot \frac{1}{2} = \frac{1}{10}, \quad \Omega_0 = 20\pi.$$

$$\begin{aligned}
c_k &= \frac{1}{T}\int_T \cos(10\pi t)\mathrm{e}^{\mathrm{j}k\Omega_0 t}dt = 5\int_{-\frac{1}{20}}^{\frac{1}{20}} \left(\mathrm{e}^{\mathrm{j}10\pi t} + \mathrm{e}^{-\mathrm{j}10\pi t}\right)\mathrm{e}^{\mathrm{j}k\cdot 2\pi t}dt \\
&= 5\int_{-\frac{1}{20}}^{\frac{1}{20}} \left(\mathrm{e}^{\mathrm{j}(2k+1)10\pi t} + \mathrm{e}^{-\mathrm{j}(2k-1)10\pi t}\right)dt \\
&= \frac{5}{\mathrm{j}(2k+1)10\pi} \cdot \mathrm{e}^{\mathrm{j}(2k+1)10\pi t}\Big|_{-\frac{1}{20}}^{\frac{1}{20}} + \frac{5}{\mathrm{j}(2k-1)10\pi} \cdot \mathrm{e}^{\mathrm{j}(2k-1)10\pi t}\Big|_{-\frac{1}{20}}^{\frac{1}{20}} \\
&= \frac{\sin\left[\frac{(2k+1)\pi}{2}\right]}{(2k+1)\pi} + \frac{\sin\left[\frac{(2k-1)\pi}{2}\right]}{(2k-1)\pi}
\end{aligned}$$



(a)



(b)

FIGURE 4.6: (a) Magnitude response of $x(t)$. (b) Phase response of $x(t)$.

6. Proof:

$$
\begin{aligned}
P_{\text{av}} &= \frac{1}{T_0}\int_{T_0} x(t)x^*(t)dt = \frac{1}{T_0}\int_{T_0}\left(\sum_k c_k \mathrm{e}^{\mathrm{j}k\Omega_0 t}\right)\left(\sum_m c_m \mathrm{e}^{\mathrm{j}m\Omega_0 t}\right)^* dt \\
&= \frac{1}{T_0}\sum_k\sum_m c_k c_m^* \int_{T_0} \mathrm{e}^{\mathrm{j}k\Omega_0 t}\mathrm{e}^{-\mathrm{j}m\Omega_0 t}
\end{aligned}
$$

$$
\int_{T_0} \mathrm{e}^{\mathrm{j}k\Omega_0 t}\mathrm{e}^{-\mathrm{j}m\Omega_0 t} = \begin{cases} 0, & k \neq m \\ T_0, & k = m \end{cases}
$$

$$
P_{\text{av}} = \frac{1}{T_0}\sum_k T_0 \cdot c_k \cdot c_k^* = \sum_{k=-\infty}^{\infty} |c_k|^2
$$

7. Solution:

$$
\begin{aligned}
c_k &= \frac{1}{T_0}\int_{T_0} y(t)\mathrm{e}^{-\mathrm{j}k\Omega_0 t}dt = \frac{1}{T_0}\int_{T_0} h(t)x(t)\mathrm{e}^{-\mathrm{j}k\Omega_0 t}dt \\
&= \frac{1}{T_0}\int_{T_0}\left(\sum_m a_m \mathrm{e}^{\mathrm{j}m\Omega_0 t}\right)\left(\sum_n b_n \mathrm{e}^{\mathrm{j}n\Omega_0 t}\right)\mathrm{e}^{-\mathrm{j}k\Omega_0 t}dt \\
&= \sum_m\sum_n a_m b_n \cdot \frac{1}{T_0}\int_{T_0} \mathrm{e}^{\mathrm{j}(m+n)\Omega_0 t}\cdot \mathrm{e}^{-\mathrm{j}k\Omega_0 t}dt \\
&= \sum_{m+n=k} a_m b_n = \sum_{\ell=-\infty}^{\infty} a_\ell b_{k-\ell}
\end{aligned}
$$

8. (a) Solution:

$$
\begin{aligned}
X(\mathrm{j}2\pi F) &= \int_{-\infty}^{\infty} x(t)\mathrm{e}^{-\mathrm{j}2\pi F t}dt = \int_{-1}^{1} \mathrm{e}^{-t}\cdot \mathrm{e}^{-\mathrm{j}2\pi F t}dt \\
&= -\frac{\mathrm{e}^{-(\mathrm{j}2\pi F+1)t}}{\mathrm{j}2\pi F+1}\bigg|_{-1}^{1} = \frac{\mathrm{e}^{\mathrm{j}2\pi F+1} - \mathrm{e}^{-\mathrm{j}2\pi F-1}}{\mathrm{j}2\pi F+1}
\end{aligned}
$$

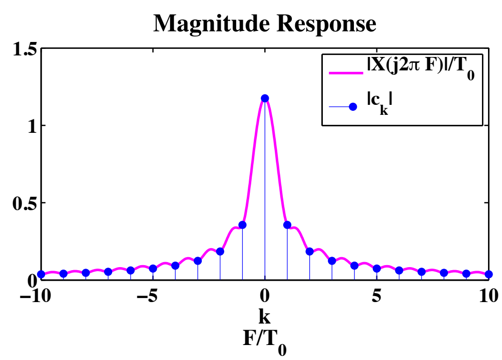
$$
T = 2, \quad F_0 = \frac{1}{T} = \frac{1}{2}, \quad \Omega_0 = 2\pi F_0 = \pi
$$

$$
\begin{aligned}
c_k &= \frac{1}{T}\int_{T} \tilde{x}(t)\mathrm{e}^{-\mathrm{j}2\pi k F_0 t}dt = \frac{1}{2}\int_{-1}^{1} \mathrm{e}^{-t}\cdot \mathrm{e}^{-\mathrm{j}\pi k t}dt \\
&= \frac{1}{2}\cdot \frac{\mathrm{e}^{-(\mathrm{j}k\pi+1)t}}{-(\mathrm{j}k\pi+1)}\Bigg|_{-1}^{1} = \frac{\mathrm{e}^{\mathrm{j}k\pi+1} - \mathrm{e}^{-\mathrm{j}k\pi-1}}{2(\mathrm{j}k\pi+1)}
\end{aligned}
$$

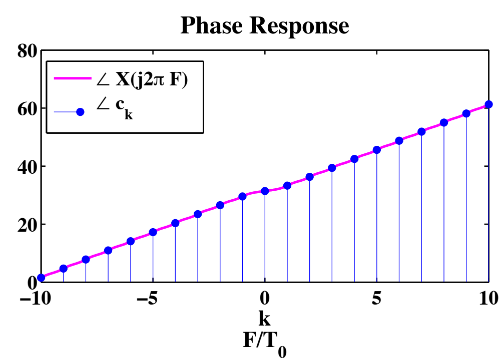
(b) Solution:

$$X(\mathrm{j}2\pi k/T_0)/T_0 = \frac{\mathrm{e}^{\mathrm{j}2\pi\frac{k}{2}+1} - \mathrm{e}^{-\mathrm{j}2\pi\frac{k}{2}-1}}{\mathrm{j}2\pi\frac{k}{2}+1} \cdot \frac{1}{2}$$
$$= \frac{\mathrm{e}^{\mathrm{j}k\pi+1} - \mathrm{e}^{-\mathrm{j}k\pi-1}}{2(\mathrm{j}k\pi+1)} = c_k$$

(c)



(a)



(b)

FIGURE 4.7: (a) $|X(\mathrm{j}2\pi F)|$ and $|c_k|$. (b) $\angle X(\mathrm{j}2\pi F)$ and $\angle c_k$.

9. (a) Solution:

$$X(\mathrm{j}2\pi F) = \int_{-\infty}^{\infty} x(t)\mathrm{e}^{-\mathrm{j}2\pi Ft}dt = \int_{-\infty}^{\infty} \frac{2\sin 2\pi t}{2\pi t}\mathrm{e}^{-\mathrm{j}2\pi Ft}dt$$
$$= \begin{cases} 1, & -1 < F < 1 \\ 0, & \text{otherwise} \end{cases}$$

(b) Solution:

$$c_k = 4\int_{-\frac{1}{80}}^{\frac{1}{80}} 1 \cdot \mathrm{e}^{-\mathrm{j}2\pi kF_0 t}dt = 4 \cdot \left.\frac{\mathrm{e}^{-\mathrm{j}2\pi kF_0 t}}{-\mathrm{j}2\pi kF_0}\right|_{-\frac{1}{80}}^{\frac{1}{80}} = \frac{\sin\frac{\pi}{10}k}{\pi k}$$

(c) Solution:

$$X_s(\mathrm{j}2\pi F) = \sum_{k=-\infty}^{\infty} \frac{\sin\frac{\pi}{10}k}{\pi k} \cdot X[\mathrm{j}2\pi(F-4k)]$$

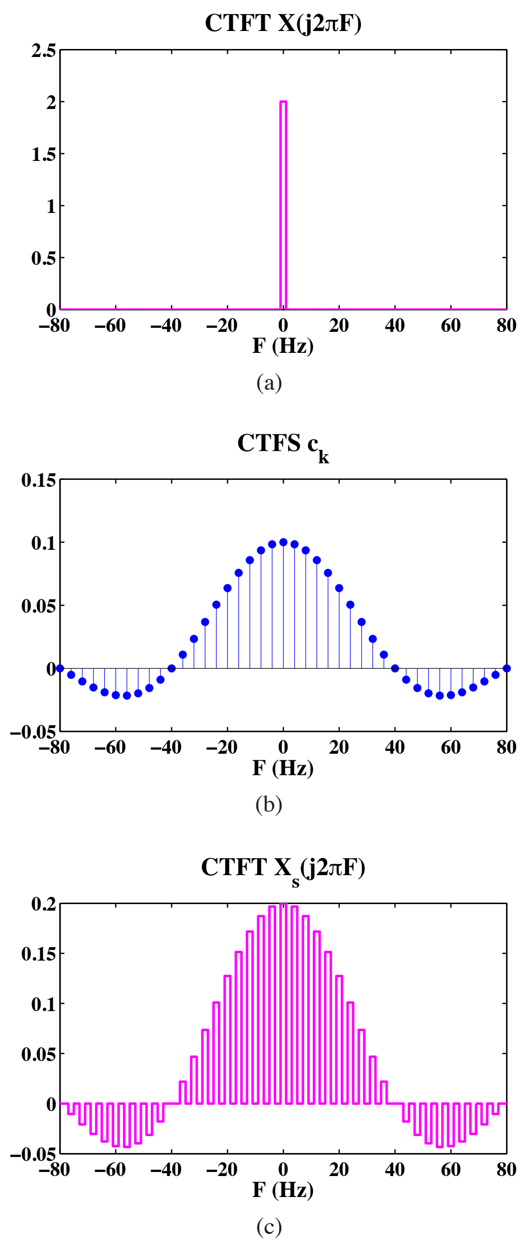


FIGURE 4.8: (a) Plot of CTFT $X(\mathrm{j}2\pi F)$. (b) Plot of CTFS coefficients c_k. (c) Plot of CTFT $X_s(\mathrm{j}2\pi F)$.

10. (a)
```
function c = dtfs0(x)
% P0410(a): Write a function c=dtfs0(x) which compute
% the DTFS coefficients (4.67) of a periodic signal
N = length(x);
x = x(:)';
k = 0:N-1;
n = 0:N-1;
nk = n'*k;
matexp = exp(-j*2*pi/N*nk);
c = x*matexp/N;
```

(b)
```
function x = idtfs0(c)
% P0410(b): Write a function x=idtfs0(c) which compute
% the inverse DTFS (4.63)
N = length(c);
c = c(:)';
k = 0:N-1;
n = 0:N-1;
kn = k'*n;
matexp = exp(j*2*pi/N*kn);
x = c*matexp;
```

(c)
```
% P0410c: Verify functions c=dtfs0(x) and x=idtfs0(c)
%         using specification in Example4.9
clc; close all;
x = [1 1 1 0 0 0 0 0 1 1]; N = length(x);
c1 = fft(x)/N;
c2 = real(dtfs0(x));
x1 = ifft(c1)*N;
x2 = real(idtfs0(c2));
```

11. (a) Solution:

$$x_1[n] = \sin[2\pi(3/10)n] = \frac{1}{2\mathrm{j}}\left[\mathrm{e}^{\mathrm{j}\frac{2\pi}{10}3\pi} - \mathrm{e}^{-\mathrm{j}\frac{2\pi}{10}3\pi}\right] = \frac{1}{2\mathrm{j}}\left[\mathrm{e}^{\mathrm{j}\frac{2\pi}{10}3\pi} - \mathrm{e}^{\mathrm{j}\frac{2\pi}{10}7\pi}\right]$$

$$c_3 = \frac{1}{2}\mathrm{e}^{-\mathrm{j}\frac{\pi}{2}}, \quad c_7 = \frac{1}{2}\mathrm{e}^{\mathrm{j}\frac{\pi}{2}}$$

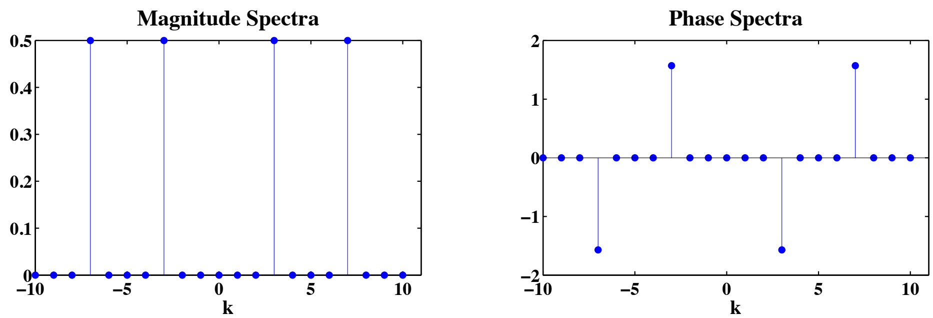


FIGURE 4.9: Magnitude and phase spectra of periodic sequence $x_1[n] = \sin[2\pi(3/10)n]$.

(b) Solution:

$$\begin{aligned}
c_k &= \frac{1}{6}\sum_{n=0}^{5} x_2[n]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{6}kn} \\
&= \frac{1}{6}\cdot\left[\mathrm{e}^{-\mathrm{j}\frac{2\pi}{6}k0} + 2\mathrm{e}^{-\mathrm{j}\frac{2\pi}{6}k1} - \mathrm{e}^{-\mathrm{j}\frac{2\pi}{6}k2} + 0 - \mathrm{e}^{-\mathrm{j}\frac{2\pi}{6}k4} + 2\mathrm{e}^{-\mathrm{j}\frac{2\pi}{6}k5}\right] \\
&= \frac{1}{6}\left[1 + 4\cos\left(\frac{2\pi}{6}k\right) - 2\cos\left(\frac{4\pi}{6}k\right)\right]
\end{aligned}$$

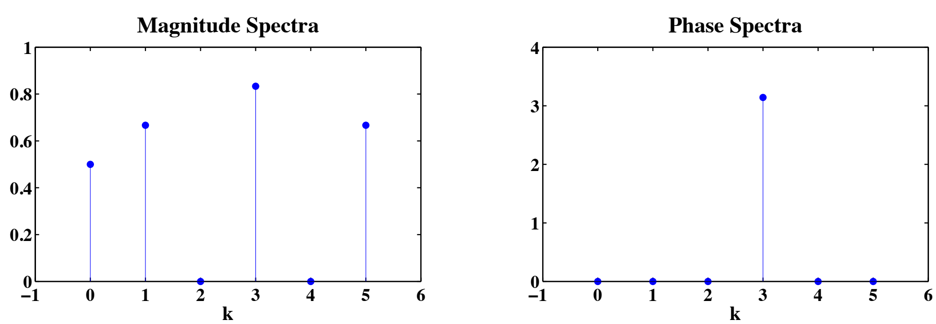


FIGURE 4.10: Magnitude and phase spectra of periodic sequence $x_2[n] = \{1, 2, -1, 0, -1, 2\}, 0 \le n \le 5$ (one period).

(c) Solution:

$$
\begin{aligned}
c_k &= \frac{1}{4}\sum_{n=0}^{3}\left[1-\sin\left(\frac{\pi n}{4}\right)\right] \mathrm{e}^{-\mathrm{j}\frac{2\pi}{4}kn} \\
&= \frac{1}{4}\left[1+(1-\sin(\frac{\pi}{4})\mathrm{e}^{-\mathrm{j}\frac{2\pi}{4}k})+0+(1-\sin(\frac{\pi}{4})\mathrm{e}^{-\mathrm{j}\frac{2\pi}{4}k3})\right] \\
&= \frac{1}{4}\left[1+(1-\sin(\frac{\pi}{4})\mathrm{e}^{-\mathrm{j}\frac{2\pi}{4}k})+0+(1-\sin(\frac{\pi}{4})\mathrm{e}^{\mathrm{j}\frac{2\pi}{4}k})\right] \\
&\frac{1}{4}\left[1+2(1-\sin(\frac{\pi}{4}))\cos(\frac{k\pi}{2})\right]
\end{aligned}
$$

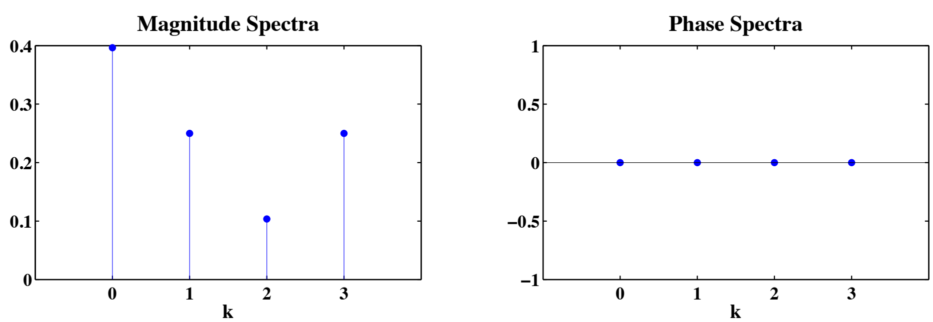


FIGURE 4.11: Magnitude and phase spectra of periodic sequence $x_3[n] = 1 - \sin(\pi n/4), 0 \leq n \leq 3$ (one period).

(d) Solution:

$$
\begin{aligned}
c_k &= \frac{1}{12}\sum_{n=0}^{11}\left[1-\sin\left(\frac{\pi n}{12}\right)\right] \mathrm{e}^{-\mathrm{j}\frac{2\pi}{4}kn} \\
&= \frac{1}{12}\left[1+(1-\sin(\frac{\pi}{4}))2\cos(\frac{k\pi}{6})+(1-\sin(\frac{3\pi}{4}))2\cos(\frac{k\pi}{2})\right. \\
&\left.+2\cos(\frac{2k\pi}{3})+(1-\sin(\frac{5\pi}{4}))2\cos(\frac{5k\pi}{6})+2\cos(k\pi)\right]
\end{aligned}
$$

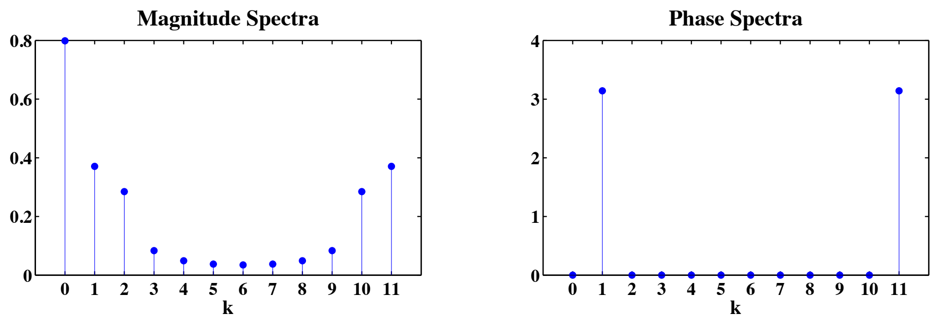


FIGURE 4.12: Magnitude and phase spectra of periodic sequence $x_4[n] = 1 - \sin(\pi n/4), 0 \leq n \leq 11$ (one period).

(e) Solution:

$$\begin{aligned} c_k &= \frac{1}{8}\sum_{n=0}^{7} x_5[n]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{8}kn} \\ &= \frac{1}{8}\left[1 + \mathrm{e}^{-\mathrm{j}\frac{2\pi}{8}k} + \mathrm{e}^{-\mathrm{j}\frac{2\pi}{8}k3} + \mathrm{e}^{-\mathrm{j}\frac{2\pi}{8}k4} + \mathrm{e}^{-\mathrm{j}\frac{2\pi}{8}k5} + \mathrm{e}^{-\mathrm{j}\frac{2\pi}{8}k7}\right] \\ &= \frac{1}{8}\left[1 + 2\cos(\frac{k\pi}{4}) + 2\cos(\frac{3k\pi}{4}) + \cos(k\pi)\right] \end{aligned}$$

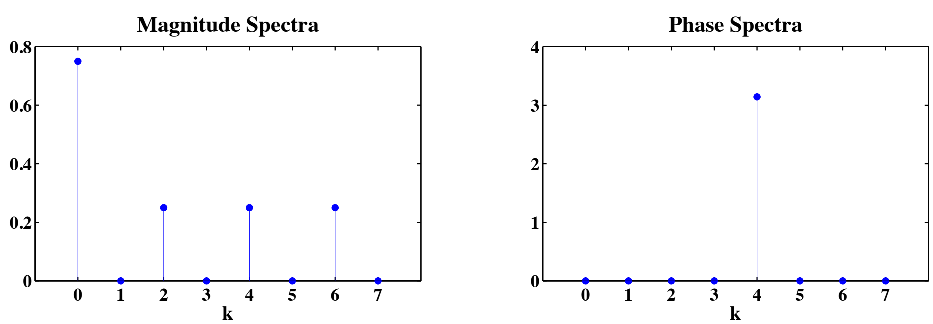


FIGURE 4.13: Magnitude and phase spectra of periodic sequence $x_5[n] = \{1, 1, 0, 1, 1, 1, 0, 1\}, 0 \leq n \leq 7$ (one period).

(f) Solution:

$$c_k = \frac{1}{N_0}\sum_{n=0}^{N_0-1} 1 \cdot \mathrm{e}^{-\mathrm{j}\frac{2\pi}{N_0}kn} = \delta[k]$$

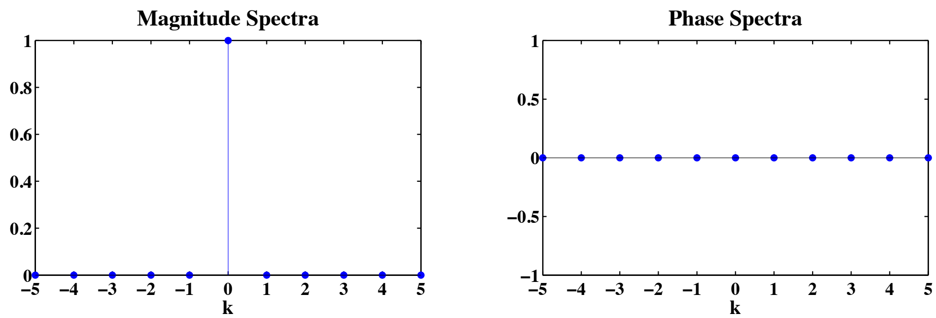


FIGURE 4.14: Magnitude and phase spectra of periodic sequence $x_6[n] = 1$ for all n.

12. Solution:

(a)

$$X_1(\omega) = \frac{1}{1 - \mathrm{e}^{-\mathrm{j}\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega - 2k\pi)$$

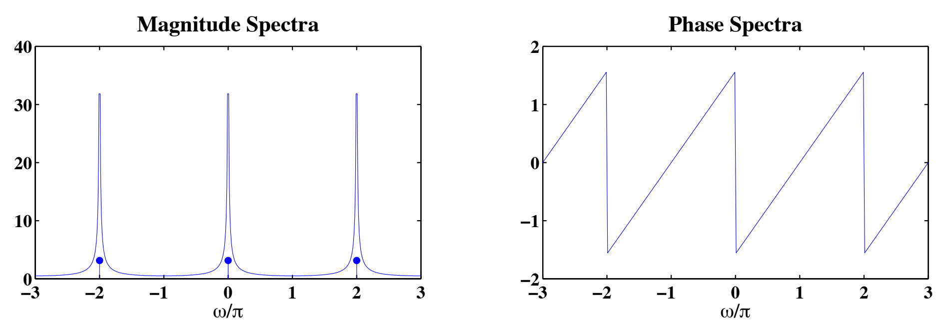


FIGURE 4.15: Magnitude and phase response for sequence $x_1[n] = u[n]$.

(b)

$$x_2[n] = \frac{1}{2}\left(e^{j\omega_0 n} + e^{-j\omega_0 n}\right)u[n]$$
$$= \frac{1/2}{1-e^{-j(\omega-\frac{\pi}{3})}} + \frac{1}{2}\sum_{k=-\infty}^{\infty}\pi\delta(\omega-\frac{\pi}{3}-2k\pi)$$
$$\frac{1/2}{1-e^{-j(\omega+\frac{\pi}{3})}} + \frac{1}{2}\sum_{k=-\infty}^{\infty}\pi\delta(\omega+\frac{\pi}{3}-2k\pi)$$

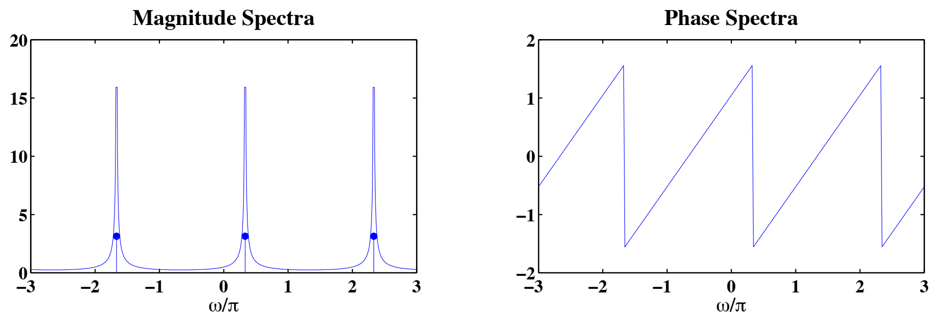


FIGURE 4.16: Magnitude and phase response for sequence $x_2[n] = \cos(\omega_0 n)u[n]$, $\omega_0 = \pi/3$.

13. (a) Solution:

$$x_1[n] = (1/2)^{|n|}\left(\frac{1}{2}e^{j\pi(n-1)/8} + \frac{1}{2}e^{-j\pi(n-1)/8}\right)$$
$$\text{DTFT}\left\{(1/2)^{|n|}\right\} = \sum_{n=-\infty}^{\infty}(1/2)^{|n|}e^{-j\omega n}$$
$$= \sum_{n=-\infty}^{-1}(1/2)^{-n}e^{-j\omega n} + 1 + \sum_{n=1}^{\infty}(1/2)^{n}e^{-j\omega n}$$
$$= \frac{3/2}{5/4-\cos\omega}$$
$$X_1(\omega) = \frac{1}{2}e^{j\pi/8}\frac{3/2}{5/4-\cos(\omega-\pi/8)} + \frac{1}{2}e^{-j\pi/8}\frac{3/2}{5/4-\cos(\omega+\pi/8)}$$

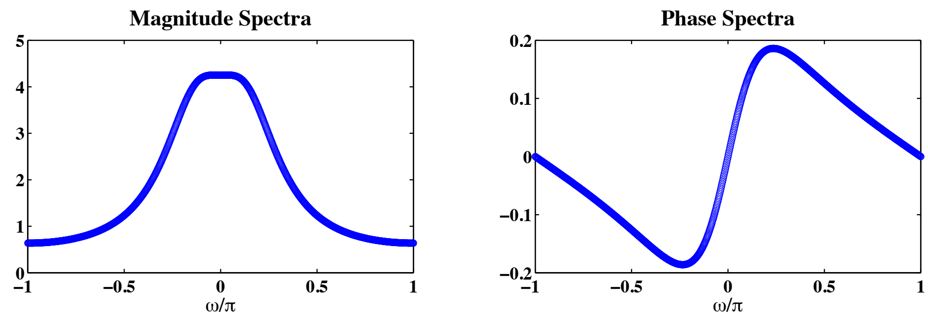


FIGURE 4.17: Magnitude and phase spectra of signal $x_1[n] = (1/2)^{|n|}\cos(\pi(n-1)/8)$.

(b) Solution:

$$X_2(\omega) = \sum_{n=-3}^{3} n\mathrm{e}^{-\mathrm{j}\omega n} = -2\mathrm{j}\sin(\omega) - 4\mathrm{j}\sin(2\omega) - 6\mathrm{j}\sin(3\omega)$$

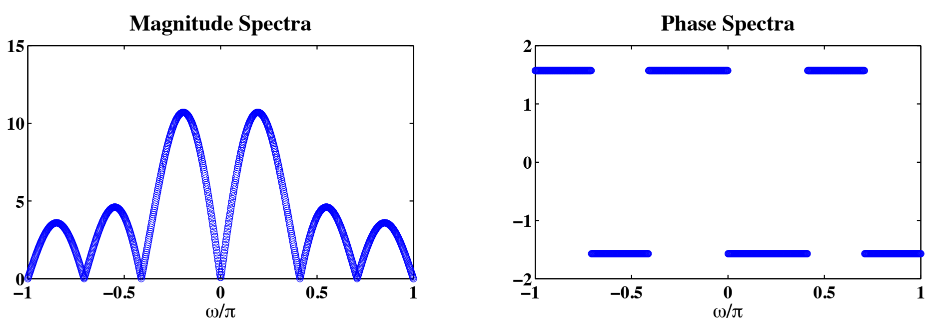


FIGURE 4.18: Magnitude and phase spectra of signal $x_2[n] = n(u[n+3] - u[n-4])$.

(c) Solution:

$$\begin{aligned} X_3(\omega) &= \sum_{n=-4}^{4} (2 - n/2)\mathrm{e}^{-\mathrm{j}\omega n} \\ &= 4\mathrm{e}^{4\mathrm{j}\omega} + \frac{7}{2}\mathrm{e}^{3\mathrm{j}\omega} + 3\mathrm{e}^{2\mathrm{j}\omega} + \frac{5}{2}\mathrm{e}^{\mathrm{j}\omega} + 2 + \frac{3}{2}\mathrm{e}^{-\mathrm{j}\omega} + \mathrm{e}^{-2\mathrm{j}\omega} + \frac{1}{2}\mathrm{e}^{-3\mathrm{j}\omega} \end{aligned}$$

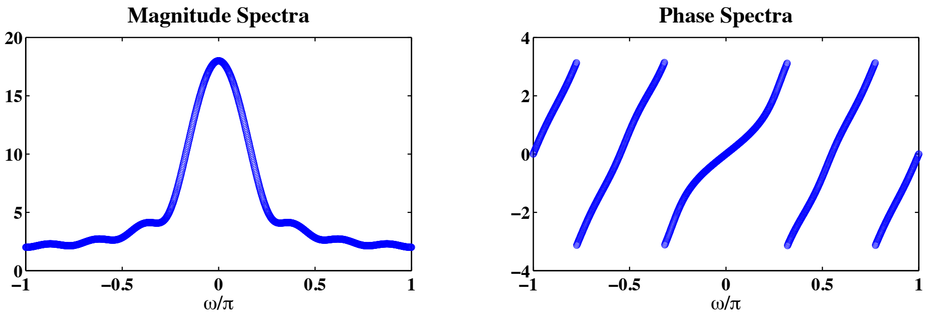


FIGURE 4.19: Magnitude and phase spectra of signal $x_3[n] = (2 - n/2)(u[n+4] - u[n-5])$.

14. (a) Solution:

$$X_1(\mathrm{e}^{\mathrm{j}\omega}) = \cos^2(\omega) + \sin^2(3\omega)$$
$$= 1 + \frac{1}{4}\left(\mathrm{e}^{2\mathrm{j}\omega} + \mathrm{e}^{-2\mathrm{j}\omega}\right) - \frac{1}{4}\left(\mathrm{e}^{6\mathrm{j}\omega} + \mathrm{e}^{-6\mathrm{j}\omega}\right)$$
$$x_1[n] = \{-\frac{1}{4}, 0, 0, 0, \frac{1}{4}, 0, \underset{\uparrow}{1}, 0, \frac{1}{4}, 0, 0, 0, -\frac{1}{4}\}$$

(b) Solution:

$$x_2[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_2(\mathrm{e}^{\mathrm{j}\omega})\mathrm{e}^{\mathrm{j}\omega n}d\omega$$
$$= \frac{1}{2\pi}\left(\int_{-\pi}^{-\omega_c} \mathrm{e}^{\mathrm{j}\omega n}d\omega + \int_{\omega_c}^{\pi} \mathrm{e}^{\mathrm{j}\omega n}d\omega\right) = \frac{-\sin\omega_c n}{\pi n}$$

(c) Solution:

$$x_3[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_3(\mathrm{e}^{\mathrm{j}\omega})\mathrm{e}^{\mathrm{j}\omega n}d\omega$$
$$= \frac{1}{2\pi}\left[\int_{-\pi/2}^{0}(1 + 2\omega/\pi)\mathrm{e}^{\mathrm{j}\omega n}d\omega + \int_{0}^{\pi/2}(1 - 2\omega/\pi)\mathrm{e}^{\mathrm{j}\omega n}d\omega\right]$$
$$= \frac{-2\sin(\frac{\pi}{2}n)}{(\pi n)^2}$$

(d) Solution:

$$\begin{aligned} x_4[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X_4\big(\mathrm{e}^{\mathrm{j}\omega}\big)\mathrm{e}^{\mathrm{j}\omega n}d\omega \\ &= \frac{1}{2\pi}\left[\int_{-\omega_c-\frac{\Delta}{2}}^{-\omega_c+\frac{\Delta}{2}} \mathrm{e}^{\mathrm{j}\omega n}d\omega + \int_{\omega_c-\frac{\Delta}{2}}^{\omega_c+\frac{\Delta}{2}} \mathrm{e}^{\mathrm{j}\omega n}d\omega\right] \\ &= \frac{2\sin(\frac{\Delta}{2}n)\cos(\omega_c n)}{\pi n} \end{aligned}$$

15. (a) Solution:

Time-shifting, Folding, and Linearity

$$X_1(\omega) = \mathrm{e}^{\mathrm{j}\omega}X(\omega) + \mathrm{e}^{\mathrm{j}\omega}X(-\omega)$$

(b) Solution:

Conjugation and Linearity

$$X_2(\omega) = \left(X(\omega) + X^*(-\omega)\right)/2$$

(c) Solution:

Differentiation and Linearity

$$X_3(\omega) = X(\omega) + 2\mathrm{j}\frac{dX(\omega)}{d\omega} + \frac{d^2X(\omega)}{d\omega^2}$$

16. Solution:

(a)

$$X\big(\mathrm{e}^{\mathrm{j}0}\big) = \sum_n x[n] = -1$$

(b)

$$x[n] \text{ real and even} \implies X\big(\mathrm{e}^{\mathrm{j}\omega}\big) \text{ real and even} \implies \angle X\big(\mathrm{e}^{\mathrm{j}\omega}\big) = 0$$

(c)

$$\int_{-\pi}^{\pi} X\big(\mathrm{e}^{\mathrm{j}\omega}\big)\,\mathrm{d}\omega = 2\pi x[0] = -6\pi$$

(d)

$$X(\mathrm{e}^{\mathrm{j}\pi}) = \sum_n x[n]\mathrm{e}^{-\mathrm{j}\pi n} = \sum_n x[n]\cos(\pi n) = -1 - 2 - 3 - 4 - 1 = -9$$

(e)

$$\int_{-\pi}^{\pi} |X(\mathrm{e}^{\mathrm{j}\omega})|^2 \mathrm{d}\omega = 2\pi \sum_n |x[n]|^2 = 38\pi, \quad \text{Parseval's Theorem}$$

17. (a) Solution:

$$\begin{aligned} r_{xy}[\ell] &= \sum_{n=-\infty}^{\infty} x[n]y[n-\ell] \\ x[n] &= [1, 2, \underset{\uparrow}{3}, 2, 1] \\ y[n] &= [2, 1, \underset{\uparrow}{0}, -1, -2] \\ \ell = 1, y[n-1] &= [2, \underset{\uparrow}{1}, 0, -1, -2], \quad r_{xy}[1] = 6 \end{aligned}$$

Compute $r_{xy}[\ell]$ for $\ell \in [-4, 4]$, we have

$$r_{xy}[\ell] = [-2, -5, -8, -6, \underset{\uparrow}{0}, 6, 8, 5, 2]$$

(b) Solution:

$$\begin{aligned} \rho_{xy}[\ell] &= \frac{r_{xy}[\ell]}{\sqrt{E_x}\sqrt{E_y}} \\ E_x &= \sum_n |x[n]|^2 = 19, \quad E_y = \sum_n |y[n]|^2 = 10 \\ \rho_{xy}[\ell] &= \frac{1}{\sqrt{190}}[-2, -5, -8, -6, \underset{\uparrow}{0}, 6, 8, 5, 2] \end{aligned}$$

(c) Comments:
The two signal has exactly the same shape and only differs by a scale factor.

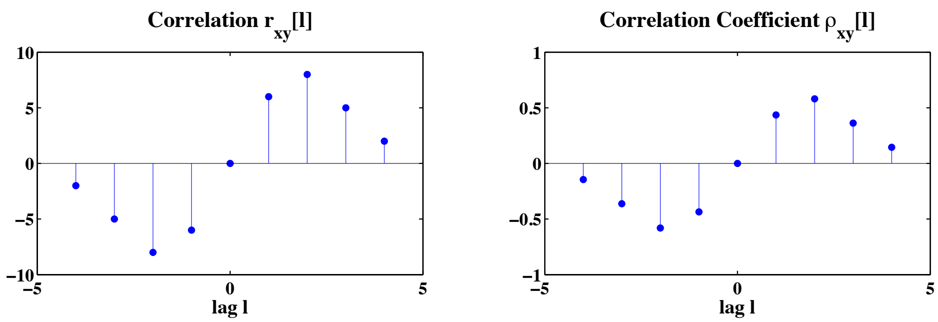


FIGURE 4.20: Plot of the correlation $r_{xy}[\ell]$ and correlation coefficient $\rho_{xy}[\ell]$ between the two signals.

18. (a) Solution:

$$\begin{aligned}
r_{xy}[\ell] &= \sum_{n=-\infty}^{\infty} (0.9)^n u[n](0.9)^{n-\ell}u[n-\ell] \\
&= u[-\ell-1]\sum_{n=0}^{\infty}(0.9)^{2n-\ell} + u[\ell]\sum_{n=0}^{\ell}(0.9)^{\ell} \\
&= \frac{1}{1-0.9^2}\left(0.9^{-\ell}u[-\ell-1] + 0.9^{\ell}u[\ell]\right) \\
E_x &= \sum_{n=0}^{\infty}|(0.9)^n|^2 = \frac{1}{1-0.9^2}, \quad E_y = E_x \\
\rho_{xy}[\ell] &= \frac{r_{xy}[\ell]}{\sqrt{E_x}\sqrt{E_y}} = 0.9^{-\ell}u[-\ell-1] + 0.9^{\ell}u[\ell]
\end{aligned}$$

(b) Solution:

$$\begin{aligned}
r_{xy}[\ell] &= \sum_{n=-\infty}^{\infty} (0.9)^n u[n](0.9)^{-n+\ell}u[-n+\ell] \\
&= u[\ell]\sum_{n=\ell}^{\infty}(0.9)^{2n-\ell} = (\ell+1)(0.9)^{\ell}u[\ell] \\
E_x &= \sum_{n=0}^{\infty}|(0.9)^n|^2 = \frac{1}{1-0.9^2}, \quad E_y = E_x \\
\rho_{xy}[\ell] &= \frac{r_{xy}[\ell]}{\sqrt{E_x}\sqrt{E_y}} = (1-0.9)^2(\ell+1)(0.9)^{\ell}u[\ell]
\end{aligned}$$

(c) Solution:

$$
\begin{aligned}
r_{xy}[\ell] &= \sum_{n=-\infty}^{\infty} (0.9)^n u[n] (0.9)^{n+5-\ell} u[n+5-\ell] \\
&= u[-\ell+4] \sum_{n=0}^{\infty} (0.9)^{2n+5-\ell} + u[\ell-5] \sum_{n=\ell-5}^{\infty} (0.9)^{2n+5-\ell} \\
&= \frac{1}{1-0.9^2} \left(0.9^{5-\ell} u[-\ell+4] + 0.9^{\ell-5} u[\ell-5] \right) \\
E_x &= \sum_{n=0}^{\infty} |(0.9)^n|^2 = \frac{1}{1-0.9^2}, \quad E_y = E_x \\
\rho_{xy}[\ell] &= \left(0.9^{5-\ell} u[-\ell+4] + 0.9^{\ell-5} u[\ell-5] \right)
\end{aligned}
$$

19.
```
function [rxy,l] = ccrs(x,nx,y,ny)
% P0419: Define function computing correlation rxy
%        between two finite length signals
% % Verification:
% nx = -2:2;
% ny = -2:2;
% x = [1 2 3 2 1];
% y = [2 1 0 -1 -2];
[rxy l] = conv0(x(:),nx(:),flipud(y(:)),sort(-ny));
```