CHAPTER 13

Random Signals

Tutorial Problems

1. (a) Solution:
   $P\{R = 1, B = 1, G = 1\} = \frac{1}{6\times6\times6} = \frac{1}{216}$

   (b) Solution:
   We can suppose the observed one is green, that is $P\{R = 1, B = 1|G = 1\} = \frac{1}{6\times6} = \frac{1}{36}$

   (c) Solution:
   $P\{G = 1, B = 1|R = 1\} = \frac{1}{6\times6} = \frac{1}{36}$

   (d) Solution: tba

2. MATLAB script:

```
% P1302: Figure 13.4 reproduction
clc; close all
load f16.mat
N = 20000; Fs = 19.98e3;
x = f16(1:N);
clear f16
[xo px] = epdf(x,50);
%% Plot
hfa = figconfg('P1302a','long');
plot((1:N)/Fs,x)
xlim([1 N]/Fs)
xlabel('Time (sec)','fontsize',LFS)
ylabel('Amplitude','fontsize',LFS)
title(['N = ',num2str(N),' samples'],'fontsize',TFS)
hfb = figconfg('P1302b','long');
```

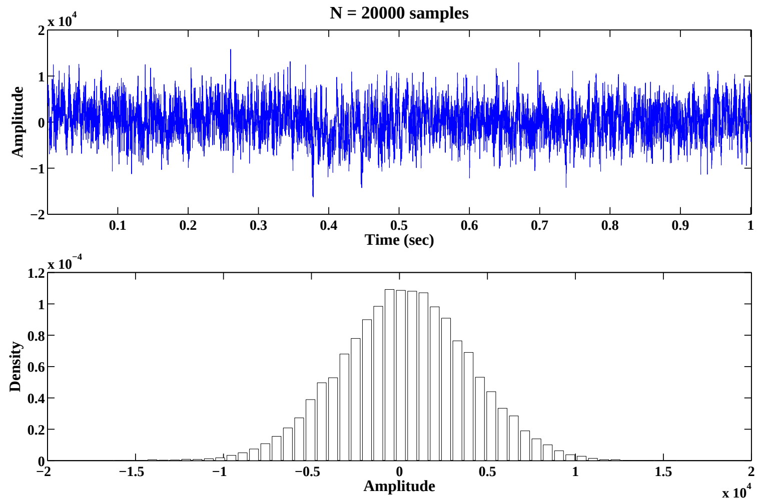

FIGURE 13.1: Waveform of F-16 noise recorded at co-pilot's seat and its empirical pdf using function `epdf` with 50 bins.

```
bar(xo,px,'w')
ylim([0 1.2e-4])
xlabel('Amplitude','fontsize',LFS)
ylabel('Density','fontsize',LFS)
```

3. (a) See plot below.

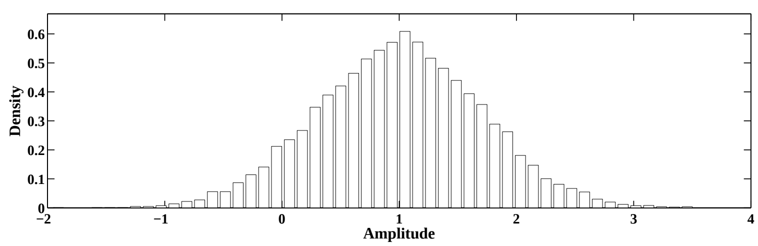

FIGURE 13.2: Plot of empirical pdf of X in part (a) using function `epdf` with 50 bins.

   (b) See plot below.

   (c) See plot below.

   (d) tba.

   MATLAB script:

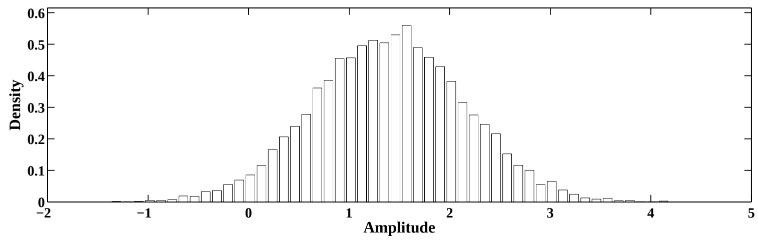

FIGURE 13.3: Plot of empirical pdf of X in part (b) using function `epdf` with 50 bins.

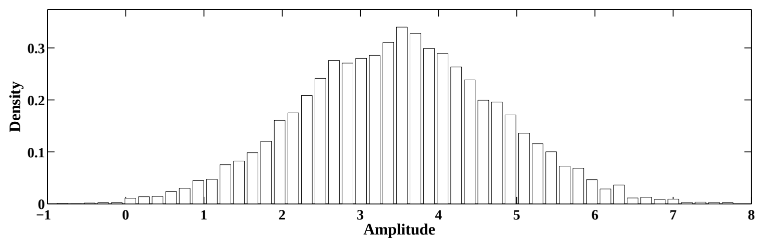

FIGURE 13.4: Plot of empirical pdf of X in part (c) using function `epdf` with 50 bins.

```
% P1303: Mixture of two Gaussians
clc; close all
N = 10000;
% a1 = 0.5; a2 = 0.5; % part a
a1 = 0.3; a2 = 0.7; % part b & c
mu1 = 0; sigma1 = 1;
% mu2 = 2; sigma2 = 1; % part a & b
mu2 = 5; sigma2 = sqrt(3); % part c
fx = a1*(sigma1*randn(1,N)+mu1) + a2*(sigma2*randn(1,N)+mu2);
[xo px] = epdf(fx,50);
disp('The mean is:')
disp(mean(fx))
disp('The standard deviation is:')
disp(std(fx))
%% Plot
hfa = figconfg('P1303a','long');
bar(xo,px,'w')
ylim([0 1.1*max(px)])
```

```
xlabel('Amplitude','fontsize',LFS)
ylabel('Density','fontsize',LFS)
```

4. (a) Proof: If we suppose $a > 0$, we have

$$F_Y(y) = P\{Y \leq y\} = P\{aX+b \leq y\} = P\{x \leq \frac{y-b}{a}\} = F_X\left(\frac{y-b}{a}\right)$$

Take the derivative of cdf of Y, we have

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \frac{d}{dy}F_X\left(\frac{y-b}{a}\right) = \frac{1}{a}f_X\left(\frac{y-b}{a}\right)$$

We can similar prove when $a < 0$,

$$f_Y(y) = -\frac{1}{a}f_X\left(\frac{y-b}{a}\right)$$

Combination of the two cases above yields,

$$f_Y(y) = \frac{1}{|a|}f_X\left(\frac{y-b}{a}\right)$$

(b) Proof:
The affine transformation of a Gaussian distributed rv is still Gaussian. We will prove the mean and variance rules under such affine transformation.

$$E[Y] = E[aX+b] = aE[X] + b = a\mu + b$$

$$\begin{aligned} E[Y^2] &= E[(aX+b)^2] = a^2E[X^2] + 2abE[X] + b^2 \\ &= a^2(\mu^2+\sigma^2) + 2ab\mu + b^2 = a^2\sigma^2 + (a\mu+b)^2 \end{aligned}$$

$$\sigma_y^2 = E[Y^2] - E^2[Y] = a^2\sigma^2$$

(c) See script below.

(d) See plots below.

(e) Comments: The numerical computation of mean and variance of Y can be verified by theoretical results. See the script below for details.

MATLAB script:

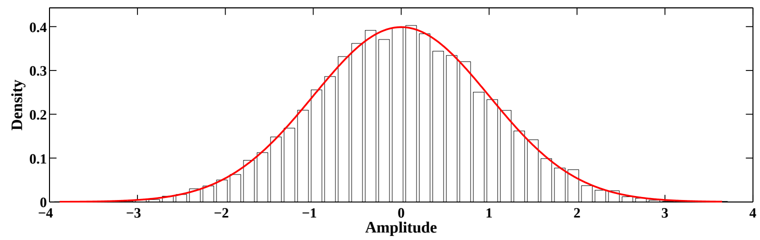

FIGURE 13.5: Plots of empirical and theoretical pdf of X.

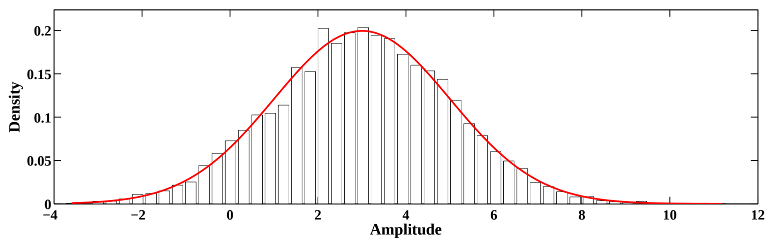

FIGURE 13.6: Plots of empirical and theoretical pdf of Y.

```
% P1304: Affine transformation of Gaussian
clc; close all
N = 10000; a = 2; b = 3;
x = randn(1,N);
y = a*x + b;
[xo px] = epdf(x,50);
[yo py] = epdf(y,50);
disp('The mean is:')
disp(mean(y))
disp('The variance is:')
disp(var(y))
xp = linspace(min(xo),max(xo),1000);
px_ref = pdf('normal',xp,0,1);
yp = linspace(min(yo),max(yo),1000);
py_ref = pdf('normal',yp,b,a);
%% Plot
hfa = figconfg('P1304a','long');
bar(xo,px,'w'); hold on
plot(xp,px_ref,'r','linewidth',2)
ylim([0 1.1*max(px)])
```

```
xlabel('Amplitude','fontsize',LFS)
ylabel('Density','fontsize',LFS)

hfb = figconfg('P1304b','long');
bar(yo,py,'w'); hold on
plot(yp,py_ref,'r','linewidth',2)
ylim([0 1.1*max(py)])
xlabel('Amplitude','fontsize',LFS)
ylabel('Density','fontsize',LFS)
```

5. (a) See plot below.

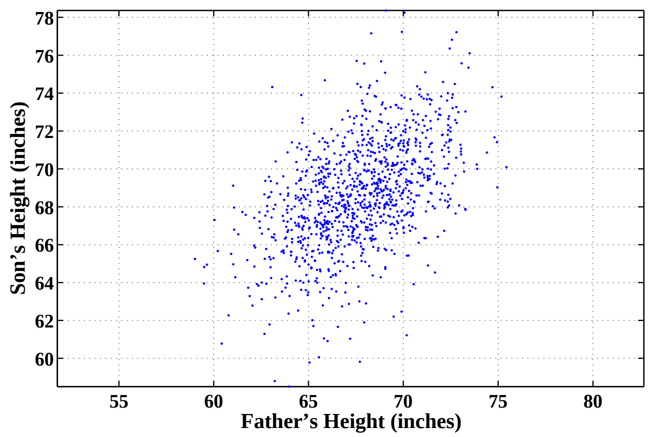

FIGURE 13.7: Scatter plot of the data between father and son heights.

(b) See plot below.

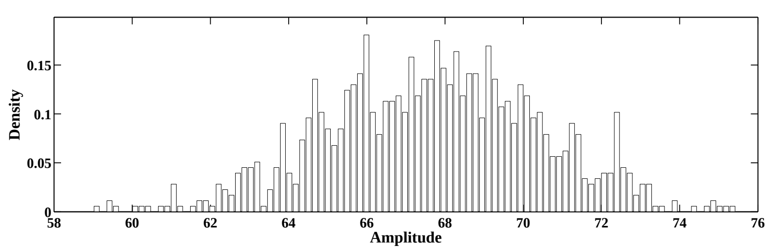

FIGURE 13.8: Normalized bar-graph for the father-height data.

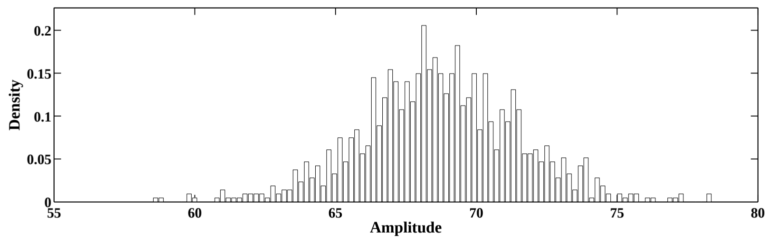

FIGURE 13.9: Normalized bar-graph for the son-height data.

(c) See plot below.

(d) See plot below.

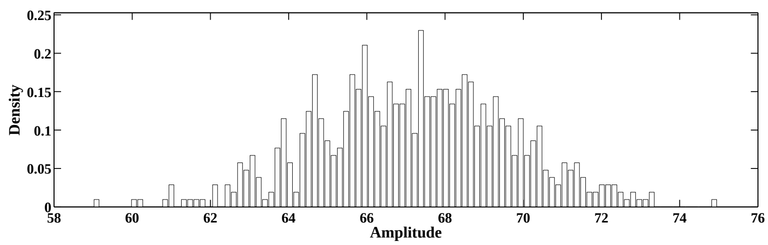

FIGURE 13.10: Normalized bar-graph for the conditional father-height data when son's heights are between 65 inches and 70 inches.

(e) See plot below.

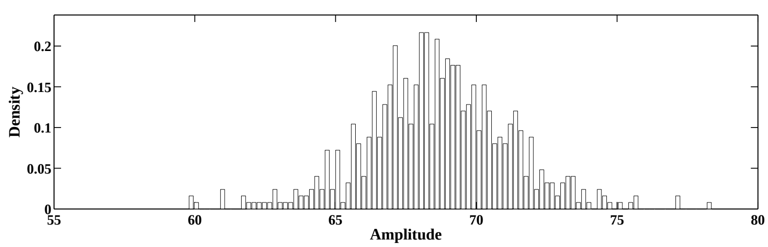

FIGURE 13.11: Normalized bar-graph for the conditional father-height data when father's heights are between 65 inches and 70 inches.

MATLAB script:

```
% P1305: fatherson
clc; close all
```

```
xx = load('fatherson.txt');
f = xx(:,1); s = xx(:,2);
%% Part a
hfa = figconfg('P1305a');
plot(f,s,'.'); axis equal; grid
xlabel('Father''s Height (inches)','fontsize',LFS)
ylabel('Son''s Height (inches)','fontsize',LFS)
%% Part b
[fo pf] = epdf(f,100);
hfb = figconfg('P1305b','long');
bar(fo,pf,'w');
ylim([0 1.1*max(pf)])
xlabel('Amplitude','fontsize',LFS)
ylabel('Density','fontsize',LFS)
%% Part c
[so ps] = epdf(s,100);
hfbc = figconfg('P1305c','long');
bar(so,ps,'w');
ylim([0 1.1*max(ps)])
xlabel('Amplitude','fontsize',LFS)
ylabel('Density','fontsize',LFS)
%% Part d
ind = (s >= 65 & s <= 70);
fc = f(ind);
[fco pfc] = epdf(fc,100);
hfd = figconfg('P1305d','long');
bar(fco,pfc,'w');
ylim([0 1.1*max(pfc)])
xlabel('Amplitude','fontsize',LFS)
ylabel('Density','fontsize',LFS)
%% Part e
ind = (f >= 65 & f <= 70);
sc = s(ind);
[sco psc] = epdf(sc,100);
hfe = figconfg('P1305e','long');
bar(sco,psc,'w');
ylim([0 1.1*max(psc)])
xlabel('Amplitude','fontsize',LFS)
ylabel('Density','fontsize',LFS)
```

6. (a) Solution:

$$f(y) = \int_{-\infty}^{\infty} f(x,y)dx$$

if $0 \leq y \leq 1$, the general integral is

$$f(y) = \int_{1-y}^{1+y} \frac{1}{2}dy = y$$

if $1 \leq y \leq 2$, the general integral is

$$f(y) = \int_{y-1}^{3-y} \frac{1}{2}dy = 2 - y$$

Combination of the two cases above yields that

$$f(y) = \begin{cases} y, & 0 \leq y \leq 1 \\ 2-y, & 1 \leq y \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

Since, the symmetric behavior of the two random variables, we can conclude that

$$f(x) = \begin{cases} x, & 0 \leq x \leq 1 \\ 2-x, & 1 \leq x \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

(b) Proof:
The expectations of X and Y are equal, and we will calculate one of them,

$$E[x] = \int_{-\infty}^{\infty} xf(x)dx = \int_0^1 x^2dx + \int_1^2 (2x - x^2)dx = 1$$

We then calculate the expectation of XY,

$$\begin{aligned} E[XY] &= \int\int xyf(x,y)dxdy = \int_0^1 y\left(\int_{1-y}^{1+y} \frac{x}{2}dx\right)dy + \int_1^2 y\left(\int_{y-1}^{3-y} \frac{x}{2}dx\right)dy \\ &= \int_0^1 y^2dy + \int_1^2 (2y - y^2)dy = 1 \end{aligned}$$

Hence, we can make the conclusion that X and Y are uncorrelated since $E[XY] = E[X]E[Y]$.

(c) Proof:
Since we have $f(x, y) \neq f(x)f(y)$, that implies X and Y are not independent.

7. (a) See plot below.

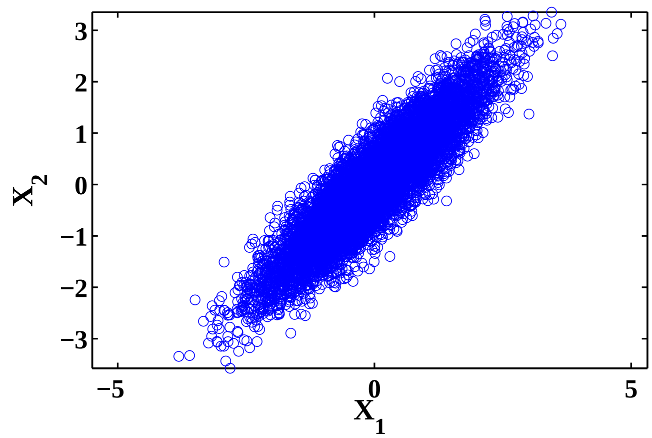

FIGURE 13.12: Scatter plot for $\rho = 0.9$.

(b) See plot below.

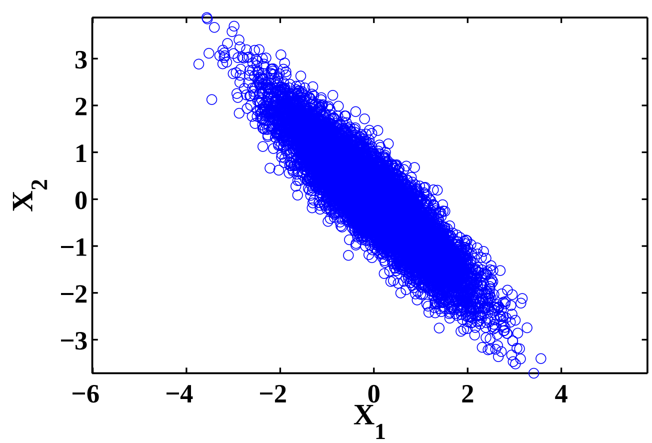

FIGURE 13.13: Scatter plot for $\rho = -0.9$.

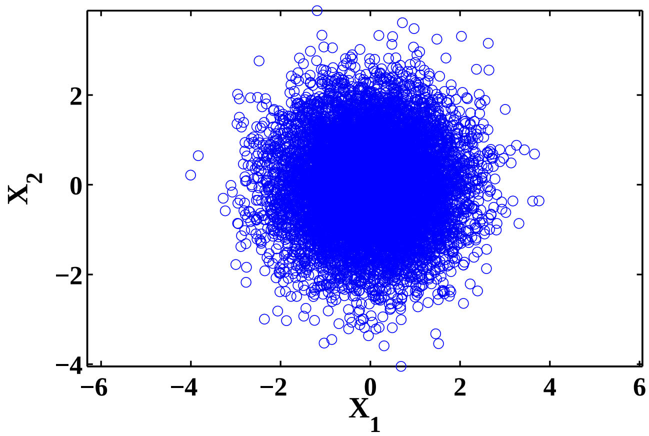

FIGURE 13.14: Scatter plot for $\rho = 0$.

(c) See plot below.

(d) tba.

MATLAB script:

```
% P1307: Investigation of correlation coefficient
clc; close all
% rho = 0.9; % part a
% rho = -0.9; % part b
rho = 0; % part c
N = 10000;
C = [1 rho;rho 1];
L = chol(C)';
x = L*randn(2,N);
%% Plot
hfa = figconfg('P1307a','small');
scatter(x(1,:),x(2,:)); axis equal; box on
xlabel('X_1','fontsize',LFS)
ylabel('X_2','fontsize',LFS)
```

8. (a) Solution:
Integrate the pdf function with respect to x_1, x_2, and x_3, we have

$$\iiint f(x_1,x_2,x_3)dx_1dx_2dx_3 = \int_0^1 x_1\left[\int_0^{x_1} x_2\left(\int_0^{x_2} x_3dx_3\right)dx_2\right]dx_1 = \frac{1}{48}$$

Since the integral of a valid pdf equals one, we have $K = 48$.
The mean of each random variable x_i, $i = 1, 2, 3$, can be computed as

$$E[x_i] = \iiint K x_i f(x_1, x_2, x_3) dx_1 dx_2 dx_3$$

Using the codes at the end of this problem, we compute that

$$\boldsymbol{\mu} = \begin{bmatrix} \frac{6}{7} & \frac{24}{35} & \frac{16}{35} \end{bmatrix}^T$$

(b) Solution:
The i, jth element r_{ij} of autocorrelation matrix $\boldsymbol{R}$ can be computed as

$$E[x_i x_j] = \iiint K x_i x_j f(x_1, x_2, x_3) dx_1 dx_2 dx_3$$

Using the codes at the end of this problem, we compute that

$$\boldsymbol{R} = \begin{bmatrix} \frac{3}{4} & \frac{3}{5} & \frac{2}{5} \\ \frac{3}{5} & \frac{1}{2} & \frac{1}{3} \\ \frac{2}{5} & \frac{1}{3} & \frac{1}{4} \end{bmatrix}$$

(c) Solution:
The i, jth element c_{ij} of autocorrelation matrix $\boldsymbol{C}$ can be computed as

$$E[(x_i - \mu_i)(x_j - \mu_j)] = \iiint K(x_i - \mu_i)(x_j - \mu_j) f(x_1, x_2, x_3) dx_1 dx_2 dx_3$$

Using the codes at the end of this problem, we compute that

$$\boldsymbol{C} = \begin{bmatrix} \frac{3}{196} & \frac{3}{245} & \frac{2}{245} \\ \frac{3}{245} & \frac{73}{2450} & \frac{73}{3675} \\ \frac{2}{245} & \frac{73}{3675} & \frac{21}{4900} \end{bmatrix}$$

MATLAB script:

```
% P1308: Compute mean vector and autocorrelation matrix and autocovariance
% matirx
clc; close all
syms x1 x2 x3
disp(int(int(int(x1*x2*x3,x3,0,x2),x2,0,x1),x1,0,1))
K = 48;
%% Mean
disp('X_1 mean is: ')
disp(int(int(int(K*x1^2*x2*x3,x3,0,x2),x2,0,x1),x1,0,1))
disp('X_2 mean is: ')
disp(int(int(int(K*x1*x2^2*x3,x3,0,x2),x2,0,x1),x1,0,1))
disp('X_3 mean is: ')
disp(int(int(int(K*x1*x2*x3^2,x3,0,x2),x2,0,x1),x1,0,1))
%% Autocorrelation
disp('R_11 mean is: ')
disp(int(int(int(K*x1^3*x2*x3,x3,0,x2),x2,0,x1),x1,0,1))
disp('R_12 mean is: ')
disp(int(int(int(K*x1^2*x2^2*x3,x3,0,x2),x2,0,x1),x1,0,1))
disp('R_13 mean is: ')
disp(int(int(int(K*x1^2*x2*x3^2,x3,0,x2),x2,0,x1),x1,0,1))
disp('R_21 mean is: ')
disp(int(int(int(K*x1^2*x2^2*x3,x3,0,x2),x2,0,x1),x1,0,1))
disp('R_22 mean is: ')
disp(int(int(int(K*x1*x2^3*x3,x3,0,x2),x2,0,x1),x1,0,1))
disp('R_23 mean is: ')
disp(int(int(int(K*x1*x2^2*x3^2,x3,0,x2),x2,0,x1),x1,0,1))
disp('R_31 mean is: ')
disp(int(int(int(K*x1^2*x2*x3^2,x3,0,x2),x2,0,x1),x1,0,1))
disp('R_32 mean is: ')
disp(int(int(int(K*x1*x2^2*x3^2,x3,0,x2),x2,0,x1),x1,0,1))
disp('R_33 mean is: ')
disp(int(int(int(K*x1*x2*x3^3,x3,0,x2),x2,0,x1),x1,0,1))
%% Autocovariance
disp('C_11 mean is: ')
disp(int(int(int(K*x1*x2*x3*(x1-6/7)^2,x3,0,x2),x2,0,x1),x1,0,1))
disp('C_12 mean is: ')
disp(int(int(int(K*x1*x2*x3*(x1-6/7)*(x2-24/35),x3,0,x2),x2,0,x1),x1,0,1))
disp('C_13 mean is: ')
disp(int(int(int(K*x1*x2*x3*(x1-6/7)*(x3-16/35),x3,0,x2),x2,0,x1),x1,0,1))
disp('C_21 mean is: ')
```

```
disp(int(int(int(K*x1*x2*x3*(x1-6/7)*(x2-24/35),x3,0,x2),x2,0,x1),x1,0,1))
disp('C_22 mean is: ')
disp(int(int(int(K*x1*x2*x3*(x2-24/35)^2,x3,0,x2),x2,0,x1),x1,0,1))
disp('C_23 mean is: ')
disp(int(int(int(K*x1*x2*x3*(x2-24/35)*(x3-16/35),x3,0,x2),x2,0,x1),x1,0,1))
disp('C_31 mean is: ')
disp(int(int(int(K*x1*x2*x3*(x1-6/7)*(x3-16/35),x3,0,x2),x2,0,x1),x1,0,1))
disp('C_32 mean is: ')
disp(int(int(int(K*x1*x2*x3*(x2-24/35)*(x3-16/35),x3,0,x2),x2,0,x1),x1,0,1))
disp('C_33 mean is: ')
disp(int(int(int(K*x1*x2*x3*(x3-16/35)^2,x3,0,x2),x2,0,x1),x1,0,1))
```

9. (a) Solution:

$$\begin{aligned}\mu_x[n] &= E\left[x[n]\right] = E[A\cos(\Omega n+\Theta)]\\ &= E[A\cos(\Omega n)\cos\Theta - A\sin(\Omega n)\sin\Theta]\\ &= E[A]E[\cos(\Omega n)]E[\cos\Theta] - E[A]E[\sin(\Omega n)]E[\sin\Theta]\\ &= 0\end{aligned}$$

(b) Solution:

$$\begin{aligned}c_X[m,n] &= r_X[m,n] = E\left[A\cos(\Omega m+\Theta)A\cos(\Omega n+\Theta)\right]\\ &= E\left[\frac{A^2}{2}\left\{\cos[\Omega(m+n)+2\Theta]+\cos[\Omega(m-n)]\right\}\right]\\ &= \frac{1}{2}E[A^2]\left\{E[\cos\Omega(m+n)]E[\cos 2\Theta] - E[\sin\Omega(m+n)]E[\sin 2\Theta]\right.\\ &\quad \left.+\ E[\cos\Omega(m-n)]\right\}\\ &= \frac{1}{2}E[A^2]E[\cos\Omega(m-n)]\end{aligned}$$

$$E[A^2] = \int_0^1 a^2 da = \frac{1}{3}$$

$$E[\cos(m-n)\Omega] = \frac{1}{2}\cos 10(m-n) + \frac{1}{2}\cos 20(m-n)$$

Hence, the ACVS $c_X[m,n]$ is

$$c_X[m,n] = \frac{1}{12}\cos 10(m-n) + \frac{1}{12}\cos 20(m-n)$$

(c) Comment:
$x[n]$ is wide-sense stationary, since its mean is constant and its second order statistic is only dependent on the lag.

10. (a) Solution:
We first note that for an exponential distributed random variable with parameter λ, that is

$$f(x) = \lambda \mathrm{e}^{-\lambda x},\ x \geq 0$$

Its first order and second order statistics are

$$E[x] = \frac{1}{\lambda}, \quad Var[x] = \frac{1}{\lambda^2}$$

$$\begin{aligned}\mu_y[n] &= E\left[y[n]\right] = E\left[x[n] + x[n-1] + v[n]\right] \\ &= E\left[x[n]\right] + E\left[x[n-1]\right] + E\left[v[n]\right] = 1 + 1 + \frac{1}{2} = \frac{5}{2}\end{aligned}$$

(b) Solution:
Without loss of generality, we can first suppose $m \geq n$. The ACRS $r_y[m,n]$ is

$$\begin{aligned}r_y[m,n] &= E[y[m]y[n]] = E[(x[m] + x[m-1] + v[m])(x[n] + x[n-1] + v[n])] \\ &= E[x[m]x[n]] + E[x[m-1]x[n]] + E[v[m]]E[x[n]] + E[x[m]x[n-1]] \\ &\quad + E[x[m-1]x[n-1]] + E[v[m]]E[x[n-1]] + E[x[m]]E[v[n]] \\ &\quad + E[x[m-1]]E[v[n]] + E[v[m]v[n]]\end{aligned}$$

If $m - 1 > n$, we have

$$r_y[m,n] = 1 + 1 + \frac{1}{2} + 1 + 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{4} = \frac{25}{4}$$

If $m - 1 = n$, we have

$$r_y[m,n] = 1 + (1+1) + \frac{1}{2} + 1 + 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{4} = \frac{29}{4}$$

If $m = n$, we have

$$r_y[m,n] = (1+1) + 1 + \frac{1}{2} + 1 + (1+1) + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + (\frac{1}{4} + \frac{1}{4}) = \frac{34}{4}$$

Combine all the cases above and also include $m < n$, we conclude that

$$r_y[m,n] = \frac{25}{4} + \delta[|m-n| - 1] + \frac{9}{4}\delta[m-n]$$

(c) Solution:
We first note a theorem that ”The density of the sum of two independent random variables is the convolution of the two pdfs.” Hence, the marginal density of $f(y)$ can be obtained by

$$f_y(x) = f_x(x) * f_x(x) * f_v(x)$$

Define $z[n] = x[n] + x[n-1]$, hence the density $f_z(x)$ is

$$f_z(z) = f_x(z) * f_x(z) = \int_0^z \mathrm{e}^{-(z-x)} \cdot \mathrm{e}^{-x} dx = \int_0^z \mathrm{e}^{-z} dx = z\mathrm{e}^{-z}, \quad z \geq 0$$

The density of $f_y(x)$ is

$$\begin{aligned} f_y(y) = f_z(y) * f_v(y) &= \int f_z(y-z) f_v(z) dz = \int_0^y (y-z)\mathrm{e}^{-(y-z)} \cdot 2\mathrm{e}^{-2z} dz \\ &= \int_0^y 2(y-z)\mathrm{e}^{-(y+z)} dz = 2y\mathrm{e}^{-y} - 2\mathrm{e}^{-y} + 2\mathrm{e}^{-2y}, \quad y \geq 0 \end{aligned}$$

That is

$$f_y(y) = \begin{cases} 2y\mathrm{e}^{-y} - 2\mathrm{e}^{-y} + 2\mathrm{e}^{-2y}, & y \geq 0 \\ 0, & y < 0 \end{cases}$$

11. Proof:
The output $y[n]$ is defined as

$$y[n] = h[n] * x[n] = \sum_{m=0}^{M} h[m]x[n-m]$$

Hence, the average power is

$$\begin{aligned} E[Y^2[n]] &= E\left[\sum_{m=0}^{M} h[m]x[n-m] \sum_{k=0}^{M} h[k]x[n-k]\right] \\ &= E\left[\sum_{m=0}^{M}\sum_{k=0}^{M} h[k]h[m]x[n-k]x[n-m]\right] \\ &= \sum_{m=0}^{M}\sum_{k=0}^{M} h[k]h[m]E[x[n-k]x[n-m]] \\ &= \sum_{m=0}^{M}\sum_{k=0}^{M} h[k]h[m]r_x[m-k] \\ &= \boldsymbol{h}^T \boldsymbol{R}_x \boldsymbol{h} \quad \text{(in matrix form)} \end{aligned}$$

The ACRS of $y[n]$ is

$$r_y[\ell] = h[\ell] * h[-\ell] * r_x[\ell]$$

and

$$S_{yy}(\omega) = |H(\mathrm{e}^{\mathrm{j}\omega})|^2 S_{xx}(\omega)$$

Use the inverse DTFT relation,

$$\begin{aligned} E[Y^2[n]] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} S_{yy}(\omega)\cdot \mathrm{e}^{\mathrm{j}\omega\cdot 0} d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_{yy}(\omega) d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|H\left(\mathrm{e}^{\mathrm{j}\omega}\right)\right|^2 S_{xx}(\omega) d\omega \end{aligned}$$

12. Proof:
We firstly prove the covariance between $x[m]$ and $y[n]$, $\ell = m - n$, that is

$$c_{xy}[\ell] = \sum_{k=-\infty}^{\infty} h[k] c_{xx}[\ell + k] = h[-\ell] * c_{xx}[\ell]$$

$$\begin{aligned} c_{xy}[\ell] &= E[(x[m] - E(x[m]))(y[n] - E(y[n]))] \\ &= E\left[(x[m] - E(x[m])) \sum_{k=-\infty}^{\infty} h[k](x[n-k] - E(x[n-k]))\right] \\ &= \sum_{k=-\infty}^{\infty} h[k] \cdot E[(x[m] - E(x[m]))(x[n-k] - E(x[n-k]))] \\ &= \sum_{k=-\infty}^{\infty} h[k] c_{xx}[(m-n) + k] \\ &= \sum_{k=-\infty}^{\infty} h[k] c_{xx}[\ell + k] = h[-\ell] * c_{xx}[\ell] \end{aligned}$$

We secondly prove the covariance between $y[m]$ and $x[n]$, $\ell = m - n$, that is

$$c_{yx}[\ell] = \sum_{k=-\infty}^{\infty} h[k] c_{xx}[\ell - k] = h[\ell] * c_{xx}[\ell]$$

$$
\begin{aligned}
c_{yx}[\ell] &= E\left[(y[m]-E(y[m]))(x[n]-E(x[n]))\right] \\
&= E\left[\sum_{k=-\infty}^{\infty} h[k](x[m-k]-E(x[m-k]))(x[n]-E(x[n]))\right] \\
&= \sum_{k=-\infty}^{\infty} h[k]\cdot E\left[(x[m-k]-E(x[m-k]))(x[n]-E(x[n]))\right] \\
&= \sum_{k=-\infty}^{\infty} h[k]\cdot c_{xx}[m-k-n] \\
&= \sum_{k=-\infty}^{\infty} h[k]c_{xx}[\ell-k] = h[\ell]*c_{xx}[\ell]
\end{aligned}
$$

We thirdly prove the covariance between $y[m]$ and $y[n]$, $\ell = m-n$, that is

$$
c_{yy}[\ell] = \sum_{k=-\infty}^{\infty} h[k]c_{xy}[\ell-k] = h[\ell]*c_{xy}[\ell]
$$

$$
\begin{aligned}
c_{yy}[\ell] &= E\left[(y[m]-E(y[m]))(y[n]-E(y[n]))\right] \\
&= \sum_{k=-\infty}^{\infty} h[k]\cdot E\left[(x[m-k]-E(x[m-k]))(y[n]-E(y[n]))\right] \\
&= \sum_{k=-\infty}^{\infty} h[k]\cdot c_{xy}[m-k-n] \\
&= \sum_{k=-\infty}^{\infty} h[k]c_{xy}[\ell-k] = h[\ell]*c_{xy}[\ell]
\end{aligned}
$$

Finally, the fourth expression can be easily proved by previous results, that is

$$
\begin{aligned}
c_{yy}[\ell] &= h[\ell]*c_{xy}[\ell] = h[\ell]*(h[-\ell]*c_{xx}[\ell]) = (h[\ell]*(h[-\ell)]*c_{xx}[\ell] \\
&= r_{hh}[\ell]*c_{xx}[\ell] = \sum_{m=-\infty}^{\infty} r_{hh}[m]c_{xx}[\ell-m]
\end{aligned}
$$

13. (a) Solution:

$$\begin{aligned} J(a,b) &= E[(Y - aX - b)(Y - aX - b)] \\ &= E[Y^2] + a^2E[X^2] + b^2 - 2aE[XY] - 2bE[Y] + 2abE[X] \end{aligned}$$

(b) Solution:

$$\frac{\partial J(a,b)}{\partial a} = 2aE[X^2] - 2E[XY] + 2bE[X] = 0 \qquad \text{(P13A)}$$

$$\frac{\partial J(a,b)}{\partial b} = 2b - 2E[Y] + 2aE[X] = 0 \qquad \text{(P13B)}$$

Solving Eq. (P13B) for b, we result in Eq. (13.58), that is

$$b = E[Y] - aE[X]$$

Plug the above equation into Eq. (P13A), and solve for a, we have

$$a = \frac{E[XY] - E[X]E[Y]}{\sigma_x^2} = \frac{c_{xy}}{\sigma_x^2} = \rho_{xy}\frac{\sigma_y}{\sigma_x}$$

which is exactly Eq. (13.62).

14. Proof:
Using the results from Problem 13-12, we have

$$c_{xy}[\ell] = h[-\ell] * c_{xx}[\ell]$$

Apply z-transform to both sides of the equation above will result in

$$C_{xy}(z) = H(1/z)C_{xx}(z)$$

$$c_{yx}[\ell] = h[\ell] * c_{xx}[\ell]$$

Apply z-transform to both sides of the equation above will result in

$$C_{yx}(z) = H(z)C_{xx}(z)$$

$$c_{yy}[\ell] = h[\ell] * h[-\ell] * c_{xx}[\ell]$$

Apply z-transform to both sides of the equation above will result in

$$C_{yy}(z) = H(z)H(1/z)C_{xx}(z)$$

15. Solution:

$$R_{yx}(z) = H(z)R_{xx}(z)$$

which implies

$$H(z) = \frac{R_{xx}(z)}{R_{yx}(z)}$$

16. Solution:

$$r_{yy}[0]a_1 + r_{yy}[1]a_2 = -r_{yy}[1] \tag{13.147}$$

$$r_{yy}[1]a_1 + r_{yy}[0]a_2 = -r_{yy}[2] \tag{13.148}$$

$$\sigma_x^2 = r_{yy}[0] + a_1 r_{yy}[1] + a_2 r_{yy}[2] \tag{13.150}$$

Plug $a_1 = -5/6$, $a_2 = 1/6$, and $\sigma_x^2 = 2$ into the three equations above and solve for $r_{yy}[0]$, $r_{yy}[1]$, and $r_{yy}[2]$, we have

$$r_{yy}[0] = \frac{21}{5}, \quad r_{yy}[1] = 3, \quad r_{yy}[2] = \frac{9}{5}$$

We can also conclude that $r_{yy}[\ell] = 0$, for $\ell > 2$.

17. (a) Proof:

$$E[x[n]] = E\left[\sum_{k=1}^{p} A_k \cos(\omega_k n + \phi_k)\right]$$
$$= \sum_{k=1}^{p} A_k E[\cos(\omega_k n + \phi_k)]$$

Since ϕ_k is uniformly distributed in the interval $(0, 2\pi)$, we have

$$E[\cos(\phi_k)] = E[\sin\phi_k] = 0$$

which implies

$$E[\cos(\omega_k n + \phi_k)] = E[\cos(\omega_k n)\cos(\phi_k) - \sin(\omega_k n)\sin(\phi_k)] = 0$$

Hence, we conclude that

$$E[x[n]] = 0$$

(b) Proof:

Since the random process has zero mean, its autocorrelation sequence equals its autocovariance sequence, that is $r_{xx}[\ell] = c_{xx}[\ell]$. For the rest of this proof, we only verify the expression for $r_{xx}[\ell]$.

$$\begin{aligned}
r_{xx}[\ell] = E[x[m]x[n]] &= E\left[\sum_{k=1}^{p} A_k \cos(\omega_k m + \phi_k) \sum_{q=1}^{p} A_q \cos(\omega_q n + \phi_q)\right] \\
&= \sum_{k=1}^{p}\sum_{q=1}^{p} A_q A_k E[\cos(\omega_k m + \phi_k)\cos(\omega_q n + \phi_q)] \\
&= \frac{1}{2}\sum_{k=1}^{p}\sum_{q=1}^{p} A_q A_k E[\cos(\omega_k m + \phi_k + \omega_q n + \phi_q) + \cos(\omega_k m + \phi_k - \omega_q n - \phi_q)]
\end{aligned}$$

Since we have

$$E[\cos(\omega_k m + \phi_k + \omega_q n + \phi_q)] = 0$$

$$E[\cos(\omega_k m + \phi_k - \omega_q n - \phi_q)] = 0, \quad k \neq q$$

The double summation can be simplified by removing all zero terms and written as

$$\begin{aligned}
r_{xx}[\ell] &= \frac{1}{2}\sum_{k=1}^{p} A_k^2 E[\cos(\omega_k m - \omega_k n)] \\
&= \frac{1}{2}\sum_{k=1}^{p} A_k^2 \cos\omega_k \ell, \quad \text{where} \quad \ell = m - n
\end{aligned}$$

Basic Problems

18. (a) Solution:
The sample space S of the experiment is:

$$S = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$$

(b) Solution:

$$P(A) = P(\text{sum} = 7) = \frac{2(1+1+1)}{36} = \frac{1}{6}$$

$$P(B) = P(\text{sum} = 9)+P(\text{sum} = 10)+P(\text{sum} = 11) = \frac{4+3+2}{36} = \frac{1}{4}$$

$$P(C) = P(\text{sum} = 11) + P(\text{sum} = 12) = \frac{2+1}{36} = \frac{1}{12}$$

19. (a) Solution:

$$P_1 = \frac{\begin{pmatrix} 12 \\ 4 \end{pmatrix}}{\begin{pmatrix} 15 \\ 4 \end{pmatrix}} = \frac{12 \times 11 \times 10 \times 9}{15 \times 14 \times 13 \times 12} = \frac{33}{91}$$

(b) Solution:

$$P_2 = \frac{\begin{pmatrix} 3 \\ 1 \end{pmatrix}\begin{pmatrix} 12 \\ 3 \end{pmatrix}}{\begin{pmatrix} 15 \\ 4 \end{pmatrix}} = \frac{3 \times 12 \times 11 \times 10}{15 \times 14 \times 13 \times 12} = \frac{11}{91}$$

(c) Solution:

$$P_3 = \frac{\begin{pmatrix} 9 \\ 4 \end{pmatrix}}{\begin{pmatrix} 15 \\ 4 \end{pmatrix}} = \frac{9 \times 8 \times 7 \times 6}{15 \times 14 \times 13 \times 12} = \frac{6}{65}$$

20. (a) Solution:

A valid pdf will integrate to 1, that is

$$\iint f_{X,Y}(x,y)dxdy = K_1\int_0^\infty\int_0^\infty (x+y)\mathrm{e}^{-x-y}dxdy = 2K_1 = 1$$

Hence,

$$K_1 = \frac{1}{2}$$

(b) Solution:

We first compute the marginal pdfs of X and Y.

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy = \int_0^\infty \frac{1}{2}(x+y)\mathrm{e}^{-x-y}u(x)dy \\ &= \frac{1}{2}(x+1)\mathrm{e}^{-x}u(x) \end{aligned}$$

Since X and Y are interchangeable, we can conclude that

$$f_Y(y) = \frac{1}{2}(y+1)\mathrm{e}^{-y}u(y)$$

The conditional pdf can be obtained by

$$\begin{aligned} f_{Y|X}(y|x) &= \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{\frac{1}{2}(x+y)e^{-(x+y)}u(x)u(y)}{\frac{1}{2}(x+1)\mathrm{e}^{-x}u(x)} \\ &= \frac{x+y}{x+1}\cdot \mathrm{e}^{-y}u(y) \end{aligned}$$

$$f_{X|Y}(x|y) = \frac{x+y}{y+1}\cdot \mathrm{e}^{-x}u(x) \quad \text{Interchangeability}$$

(c) Solution:

Random variables X and Y are NOT independent because

$$f_{Y|X}(y|x) \neq f_Y(y)$$

21. (a) Proof:

$$\begin{aligned} F_Y(y) &= P\{Y \leq y\} = P\{X^2 \leq y\} = P\{-\sqrt{y} \leq x \leq \sqrt{y}\} \\ &= F_X(\sqrt{y}) - F_X(-\sqrt{y}), \quad y \geq 0 \end{aligned}$$

$$\begin{aligned} f_Y(y) &= \frac{dF_Y(y}{dy} = \frac{dF_X(\sqrt{y}) - dF_X(-\sqrt{y})}{dy} \\ &= f_X(\sqrt{y}) \cdot \frac{1}{2\sqrt{y}} - f_X(-\sqrt{y}) \cdot \left(-\frac{1}{2\sqrt{y}}\right) \\ &= \frac{1}{2\sqrt{y}}[f_X(\sqrt{y}) - f_X(-\sqrt{y}) \end{aligned}$$

(b) Proof:

The pdf of a chi-square distribution with one degree of freedom is,

$$f(x) = \frac{1}{\sqrt{2} \cdot \Gamma(1/2)} x^{-\frac{1}{2}} \cdot \mathrm{e}^{-\frac{x}{2}} u(x)$$

The pdf of $X \sim N(0,1)$ is,

$$f(x) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{x^2}{2}}$$

Using the relationship in last proof, we obtain the pdf of $Y = X^2$ as

$$f_Y(y) = \frac{1}{2\sqrt{y}} \left(\frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{y}{2}} + \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{y}{2}}\right) = \frac{1}{\sqrt{2\pi y}} \mathrm{e}^{-\frac{y}{2}}$$

Note that

$$\sqrt{\pi} = \Gamma\left(\frac{1}{2}\right)$$

We can see that the pdf of Y is exactly the same as the pdf of a chi-square distribution with one degree of freedom.

(c) See script below.

(d) See plots below.

MATLAB script:

```
% P1320: Affine transformation of Gaussian
clc; close all
N = 10000;
```

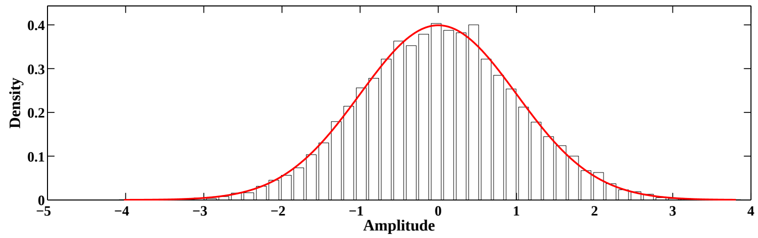

FIGURE 13.15: Empirical and theoretical pdfs of X.

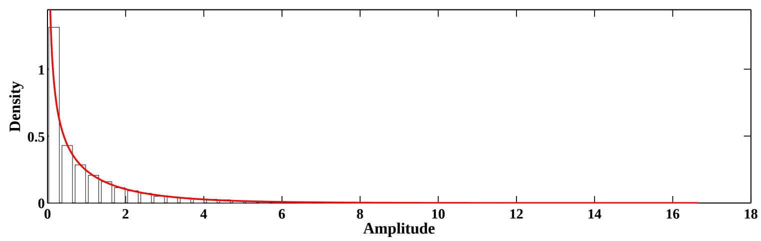

FIGURE 13.16: Empirical and theoretical pdfs of Y.

```
x = randn(1,N);
y = x.^2;
[xo px] = epdf(x,50);
[yo py] = epdf(y,50);
xp = linspace(min(xo),max(xo),1000);
px_ref = pdf('normal',xp,0,1);
yp = linspace(0,max(yo),1000);
py_ref = chi2pdf(yp,1);
%% Plot
hfa = figconfg('P1320a','long');
bar(xo,px,'w'); hold on
plot(xp,px_ref,'r','linewidth',2)
ylim([0 1.1*max(px)])
xlabel('Amplitude','fontsize',LFS)
ylabel('Density','fontsize',LFS)

hfb = figconfg('P1320b','long');
bar(yo,py,'w'); hold on
plot(yp,py_ref,'r','linewidth',2)
ylim([0 1.1*max(py)])
```

```
xlabel('Amplitude','fontsize',LFS)
ylabel('Density','fontsize',LFS)
```

22. (a) Solution:

$$f(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_0^x 8xydy = 4x^3, \quad 0 \le x \le 1$$

$$f(y) = \int_{-\infty}^{\infty} f(x,y)dx = \int_y^1 8xydx = 4y(1-y^2), \quad 0 \le y \le 1$$

$$f(x|y) = \frac{f(x,y)}{f(y)} = \frac{8xy}{4y(1-y^2)} = \frac{8x}{1-y^2}, \quad y \le x \le 1$$

$$f(y|x) = \frac{f(x,y)}{f(x)} = \frac{8xy}{4x^3} = \frac{2y}{x^2}, \quad 0 \le y \le x$$

(b) Solution:
$f(x|y) \neq f(x)$ which implies X and Y are NOT independent.

23. Solution:
$X \sim N(0,9)$ implies $X/3 \sim N(0,1)$, $y = 5x^2$ equals $y = 45(x/3)^2$. Define $z = (x/3)^2$, hence we have $Z \sim \chi_1^2$. The mth moments for Z can be written as

$$E[Z^m] = 2^m \frac{\Gamma(m+1/2)}{\Gamma(1/2)}$$

From which we can compute the first two moments as

$$E[z] = 2 \cdot \frac{\Gamma(1+1/2)}{\Gamma(1/2)} = 1$$

$$E[z^2] = 2^2 \cdot \frac{\Gamma(2+1/2)}{\Gamma(1/2)} = 3$$

Hence, we can compute the mean and variance of Y as

$$E[Y] = 45E[Z] = 45$$

$$\sigma_Y^2 = 45^2(E[Z^2] - E^2[Z]) = 45^2 \cdot (3-1) = 4050$$

24. (a) Solution:
The probability mass function (pmf) for the random variable $x[3]$ is

$$P\{x[3]=0\}=\frac{1}{2}, \quad P\{x[3]=1\}=\frac{1}{2}$$

(b) Solution:

$$m_x[n]=\frac{1}{2}\times 0+\frac{1}{2}\times 1=0.5$$

(c) Solution:
When $m\neq n$, the pmf of $x[m]x[n]$ is

$$P\{x[m]x[n]=0\}=\frac{3}{4}, \quad P\{x[m]x[n]=1\}=\frac{1}{4}$$

$$r_x[m,n]=E[x[m]x[n]]=0\times\frac{3}{4}+1\times\frac{1}{4}=\frac{1}{4}$$

When $m=n$, the pmf of $x^2[n]$ is

$$P\{x^2[n]=0\}=\frac{1}{2}, \quad P\{x^2[n]=1\}=\frac{1}{2}$$

$$r_x[n,n]=E[x[n]x[n]]=0\times\frac{1}{2}+1\times\frac{1}{2}=\frac{1}{2}$$

Combining the results of the two cases, we conclude that

$$r_x[m,n]=\frac{1}{4}+\frac{1}{4}\delta[m-n]$$

25. (a) Solution:

$$\mu_y[n]=E[y[n]]=E\left[\sum_{i=1}^{n}x[i]\right]=\sum_{i=1}^{n}E[x[i]]$$

Since we have

$$E[x[n]]=0.6\times 1+0.4\times 0=0.6$$

The mean of $y[n]$ is

$$\mu_y[n]=\sum_{i=1}^{n}0.6=0.6n$$

We can also compute the second moment of $x[n]$ as

$$E[x^2[n]] = 0.6 \times 1^2 + 0.4 \times 0^2 = 0.6$$

Hence, the second moment of $y[n]$ is

$$\begin{aligned} E[y^2[n]] = E\left[\sum_{i=1}^{n} x[i] \cdot \sum_{j=1}^{n} x[j]\right] &= \sum_{i=1}^{n}\sum_{j=1}^{n} E[x[i]x[j]] \\ &= n \times 0.6 + (n^2 - n) \times 0.6^2 = 0.24n + 0.36n^2 \end{aligned}$$

The variance of $y[n]$ is

$$\sigma_y^2[n] = E[y^2[n]] - \mu_y^2[n] = 0.24n$$

(b) Solution:
Without loss of generality, suppose $m \geq n$, the covariance function $\gamma_y[m,n]$ can be computed by

$$\gamma_y[m,n] = E[y[m]y[n]] - \mu_y[m]\mu_y[n]$$

The first item on the right side of the equation above, which is the correlation function, can be calculated as

$$\begin{aligned} E[y[m]y[n]] = E\left[\sum_{i=1}^{m} x[i] \cdot \sum_{j=1}^{n} x[j]\right] &= \sum_{i=1}^{m}\sum_{j=1}^{n} E[x[i]x[j]] \\ &= n \times 0.6 + (mn - n) \times 0.6^2 = 0.24n + 0.36mn \end{aligned}$$

Thus, the covariance function is

$$\gamma_y[m,n] = 0.24n + 0.36mn - 0.6m \times 0.6n = 0.24n$$

In general, we can conclude that

$$\gamma_y[m,n] = 0.24 \min(m,n)$$

(c) Solution:
Suppose $m \geq n$, and we have

$$A = \sum_{i=n+1}^{m} x[i] \implies \sum_{j=1}^{m-n} x[j]$$

Using the previous results, we have

$$\sigma_A^2 = 0.24(m-n)$$

26. (a) Solution:

$$\mu_w[n] = E[w[n]] = -4 \times \frac{1}{4} + 0 \times \frac{1}{4} + \frac{1}{2} \times 4 = 1$$

When $m = n$, the pmf of $w^2[n]$ is

$$P\{w^2[n] = 0\} = \frac{1}{4}, \quad P\{w^2[n] = 16\} = \frac{3}{4}$$

$$E[w^2[n]] = 0 \times \frac{1}{4} + 16 \times \frac{3}{4} = 12$$

When $m \neq n$, the pmf of $w[m]w[n]$ is

$$P\{w[m]w[n] = -16\} = \frac{1}{4}, \quad P\{w[m]w[n] = 0\} = \frac{7}{16}, \quad P\{w[m]w[n] = 16\} = \frac{5}{16}$$

$$E[w[m]w[n]] = -16 \times \frac{1}{4} + 0 \times \frac{7}{16} + 16 \times \frac{5}{16} = 1$$

Hence, the autocorrelation $r_w[m, n]$ is

$$r_w[m, n] = 11\delta[m - n] + 1$$

(b) Solution:

$$\mu_v[n] = E[v[n]] = \int_{-5}^{7} \frac{v}{12} dv = 1$$

When $m \neq n$

$$E[v[m]v[n]] = E[v[m]]E[v[n]] = 1$$

When $m = n$

$$E[v^2[n]] = \int_{-5}^{7} \frac{v^2}{12} dv = 13$$

Combining the two cases above, we conclude the autocorrelation $r_v[m, n]$ as

$$r_v[m, n] = 12\delta[m - n] + 1$$

(c) Solution:

$$r_{w,v}[m, n] = E[w[m]v[n]] = E[w[m]]E[v[n]] = 1$$

(d) Solution:

$$\mu_X[n] = E[x[n]] = E[w[n] + v[n - 1]] = E[w[n]] + E[v[n - 1]] = 2$$

(e) Proof:

$$\begin{aligned}
r_x[m,n] &= E[x[m]x[n]] = E[(w[m]+v[m-1])(w[n]+v[n-1])] \\
&= E[w[m]w[n]] + E[w[m]v[n-1]] + E[v[m-1]w[n]] + E[v[m-1]v[n-1]] \\
&= (11\delta[m-n]+1) + 1 + 1 + (12\delta[m-n]+1) \\
&= 4 + 23\delta[m-n]
\end{aligned}$$

Assessment Problems

27. (a) Solution:

A valid pdf must integrate to one, we have

$$\int_{-\infty}^{\infty} f_Y(y)dy = \int_0^{\infty} a\mathrm{e}^{-by}dy = \frac{a}{b} = 1$$

Hence,

$$a = b$$

(b) Solution:

Since

$$P[y > c] = 1 - \int_0^c \mathrm{e}^{-by}dy = \mathrm{e}^{-bc}$$

$$f(y|y > c) = \frac{f(y, y > c)}{P[y > c]} = \frac{a\mathrm{e}^{-by}}{\mathrm{e}^{-bc}} = b\mathrm{e}^{-b(y-c)}, \quad y > c$$

Hence, we can write

$$f(y|y > c) = \begin{cases} b\mathrm{e}^{-b(y-c)}, & y > c \\ 0, & y \leq c \end{cases}$$

(c) Solution:

$$P[c < y < d] = \int_c^d b\mathrm{e}^{-by}dy = \mathrm{e}^{-bc}(1 - \mathrm{e}^{-b(d-c)})$$

$$f(y|c < y < d) = \frac{f(y, c < y < d)}{P[c < y < d]} = \begin{cases} \frac{b\mathrm{e}^{-b(y-c)}}{1-\mathrm{e}^{-b(d-c)}}, & c \leq y \leq d \\ 0, & \text{otherwise} \end{cases}$$

28. (a) Solution:

$$f_{X,Y}(x, y) = f(x|y)f(y) = \begin{cases} y\mathrm{e}^{-yx}, & x \leq 0, 1 \leq y \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

(b) Solution:

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f(x, y)dy = \int_1^2 y\mathrm{e}^{-yx}dy \\ &= \begin{cases} -\frac{2}{x}\mathrm{e}^{-2x} + \frac{\mathrm{e}^{-x}}{x} - \frac{\mathrm{e}^{-2x}}{x^2} + \frac{\mathrm{e}^{-x}}{x^2}, & x \leq 0 \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

$$f_Y(y) = \begin{cases} 1, & 1 \leq y \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

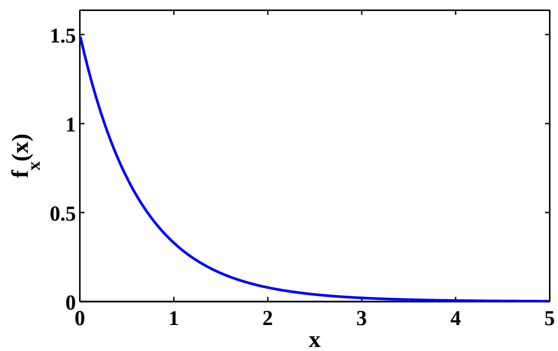

FIGURE 13.17: Marginal density of $f_X(x)$.

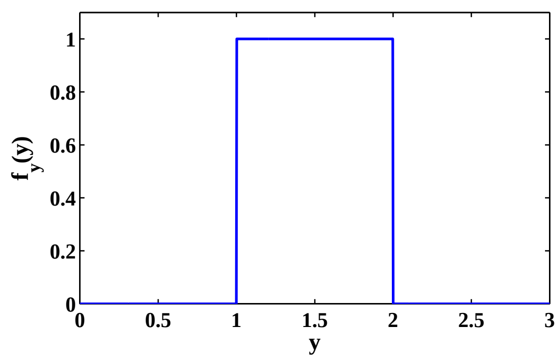

FIGURE 13.18: Marginal density of $f_Y(y)$.

(c) Solution:

$$f_{X|Y}(x\,|\,y=1.5)=\begin{cases}1.5\mathrm{e}^{-1.5x}, & x\le 0\\ 0, & x<0\end{cases}$$

$$f_{Y|X}\left(y\mid x=1\right)=\frac{f(x=1,y)}{f(x=1)}=\begin{cases}\frac{y\mathrm{e}^{-y}}{2\mathrm{e}^{-1}-3\mathrm{e}^{-2}}, & 1\le y\le 2\\ 0, & \text{otherwise}\end{cases}$$

MATLAB script:

```
% P1328: Density plot
clc; close all
x = linspace(0,5,1001);
fx = -2*exp(-2*x)./x + exp(-x)./x - exp(-2*x)./(x.^2) + exp(-x)./(x.^2);
```

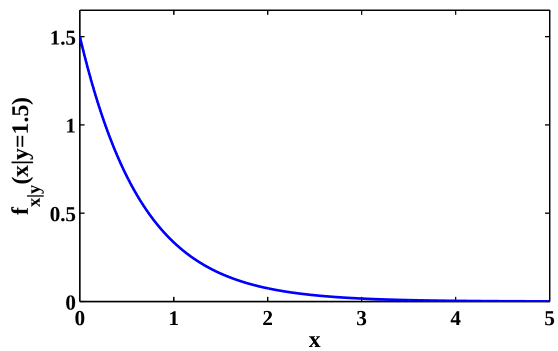

FIGURE 13.19: Conditional density of $f_{X|Y}(x \,|\, y = 1.5)$.

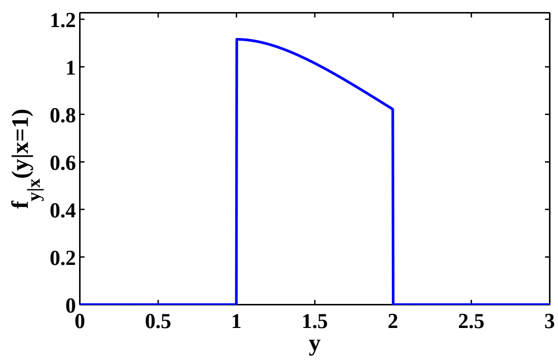

FIGURE 13.20: Conditional density of $f_{Y|X}\,(y \mid x = 1)$.

```
y = linspace(0,3,1001);
fy = zeros(size(y));
fyx = zeros(size(y));
ind = (y>=1 & y<=2);
fy(ind) = 1;
fxy = 1.5*exp(-1.5*x);
fyx(ind) = y(ind).*exp(-y(ind))/(2*exp(-1)-3*exp(-2));
%% Plot
hfa = figconfg('P1328a','small');
plot(x,fx,'linewidth',2)
ylim([0 1.1*max(fx)])
xlabel('x','fontsize',LFS)
ylabel('f_x(x)','fontsize',LFS)
```

```
hfb = figconfg('P1328b','small');
plot(y,fy,'linewidth',2)
ylim([0 1.1*max(fy)])
xlabel('y','fontsize',LFS)
ylabel('f_y(y)','fontsize',LFS)
hfc = figconfg('P1328c','small');
plot(x,fxy,'linewidth',2)
ylim([0 1.1*max(fxy)])
xlabel('x','fontsize',LFS)
ylabel('f_{x|y}(x|y=1.5)','fontsize',LFS)
hfd = figconfg('P1328d','small');
plot(y,fyx,'linewidth',2)
ylim([0 1.1*max(fyx)])
xlabel('y','fontsize',LFS)
ylabel('f_{y|x}(y|x=1)','fontsize',LFS)
```

29. (a) Proof:

$$E[X] = E[\sin(2\pi Z)] = \int_0^1 \sin(2\pi z)dz = 0$$

$$E[Y] = E[\cos(2\pi Z)] = \int_0^1 \cos(2\pi z)dz = 0$$

$$\begin{aligned} E[XY] &= E[\sin(2\pi z)\cos(2\pi z)] = \frac{1}{2}E[\sin(4\pi z) + 0] \\ &= \frac{1}{2}E[\sin(4\pi z)] = \frac{1}{2}\int_0^1 \sin(4\pi z)dz = 0 \end{aligned}$$

Hence,

$$\text{cov}(X, Y) = E[XY] - E[X]E[Y] = 0$$

(b) See plot below.

(c) Comments:
The shape of the scatter diagram is symmetric with respect to x and y axes and there is no trend of linear dependence which explain that the two variables X and Y are uncorrelated.

MATLAB script:

```
% P1329: UnCorrelated but dependent random variables
clc; close all
```

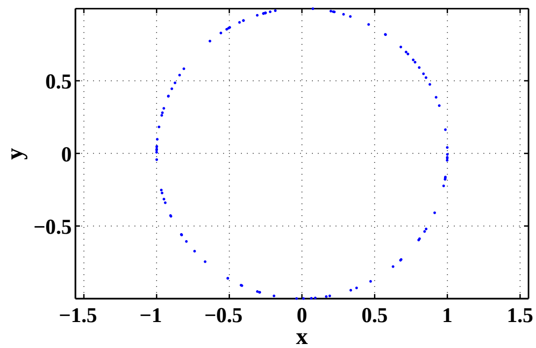

FIGURE 13.21: Scatter diagram of dependent but uncorrelated random variable samples X and Y.

```
z = rand(100,1);
x = sin(2*pi*z);
y = cos(2*pi*z);
cov([x y])
%% Plot
hfa = figconfg('P1329a','small');
plot(x,y,'.'); axis equal; grid
xlabel('x','fontsize',LFS)
ylabel('y','fontsize',LFS)
```

30. Proof: The pdf of a multivariate normal distribution is given by

$$f(\boldsymbol{x}) = \frac{1}{(2\pi)^{-\frac{K}{2}}|C|^{\frac{1}{2}}} \exp\left\{-\frac{(\boldsymbol{x}-\boldsymbol{m})^T C^{-1}(\boldsymbol{x}-\boldsymbol{m})}{2}\right\}$$

Since $z_i = (x_i - \mu_i)/\sigma_i$, $i = 1, 2$, we have $z_i \sim N(0,1)$, $i = 1, 2$.

$$\boldsymbol{z} = \begin{bmatrix} z_1 & z_2 \end{bmatrix}^T, \quad C_z = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$$

The determinant of C is

$$|C| = \begin{vmatrix} 1 & \rho \\ \rho & 1 \end{vmatrix} = 1 - \rho^2$$

The inverse of C is

$$C^{-1} = \frac{1}{\sqrt{1-\rho^2}} \begin{bmatrix} 1 & -\rho \\ -\rho & 1 \end{bmatrix}$$

Hence, we have

$$\begin{bmatrix} z_1 & z_2 \end{bmatrix} \begin{bmatrix} 1 & -\rho \\ -\rho & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = z_1^2 - 2\rho z_1 z_2 + z_2^2$$

Plug all equations back to the multivariate normal pdf yields

$$f(z_1, z_2) = \frac{1}{2\pi\sqrt{1-\rho^2}} \times \exp\left[-\frac{z_1^2 - 2\rho z_1 z_2 + z_2^2}{2\sqrt{1-\rho^2}}\right]$$

31. (a) Solution:

$$f_{Y_2}(y) = f_{X_1}(y) * f_{X_2}(y) = \int_{-\infty}^{\infty} f_{X_1}(x) f_{X_2}(y-x)dx$$

If $-1 \leq y \leq 0$, we have

$$f_{Y_2}(y) = \int_{-\frac{1}{2}}^{y+\frac{1}{2}} dx = y + 1$$

If $0 \leq y \leq 1$, we have

$$f_{Y_2}(y) = \int_{y-\frac{1}{2}}^{\frac{1}{2}} dx = 1 - y$$

Hence, we conclude that

$$f_{Y_2}(y) = \begin{cases} 1 - |y|, & |y| \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

(b) Solution:

$$f_{Y_3}(y) = f_{Y_2}(y) * f_{X_3}(y) = \int_{-\infty}^{\infty} f_{Y_2}(x) f_{X_3}(y-x)dx$$

If $-\frac{3}{2} \leq y \leq -\frac{1}{2}$, we have

$$f_{Y_3}(y) = \int_{-1}^{\frac{1}{2}+y} (x+1)dx = \frac{y^2}{2} + \frac{3y}{2} + \frac{9}{8}$$

If $-\frac{1}{2} \leq y \leq \frac{1}{2}$, we have

$$f_{Y_3}(y) = \int_{y-\frac{1}{2}}^{0} (x+1)dx + \int_{0}^{y+\frac{1}{2}} (-x+1)dx = \frac{3}{4} - y^2$$

If $\frac{1}{2} \leq y \leq \frac{3}{2}$, we have

$$f_{Y_3}(y) = \int_{y-\frac{1}{2}}^{1} (-x+1)dx = \frac{y^2}{2} - \frac{3y}{2} + \frac{9}{8}$$

Hence, we can conclude that

$$f_{Y_3}(y) = \begin{cases} \frac{y^2}{2} - \frac{3|y|}{2} + \frac{9}{8}, & \frac{1}{2} \leq |y| \leq \frac{3}{2} \\ \frac{3}{4} - y^2, & |y| \leq \frac{1}{2} \\ 0, & \text{otherwise} \end{cases}$$

(c) Solution:

$$f_{Y_4}(y) = f_{Y_3}(y) * f_{X_4}(y) = \int_{-\infty}^{\infty} f_{Y_3}(x) f_{X_4}(y-x)dx$$

If $-2 \leq y \leq -1$,

$$f_{Y_4}(y) = \int_{-\frac{3}{2}}^{y+\frac{1}{2}} \left(\frac{x^2}{2} + \frac{3x}{2} + \frac{9}{8} \right) dx = \frac{1}{6}(y+2)^3$$

If $-1 \leq y \leq 0$,

$$\begin{aligned} f_{Y_4}(y) &= \int_{y-\frac{1}{2}}^{-\frac{1}{2}} \left(\frac{x^2}{2} + \frac{3x}{2} + \frac{9}{8} \right) dx + \int_{-\frac{1}{2}}^{y+\frac{1}{2}} (\frac{3}{4} - x^2)dx \\ &= -\frac{1}{2}y^3 - y^2 + \frac{2}{3} \end{aligned}$$

If $0 \leq y \leq 1$,

$$\begin{aligned} f_{Y_4}(y) &= \int_{y+\frac{1}{2}}^{\frac{1}{2}} \left(\frac{x^2}{2} - \frac{3x}{2} + \frac{9}{8} \right) dx + \int_{y-\frac{1}{2}}^{\frac{1}{2}} (\frac{3}{4} - x^2)dx \\ &= \frac{1}{2}y^3 - y^2 + \frac{2}{3} \end{aligned}$$

If $1 \leq y \leq 2$,

$$f_{Y_4}(y) = \int_{\frac{3}{2}}^{y-\frac{1}{2}} \left(\frac{x^2}{2} - \frac{3x}{2} + \frac{9}{8} \right) dx = -\frac{1}{6}(y-2)^3$$

Hence, we can conclude that

$$f_{Y_4}(y) = \begin{cases} \frac{1}{2}|y|^3 - y^2 + \frac{2}{3}, & |y| \leq 1 \\ -\frac{1}{6}(|y|-2)^3, & 1 \leq |y| \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

(d) Solution:
Solve for $X \sim N(0,1)$

$$P\{X \leq \frac{-2}{\sigma}\} = 0.005$$

We have $\sigma = 0.7765$

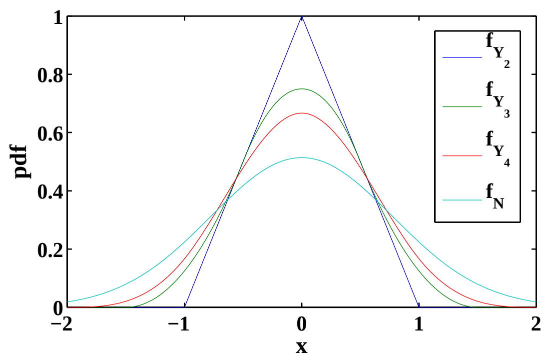

FIGURE 13.22: Plot of the pdfs of Y_2, Y_3, Y_4 and the normal distribution.

(e) Comments:
The sum of infinite number of IID random variables is a normal distribution.

MATLAB script:

```
% P1331: Sum of IID uniform distributed random variables
clc; close all
% syms x y;
%% X1+X2+X3
% int(x+1,x,-1,y+1/2)
% int(x+1,x,y-1/2,0) + int(-x+1,x,0,y+1/2)
%% X1+X2+X3+X4
% int(x^2/2+3*x/2+9/8,x,-3/2,y+1/2)
% int(x^2/2+3*x/2+9/8,x,y-1/2,-1/2) + int(3/4-x^2,x,-1/2,y+1/2)
% int(x^2/2-3*x/2+9/8,x,1/2,y+1/2) + int(3/4-x^2,x,y-1/2,1/2)
% int(x^2/2-3*x/2+9/8,x,y-1/2,3/2)

x = linspace(-2,2,1001);
%% part a
x2pdf = zeros(size(x));
```

```
ind = (abs(x)<=1);
x2pdf(ind) = 1-abs(x(ind));
%% part b
x3pdf = zeros(size(x));
ind = (abs(x)<=1/2);
x3pdf(ind) = 3/4 - x(ind).^2;
ind = (abs(x)>1/2 & abs(x)<=3/2);
x3pdf(ind) = x(ind).^2/2 - 3/2*abs(x(ind)) + 9/8;
%% part c
x4pdf = zeros(size(x));
ind = (abs(x)<=1);
x4pdf(ind) = abs(x(ind)).^3/2 - x(ind).^2 + 2/3;
ind = (abs(x)<=2 & abs(x)>1);
x4pdf(ind) = -(abs(x(ind))-2).^3/6;
%% part d
sigma = 0.77645;
xGpdf = normpdf(x,0,sigma);
%% Plot
hfa = figconfg('P1331a','small');
plot(x,x2pdf,x,x3pdf,x,x4pdf,x,xGpdf);
xlabel('x','fontsize',LFS)
ylabel('pdf','fontsize',LFS)
legend('f_{Y_2}','f_{Y_3}','f_{Y_4}','f_{N}','location','best')
```

32. (a) Solution:
The mean vector $\boldsymbol{m}_y$ is

$$\begin{aligned} E[\underline{\boldsymbol{Y}}] &= E[\boldsymbol{A}\underline{\boldsymbol{X}}] = AE[\underline{\boldsymbol{X}}] = A\boldsymbol{m} \\ &= \begin{bmatrix} 1 & 3 \\ -1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ 8 \end{bmatrix} \end{aligned}$$

(b) Solution:
The auto-covariance matrix $\boldsymbol{C}_y$ is

$$\begin{aligned} \boldsymbol{C}_y &= E[(\underline{\boldsymbol{Y}} - \boldsymbol{m}_{\underline{\boldsymbol{Y}}})(\underline{\boldsymbol{Y}} - \boldsymbol{m}_{\underline{\boldsymbol{Y}}})^T] = \boldsymbol{A}E[(\underline{\boldsymbol{X}} - \boldsymbol{m}_{\underline{\boldsymbol{X}}})(\underline{\boldsymbol{X}} - \boldsymbol{m}_{\underline{\boldsymbol{X}}})^T]\boldsymbol{A}^T \\ &= \begin{bmatrix} 1 & 3 \\ -1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 4 & 0.8 \\ 0.8 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 3 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 17.8 & 1.2 & 24.2 \\ 1.2 & 4.8 & -1.2 \\ 24.2 & -1.2 & 34.6 \end{bmatrix} \end{aligned}$$

(c) Solution:
The cross-covariance matrix $\boldsymbol{R}_{xy}$ is

$$\begin{aligned}\boldsymbol{R}_{xy} &= E[\underline{\boldsymbol{X}}\underline{\boldsymbol{Y}}^T] = E[\underline{\boldsymbol{X}}\underline{\boldsymbol{X}}^T\boldsymbol{A}^T] = E[\underline{\boldsymbol{X}}\underline{\boldsymbol{X}}^T]\boldsymbol{A}^T \\ &= \boldsymbol{R}_{xx}\boldsymbol{A}^T = \left(\begin{bmatrix} 4 & 0.8 \\ 0.8 & 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \end{bmatrix}\right)\begin{bmatrix} 1 & -1 & 2 \\ 3 & 2 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 13.4 & 0.6 & 18.4 \\ 17.8 & 7.2 & 20.6 \end{bmatrix}\end{aligned}$$

33. (a) Solution:
The mean sequence $m_x[n]$ is

$$\begin{aligned}m_x[n] &= E[x[n]] = E[A\sin(\Omega n)] \\ &= \frac{1}{5}\cdot 0 + \frac{1}{5}\sin(\frac{\pi}{2}n) + \frac{1}{5}\sin(-\frac{\pi}{2}n) - \frac{1}{5}\sin(\frac{\pi}{2}n) - \frac{1}{5}\sin(-\frac{\pi}{2}n) = 0\end{aligned}$$

(b) Solution:
The auto-correlation $r_x[m,n]$ is

$$\begin{aligned}r_x[m,n] &= E[x[m]x[n]] = E[A\sin(\Omega m)A\sin(\Omega n)] = E[A^2\sin(\Omega m)\sin(\Omega n)] \\ &= \frac{1}{5}\times 0 + \frac{1}{5}\sin(\frac{\pi}{2}m)\sin(\frac{\pi}{2}n) + \frac{1}{5}\sin(-\frac{\pi}{2}m)\sin(-\frac{\pi}{2}n) \\ &\quad + \frac{1}{5}\times(-1)^2\times\sin(\frac{\pi}{2}m)\sin(\frac{\pi}{2}n) + \frac{1}{5}\times(-1)^2\times\sin(-\frac{\pi}{2}m)\sin(-\frac{\pi}{2}n) \\ &= \frac{4}{5}\sin(\frac{\pi}{2}m)\sin(\frac{\pi}{2}n) \\ &= \frac{2}{5}\left[\sin\frac{\pi}{2}(m+n) + \sin\frac{\pi}{2}(m-n)\right]\end{aligned}$$

(c) Solution:
x_n is NOT wide-sense stationary, NOT uncorrelated, and NOT orthogonal.

34. (a) Solution:
The mean sequence $\mu_y[n]$ is

$$\mu_y[n] = \mu_x[n]\sum_{n=-\infty}^{\infty} h[n] = 4\mu_x = 16$$

(b) Solution:
The cross-covariance $c_{xy}[m,n]$ is

$$c_{xy}[m,n] = c_{xy}[\ell] = h[-\ell] * c_x[\ell]$$

ℓ	-6	-5	-4	-3	-2	-1	0	1	2	3
$c_{xy}[\ell]$	1	3	6	10	12	12	10	6	3	1

(c) Solution: The auto-covariance $c_y[m,n]$ of the output $y[n]$ is

$$c_y[m,n] = c_y[\ell] = h[\ell] * (h[-\ell] * c_x[\ell])$$

ℓ	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6
$c_y[\ell]$	1	4	10	20	31	40	44	40	31	20	10	4	1

35. (a) Solution:
The ACRS $r_x[m,n]$ of $x[n]$ is

$$r_x[m,n] = E[x[m]x[n]] = E[x[m]]E[x[n]] = \left(\frac{1}{2}\right)^m \left(\frac{1}{2}\right)^n = \left(\frac{1}{2}\right)^{m+n}$$

(b) Solution:
The mean sequence $\mu_y[n]$ of $y[n]$ is

$$\mu_y[n] = \mu_x[n] \cdot \sum_{n=-\infty}^{\infty} h[n] = \left(\frac{1}{2}\right)^n \cdot \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n = \frac{4}{3} \cdot \left(\frac{1}{2}\right)^n$$

(c) Solution:
The ACRS $r_y[m,n]$ of $y[n]$ is

$$\begin{aligned}
r_y[m,n] = r_x[m,n] * h[m] * h[-n] &= \left(\left(\frac{1}{2}\right)^m * h[m]\right)\left(\left(\frac{1}{2}\right)^n * h[-n]\right) \\
&= \left(\sum_{p=0}^{\infty} \left(\frac{1}{4}\right)^p \left(\frac{1}{2}\right)^{m-p}\right)\left(\sum_{q=-\infty}^{0} \left(\frac{1}{4}\right)^{-q} \left(\frac{1}{2}\right)^{n-q}\right) \\
&= \left(\sum_{p=0}^{\infty} \left(\frac{1}{4}\right)^p \left(\frac{1}{2}\right)^{m-p}\right)\left(\sum_{q=0}^{\infty} \left(\frac{1}{4}\right)^{q} \left(\frac{1}{2}\right)^{n+q}\right) \\
&= \left[\left(\frac{1}{2}\right)^m \sum_{p=0}^{\infty} \left(\frac{1}{2}\right)^p\right]\left[\left(\frac{1}{2}\right)^n \sum_{q=0}^{\infty} \left(\frac{1}{2}\right)^{3q}\right] \\
&= \frac{16}{7}(\frac{1}{2})^{m+n}
\end{aligned}$$

36. tba

37. (a) Solution:

$$m_x[n] = E[x[n]] = \int_0^{2\sqrt{3}} \frac{x}{2\sqrt{3}} dx = \sqrt{3}$$

$$E[x^2[n]] = \int_0^{2\sqrt{3}} \frac{x^2}{2\sqrt{3}} dx = 4$$

$$\sigma_x^2[n] = E[x^2[n]] - m_x^2[n] = 1$$

(b) Solution:

$$m_y[n] = m_x[n] \sum_{n=-\infty}^{\infty} h[n] = m_x[n] \cdot 0 = 0$$

(c) Solution:

$$\begin{aligned}
c_y[\ell] &= c_x[\ell] * h[\ell] * h[-\ell] = \delta[\ell] * h[\ell] * h[-\ell] = *h[\ell] * h[-\ell] \\
&= \sum_{n=-\infty}^{\infty} h[n]h[\ell+n] \\
&= \sum_{n=-\infty}^{\infty} (\delta[n-1] - \delta[n+1]) (\delta[\ell+n-1] - \delta[\ell+n+1]) \\
&= 2\delta[\ell] - \delta[\ell-2] - \delta[\ell+2]
\end{aligned}$$

38. Solution:

$$r_{yy}[0]a_1 + r_{yy}[1]a_2 = -r_{yy}[1] \tag{13.147}$$

$$r_{yy}[1]a_1 + r_{yy}[0]a_2 = -r_{yy}[2] \tag{13.148}$$

$$\sigma_x^2 = r_{yy}[0] + a_1 r_{yy}[1] + a_2 r_{yy}[2] \tag{13.150}$$

Plug $a_1 = -5/6$, $a_2 = 1/6$, and $\sigma_x^2 = 2$ into the three equations above and solve for $r_{yy}[0]$, $r_{yy}[1]$, and $r_{yy}[2]$, we have

$$r_{yy}[0] = \frac{21}{5}, \quad r_{yy}[1] = 3, \quad r_{yy}[2] = \frac{9}{5}$$

We can also conclude that $r_{yy}[\ell] = 0$, for $\ell > 2$.