# A Step-indexed Semantics of Imperative Objects

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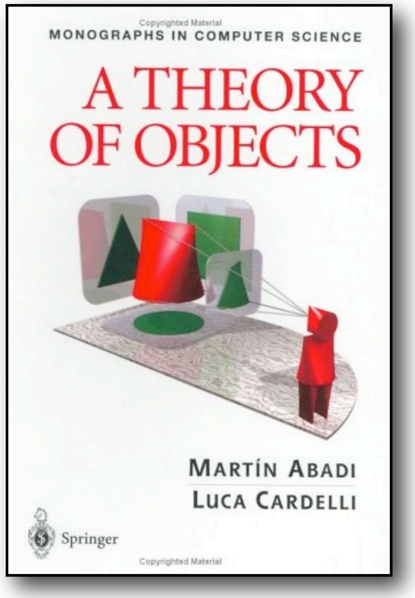
# Imperative object calculus

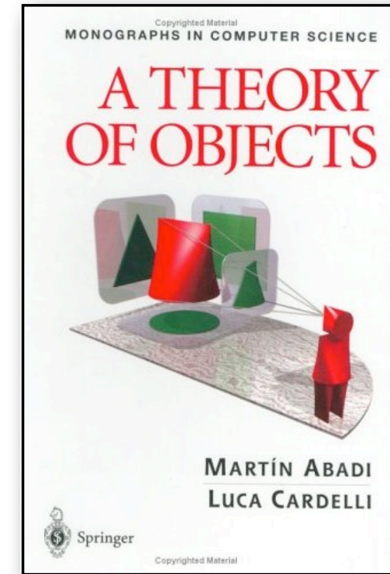
[Abadi and Cardelli, '96]




$$
\begin{aligned}
a, b ::= \ & x \mid [\mathrm{m}_d {=} \varsigma(x_d) b_d]_{d \in D} \\
& \mid a.\mathrm{m} \mid a.\mathrm{m} \coloneqq \varsigma(x) b \\
& \mid \text{clone } a \mid \lambda(x) b \mid a \ b \\
v ::= \ & \{\mathrm{m}_d {=} l_d\}_{d \in D} \mid \lambda(x) b
\end{aligned}
$$

# Imperative object calculus

[Abadi and Cardelli, '96]




$$
\begin{aligned}
a, b ::=\ & x \mid [\mathrm{m}_d{=}\varsigma(x_d)b_d]_{d\in D} \\
& \mid a.\mathrm{m} \mid a.\mathrm{m} := \varsigma(x)b \\
& \mid \mathrm{clone}\ a \mid \lambda(x)b \mid a\ b \\
v ::=\ & \{\mathrm{m}_d{=}l_d\}_{d\in D} \mid \lambda(x)b
\end{aligned}
$$

dynamic allocation

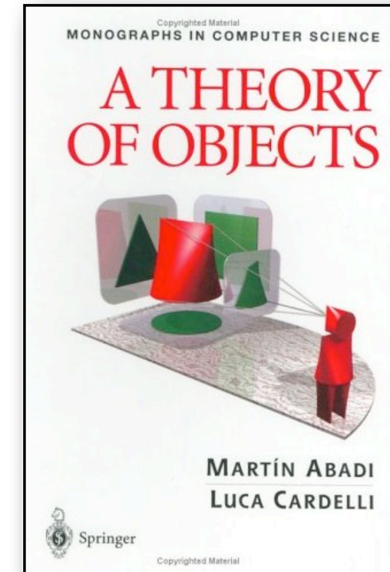
heap stores code

recursion involves heap

$$
\begin{aligned}
& \langle \emptyset, [\mathrm{m}{=}\varsigma(x)x.\mathrm{m}].\mathrm{m}\rangle \xrightarrow{\textsc{Obj}} \\
& \to \langle \{\!\{ l \mapsto \lambda(x)x.\mathrm{m} \}\!\}, \{\mathrm{m}{=}l\}.\mathrm{m}\rangle \xrightarrow{\textsc{Inv}} \\
& \to \langle \{\!\{ l \mapsto \lambda(x)x.\mathrm{m} \}\!\}, (\lambda(x)x.\mathrm{m})\ \{\mathrm{m}{=}l\}\rangle \xrightarrow{\textsc{Beta}} \\
& \to \langle \{\!\{ l \mapsto \lambda(x)x.\mathrm{m} \}\!\}, \{m{=}l\}.\mathrm{m}\rangle \xrightarrow{\textsc{Inv}} \dots
\end{aligned}
$$

# Imperative object calculus

[Abadi and Cardelli, '96]




$$\begin{aligned} a, b ::= \ & x \mid [\mathrm{m}_d{=}\varsigma(x_d)b_d]_{d\in D} \\ & \mid a.\mathrm{m} \mid a.\mathrm{m} := \varsigma(x)b \\ & \mid \mathrm{clone}\ a \mid \lambda(x)b \mid a\ b \\ v ::= \ & \{\mathrm{m}_d{=}l_d\}_{d\in D} \mid \lambda(x)b \end{aligned}$$

dynamically-allocated,
higher-order store

+ expressive type system

- Object types and subtyping
- Impredicative second-order types
- Recursive types

# Hard to find good semantic models

- For domain-theoretic models ...
- Higher-order store
  - Solving recursive domain equations
- + Dynamic allocation - possible-worlds models
  - Recursively defined functor categories over CPOs
- Existing domain-theoretic models
  [Levi, '02] [Reus & Schwinghammer, '06]
  - Despite being complex are not abstract enough
- + Polymorphic values on the heap (impredicative)
  - No domain-theoretic models known, in general!

# Types and heap typings

- In a set-theoretic term model of our calculus
  are types just sets of values?
- No! Our values depend on the heap, e.g. $\{\mathrm{m}_d = l_d\}_{d \in D}$
  - so semantic types depend on heap typings
  - heap typings are maps from locations to semantic types
- Model types as sets of pairs?

$$Type = \mathcal{P}(\mathit{HeapTyping} \times \mathit{CVal})$$
$$\mathit{HeapTyping} = \mathit{Loc} \rightharpoonup_{\mathit{fin}} Type$$

  - There are no set-theoretic solutions to this!

# Step-indexed models

- Alternative to subject-reduction [Appel & Felty, '00]
  - Simpler machine-checkable proofs of type soundness
- Much simpler than the domain-theoretic models
  - Only based on a small-step operational semantics
- Model of types for the lambda calculus with recursive types [Appel & McAllester, '01]
- Later extended to general references and impredicative polymorphism [Ahmed, '04]
  - We further extended it with object types and subtyping
  - Used it to prove the soundness of an expressive, standard type system for the imperative object calculus

# Types and heap typings

- Circular definition $\mathit{Type} = \mathcal{P}(\mathit{HeapTyping} \times \mathit{CVal})$

$$\mathit{HeapTyping} = \mathit{Loc} \rightharpoonup_{\mathit{fin}} \mathit{Type}$$

- We can solve this by a stratified construction

$$\mathit{Type}_{k+1} = \mathcal{P}(j \in [0, k] \times \mathit{HeapTyping}_j \times \mathit{CVal})$$

$$\mathit{HeapTyping}_j = \mathit{Loc} \rightharpoonup_{\mathit{fin}} \mathit{Type}_j$$

- *k*-th approximation: $\lfloor \tau \rfloor_k = \{\langle j, \Psi, v \rangle \in \tau \mid j < k\}$
  - We have that $\lfloor \tau \rfloor_k \in \mathit{Type}_k$
- Stratification invariant:
  - $\lfloor \alpha \rfloor_{k+1}$ is only defined in terms of $\lfloor \Psi \rfloor_k$ and $\lfloor \tau \rfloor_k$

# Semantic approximation

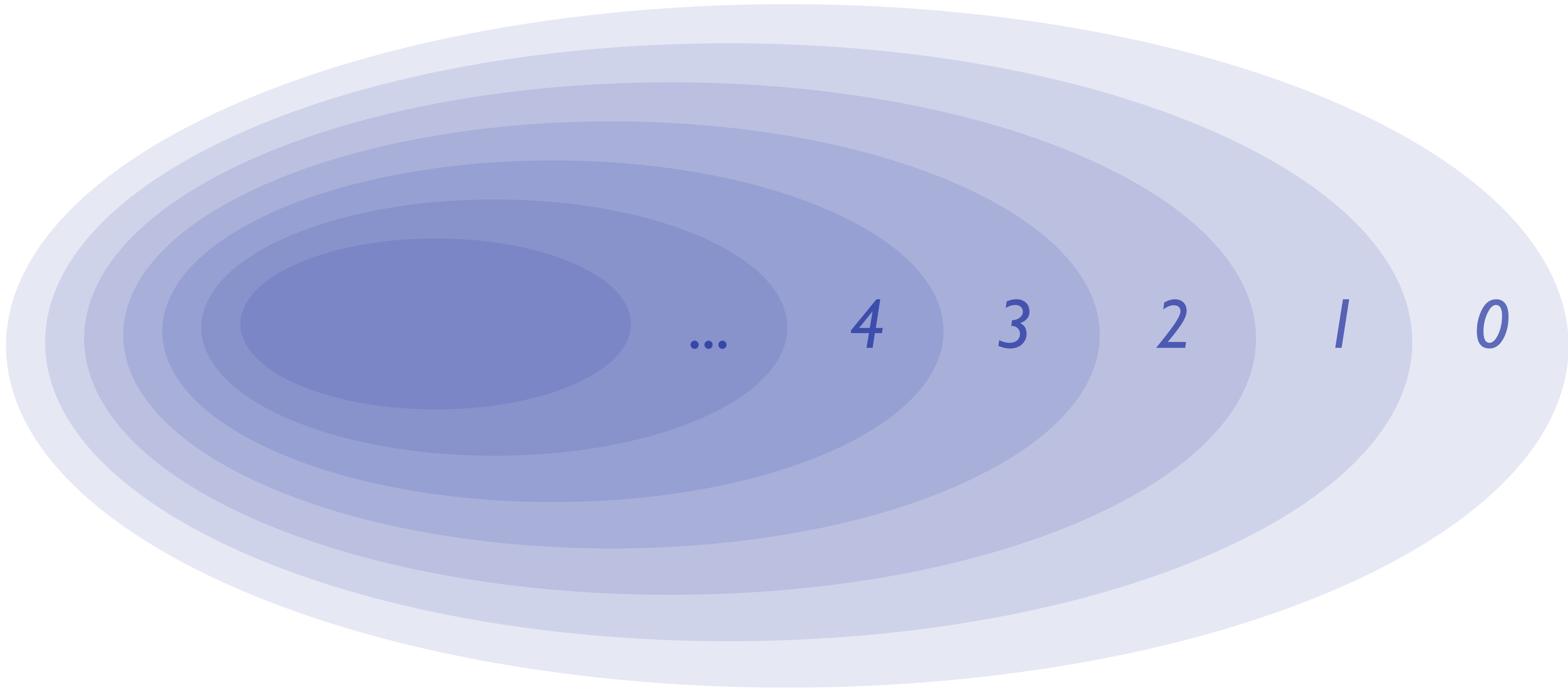
- Semantic types are sets of triples
- $\langle k, \Psi, v \rangle \in \tau$ if $v$ executes for at least $k$ steps without getting stuck in every context of type $\tau$, for every $h :_k \Psi$
- Example: $\langle 1, \emptyset, (\lambda x.\, true) \rangle \in Nat \to Nat$

$$\langle 2, \emptyset, (\lambda x.\, true) \rangle \notin Nat \to Nat,$$

$$C[\cdot] = ([\cdot]\ 42) + 2$$

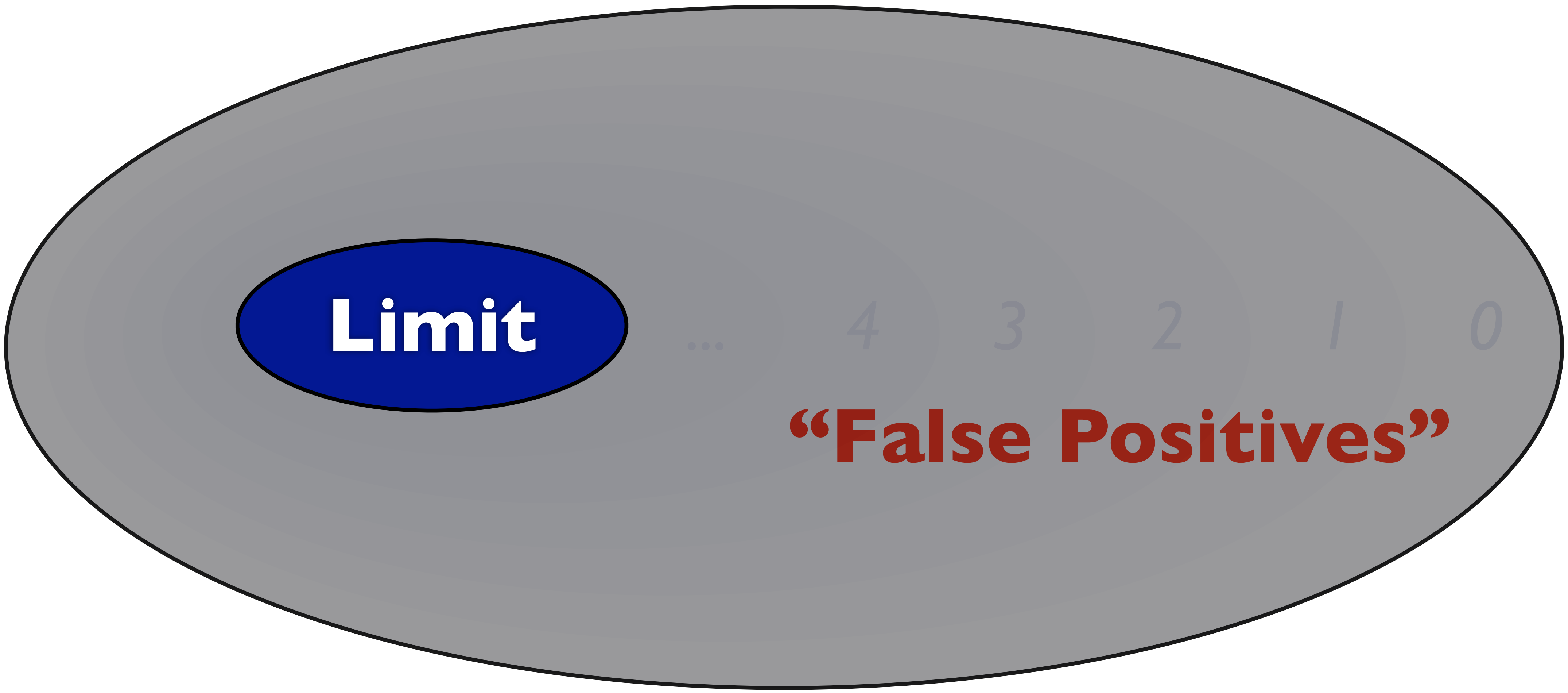
# Semantic types

- Sequences of increasingly accurate approximations



# Semantic types

- Sequences of increasingly accurate approximations




- In the end we are only interested in the limit
- Approximation crucial for well-founded construction

  + Extremely useful when giving recursive definitions of types

# State extension

- Heaps evolve during computation
  - Dynamic allocation, no deallocation, weak updates
    - ➡ Heap typings can only "grow"
- The precision of our approximation decreases with each reduction step
- State extension relation: $(k, \Psi) \sqsubseteq (j, \Psi')$
- Closure under state extension (Kripke monotonicity)

$$\langle k, \Psi, v \rangle \in \alpha \ \wedge \ (k, \Psi) \sqsubseteq (j, \Psi') \ \Rightarrow \ \langle j, \Psi', v \rangle \in \alpha$$

- Semantic types must be closed under state extension
- Possible-worlds model

# The type of arbitrary terms

- For a closed term $a$, $a :_{k,\Psi} \tau$ iff

  $$\langle h, a\rangle \to^{j} \langle h', b\rangle \not\to, \text{ for any } j < k,\ h :_k \Psi,\ b, \text{ and } h'$$
  $$\Rightarrow \langle k-j, \Psi', b\rangle \in \tau, \text{ for some } \Psi' \text{ such that}$$
  $$(k, \Psi) \sqsubseteq (k-j, \Psi') \text{ and } h' :_{k-j} \Psi'$$

- Semantic typing judgement

  $$\Sigma \models a : \alpha \;\Leftrightarrow\; \forall k \geq 0.\ \forall \Psi.\ \forall \sigma :_{k,\Psi} \Sigma.\ \sigma(a) :_{k,\Psi} \alpha$$

  - Typing open terms; not approximative

- This definition directly enforces type safety
  - Still need to prove the soundness of the typing rules

# Simple semantic types

- Base types

$$Bool \triangleq \{\langle k, \Psi, v\rangle \mid k \in \mathbb{N}, \Psi \in \mathit{HeapTyping}_k, v \in \{\mathrm{true}, \mathrm{false}\}\}$$

$$Nat \triangleq \{\langle k, \Psi, \underline{n}\rangle \mid k \in \mathbb{N}, \Psi \in \mathit{HeapTyping}_k, n \in \mathbb{N}\}$$

- Procedure types

$$\alpha \rightarrow \beta \triangleq \{\langle k, \Psi, \lambda(x)b\rangle \mid \forall j{<}k.\ \forall \Psi'.\ \forall v.\ (k, \Psi) \sqsubseteq (j, \Psi') \ \wedge\ \langle j, v\rangle \in \alpha \Rightarrow \{\!\!\{x \mapsto v\}\!\!\}(b) :_{j,\Psi'} \beta\}$$

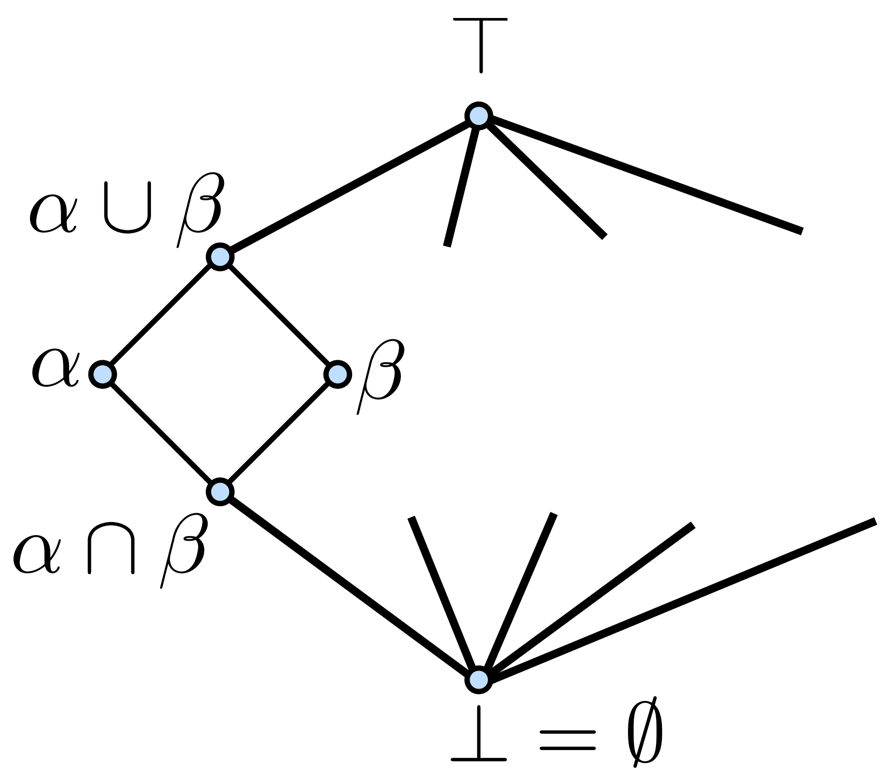
- Reference types

$$\mathrm{ref}\ \tau = \{\langle k, \Psi, l\rangle \mid \lfloor \Psi(l) \rfloor_k = \lfloor \tau \rfloor_k\}$$

# Object types and subtyping

# Subtyping

- Since types are sets, subtyping is set inclusion
- Subtyping forms a lattice on types




- Simple, but not orthogonal to the other features
  - e.g. non-trivial interaction with object types

# Definition of object types

- Methods stored in the heap as procedures and self-application semantics of method invocation suggest

$$[\mathrm{m}_d : \tau_d]_{d\in D} \approx \mu(\alpha).\{\mathrm{m}_d : \mathrm{ref}\ (\alpha \rightarrow \tau_d)\}_{d\in D}$$

- This validates all typing rules for objects
  - Let $\alpha = [m_d : \tau_d]_{d\in D}$

$$(\textsc{Obj})\ \frac{\forall d \in D.\ \Sigma[x_d \mapsto \alpha] \models b_d : \tau_d}{\Sigma \models [m_d {=} \varsigma(x_d) b_d]_{d\in D} : \alpha} \qquad (\textsc{Clone})\ \frac{\Sigma \models a : \alpha}{\Sigma \models \mathrm{clone}\ a : \alpha}$$

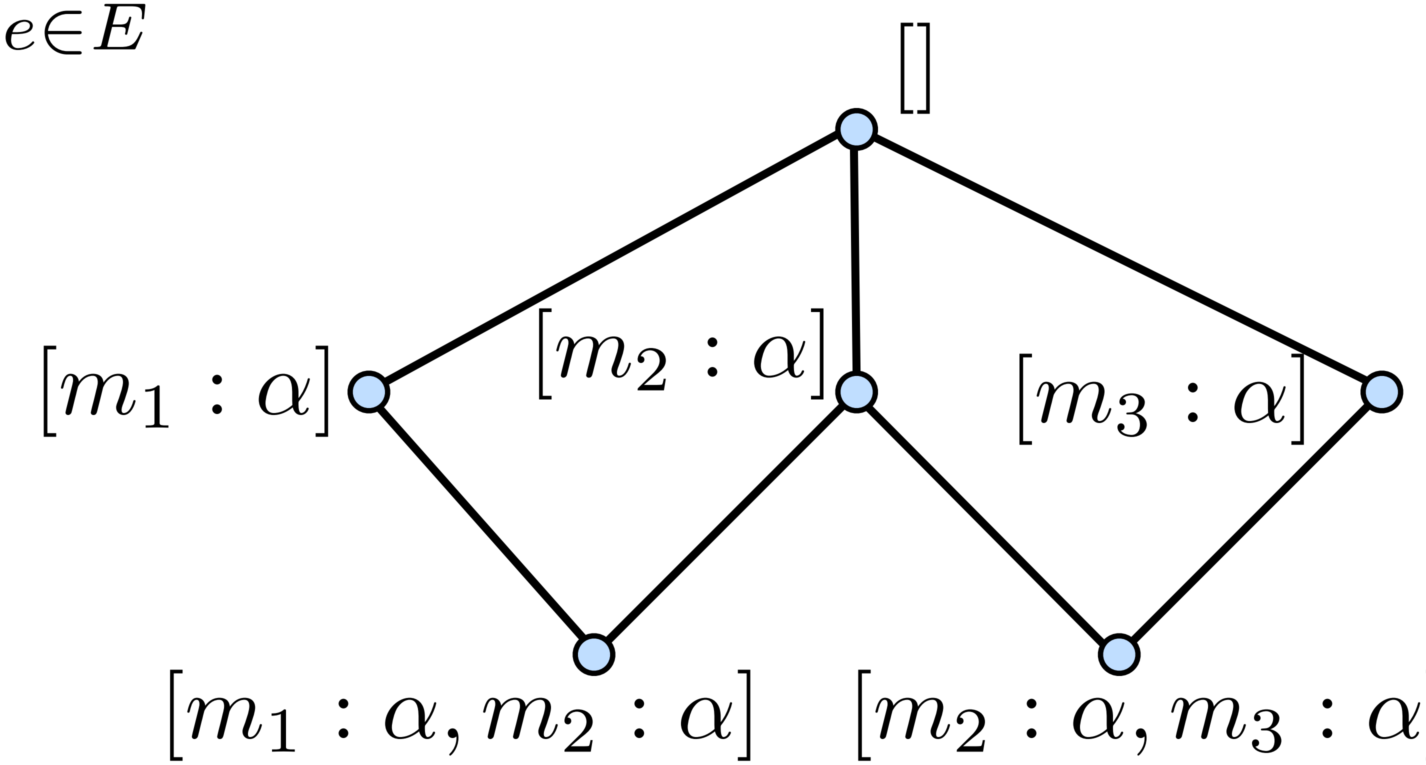
$$(\textsc{Inv})\ \frac{\Sigma \models a : \alpha \quad e \in D}{\Sigma \models a.m_e : \tau_e} \qquad (\textsc{Upd})\ \frac{\Sigma \models a : \alpha \quad e \in D \quad \Sigma[x \mapsto \alpha] \models b : \tau_e}{\Sigma \models a.m_e \coloneqq \varsigma(x)b : \alpha}$$

- But none of the subtyping rules!

# Subtyping in width

- Object types with more methods are subtypes of object types with less methods
- Assuming the same type for the common methods

$$\frac{E \subseteq D}{[m_d : \tau_d]_{d \in D} \subseteq [m_e : \tau_e]_{e \in E}}$$



# Subtyping in width

$$[\mathrm{m}_d : \tau_d]_{d \in D} \approx \mu(\alpha).\{\mathrm{m}_d : \mathrm{ref}\ (\alpha \to \tau_d)\}_{d \in D}$$

- Subtyping in width fails because:
  - positions of recursion variable are invariant
  - even without reference positions contravariant
  - they should be covariant! (see below)

# Subtyping in width

$$[\mathrm{m}_d : \tau_d]_{d \in D} \approx \mu(\alpha).\{\mathrm{m}_d : \mathrm{ref}\ (\alpha \to \tau_d)\}_{d \in D}$$

- Subtyping in width fails because:
  - positions of recursion variable are invariant
  - even without reference positions contravariant
  - they should be covariant! (see below)

$$\frac{\alpha \subseteq \beta \Rightarrow \dfrac{E \subseteq D \quad \forall d \in D.\ \mathrm{ref}\ (\alpha \to \tau_d) \subseteq \mathrm{ref}\ (\beta \to \tau_d)}{\{\mathrm{m}_d : \mathrm{ref}\ (\alpha \to \tau_d)\}_{d \in D} \subseteq \{\mathrm{m}_e : \mathrm{ref}\ (\beta \to \tau_e)\}_{e \in E}}}{\mu(\alpha).\{\mathrm{m}_d : \mathrm{ref}\ (\alpha \to \tau_d)\}_{d \in D} \subseteq \mu(\beta).\{\mathrm{m}_e : \mathrm{ref}\ (\beta \to \tau_e)\}_{e \in E}}$$

# Subtyping in width

- We force covariance for recursion variable using a bounded existential

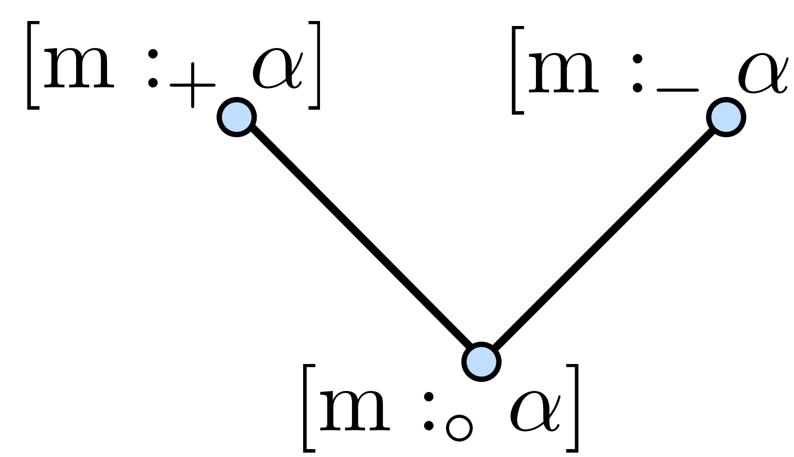
$$\frac{\alpha \subseteq \beta \quad \forall \tau \subseteq \alpha.\ F(\tau) \subseteq G(\tau)}{\exists \alpha' \subseteq \alpha . F(\alpha') \subseteq \exists \beta' \subseteq \beta . G(\beta')}$$

$$[\mathrm{m}_d : \tau_d]_{d \in D} \approx \mu(\alpha).\exists \alpha' \subseteq \alpha . \{\mathrm{m}_d : \mathrm{ref}\ (\alpha' \to \tau_d)\}_{d \in D}$$

- $\alpha'$ can be viewed as the "true" type of the object
- Similar to some encodings of the functional obj. calculus [Abadi & Cardelli, '96] and [Abadi, Cardelli & Viswanathan, '96]

# Subtyping in depth

- Our methods can be both invoked and updated
  - They need to be invariant (o)
- Still, if we mark methods with their desired variance and restrict invocations and updates accordingly
  - Covariant subtyping for invoke-only methods (+)
  - Contravariant subtyping for update-only methods (-)
- Moreover, we would like that



# Extending reference types

- However, the usual reference types are invariant

  $$\mathrm{ref}_\circ \tau = \{\langle k, \Psi, l\rangle \mid \lfloor \Psi \rfloor_k (l) = \lfloor \tau \rfloor_k\}$$

- The type of the location is precisely known
  - So both reading and writing are safe at type $\tau$
- If we only give a bound on $\Psi(l)$ then only one of these operations is safe at a meaningful type
  - Readable reference type

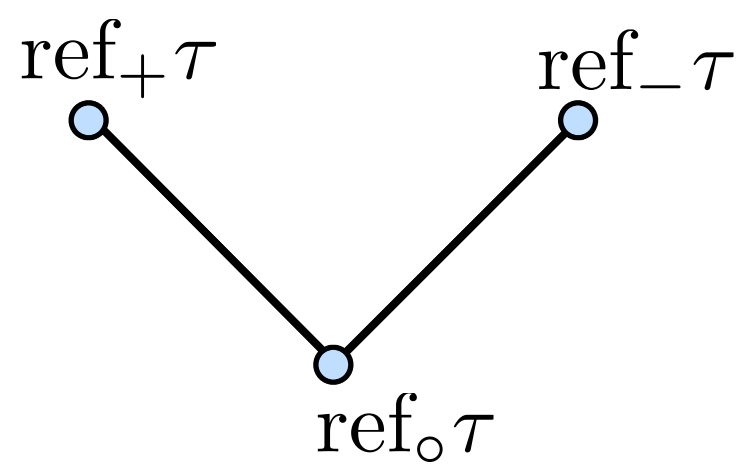
    $$\mathrm{ref}_+ \tau = \{\langle k, \Psi, l\rangle \mid \lfloor \Psi \rfloor_k (l) \subseteq \lfloor \tau \rfloor_k\}$$

    - This is **not** read-only!
  - Writable reference types

    $$\mathrm{ref}_- \tau = \{\langle k, \Psi, l\rangle \mid \lfloor \tau \rfloor_k \subseteq \lfloor \Psi \rfloor_k (l)\}$$

# Extending reference types

- Readable reference type is covariant $$\frac{\alpha \subseteq \beta}{\mathrm{ref}_+\alpha \subseteq \mathrm{ref}_+\beta}$$
- Writable reference type is contravariant $$\frac{\beta \subseteq \alpha}{\mathrm{ref}_-\alpha \subseteq \mathrm{ref}_-\beta}$$
- The usual reference types can actually be defined as $\mathrm{ref}_\circ\tau = \mathrm{ref}_+\tau \cap \mathrm{ref}_-\tau$, so clearly




- Not really new [Reynolds, '88] [Pierce & Sangiorgi, '96]

# Definition of object types

$$\langle k, \Psi, \{\mathrm{m}_e{=}l_e\}_{e\in E}\rangle \in \alpha = [\mathrm{m}_d : \tau_d]_{d\in D} \;\Leftrightarrow\; D \subseteq E$$
$$\wedge\; \exists \alpha' \subseteq \lfloor \alpha \rfloor_k.\, (\forall d \in D.\, \langle k, \Psi, l_d \rangle \in \mathrm{ref}_{\nu_d}(\alpha' \to \tau_d))$$

$$\Uparrow$$

$$[\mathrm{m}_d :_{\nu_d} \tau_d]_{d\in D} \approx \mu(\alpha).\exists \alpha' \subseteq \alpha.\{\mathrm{m}_d : \mathrm{ref}_{\nu_d}(\alpha' \to \tau_d)\}_{d\in D}$$

# Definition of object types

$$\langle k, \Psi, \{\mathrm{m}_e{=}l_e\}_{e\in E}\rangle \in \alpha = [\mathrm{m}_d : \tau_d]_{d\in D} \;\Leftrightarrow\; D \subseteq E$$
$$\wedge\;\; \exists \alpha' \subseteq \lfloor \alpha \rfloor_k .\; (\forall d \in D.\; \langle k, \Psi, l_d\rangle \in \mathrm{ref}_{\nu_d}(\alpha' \to \tau_d))$$

- But, because $\alpha'$ is kept abstract
  - invocation and cloning rules are no longer validated
- Fixing invocation
  - We need to permit self-application
  - We explicitly enforce that $\alpha'$ contains $\{\mathrm{m}_e{=}l_e\}_{e\in E}$
  - Not surprising, $\alpha'$ is the "true" type of $\{\mathrm{m}_e{=}l_e\}_{e\in E}$
- Fixing clone
  - We enforce that $\alpha'$ contains all clones of $\{\mathrm{m}_e{=}l_e\}_{e\in E}$ i.e. all objects that satisfy the same typing assumptions

# Definition of object types

$$\langle k, \Psi, \{\mathrm{m}_e{=}l_e\}_{e\in E}\rangle \in \alpha = [\mathrm{m}_d : \tau_d]_{d\in D} \;\Leftrightarrow\; D \subseteq E$$
$$\wedge\;\; \exists\alpha'{\subseteq} \lfloor\alpha\rfloor_k.\; (\forall d \in D.\; \langle k, \Psi, l_d\rangle \in \mathrm{ref}_{\nu_d}(\alpha' \to \tau_d)) \;\wedge\; \ldots$$

- This definition is well-founded (inductive on k)
  - $\lfloor\alpha\rfloor_{k+1}$ is defined in terms of $\lfloor\alpha\rfloor_k$
- Validates all typing and subtyping rules for objects
  - Most interesting proof is for object creation (nested induction on naturals)
- Main contribution of the paper

# Conclusion

- We extended the step-indexed model of Ahmed et. al. with object types and subtyping, and used it for the imperative object calculus
- Our interpretation of object types uses
  - Recursive types and bounded existentials
  - Readable and writable reference types
- Resulting model
  - is much simpler than a domain-theoretic ones
  - interprets a richer type discipline - impredicative 2nd order types, subtyping in depth wrt. variance annotations
  - However, it only deals with types and type safety

# Beyond types

- Purely syntactic argument would have sufficed for proving the safety of our type system (subject-reduction)
  - So why do we need models?
- For more expressive deduction systems, e.g. program logics
  - Meaning of assertions no longer obvious
    - They should describe the code in the (higher-order) heap
  - Subject-reduction limited to whole programs of base type
  - Proving soundness using semantic model (derivability implies validity in the model) gives much stronger guarantees
- **Future work:** Prove the soundness of a program logic for the imperative object calculus using step-indexed model

# Backup slides

# Problem 1: Semantic domains

- Higher-order store
  - Solving recursive domain equation

$$D_{Val} = (D_{Heaps} \times D_{Val} \rightharpoonup D_{Heaps} \times D_{Val}) + \ldots$$
$$D_{Heaps} = Loc \rightharpoonup_{fin} D_{Val}$$

  - For the imperative object calculus done in:
    [Kamin & Reddy, 94] [Reus & Streicher, '04]
- \+ polymorphic values stored (impredicative)
  - No domain-theoretic models known!

# Semantic typing judgement

- Typing open terms; not approximative

$$\Sigma \models a : \alpha \;\Leftrightarrow\; \forall k \geq 0.\ \forall \Psi.\ \forall \sigma :_{k,\Psi} \Sigma.\ \sigma(a) :_{k,\Psi} \alpha$$

- This definition directly enforces type safety
- But we still need to prove the typing rules sound
  - We first prove the validity of semantic typing lemmas
  - Then use these lemmas to prove the syntactic typing rules
- Example: subtyping recursive types (the Amber rule)

$$(\textsc{Semantic})\ \frac{\forall \alpha, \beta \in \mathit{Type}.\ \alpha \subseteq \beta \Rightarrow F(\alpha) \subseteq G(\beta)}{\mu F \subseteq \mu G}$$

$$(\textsc{Syntactic})\ \frac{\Gamma \vdash \mu X.\underline{A} \quad \Gamma \vdash \mu Y.\underline{B} \quad \Gamma, Y \leqslant \mathit{Top}, X \leqslant Y \vdash \underline{A} \leqslant \underline{B}}{\Gamma \vdash \mu X.\underline{A} \leqslant \mu Y.\underline{B}}$$

# Semantic soundness

- We relate the syntactic type expressions to their corresponding semantic types
- We prove that the two are in close correspondence
- Theorem: Soundness of subtyping

  $$\text{If } \Gamma \vdash A \leqslant B \text{ and } \eta \models \Gamma, \text{ then } [\![A]\!]_\eta \subseteq [\![B]\!]_\eta$$

- Theorem: Semantic soundness

  $$\text{If } \Gamma \vdash a : A \text{ and } \eta \models \Gamma, \text{ then } [\![\Gamma]\!]_\eta \models a : [\![A]\!]_\eta$$

- Corollary (Type safety)
  Well-typed terms are safe to evaluate.

# More than types (related work)

- Step-indexed PER model for lambda calculus with recursive and impredicative quantified types [Ahmed, '06]
  - Captures exactly observational equivalence, no state
- Soundness of compositional program logic for a very simple stack-based abstract machine [Benton, '05]
- Floyd-Hoare-style framework based on relational parametricity for machine code programs [Benton, '06]

# More extensions and future work

- Generalizing reference types ... and object types

$$\mathrm{ref}_{\circ}\tau = \mathrm{ref}_{+}\tau \cap \mathrm{ref}_{-}\tau$$
$$\mathrm{ref}(\alpha, \beta) = \mathrm{ref}_{-}\alpha \cap \mathrm{ref}_{+}\beta$$

- Accommodating self types (easy)
- More realistic languages