Graph Algorithms - Seminar 1

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Exercise 1.

A professor wrote p exercises and wants to give them to n students.

a) In how many ways this can be done? (One exercise to each student, but necessarely different)

b) The same question if he wants to give them different exercises ($p \geqslant n$).

c) But if he wants to give two different exercises to each of its students? (Different students can share at most one common exercise, $p \geqslant n$.)

Exercises for the 1st seminar

Exercise 2. Let $f : \mathbb{N}^* \to \mathbb{N}^*$ be a function. We define the class of complexity

$$\mathcal{O}(f(n)) = \{g : \mathbb{N}^* \to \mathbb{N}^* : \exists a, b, n_0 \in \mathbb{N}^* \text{ s. t. } g(n) \leqslant a \cdot f(n) + b, \forall n \geqslant n_0\}$$

(Note that $a = a_g, b = b_g, n_0 = n_{0,g}$.) By abuse of notation we write $g(n) = \mathcal{O}(f(n)$ instead of $g(n) \in \mathcal{O}(f(n))$.

(a) Prove that one can skip b in the above definition.

(b) Prove that if $\lim\limits_{n\to\infty} \dfrac{g(n)}{f(n)} \in \mathbb{R}$, then $g(n) = \mathcal{O}(f(n))$

Exercise 3. Prove that

(a) $\displaystyle\sum_{k=1}^{n} k = \mathcal{O}(n^2)$ and $\displaystyle\sum_{k=1}^{n} k^2 = \mathcal{O}(n^3)$;

(b) $\log_a n = \mathcal{O}(\log_b n)$ and $\displaystyle\sum_{k=1}^{n} \frac{1}{k} = \mathcal{O}(\log n)$;

Exercise 4. Legend has it that Josephus wouldn't have lived to become famous without his mathematical talents. During the Jewish-Roman war, he was among a band of 41 Jewish rebels trapped in a cave by the Romans. Preferring suicide to capture, the rebels decided to form a circle and, proceeding around it, to kill every third remaining person until no one was left. But Josephus, along with an unindicted co-conspirator, wanted none of this suicide nonsense; so he quickly calculated where he and his friend should stand in the vicious circle.

In our variation, we start with n peple numbered from 1 to n around a circle, and we eliminate every second remaining person until only one survives. Let j_n be the number of the survivor.

a) Find $j_1, j_2, \ldots, j_{10}$.

b) Write a recursive function that computes j_n.

(c) Can you give a formula for j_n? (Hint: look what hapens when n becomes a power of 2.)

Exercise 5.
What is the maximum number, r_n, of regions defined by n lines in the plane?

Exercise 6. Given a set M of $n \geqslant 1$ elements, a partition of M is a family of subsets of M, $\mathcal{P} = \{M_1, M_2, \ldots, M_k\}$, astfel încât $M_i \neq \varnothing, \forall i$, $M_i \cap M_j = \varnothing, \forall i \neq j$ and $\bigcup_{i=1}^{n} M_i = M$. If $\mathcal{P}$ and $\mathcal{P}'$ are two partitions we say that $\mathcal{P}$ is a refinement of $\mathcal{P}'$ if for every subset $M_i \in \mathcal{P}$ there exists a subset $M'_j \in \mathcal{P}'$ such that $M_i \subseteq M'_j$. We consider the following decision problem.

PARTITION

Instance: M, $|M| = n \in \mathbb{N}^*$ and two partitions $\mathcal{P}$, $\mathcal{P}'$.

Question: Is $\mathcal{P}$ a refinement of $\mathcal{P}'$?

(a) Prove that **PARTITION** $\in$ **P**.

(b) Write a recursive procedure that has to decide for two given parti-