Exerc: Fie $(A,\leq)$ și $(B,\sqsubseteq)$ poseturi nevide. Dem. că:

(1) Dacă $|A|=1$, at $\Rightarrow (A,\leq)\times(B,\sqsubseteq)\cong(B,\sqsubseteq)$ (izomorfe ca poseturi)

(comut. x, sau analog)

(2) Dacă $|B|=1$, at $\Rightarrow (A,\leq)\times(B,\sqsubseteq)\cong(A,\leq)$ (— " —)

(3) Dacă $|A|\geq 2$ și $|B|\geq 2$, at $\Rightarrow (A,\leq)\times(B,\sqsubseteq)$ nu e lanț

REZ:

(1) Fie $A=\{a\} \Rightarrow \leq=\{(a,a)\} \Rightarrow (A,\leq)\times(B,\sqsubseteq)=(\underbrace{A\times B}_{\{(a,b)\mid b\in B\}},\underbrace{\leq\times\sqsubseteq}_{\{((a,b),(a,c))\mid b,c\in B,\, b\sqsubseteq c\}})$

$\leq\times\sqsubseteq=\{((x,y),(x',y'))\mid x,x'\in A,\ x\leq x',\ y,y'\in B,\ y\sqsubseteq y'\}$

Fie $f: B\to A\times B$, $(\forall b\in B)\ f(b):=(a,b) \Rightarrow f\to$ bij. cu $f^{-1}: A\times B\to B$, $(\forall b\in B)$

$f^{-1}(a,b)=b$

Fie $b,c\in B$

Dacă $b\sqsubseteq c \Rightarrow (a,b)(\leq\times\sqsubseteq)(a,c) \Rightarrow f(b)(\leq\times\sqsubseteq)f(c) \Rightarrow f\nearrow$

Dc. $(a,b)(\leq\times\sqsubseteq)(a,c) \Rightarrow b\sqsubseteq c \Leftrightarrow f^{-1}(a,b)\sqsubseteq f^{-1}(a,c) \Rightarrow f^{-1}\nearrow$

$\Longrightarrow f\to$ izom. de poseturi

(3) $(A,\leq),(B,\sqsubseteq)$ poseturi; $|A|\geq 2$, $|B|\geq 2$

$(A,\leq)\times(B,\sqsubseteq)=(A\times B,\leq\times\sqsubseteq)$

Caz 1: $(A,\leq)$ nu e lanț $\Leftrightarrow (\exists\, a_1,a_2\in A)\ a_1\nleq a_2$ și $a_2\nleq a_1$

Fie $b\in B$

$\Rightarrow (a_1,b)\,\cancel{(\leq\times\sqsubseteq)}\,(a_2,b)$ și
$(a_2,b)\,\cancel{(\leq\times\sqsubseteq)}\,(a_1,b)$ $\Rightarrow (A,\leq)\times(B,\sqsubseteq)$ nu este lanț

Caz 2: $(B,\sqsubseteq)$ nu e lanț

Analog cazului 1 $\Rightarrow (A,\leq)\times(B,\sqsubseteq)$ nu e lanț

Caz 3: $(A,\leq),(B,\sqsubseteq)$ sunt lanțuri

$|A|\geq 2 \Leftrightarrow (\exists\, a_1,a_2\in A)\ a_1\neq a_2$ (*)

$|B|\geq 2 \Leftrightarrow (\exists\, b_1,b_2\in B)\ b_1\neq b_2$ (**)

$(A,\leq)\to$ lanț $\Rightarrow a_1\leq a_2$ sau $a_2\leq a_1$. Pp. de ex: $a_1\leq a_2 \overset{(*)}{\Longrightarrow} a_2\nleq a_1$

ex: $(\mathbb{N},|)\times(\mathbb{N},\leq)=(\mathbb{N}^2,|\times\leq)$, $|\times\leq=\{((x,y),(x',y'))\mid x,x',y,y'\in\mathbb{N},\ x\mid x',\ y\leq y'\}$

$(B,\sqsubseteq)\to$ lanț $\Rightarrow b_1\sqsubseteq b_2$ sau $b_2\sqsubseteq b_1$. Pp. de ex: $b_1\sqsubseteq b_2 \overset{(**)}{\Longrightarrow} b_2\not\sqsubseteq b_1$