

Not: $(\forall x \in \mathbb{R}) \begin{cases} \lfloor x \rfloor := \max \{ n \in \mathbb{Z} / n \leq x \} \in \mathbb{Z} \\ \text{Not parte fracționară: } \text{frac}\{x\} := x - \lfloor x \rfloor \in [0, 1) (\subset \mathbb{R}) \end{cases}$

Ex: $\lfloor -x \rfloor = -\lceil x \rceil$,

$\text{frac}\{-x\} \geq 0$

$\lfloor 8,3 \rfloor = 8 \quad \text{frac}\{8,3\} = 0,3$

$\lfloor -9,2 \rfloor = -10 \quad \text{frac}\{-9,2\} = 0,8$

Exerc: $\sim \subseteq \mathbb{R}^2$; $(\forall x, y \in \mathbb{R})$

$x \sim y \stackrel{\text{def}}{\iff} x - y \in \mathbb{Z}$

Arăm că: (1) $(\forall x, y \in \mathbb{R}) x \sim y \iff \text{frac}\{x\} = \text{frac}\{y\}$;

(2) $\sim \in \text{Equiv}(\mathbb{R})$;

(3) $\mathbb{R}/\sim \cong [0, 1) (\subset \mathbb{R})$

(Mult factor
a lui \sim)

(1) Fie $x, y \in \mathbb{R}$, arb, fixate.

$$x \sim y \iff x - y \in \mathbb{Z} \iff \underbrace{\lfloor x \rfloor - \lfloor y \rfloor}_{\in \mathbb{Z}} + \underbrace{\text{frac}\{x\} - \text{frac}\{y\}}_{\in (-1, 1)} \in \mathbb{Z} \iff \text{frac}\{x\} - \text{frac}\{y\} \in \mathbb{Z} \cap (-1, 1) = \{0\} \iff \text{frac}\{x\} = \text{frac}\{y\}$$

$$\underbrace{\text{frac}\{x\}}_{\in [0, 1)} - \underbrace{\text{frac}\{y\}}_{\in (-1, 0]} \in \mathbb{Z} \iff \text{frac}\{x\} - \text{frac}\{y\} \in \mathbb{Z} \cap (-1, 1) = \{0\} \iff \text{frac}\{x\} = \text{frac}\{y\}$$

$$\iff \text{frac}\{x\} - \text{frac}\{y\} = 0 \iff \text{frac}\{x\} = \text{frac}\{y\}$$

(2) Fie $x, y, z \in \mathbb{R}$ arb, fixate

$\text{frac}\{x\} = \text{frac}\{x\} \stackrel{(1)}{\iff} x \sim x \Rightarrow \sim \text{reflexivă}$

$\text{frac}\{x\} = \text{frac}\{y\} \stackrel{(1)}{\iff} x \sim y \stackrel{(1)}{\iff} \text{frac}\{y\} = \text{frac}\{x\} \stackrel{(1)}{\iff} y \sim x \Rightarrow \sim \text{simetrică}$

$\text{frac}\{x\} = \text{frac}\{y\} \stackrel{(1)}{\iff} x \sim y \stackrel{(1)}{\iff} \text{frac}\{y\} = \text{frac}\{z\} \stackrel{(1)}{\iff} y \sim z$

$\Rightarrow \text{frac}\{x\} = \text{frac}\{z\} \stackrel{(1)}{\iff} x \sim z \Rightarrow \sim \text{transitivă}$

$\Rightarrow \sim \in \text{Equiv}(\mathbb{R})$