CONSULTAȚII LMC

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~SERIA 13~

Exerc.: Fie $\mathcal{B}=(B,\vee,\wedge,\leq,\bar{\cdot},0,1)$ o algebră Boole.

(1) Fie $R:=\{(x,x) \mid x\in B\}\subseteq B^2$. Dem. că:

(a) R e reflexivă $\Leftrightarrow$ $\mathcal{B}$ e netrivială;

(b) R e simetrică;

(c) R e tranzitivă $\Leftrightarrow$ $\mathcal{B}$ e trivială.

(2) Fie $x,y\in B$. Dem. că:

(d) $[x)\cap[y)=[x\vee y)$;

(e) $[[x)\cup[y))=[x\wedge y)$.

(3) Fie $I\neq\emptyset$ o mulțime și $(\forall i\in I)(\emptyset\neq X_i\subseteq B)$. Dem. că:

$$\left[\bigcup_{i\in I}[X_i)\right)=\left[\bigcup_{i\in I}X_i\right).$$

REZOLVARE:

Pt. ② și ③, amintim că:

$(\forall \alpha \in B)\left([\alpha) = \{a \in B \mid \alpha \leq a\}\right)$,

$(\forall \emptyset \neq A \subseteq B)\left([A) = \{a \in B \mid (\exists n \in \mathbb{N}^*)(\exists \alpha_1, \dots, \alpha_n \in A)(\alpha_1 \wedge \dots \wedge \alpha_n \leq a)\}\right)$

$\Rightarrow (\forall n \in \mathbb{N}^*)(\forall \alpha_1, \dots, \alpha_n \in B)\left([\{\alpha_1, \dots, \alpha_n\}) = [\alpha_1 \wedge \dots \wedge \alpha_n)\right)$

Pt. ③, menționăm că:

$(\forall X \subseteq Y \subseteq B)\left(\Rightarrow [X) \subseteq [Y)\right)$, pt. că: $\left\{\begin{array}{l}[Y) \supseteq Y \supseteq X \\ \text{și} \\ [Y) \in \mathcal{F}(B)\end{array}\right. \xrightarrow{\text{def. } [X)} [Y) \supseteq [X)$

(not. uzuală) Filt(B)

① (a) „$\Rightarrow$": R e ireflexivă $\Rightarrow 0 \not R 0 \Leftrightarrow 0 \neq \bar{0} \Leftrightarrow 0 \neq 1 \Leftrightarrow B$ e netrivială.

„$\Leftarrow$": B e netrivială $\Leftrightarrow 0 \neq 1$.

Pp. prin absurd că R nu e ireflexivă $\Leftrightarrow$

$\Leftrightarrow (\exists x \in B)(xRx) \Leftrightarrow (\exists x \in B)(x = \bar{x}) \Rightarrow$

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$$\Rightarrow \begin{cases} 1 = x \vee \overline{x} = x \vee x = x \\ 0 = x \wedge \overline{x} = x \wedge x = x \end{cases} \Rightarrow 0 = 1 \text{ Abs}$$

$\Rightarrow R$ e reflexivă.

(b) $(\forall x, y \in B)$ (avem: $xRy \Leftrightarrow$
$\Leftrightarrow y = \overline{x} \Leftrightarrow \overline{y} = \overline{\overline{x}} \Leftrightarrow \overline{y} = x \Leftrightarrow$
$\Leftrightarrow yRx) \Rightarrow R$ e simetrică.

(c) $\underline{\text{“}\Leftarrow\text{”}}$: B e trivială, $\Leftrightarrow 0 = 1 \Leftrightarrow$
$\Leftrightarrow B = 0 \xRightarrow{\text{def } R} R = \{(0,\overline{0})\} = \{(0,1)\} =$
$= \{(0,0)\} = B^2 \rightarrow$ tranzitivă.

$\underline{\text{“}\Rightarrow\text{”}}$: $R \rightarrow$ tranzitivă.

$$\left.\begin{array}{l} 0R\overline{0} \Leftrightarrow 0R1, \\ 1R\overline{1} \Leftrightarrow 1R0. \end{array}\right\} \Rightarrow 0R0 \Leftrightarrow$$
$\Leftrightarrow 0 = \overline{0} \Leftrightarrow$
$\Leftrightarrow 0 = 1 \Leftrightarrow B$ e trivială.

(2) (d) Fie $a \in B$, arbitrar, fixat.

$a \in [x) \cap [y) \Leftrightarrow \begin{cases} a \in [x) \\ a \in [y) \end{cases} \Leftrightarrow \begin{cases} x \leq a \\ y \leq a \end{cases} \Leftrightarrow$
$\Leftrightarrow x \vee y \leq a \Leftrightarrow a \in [x \vee y)$. $\Rightarrow$

$\Rightarrow [x) \cap [y) = [x \vee y)$.

(e) $[x \wedge y) = [\{x, y\})$,

$F := [[x) \cup [y))$, $\boxed{F = [\{x, y\})}$, //

$\begin{cases} x \in [x) \subseteq [x) \cup [y) \subseteq [[x) \cup [y)) = F \\ y \in [y) \subseteq [x) \cup [y) \subseteq [[x) \cup [y)) = F, \end{cases}$

$\Rightarrow F \supseteq \{x, y\}$. (*),

$F \in \mathcal{F}(B)$. (**),

Fie $G \in \mathcal{F}(B)$, a.i. $G \supseteq \{x, y\} \Leftrightarrow$

$\Leftrightarrow \begin{cases} G \ni x \Rightarrow G \supseteq [x) \\ \text{și} \\ G \ni y \Rightarrow G \supseteq [y), \end{cases} \Rightarrow G \supseteq [x) \cup [y) \Rightarrow$

$\Rightarrow G \supseteq [[x) \cup [y)) = F$. (***),

(*), (**), (***) $\Rightarrow F = [\{x, y\})$,

$\boxed{[[x) \cup [y)) \quad // \quad [x \wedge y)} \Rightarrow$

$\Rightarrow [[x) \cup [y)) = [x \wedge y)$.

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③ $(\forall i \in I)(X_i \subseteq [X_i)) \Rightarrow \bigcup_{i\in I} X_i \subseteq \bigcup_{i\in I} [X_i)$

$\Rightarrow [\bigcup_{i\in I} X_i) \subseteq [\bigcup_{i\in I} [X_i))$.

Fie $a \in [\bigcup_{i\in I} [X_i)) \Leftrightarrow (\exists n \in \mathbb{N}^*)$
$(\exists \alpha_1, \dots, \alpha_n \in \bigcup_{i\in I} [X_i))(\alpha_1 \wedge \dots \wedge \alpha_n \leq a)$. (I)

$(\forall k \in \overline{1,n})(\alpha_k \in \bigcup_{i\in I} [X_i)) \Leftrightarrow$

$\Leftrightarrow (\forall k \in \overline{1,n})(\exists i_k \in I)\underbrace{(\alpha_k \in [X_{i_k}))}$

$\Updownarrow$

$(\exists m_k \in \mathbb{N}^*)(\exists \beta_{k,1}, \beta_{k,2}, \dots, \beta_{k,m_k} \in \underbrace{X_{i_k}}_{\subseteq \bigcup_{i\in I} X_i})(\bigwedge_{j=1}^{m_k} \beta_{k,j} \leq \alpha_k)$. (II)

(I), (II) $\Rightarrow \bigwedge_{k=1}^{n} \bigwedge_{j=1}^{m_k} \underbrace{\beta_{k,j}}_{\in \bigcup_{i\in I} X_i} \leq \bigwedge_{k=1}^{n} \alpha_k \leq a \Rightarrow$

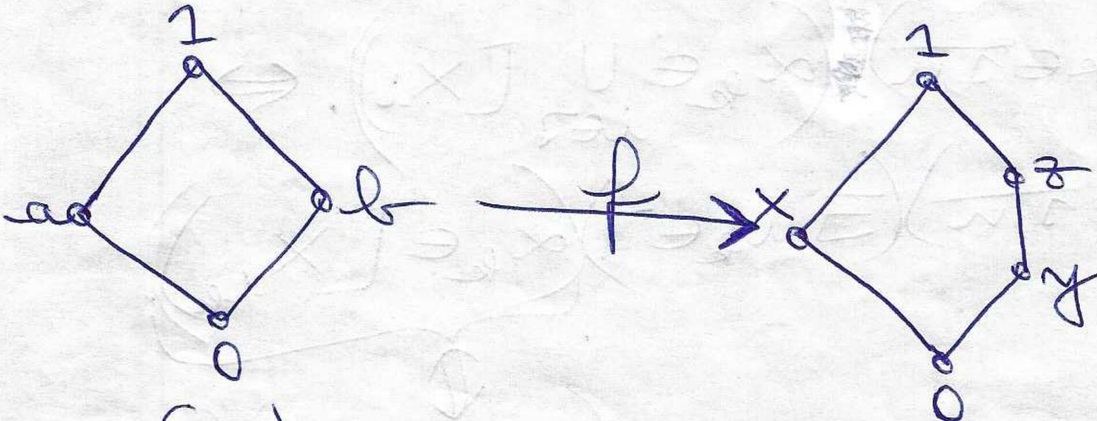
$\Rightarrow a \in [\bigcup_{i\in I} X_i) \Rightarrow [\bigcup_{i\in I} [X_i)) \subseteq [\bigcup_{i\in I} X_i)$.

$\Rightarrow [\bigcup_{i\in I} [X_i)) = [\bigcup_{i\in I} X_i)$.

Exerc.: Fie $R :=$ rombul, $P :=$ pentagonul și $f : R \to P$ un morfism de latici mărginite. Demonstrați că:

$$f(R) \neq \{0,1\} \Leftrightarrow f \text{ e injectivă.}$$

REZOLVARE:



(R) ($\mathcal{L}_2^2$)

(P)

Fie $R = \{0, a, b, 1\}$ și $P = \{0, x, y, z, 1\}$, cu elementele notate ca în figura de mai sus.

"$\Leftarrow$": Ipoteza acestei implicații este că f e injectivă.

Pp. prin absurd că $f(R) = \{0,1\} \Rightarrow$
$\Rightarrow |R| = 4 > 2 = |\{0,1\}| = |f(R)|. \Rightarrow f$ nu poate fi injectivă $\to$ ⚡. $\Rightarrow f(R) \neq \{0,1\}$.

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"$\Rightarrow$" Ipoteza acestei implicații este că $f(R) \neq \{0,1\}$.

f e morfism de latici mărginite. $\Rightarrow$
$\Rightarrow f(0)=0$ și $f(1)=1$. $\Rightarrow \{0,1\} \subseteq f(R)$ $\Rightarrow$

$\Rightarrow \{0,1\} \subsetneq f(R)$. $\Rightarrow \exists\, \beta \in f(R) \setminus \{0,1\}$.

$\beta \in f(R) \Leftrightarrow (\exists\, \alpha \in R)(f(\alpha)=\beta)$
$f(0)=0$ și $\beta \neq 0$.
$f(1)=1$ și $\beta \neq 1$. $\Rightarrow$

$\Rightarrow \alpha \in R \setminus \{0,1\} = \{a, b\}$.

Pp. că $a=\alpha$, cazul $b=\alpha$ se tratează analog.

Așadar, $f(a)=\beta \in P \setminus \{0,1\} = \{x, y, z\}$.

Pp. prin absurd că f nu e injectivă.

$f(0)=0$, $f(1)=1$ și $f(a)=\beta$ sunt 2 câte 2 distincte. $\Rightarrow$

$\Rightarrow f(b) \in \{f(0), f(1), f(a)\} = \{0, 1, \beta\}$.

Cazul 1: $f(b) = 0$.

$1 = f(a \vee b) = f(a) \vee f(b) \overset{\vee}{=} \beta \vee 0 =$
$= \beta. \Rightarrow \beta = 1 \to$ ↯.

Cazul 2: $f(b) = 1$.

$0 = f(a \wedge b) = f(a) \wedge f(b) \overset{\vee}{=}$
$= \beta \wedge 1 = \beta. \Rightarrow \beta = 0. \to$ ↯.

Cazul 3: $f(b) = \beta$.

$0 = f(a \wedge b) = f(a) \wedge f(b) \overset{\vee}{=} \beta \wedge \beta =$
$= \beta. \Rightarrow \beta = 0 \to$ ↯.

$\Rightarrow f$ e injectiv.

Exerc.: Fie $\alpha, \varphi, \psi, \chi \in E$, a.î.
$\alpha = (\varphi \to \neg(\neg\psi \to \chi)) \leftrightarrow ((\psi \to \neg\varphi) \wedge (\chi \to \neg\varphi))$. Demonstrați că:

$\vdash \alpha$: $\begin{cases} (a) & \text{algebric, S.XIII, } {}^{2}12 - {}^{2}13 \\ (b) & \text{semantic.} \end{cases}$

REZOLVARE:

(a) Fie $x:=\hat{\varphi}$, $y:=\hat{\psi}$ și $z:=\hat{\chi}$ în $E/_{\sim}$.

$$\Rightarrow \hat{\alpha} = \left(\hat{\varphi} \to \overline{\left(\overline{\hat{\psi}} \to \hat{\chi}\right)}\right) \leftrightarrow$$

$$\leftrightarrow \left(\left(\hat{\psi} \to \overline{\hat{\varphi}}\right) \wedge \left(\hat{\chi} \to \overline{\hat{\varphi}}\right)\right) =$$

$$= \left(x \to \overline{\left(\overline{y} \to z\right)}\right) \leftrightarrow \left(\left(y \to \overline{x}\right) \wedge \left(z \to \overline{x}\right)\right) =$$

$$= \left(x \to \overline{\left(\overline{\overline{y}} \vee z\right)}\right) \leftrightarrow \left(\left(\overline{y} \vee \overline{x}\right) \wedge \left(\overline{z} \vee \overline{x}\right)\right) =$$

$$= \left(\overline{x} \vee \overline{\left(y \vee z\right)}\right) \leftrightarrow \left(\left(\overline{y} \wedge \overline{z}\right) \vee \overline{x}\right) \overset{\text{de Morgan}}{=}$$

$$= \left(\overline{x} \vee \left(\overline{y} \wedge \overline{z}\right)\right) \leftrightarrow \left(\overline{x} \vee \left(\overline{y} \wedge \overline{z}\right)\right) = 1,$$

fiind o echivalență booleană între doi termeni egali,

deci $\hat{\alpha} = 1 \Leftrightarrow \vdash \alpha$.

(b) Demonstrăm că $\models \alpha$.

Fie $h: V \to \mathcal{L}_2$, arbitrară, fixată.

$$\tilde{h}(\alpha) = \left(\tilde{h}(\varphi) \to \overline{\left(\overline{\tilde{h}(\psi)} \to \tilde{h}(\chi)\right)}\right) \leftrightarrow$$

$$\leftrightarrow \left(\left(\tilde{h}(\psi) \to \overline{\tilde{h}(\varphi)}\right) \wedge \left(\tilde{h}(\chi) \to \overline{\tilde{h}(\varphi)}\right)\right).$$

De aici putem continua cu un

calcul boolean, ca la punctul (a); sau putem proceda astfel: notăm

$\beta := \varphi \rightarrow \neg(\neg\psi \rightarrow \chi) \in E$

$\gamma := (\psi \rightarrow \neg\varphi) \wedge (\chi \rightarrow \neg\varphi) \in E$ şi

$$\tilde{h}(\alpha) = \tilde{h}(\beta) \leftrightarrow \tilde{h}(\gamma),$$

aşadar: $\tilde{h}(\alpha) = 1 \Leftrightarrow \tilde{h}(\beta) = \tilde{h}(\gamma)$.

Caz 1: $\tilde{h}(\beta) = 0 \Leftrightarrow$

$\Leftrightarrow \tilde{h}(\varphi) \rightarrow \overline{\left(\overline{\tilde{h}(\psi)} \rightarrow \tilde{h}(\chi)\right)} = 0 \Leftrightarrow$

$$\Leftrightarrow \begin{cases} \tilde{h}(\varphi) = 1 \\ \text{şi} \\ \overline{\overline{\tilde{h}(\psi)} \rightarrow \tilde{h}(\chi)} = 0 \end{cases} \Leftrightarrow$$

$\Leftrightarrow \overline{\tilde{h}(\psi)} \rightarrow \tilde{h}(\chi) = 1 \Leftrightarrow$

$$\Leftrightarrow \begin{cases} \overline{\tilde{h}(\psi)} = 0 \Leftrightarrow \tilde{h}(\psi) = 1. \\ \text{sau} \\ \tilde{h}(\chi) = 1. \end{cases}$$

Aşadar: $\tilde{h}(\beta) = 0 \Leftrightarrow \begin{cases} \tilde{h}(\varphi) = \tilde{h}(\psi) = 1 \\ \text{sau} \\ \tilde{h}(\varphi) = \tilde{h}(\chi) = 1 \end{cases} \Leftrightarrow$

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$$\Leftrightarrow \begin{cases} \tilde{h}(\psi)=1 \text{ și } \overline{\tilde{h}(\varphi)}=0 \\ \text{sau} \\ \tilde{h}(\chi)=1 \text{ și } \overline{\tilde{h}(\varphi)}=0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \tilde{h}(\psi)\rightarrow\overline{\tilde{h}(\varphi)}=0 \\ \text{sau} \\ \tilde{h}(\chi)\rightarrow\overline{\tilde{h}(\varphi)}=0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \left(\tilde{h}(\psi)\rightarrow\overline{\tilde{h}(\varphi)}\right)\wedge\left(\tilde{h}(\chi)\rightarrow\overline{\tilde{h}(\varphi)}\right)=$$

$$=0. \Leftrightarrow \tilde{h}(\gamma)=0.$$

<u>Caz 2</u>: $\tilde{h}(\beta)=1. \overset{\text{caz } 1}{\Longleftrightarrow} \tilde{h}(\gamma)=1.$

$$\Rightarrow \tilde{h}(\beta)=\tilde{h}(\gamma). \Leftrightarrow \tilde{h}(\alpha)=1.$$

$$\Rightarrow \vDash\alpha. \overset{(TC)}{\Longleftrightarrow} \vdash\alpha.$$

<u>Exerc.</u>: Să se dem. semantic următoarea regulă de deducție: pt. orice $\Gamma\subseteq E$ și orice $\varphi,\psi,\gamma\in E$,

$$\frac{\Gamma\cup\{\varphi\}\vdash\gamma,\ \Gamma\cup\{\psi\}\vdash\gamma}{\Gamma\cup\{\varphi\vee\psi\}\vdash\gamma}.$$

REZOLVARE:

Fie $\Gamma\subseteq E$ și $\varphi,\psi,\gamma\in E$, a.î.

$$\begin{cases} \Gamma\cup\{\varphi\}\vdash\gamma \\ \text{și} \\ \Gamma\cup\{\psi\}\vdash\gamma. \end{cases} \overset{(TCT)}{\Longleftrightarrow} \begin{cases} \Gamma\cup\{\varphi\}\vDash\gamma \ (1) \\ \text{și} \\ \Gamma\cup\{\psi\}\vDash\gamma \ (2). \end{cases}$$

$\Gamma \cup \{\varphi \vee \psi\} \models \chi$. — // —

Fie $h : V \to L_2$, a.î.

$h \models \Gamma \cup \{\varphi \vee \psi\} \Leftrightarrow \begin{cases} h \models \Gamma \; (*) \\ \text{şi} \\ \tilde{h}(\varphi \vee \psi) = 1 \Leftrightarrow \end{cases}$

$\Leftrightarrow \tilde{h}(\varphi) \vee \tilde{h}(\psi) = 1$.

$\left.\begin{array}{l} \tilde{h}(\varphi) \vee \tilde{h}(\psi) = 1 \\ \tilde{h}(\varphi), \tilde{h}(\psi) \in L_2 = \{0, 1\} \end{array}\right] \Rightarrow$

$\Rightarrow \tilde{h}(\varphi) = 1$ sau $\tilde{h}(\psi) = 1$.

<u>Caz 1</u>: dacă $\tilde{h}(\varphi) = 1 \overset{(*)}{\Rightarrow}$

$\overset{(*)}{\Rightarrow} h \models \Gamma \cup \{\varphi\} \overset{(1)}{\Rightarrow} \tilde{h}(\chi) = 1$.

<u>Caz 2</u>: dacă $\tilde{h}(\psi) = 1 \overset{(*)}{\Rightarrow}$

$\overset{(*)}{\Rightarrow} h \models \Gamma \cup \{\psi\} \overset{(2)}{\Rightarrow} \tilde{h}(\chi) = 1$.

$\Rightarrow \Gamma \cup \{\varphi \vee \psi\} \models \chi \overset{(TCT)}{\Leftrightarrow}$

$\overset{(TCT)}{\Leftrightarrow} \Gamma \cup \{\varphi \vee \psi\} \vdash \chi$.

ALTĂ METODĂ LA (b) LA EXERCIŢIUL ENUNŢAT PE VERSO-UL PAGINII 4:

Amintesc că, în orice algebră Boole B, pt. orice $a \in B$, avem:

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$a = a \iff a \leftrightarrow a = 1;$ (1)

$0 \leq a \iff 0 \to a = 1;$ (*)

$a \leq 1 \iff a \to 1 = 1;$ (**)

$a \to 0 = \bar{a} \vee 0 = \bar{a};$ (***)

$1 \to a = \bar{1} \vee a = 0 \vee a = a.$ (****)

Fie $h : V \to L_2 = \{0,1\}$ arbitrară fixată.

$$\tilde{h}(\alpha) = \left(\tilde{h}(\varphi) \to \left(\tilde{h}(\psi) \to \tilde{h}(\chi)\right)\right) \leftrightarrow \overline{\left(\left(\tilde{h}(\psi) \to \overline{\tilde{h}(\varphi)}\right) \wedge \left(\tilde{h}(\chi) \to \overline{\tilde{h}(\varphi)}\right)\right)}.$$

Caz 1: Dacă $\tilde{h}(\varphi) = 0 \Rightarrow \overline{\tilde{h}(\varphi)} = \bar{0} = 1 \Rightarrow \tilde{h}(\alpha) = \left(0 \to \left(\tilde{h}(\psi) \to \tilde{h}(\chi)\right)\right) \leftrightarrow \left(\left(\tilde{h}(\psi) \to 1\right) \wedge \left(\tilde{h}(\chi) \to 1\right)\right) \overset{(*),(**)}{=} 1 \leftrightarrow (1 \wedge 1) = 1 \leftrightarrow 1 \overset{(1)}{=} 1.$

Caz 2: Dacă $\tilde{h}(\varphi) = 1 \Rightarrow \overline{\tilde{h}(\varphi)} = \bar{1} = 0 \Rightarrow \tilde{h}(\alpha) = \left(1 \to \left(\tilde{h}(\psi) \to \tilde{h}(\chi)\right)\right) \leftrightarrow \overline{\left(\left(\tilde{h}(\psi) \to 0\right) \wedge \left(\tilde{h}(\chi) \to 0\right)\right)} \overset{(****),(***)}{=}$

$$= \left(\tilde{h}(\psi) \to \tilde{h}(\chi)\right) \leftrightarrow \overline{\left(\overline{\tilde{h}(\psi)} \wedge \overline{\tilde{h}(\chi)}\right)} =$$

$$= \overline{\overline{\tilde{h}(\psi)} \vee \tilde{h}(\chi)} = \overline{\overline{\tilde{h}(\psi)}} \wedge \overline{\tilde{h}(\chi)} = \tilde{h}(\psi) \wedge \overline{\tilde{h}(\chi)}$$

$$= \left(\tilde{h}(\psi) \wedge \overline{\tilde{h}(\chi)}\right) \leftrightarrow \left(\tilde{h}(\psi) \wedge \overline{\tilde{h}(\chi)}\right) \overset{(1)}{=} 1.$$

$\Rightarrow \left(\forall h : V \to L_2\right)\left(h \vDash \alpha\right) \overset{(\text{def})}{\iff} \vDash \alpha \overset{(\text{TC})}{\iff} \vdash \alpha.$

Exercițiu: Să se demonstreze semantic următoarea regulă de deducție: pentru orice $\Sigma_1, \Sigma_2, \Sigma_3 \in \mathcal{P}(E)$ și orice $\varphi, \psi, \chi \in E$:

$$\frac{\Sigma_1 \cup \{\varphi\} \vdash \psi,\ \Sigma_2 \cup \{\psi \wedge \chi\} \vdash \varphi,\ \Sigma_3 \cup \{\psi\} \vdash \chi}{\Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \vdash \varphi \leftrightarrow (\psi \wedge \chi)}.$$

REZOLVARE: Fie $\Sigma_1, \Sigma_2, \Sigma_3 \in \mathcal{P}(E)$ și $\varphi, \psi, \chi \in E$ a.î. au loc deducțiile sintactice:

$$\begin{cases} \Sigma_1 \cup \{\varphi\} \vdash \psi \overset{(TCT)}{\Longleftrightarrow} \Sigma_1 \cup \{\varphi\} \vDash \psi; & (1) \\ \Sigma_2 \cup \{\psi \wedge \chi\} \vdash \varphi \overset{(TCT)}{\Longleftrightarrow} \Sigma_2 \cup \{\psi \wedge \chi\} \vDash \varphi; & (2) \\ \Sigma_3 \cup \{\psi\} \vdash \chi \overset{(TCT)}{\Longleftrightarrow} \Sigma_3 \cup \{\psi\} \vDash \chi. & (3) \end{cases}$$

Fie $h : V \to \mathcal{L}_2$ a.î. $h \vDash \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \Leftrightarrow h \vDash \Sigma_1,\ h \vDash \Sigma_2,\ h \vDash \Sigma_3$;

Caz 1: Dacă $\tilde{h}(\varphi) = 1 \xRightarrow{(h \vDash \Sigma_1)}$

$\Rightarrow h \vDash \Sigma_1 \cup \{\varphi\} \xRightarrow{(1)} \tilde{h}(\psi) = 1 \xRightarrow{(h \vDash \Sigma_3)}$

$\Rightarrow h \vDash \Sigma_3 \cup \{\psi\} \xRightarrow{(3)} \tilde{h}(\chi) = 1. \Rightarrow$

$\Rightarrow \tilde{h}(\varphi \leftrightarrow (\psi \wedge \chi)) = \tilde{h}(\varphi) \leftrightarrow (\tilde{h}(\psi) \wedge \tilde{h}(\chi)) =$

$= 1 \leftrightarrow (1 \wedge 1) = 1 \leftrightarrow 1 = 1.$

Caz 2: Dacă $\tilde{h}(\varphi) = 0$:

Presupunem prin absurd că $\tilde{h}(\psi \wedge \chi) = 1 \Rightarrow$

$\xRightarrow{(h \vDash \Sigma_2)} h \vDash \Sigma_2 \cup \{\psi \wedge \chi\} \xRightarrow{(2)} \tilde{h}(\varphi) = 1$ ↯ $\Rightarrow$

$\Rightarrow \tilde{h}(\psi \wedge \chi) = 0 \Rightarrow \tilde{h}(\varphi \leftrightarrow (\psi \wedge \chi)) =$

$= \tilde{h}(\varphi) \leftrightarrow \tilde{h}(\psi \wedge \chi) = 0 \leftrightarrow 0 = 1.$

$\Rightarrow \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \vDash \varphi \leftrightarrow (\psi \wedge \chi) \overset{(TCT)}{\Longleftrightarrow} \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \vdash \varphi \leftrightarrow (\psi \wedge \chi).$

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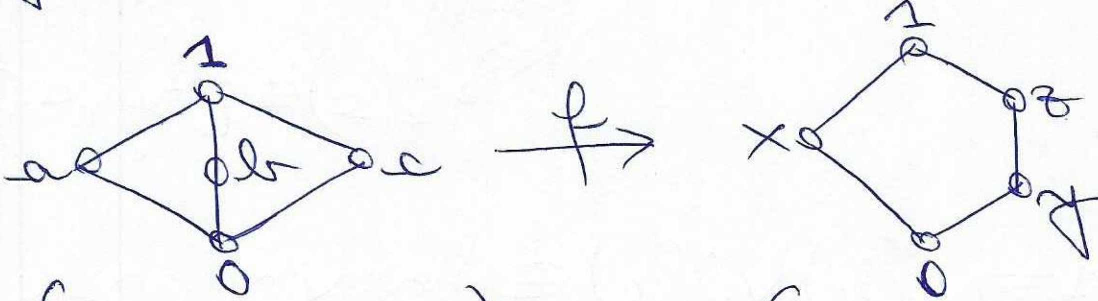
<u>Exercițiu:</u> Să se demonstreze că nu există morfisme de latici mărginite de la diamant la pentagon.

REZOLVARE:

Notăm elementele diamantului și pentagonului ca în următoarele diagrame Hasse:



$\mathcal{D}=(D,\vee,\wedge,\leq,0,1)$ $\quad$ $\mathcal{P}=(P,\vee,\wedge,\leq,0,1)$

Presupunem prin absurd că $f: D \to P$ este un morfism de latici mărginite. Rezultă:

$$f(0)=0 \text{ și } f(1)=1;$$

$$\begin{cases} a\vee b=1 \Rightarrow f(a)\vee f(b)=f(a\vee b)=f(1)=1; \\ a\wedge b=0 \Rightarrow f(a)\wedge f(b)=f(a\wedge b)=f(0)=0. \end{cases} \quad (*)$$

Analog rezultă: $\begin{cases} f(a)\vee f(c)=1; \\ f(a)\wedge f(c)=0; \end{cases} \quad (**)$

și: $\begin{cases} f(b)\vee f(c)=1; \\ f(b)\wedge f(c)=0. \end{cases} \quad (***)$

<u>Caz 1:</u> $f(a)=0 \xRightarrow{(*)} f(b)=1$, $\quad f(a)=0 \xRightarrow{(**)} f(c)=1$ $\quad \xrightarrow{(***)}$

$\Rightarrow 0 = f(b) \wedge f(c) = 1 \wedge 1 = 1.$ $\cancel{\times}$

<u>Caz 2:</u> $f(a) = 1, \begin{cases} \xrightarrow{(*)} f(b) = 0 \\ \xrightarrow{(**)} f(c) = 0 \end{cases} \xrightarrow{(*)\,(**)}$

$\Rightarrow 1 = f(b) \vee f(c) = 0 \vee 0 = 0.$ $\cancel{\times}$

<u>Caz 3:</u> $f(a) = x, \begin{cases} \xrightarrow{(*)} f(b) \in \{y, z\} \\ \xrightarrow{(**)} f(c) \in \{y, z\} \end{cases} \Rightarrow$

$\xrightarrow{(*)\,(**)} 1 = f(b) \vee f(c) \in \{y \vee y, y \vee z, z \vee y, z \vee z\} = \{y, z\}.$ $\cancel{\times}$

<u>Caz 4:</u> $f(a) \in \{y, z\}, \begin{cases} \xrightarrow{(*)} f(b) = x \\ \xrightarrow{(**)} f(c) = x \end{cases} \Rightarrow$

$\xrightarrow{(*)\,(**)} 1 = f(b) \vee f(c) = x \vee x = x.$ $\cancel{\times}$

$\Rightarrow$ Nu există morfisme de latici mărginite de la D la P.

<u>Exercițiu temă:</u>

Demonstrați că morfismele de latici mărginite $h: P \to D$ sunt:

u	0	x	y	z	1
$h(u)$	0	0	1	1	1
	0	1	0	0	1
	0	a	b	b	1
	0	a	c	c	1
	0	b	c	c	1
	0	b	a	a	1
	0	c	a	a	1
	0	c	b	b	1