(S 13, 14–15)

LOGICĂ MATEMATICĂ ŞI COMPUTAŢIONALĂ, SEMINARUL XIII

~ MATERIAL PENTRU STUDENŢI ~

~ 2014 – 2015 ~

Considerăm algebra Boole standard:

$\mathcal{L}_2 = (L_2, \vee, \wedge, \bar{\ }, \leq, 0, 1)$, $L_2 = \{0, 1\}$, cu $0 \neq 1$.

Def.: Interpretare (evaluare, semantică): $h : V \to L_2$, → „dă valori de adevăr" variabilelor propoziţionale.

Prop.: $\forall h : V \to L_2$, $\exists!\, \tilde{h} : E \to L_2$ a.î.:

$\tilde{h} \stackrel{not.}{=} h$, numită tot interpretare (evaluare, semantică)

$$\begin{tikzcd} V \arrow[r, hook] \arrow[dr, "h"'] & E \arrow[d, "\tilde{h}"] \\ & L_2 \end{tikzcd}$$

(a) $(\forall u \in V)\big(\tilde{h}(u) = h(u)\big)$, (i.e. $\tilde{h}|_V = h$);

(b) $(\forall \varphi \in E)\big(\tilde{h}(\neg\varphi) = \overline{\tilde{h}(\varphi)}\big)$;

(c) $(\forall \varphi, \psi \in E)\big(\tilde{h}(\varphi \to \psi) = \tilde{h}(\varphi) \to \tilde{h}(\psi)\big)$

Obs.: $\forall h : V \to L_2$, $\forall \varphi, \psi \in E$, au loc:

(d) $\tilde{h}(\varphi \vee \psi) = \tilde{h}(\varphi) \vee \tilde{h}(\psi)$;

(e) $\tilde{h}(\varphi \wedge \psi) = \tilde{h}(\varphi) \wedge \tilde{h}(\psi)$;

(f) $\tilde{h}(\varphi \leftrightarrow \psi) = \tilde{h}(\varphi) \leftrightarrow \tilde{h}(\psi)$.

Dem.:

(d) $\tilde{h}(\varphi \vee \psi) = \tilde{h}(\neg\varphi \to \psi) \overset{(c)}{=}$
$= \tilde{h}(\neg\varphi) \to \tilde{h}(\psi) \overset{(b)}{=}$
$= \overline{\tilde{h}(\varphi)} \to \tilde{h}(\psi) = \overline{\overline{\tilde{h}(\varphi)}} \vee \tilde{h}(\psi) =$
$= \tilde{h}(\varphi) \vee \tilde{h}(\psi)$.

(e) $\tilde{h}(\varphi \wedge \psi) = \tilde{h}(\neg(\varphi \to \neg\psi)) \overset{(b),(c)}{=}$
$= \overline{\tilde{h}(\varphi) \to \overline{\tilde{h}(\psi)}} = \overline{\overline{\tilde{h}(\varphi)} \vee \overline{\tilde{h}(\psi)}} =$
$= \overline{\overline{\tilde{h}(\varphi)}} \wedge \overline{\overline{\tilde{h}(\psi)}} = \tilde{h}(\varphi) \wedge \tilde{h}(\psi)$.

(f) $\tilde{h}(\varphi \leftrightarrow \psi) = \tilde{h}((\varphi \to \psi) \wedge (\psi \to \varphi))$
$\overset{(e)}{=} \tilde{h}(\varphi \to \psi) \wedge \tilde{h}(\psi \to \varphi) \overset{(c)}{=}$
$= (\tilde{h}(\varphi) \to \tilde{h}(\psi)) \wedge (\tilde{h}(\psi) \to \tilde{h}(\varphi)) =$
$= \tilde{h}(\varphi) \leftrightarrow \tilde{h}(\psi)$

Def.: $h : V \to L_2$; $\varphi \in E$; $\Sigma \subseteq E$.

$h \vDash \varphi \overset{def.}{\Longleftrightarrow} \tilde{h}(\varphi) = 1$.

$\vDash \varphi \overset{def.}{\Longleftrightarrow} (\forall f : V \to L_2)(f \vDash \varphi)$.

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$h \models \Sigma \Leftrightarrow (\forall \gamma \in \Sigma)(h \models \gamma)$.

$\Sigma \models \varphi \Leftrightarrow (\forall f : V \to L_2)$
$(f \models \Sigma \Rightarrow f \models \varphi)$.

__Obs.:__ $\varphi \in E$. Atunci: $\models \varphi \Leftrightarrow \emptyset \models \varphi$.

__Dem.:__

$\emptyset \models \varphi \overset{(def)}{\Leftrightarrow} (\forall f : V \to L_2)(f \models \emptyset \Rightarrow f \models \varphi)$. (I)

$(\forall f : V \to L_2)(f \models \emptyset)$. //

Fie $h : V \to L_2$.

$h \models \emptyset \overset{(def)}{\Leftrightarrow} (\forall \gamma \in \emptyset)(h \models \gamma) \Leftrightarrow$

$\Leftrightarrow \underbrace{(\forall \gamma)(\underbrace{\gamma \in \emptyset}_{\text{fals}, \forall \gamma} \Rightarrow h \models \gamma)}_{\text{adevărat}, \forall \gamma} \to$ adevărat (*)

Cf. (I), avem (scriem echivalent "$\Rightarrow$"):

$\emptyset \models \varphi \Leftrightarrow (\forall f : V \to L_2)$
$(\underbrace{f \not\models \emptyset}_{\text{fals}, \forall f : V \to L_2}$ sau $f \models \varphi) \Leftrightarrow$

$\Leftrightarrow (\forall f : V \to L_2)(f \vDash \varphi) \overset{(def.)}{\Leftrightarrow} \vDash \varphi$.

__Teor.:__ $\varphi \in E$, $\Sigma \subseteq E$. Atunci:

(TC) $\vdash \varphi \Leftrightarrow \vDash \varphi$;

(TCT) $\Sigma \vdash \varphi \Leftrightarrow \Sigma \vDash \varphi$.

__Obs.:__ Știm că, pt. orice $\varphi \in E$, $\vdash \varphi \Leftrightarrow \emptyset \vdash \varphi$. Din acest fapt și obs. anterioară, se observă că (TC) este caz particular al (TCT) (cazul $\Sigma = \emptyset$).

__Corolar al (TC):__ __Noncontradicția logicii propoziționale clasice:__

$(\nexists\, \varphi \in E)(\vdash \varphi$ și $\vdash \neg\varphi)$.

__Def.:__ $\Sigma \subseteq E$, __ex.:__ $\{\varphi, \neg\varphi\}$, $\varphi \in E$; __ex.:__ E

Σ → __inconsistentă__ $\overset{def}{\Leftrightarrow} (\forall \varphi \in E)(\Sigma \vdash \varphi)$;

Σ → __consistentă__ $\overset{def}{\Leftrightarrow} (\exists \varphi \in E)(\Sigma \nvdash \varphi)$.

__ex.:__ $\emptyset$, mulț. axiomelor, T. (v. corolarul anterior) → (și __obs.:__ Σ → consist. $\Leftrightarrow (\nexists\, \varphi \in E)(\Sigma \vdash \varphi$ și $\Sigma \vdash \neg\varphi)$).

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Prop.*: $\Sigma \subset E$, $\varphi \in E$. Atunci:

(1) $\Sigma \cup \{\varphi\} \to$ inconsistentă $\Leftrightarrow$ $\Leftrightarrow \Sigma \vdash \neg\varphi$

(2) $\Sigma \cup \{\neg\varphi\} \to$ inconsistentă $\Leftrightarrow$ $\Leftrightarrow \Sigma \vdash \varphi$.

Prop.:** $\Sigma \subset E$. Atunci: $\Sigma \to$ consistentă $\Leftrightarrow$ $\Leftrightarrow (\exists h : V \to L_2)(h \vDash \Sigma)$.

Exerc.: Considerăm următoa

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Exerc.: Considerăm următorul text:

(a) dacă nu plouă, atunci, în cazul când ies la plimbare, nu trec pe la cafenea.

(b) dacă nu plouă, atunci ies la plimbare.

(c) Trec pe la cafenea.

(d) Plouă.

Să se dem. că din (a), (b), (c) se deduce (d).

REZOLVARE: (V. ALTE DOUĂ METODE LA SF. ACESTUI S!)

Considerăm trei variabile propoziționale, $p, q, r \in V$, cărora le dăm următoarele valori:

p: "plouă"; q: "ies la plimbare"; r: "trec pe la cafenea".

Atunci frazelor de la punctele (a), (b), (c), (d) le corespund urm. enunțuri:

(a): $\varphi := \neg p \to (q \to \neg r) \in E$

(b): $\psi := \neg p \to q \in E$

(c): $r \in E$

(d): $p \in E$.

Avem de dem. că:

$\{\varphi, \psi, r\} \vDash p$. Not. $\Sigma :=$ $:= \{\varphi, \psi, r\} \subset E$.

$\underline{\Sigma \vDash p.}$

Fie $h : V \to \mathcal{L}_2$, a.î. $h \vDash \Sigma \Leftrightarrow$

$$\Leftrightarrow \begin{cases} 1 = \tilde{h}(\varphi) = \overline{\tilde{h}(p)} \to (\tilde{h}(q) \to \overline{\tilde{h}(r)}) \\ 1 = \tilde{h}(\psi) = \overline{\tilde{h}(p)} \to \tilde{h}(q) \\ 1 = \tilde{h}(r) \end{cases}$$

$$\Rightarrow 1 = \overline{\tilde{h}(p)} \to (\tilde{h}(q) \to \overline{1}) =$$

$$= \overline{\tilde{h}(p)} \to (\tilde{h}(q) \to 0) = \overline{\overline{\tilde{h}(p)}} \vee$$

$$\vee \overline{\tilde{h}(q)} \vee 0 = \tilde{h}(p) \vee \overline{\tilde{h}(q)}. \quad (*)$$

Dar avem și:

$$\overline{1 = \tilde{h}(p)} \rightarrow \tilde{h}(q) = \overline{\overline{\tilde{h}(p)}} \vee$$
$$\vee\, \tilde{h}(q) = \tilde{h}(p) \vee \tilde{h}(q) \quad (**)$$

$$(*), (**) \Rightarrow 1 = 1 \wedge 1 =$$
$$= (\tilde{h}(p) \vee \overline{\tilde{h}(q)}) \wedge (\tilde{h}(p) \vee \tilde{h}(q)) =$$
$$= \tilde{h}(p) \vee (\overline{\tilde{h}(q)} \wedge \tilde{h}(q)) = \tilde{h}(p) \vee$$
$$\vee\, 0 = \tilde{h}(p).$$ Deci $\tilde{h}(p) = 1$, prin urmare $\Sigma \models p$.

— 14.10 (MS) — MS PT. LMC (S XIV)

METODA II DE REZOLVARE PENTRU Exerc. / pg. 14, 1:

Folosind notațiile p, q, r, φ, ψ din prima metodă:
Fie $h: V \to L_2$, arbitrară. După cum arată tabelul semantic alăturat, dacă $h \vDash \{\varphi, \psi, r\}$, adică $\tilde{h}(\varphi) = \tilde{h}(\psi) = \tilde{h}(r) = 1$, atunci $\tilde{h}(p) = 1$.

Așadar: $\{\varphi, \psi, r\} \vDash p$

(ne interesează doar valorile lui h în $p, q, r \in V$)

($\exists$ o infinitate de interpretări h $(h: V \to L_2)$ cu $\tilde{h}(\varphi) = \tilde{h}(\psi) = \tilde{h}(r) = 1$. Toate acestea au $\tilde{h}(p) = 1$).

1	1	1	1	0	0	0	0	$\tilde{h}(p)$
1	1	0	0	1	1	0	0	$\tilde{h}(q)$
1	0	1	0	1	0	1	0	$\tilde{h}(r)$
0	0	0	0	1	1	1	1	$\tilde{h}(\neg p)$
0	1	0	1	0	1	0	1	$\tilde{h}(\neg r)$
0	1	1	1	0	1	1	1	$\tilde{h}(q \to \neg r)$
1	1	1	1	1	1	0	0	$\tilde{h}(\varphi)$
1	1	1	1	0	1	1	1	$\tilde{h}(\psi)$

METODA III DE REZOLVARE PENTRU ~~EXERCIŢIU~~ PG. 14.1: Folosind notaţiile p, q, r, φ, ψ din prima metodă, să demonstrăm sintactic ~~că~~ $\{\varphi, \psi, r\} \vdash p$.

$\{\varphi, \psi, r\} \vdash \varphi = \neg p \to (q \to \neg r)$

$\{\varphi, \psi, r\} \vdash \psi = \neg p \to q$

$\{\varphi, \psi, r\} \vdash (\neg p \to (q \to \neg r)) \to ((\neg p \to q) \to (\neg p \to \neg r))$ (A_2)

$\{\varphi, \psi, r\} \vdash (\neg p \to q) \to (\neg p \to \neg r)$ (MP)

$\{\varphi, \psi, r\} \vdash \neg p \to \neg r$ (MP)

$\{\varphi, \psi, r\} \vdash (\neg p \to \neg r) \to (r \to p)$ (A_3)

$\{\varphi, \psi, r\} \vdash r \to p$ (MP)

$\{\varphi, \psi, r\} \vdash r$.

$\{\varphi, \psi, r\} \vdash p$ (MP).

Aşadar, într-adevăr, avem:

$\{\varphi, \psi, r\} \vdash p. \overset{(TCT)}{\Leftrightarrow} \{\varphi, \psi, r\} \vDash p.$