Leopea Cătălina, grupa 243

Tema 3

Exercițiul 1

5 tranzistori din care 2 sunt defecți

N_1 = nr de teste pt identificarea primului tranzistor defect

N_2 = —"— celui de-al doilea —"—

$N_1 \backslash N_2$	1	2	3	4	5	$f_{N_1}(x)$
1	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	0	$\frac{4}{10}$
2	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	0	0	$\frac{3}{10}$
3	$\frac{1}{10}$	$\frac{1}{10}$	0	0	0	$\frac{2}{10}$
4	$\frac{1}{10}$	0	0	0	0	$\frac{1}{10}$
5	0	0	0	0	0	0
$f_{N_2}(x)$	$\frac{4}{10}$	$\frac{3}{10}$	$\frac{2}{10}$	$\frac{1}{10}$	0	

f_{N_1} = funcția probabilității de masă pentru N_1

f_{N_2} = funcția probabilității de masă pentru N_2

$$E[N_1] = \sum_{i=1}^{5} i f_{N_1}(i)$$

$$= \frac{1\cdot 4}{10} + \frac{2\cdot 3}{10} + \frac{3\cdot 2}{10} + \frac{4\cdot 1}{10} + 5\cdot 0$$

$$= \frac{20}{10} = 2$$

$$E[N_2] = \sum_{i=1}^{5} i f_{N_2}(i)$$

$$= 1\cdot\frac{4}{10} + 2\cdot\frac{3}{10} + 3\cdot\frac{2}{10} + 4\cdot\frac{1}{10} + 5\cdot 0$$

$$= \frac{20}{10} = 2$$

Exercițiul 2

X\Y	2	4	6	P_X
0	0.1	0.2	0.1	0.4
1	0.1	0.1	0.1	0.3
2	0.1	0.1	0	0.2
3	0.05	0	0.05	0.1
$f_Y(x)$	0.035	0.4	0.25	

a) $E[Y] = 2\cdot 0.35 + 4\cdot 0.4 + 6\cdot 0.25$
$= 0.7 + 1.6 + 1.5$
$= 3.8$

$E[Y^2] = 4\cdot 0.35 + 16\cdot 0.4 + 36\cdot 0.25$
$= 1.4 + 6.4 + 9$
$= 16.8$

$Var(Y) = E[Y^2] - E[Y]^2 = 16.8 - (3.8)^2 = 16.8 - 14.44$
$= 2.36$

b) $E[Y|X]$

$P(X=0) = 0.1 + 0.2 + 0.1 = 0.4$

$E[Y|X=0] = \sum_{y\in\{2,4,6\}} y\cdot P_{Y|X=0}(y)$

$= 2\frac{0.1}{0.4} + 4\frac{0.2}{0.4} + 6\frac{0.1}{0.4}$

$= 2\frac{1}{4} + 4\frac{2}{4} + 6\frac{1}{4} = \frac{1}{2} + \frac{4}{2} + \frac{3}{2} = \frac{8}{2} = 4$

$P(X=1) = 0.1 + 0.1 + 0.1 = 0.3$

$E[Y|X=1] = 2\frac{0.1}{0.3} + 4\frac{0.1}{0.3} + 6\frac{0.1}{0.3} = \frac{12}{3} = 4$

$P(X=2) = 0.1 + 0.1 + 0 = 0.2$

$E[Y|X=2] = 2\frac{0.1}{0.2} + 4\frac{0.1}{0.2} + 6\frac{0}{0.2} = 3$

$P(X=3) = 0.05 + 0 + 0.05 = 0.1$

$E[Y|X=3] = 2\frac{0.05}{0.1} + 4\frac{0}{0.1} + 6\frac{0.05}{0.1} = \frac{8}{2} = 4$

$E[Y|X=0] = 4$, $p(X=0) = 0.4$
$E[Y|X=1] = 4$, $p(X=1) = 0.3$
$E[Y|X=2] = 3$, $p(X=2) = 0.2$
$E[Y|X=3] = 4$, $p(X=3) = 0.1$

$E[Y\|X]$	3	4
$P(E[Y\|X])$	0.2	0.8

$E[Y^2|X=0] = 4\frac{1}{4} + 16\frac{2}{4} + 36\frac{1}{4} = \frac{72}{4} = 18$
$[Y|X=0] = 18 - 16 = 2$

$E[Y^2|X=1] = 4\frac{1}{3} + 16\frac{1}{3} + 36\frac{1}{3} = \frac{56}{3}$
$[Y|X=1] = \frac{56}{3} - \frac{48}{3} = \frac{8}{3} = 2.66$

$E[Y^2|X=2] = 4\frac{1}{2} + 16\frac{1}{2} = 10$

$V(Y|X=2) = 10 - 9 = 1$

$E[Y^2|X=3] = 4\frac{1}{2} + 36\frac{1}{2} = \frac{40}{2} = 20$

$V(Y|X=3) = 20 - 4^2 = 4$

$V(Y\|X)$	1	2	2.66	4
P	0.2	0.4	0.3	0.1

$E[V(Y|X)] = 0.2 + 0.8 + 0.3\frac{8}{3} + 0.4$

$= 0.2 + 0.8 + 0.8 + 0.4 = 2.2$

$E[E[Y|X]] = 0.2 \cdot 3 + 0.8 \cdot 4 = 2 + 1.8 = 3.8$

$E[E[Y|X]^2] = 0.2 \cdot 9 + 0.8 \cdot 16 = 14.6$

$V[E[Y|X]] = 14.6 - 14.44 = 0.16$

$$\left.\begin{array}{l} Var(Y) = 2.36 \\ E[V[Y|X]] = 2.2 \\ V[E[Y|X]] = 0.16 \end{array}\right\} \Rightarrow Var(Y) = E[Var(Y|X)] + V(E[Y|X])$$

Exercițiul 3

a) X v.a. cu valori în $\mathbb{N}$

$\Rightarrow E[X] = \sum_{m\geq 1} P(X \geq m)$

$E[X] = \sum_{m\geq 0} mP(m) = 0 \cdot P(0) + 1 \cdot P(1) + 2 \cdot P(2) + \dots$

$= 1 \cdot P(1) + 2P(2) + 3P(3) + \dots$

Dem: $E[X] = \sum_{m\geq 1} P(X \geq m)$

$P(X \geq m) = \sum_{i=m}^{\infty} P(i) = P(m) + P(m+1) + \dots$

$$\begin{aligned} \sum_{m\geq 1} P(X\geq m) &= P(X\geq 1) + P(X \geq 2) + \dots \\ &= P(1) + P(2) + P(3) + \dots + P(m) + \dots + \\ &\quad\ P(2) + P(3) + \dots\dots + P(m) + \dots + \\ &\quad\ P(3) + \dots\dots + P(m) + \dots + \\ &\quad\ \vdots \\ &\quad\ + P(m) + \dots + \dots \\ &= P(1) + 2P(2) + 3P(3) + \dots + mP(m) + \dots \\ &= E[X] \end{aligned}$$

b) X v.a cu valori pozitive $\Rightarrow E[X] = \int_0^\infty P(X \geq x)\,dx$

Dem $E[X] = \int_0^\infty x\,P(x)\,dx$

$\int_0^\infty P(X \geq x) = \int_0^\infty \left(\int_x^\infty P(y)\,dy\right)dx \quad , P(X \geq x) = \int_x^\infty P(y)\,dy$

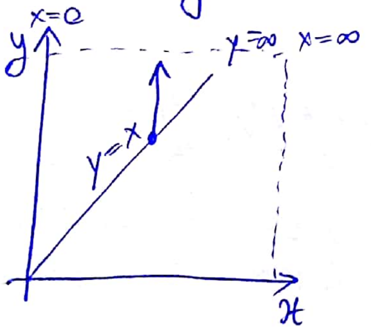
$\overset{(*)}{=} \int_0^\infty \left(\int_0^y p(y)\,dx\right)dy = \int_0^\infty (x\,p(y)) \Big|_0^y\,dy$

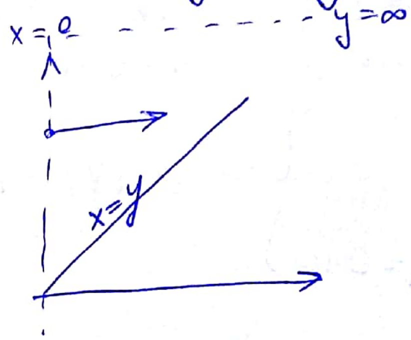
$= \int_0^\infty (y\,p(y) - 0 \cdot p(y))\,dy = \int_0^\infty y\,p(y)\,dy$

$= \int_0^\infty x\,p(x)\,dx = E[X]$

(*) Schimbarea ordinii de integrare

$\int_{0=x}^{\infty=x} \int_{x=y}^{\infty=y} p(y)\,dy\,dx \qquad\qquad \int_0^\infty \int_0^y p(y)\,dx\,dy$





Exercițiul 4

X v.a cu densitatea de probabilitate $f(x) = \begin{cases} \alpha x^2 e^{-kx}, & x \geq 0 \\ 0, & x < 0 \end{cases}, k > 0$

a) $\alpha = ?$

$\int_{-\infty}^{\infty} \alpha x^2 e^{-kx} = 1$

$\int_{-\infty}^{\infty} \alpha x^2 e^{-kx} = \int_0^\infty \alpha x^2 e^{-kx} = -\alpha \dfrac{e^{-kx}(kx(kx+2)+2)}{k^3} + c$

$\Rightarrow \int_0^\infty \alpha x^2 e^{-kx} = \lim_{v \to \infty} \left(-\dfrac{\alpha(kx(kx+2)+2)e^{-kx}}{k^3} + c\right)\Big|_0^v$

$= \lim_{v \to \infty} \left(-\dfrac{\alpha(kv(kv+2)+2)e^{-kv}}{k^3} + \dfrac{2\alpha \cdot e^0}{k^3}\right)$

$= \dfrac{2\alpha}{k^3} = 1 \Rightarrow \alpha = \dfrac{k^3}{2}$

b) funcția de repartiție

$$F(x) = P(X \leq x) = \int_0^{\infty} f(x) = 1 - \frac{kx(kx+2)+2)e^{-kx}}{2}$$

c) $$P(0 < X < k^{-1}) = F(k^{-1}) = 1 - \frac{(k \cdot k^{-1}(k \cdot k^{-1}+2)+2)e^{-k \cdot k^{-1}}}{2}$$
$$= 1 - \frac{(1 \cdot (1+2)+2)e^{-1}}{2} = 1 - \frac{5}{2e} = \frac{2e-5}{2e}$$

Exercițiul 5

a) X v. repartizată exponențial $\Rightarrow f(x) = \lambda e^{-\lambda x}$

$P(X > s+t \mid X > s) = P(X > t)$

$$P(X > s+t \mid X > s) = \frac{P(X > s+t)}{P(X > s)}$$

$P(X > s) = 1 - P(X \leq s)$

$P(X > s) = 1 - F(s)$

$$F(s) = \int_0^s \lambda e^{-\lambda s} = \lambda \left(-\frac{1}{\lambda} e^{-\lambda s}\Big|_0^s\right) = -e^{-\lambda s} + e^{-\lambda \cdot 0} = 1 - e^{-\lambda \cdot s}$$

$\Rightarrow P(X > s) = e^{-\lambda \cdot s}$

$\Rightarrow P(X > s+t) = e^{-\lambda(s+t)}$

$$\Rightarrow P(X > s+t \mid X > s) = \frac{e^{-\lambda(s+t)}}{e^{-\lambda \cdot s}} = \frac{\cancel{e^{-\lambda s}} \cdot e^{-\lambda t}}{\cancel{e^{-\lambda s}}} = e^{-\lambda t} = P(X > t)$$

$\Leftrightarrow P(X > s+t \mid X > s) = P(X > t)$

b) X v.a care verifică $P(X > s+t \mid X > s) = P(X > t)$ atunci X e repartizată exponențial

$$P(X > s+t \mid X > s) = P(X > t) \Leftrightarrow \frac{P(X > s+t)}{P(X > s)} = P(X > t) \Leftrightarrow$$

$\Leftrightarrow P(X > s+t) = P(X > s) \cdot P(X > t)$

notăm $P(X > y) \overset{not}{=} \mathcal{F}(y)$

$\Rightarrow \mathcal{F}(s+t) = \mathcal{F}(s) \cdot \mathcal{F}(t)$

$F(1)=F(\frac{1}{2})F(\frac{1}{2}) \Rightarrow F(\frac{1}{2})=F(1)^{\frac{1}{2}}$

$F(2)=F(1)F(1) \Leftrightarrow F(2)=F(1)^2$

. . . . .

$F(\underbrace{X_1+X_2+\ldots+X_n}_{a})=F(X_1)F(X_2)F(X_3)\cdot\ldots\cdot F(X_n) \Leftrightarrow$

$\Leftrightarrow F(a)=F(1)^a \quad (*)$

Fie $x=\frac{a}{b}$ un nr rational presupunem $F(\frac{a}{b})^{\frac{b}{a}}=F(1)$

$F(\frac{a}{b})^{\frac{b}{a}}=F(1) \;\Big|^{a} \Leftrightarrow F(\frac{a}{b})^{b}=F(1)^a \overset{(*)}{\Leftrightarrow} F(\frac{a}{b}\cdot b)=F(a) \Leftrightarrow$

$F(a)=F(a)$ A $\Rightarrow F(x)=F(1)^x$

În cazul în care x este un număr real, $F(x)=F(1)^x$ este adevărată doar în cazul $F(1)^x=e^{\ln(F(1))\cdot x}=e^{-\lambda\cdot x}$ unde $\lambda=-\ln(F(1))$.

Deoarece F(X) este o probabilitate și $\lambda>0$, orice funcție cu proprietatea lipsei de memorie trebuie să fie exponențială.

Exercițiul 6

X v.a cu densitatea $f(x)=\frac{1}{2\beta}e^{-\frac{|x-\alpha|}{\beta}}$, $-\infty<x<\infty$, $\beta>0$

$E[X]$, $Var(X)=?$ notăm $E[X]=a$, $Var(X)=b$

$E[X]=\int_{-\infty}^{\infty}xf(x)=\int_{-\infty}^{\infty}\frac{1}{2\beta}x e^{\frac{-|X-\alpha|}{\beta}}$

$=\frac{e^{\frac{-|X-\alpha|}{\beta}}(b|X-a|+X(X-a))}{2|X-a|}\Big|_{-\infty}^{\infty}$

Notăm $N(\alpha,\beta)$ distribuția cu parametrii α și β

Dem $X\sim N(\alpha,\beta)$

$Z=\frac{X-\alpha}{\beta}\sim N(0,1)$

$z=\frac{x-\alpha}{\beta} \Rightarrow dz=\frac{dx}{\beta} \Rightarrow dx=\beta dz$

$f(x)dx=\frac{1}{2\beta}e^{\frac{-|X-\alpha|}{\beta}}dx=\frac{1}{2\beta}e^{-\frac{|\beta z+\alpha-\alpha|}{\beta}}\beta dz$

$$f_X(x)\,dx = \frac{1}{2} e^{-\frac{|bz|}{b}}\, dz = \frac{e^{-|z|}\, dz}{2} \Rightarrow z \sim N(0,1)$$

$$E[N(0,1)] = \int_{-\infty}^{\infty} \frac{1}{2} z e^{-|z|}\, dz$$

$$\int_0^{\infty} \frac{1}{2} z e^{-z}\, dz = \frac{1}{2} \int_0^{\infty} z e^{-z}\, dz$$

$$= \frac{1}{2}\left(-xe^{-x} - \int -e^{-x}\, dx\right)$$

$$= \frac{1}{2}\left(-xe^{-x} - e^{-x}\right)$$

$$= \frac{1}{2} e^{-x}(-x-1)$$

$$\int_0^{\infty} \frac{1}{2} x e^{-x}\, dx = \lim_{c \to \infty} \frac{1}{2} e^{-x}(-x-1) \Big|_0^c$$

$$= \frac{1}{2}$$

$$\int_{-\infty}^{0} \frac{1}{2} x e^{-x}\, dx = \lim_{c \to -\infty} \frac{1}{2} e^{-x}(-x-1) \Big|_c^0$$

$$= -\frac{1}{2}$$

$$\Rightarrow E[N(0,1)] = -\frac{1}{2} + \frac{1}{2} = 0$$

Fie $N(a,b)$, $N(0,1) \sim \frac{N(a,b) - a}{b}$

$$N(a,b) \sim b \cdot (N(0,1)) + a$$

$$\Rightarrow E[N(a,b)] = E[b N(0,1) + a]$$

$$= b\, E[N(0,1)] + a = 0 + a = a$$

$$\text{Var}(N(0,1)) = E[N(0,1)^2] - E[N(0,1)]^2$$

$$= E[N(0,1)^2] - 0$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} z^2 e^{-|z|}$$

$$= \frac{1}{2} \int_{-\infty}^{0} z^2 e^{z} + \frac{1}{2} \int_0^{\infty} z^2 e^{-z}$$

$$= \frac{1}{2}\left(-(x^2+2x+2)e^{-x}\Big|_0^{\infty}\right) + \frac{1}{2}\left((x^2+2x+2)e^{x}\Big|_0^{\infty}\right)$$

$$= \frac{1}{2}2 + \frac{1}{2}2 = 2 \Rightarrow Var(N(0,1)) = 2$$

$$z = \frac{X-a}{b}$$

$$Var(X) = E[(X-a)^2] = \int_{-\infty}^{\infty} (x-a)^2 f(x)$$

$$= \int_{-\infty}^{\infty} (x-a)^2 \frac{1}{2\beta} e^{-\frac{|x-a|}{\beta}} dx$$

$$= \int_{-\infty}^{\infty} (bz+\cancel{\alpha}-\cancel{\alpha})^2 \frac{1}{2\cancel{\beta}} e^{-\frac{|bz+\cancel{\alpha}-\cancel{\alpha}|}{\beta}} \cancel{\beta} dz$$

$$= \int_{-\infty}^{\infty} b^2 z^2 \frac{1}{2} e^{-|z|} dz$$

$$= b^2 \int_{-\infty}^{\infty} z^2 e^{-|z|} dz = 2b^2 = Var(X)$$

Exercițiul 7

Fie X v.a ce reprezintă nr de clienți, $E[X] = 50$

S v.a ce reprezintă suma cheltuită de un client,

$E[S] = 30$

C v.a ce reprezintă cifra de afaceri din acea zi,

$E[C] = ?$

$$E[C] = E[X \cdot S] = E[X] \cdot E[S]$$

$$= 50 \cdot 30 = 1500$$