Course 1

Relations

Prof. dr. Septimiu Crivei

Algebra - First year - CS & AI

- Chapter 1: Preliminaries
- Chapter 2: Vector Spaces
- Chapter 3: Matrices and Linear Systems
- Chapter 4: Introduction to Coding Theory

- S. Crivei, *Basic Linear Algebra*, Presa Universitară Clujeană, Cluj-Napoca, 2022.
- W. J. Gilbert, W. K. Nicholson, *Modern Algebra with Applications*, John Wiley, 2004.
- J. S. Golan, *The Linear Algebra a Beginning Graduate Student Ought to Know*, Springer, Dordrecht, 2007.
- P. N. Klein, *Coding the Matrix. Linear Algebra through Applications to Computer Science*, Newtonian Press, 2013.
- R. Lidl, G. Pilz, *Applied Abstract Algebra*, Springer-Verlag, 1998.
- I. Purdea, C. Pelea, *Probleme de algebră*, Eikon, Cluj-Napoca, 2008.
- L. Robbiano, *Linear Algebra for Everyone*, Springer, Milan, 2011.
- G. Strang, *Linear Algebra and its Applications*, Brooks/Cole, 1988.

Coordinates: course and seminar

- Course materials will be available in MS Teams *Algebra-CS-AI (2024-2025)* (code: 8jou114).
- Students may get up to 1 bonus point from course projects to the final grade: up to 5 projects, each for 0.2 points [you will receive details in due time...].
- Minimum attendance: 75% for seminar classes in order to be allowed to participate in the second partial exam and pass the course.
- Problems for the next week will be available in MS Teams *Algebra-CS-AI (2024-2025)* (code: 8jou114).
- Students may get up to 0.5 bonus points from seminar to the final grade: 5 problems solved during the seminar, each for 0.1 points [you will receive details during seminars...].

Coordinates: exam

- Written partial exams in:
  Week 7 (Chapters 1-2): Saturday, November 16, 2024
  Week 14 (Chapters 3-4): Saturday, January 18, 2025
- The final grade is computed as follows:

$$G = 1 + P_1 + P_2 + B,$$

  where:
  G = the final grade
  P_1 = the points from the first partial exam (max. 4.5)
  P_2 = the points from the second partial exam (max. 4.5)
  B = bonus points from seminar or course (max. 1.5)
- Students may not pass the exam unless they participate in the second partial exam.

Computer Science topics using Linear Algebra I

The Association for Computing Machinery (ACM) has developed the 2012 ACM Computing Classification System for the research topics in the field of Computer Science (www.acm.org) under the form of a multi-level tree.
We mention some higher level branches of this tree in which Linear Algebra has important applications.

- **Networks**
  - Network architectures
    - Network design principles
  - Network types
    - Public Internet
- **Theory of Computation**
  - Models of computation
    - Quantum computation theory
  - Computational complexity and cryptography
    - Cryptographic protocols

Computer Science topics using Linear Algebra II

- Randomness, geometry and discrete structures
  - Error-correcting codes
- Theory and algorithms for application domains
  - Machine learning theory

- **Mathematics of Computing**
  - Information theory
    - Coding theory
  - Mathematical analysis
    - Mathematical optimization
- **Information Systems**
  - World Wide Web
    - Web searching and information discovery
  - Information retrieval
    - Retrieval models and ranking
- **Security and Privacy**
  - Cryptography
    - Symmetric cryptography and hash functions

Computer Science topics using Linear Algebra III

- Network security
    - Security protocols
- **Computing Methodologies**
    - Machine learning
        - Machine learning approaches
    - Computer graphics
        - Image manipulation
- **Applied Computing**
    - Electronic commerce
        - Online banking
    - Operations research
        - Decision analysis

Application: relational database

ID (Integer)	**Surname** (String)	**Name** (String)	**Grade** (Integer)
7	Ionescu	Alina	9
11	Ardelean	Cristina	10
23	Ionescu	Dan	7

Definition

A triple $r = (A, B, R)$, where A, B are sets and

$$R \subseteq A \times B = \{(a, b) \mid a \in A, b \in B\},$$

is called a *(binary) relation*.
The set A is called the *domain*, the set B is called the *codomain* and the set R is called the *graph* of the relation r.
If $A = B$, then the relation r is called *homogeneous*.
If $(a, b) \in R$, then we sometimes write $a\,r\,b$ and we say that *a has the relation r to b* or *a and b are related with respect to the relation r*.

Relation classes

Definition

Let $r = (A, B, R)$ be a relation and let $X \subseteq A$. Then the set

$$r(X) = \{b \in B \mid \exists x \in X : x\, r\, b\}$$

is called the *relation class of X with respect to r*.
If $x \in X$, then we denote

$$r < x >= r(\{x\}) = \{b \in B \mid x\, r\, b\}.$$

Notice that

$$r(X) = \bigcup_{x \in X} r < x > \ .$$

Relation representation

If A, B are finite sets, then $r = (A, B, R)$ may be represented by a diagram consisting of two sets with elements and connecting arrows. For instance, let $r = (A, B, R)$, where $A = \{1, 2, 3\}$, $B = \{1, 2\}$ and

$$R = \{(1,1), (1,2), (3,1)\}.$$

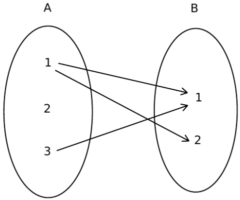

Figure: Diagram of a relation.

Also note that $r < 1 >= \{1, 2\} = r(A)$.

(a) Let C be the set of all children and let P be the set of all parents. Then we may define the relation $r = (C, P, R)$, where

$$R = \{(c, p) \in C \times P \mid c \text{ is a child of } p\}.$$

(b) The triple $r = (\mathbb{R}, \mathbb{R}, R)$, where

$$R = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x \leq y\}$$

is a homogeneous relation, called the *inequality relation* on $\mathbb{R}$. We have

$$r < 1 >= [1, \infty) = r([1, 2]).$$

(c) There are several examples from Number Theory, such as divisibility on $\mathbb{N}$ or on $\mathbb{Z}$, and Geometry, such as parallelism of lines, perpendicularity of lines, congruence of triangles, similarity of triangles.

Examples of relations II

(d) Let A and B be two sets. Then the triples

$$o = (A, B, \emptyset), \quad u = (A, B, A \times B)$$

are relations, called the *void relation* and the *universal relation* respectively.

(e) Let A be a set. Then the triple $\delta_A = (A, A, \Delta_A)$, where

$$\Delta_A = \{(a, a) \mid a \in A\}$$

is a relation called the *equality relation* on A.

(f) Every function is a relation. Indeed, a function $f : A \to B$ is determined by its domain A, its codomain B and its graph

$$G_f = \{(x, y) \in A \times B \mid y = f(x)\}.$$

Then the triple (A, B, G_f) is a relation.

(g) Every directed graph is a relation.
For instance, the directed graph

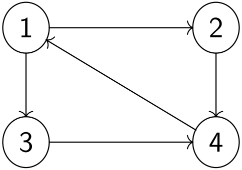

Figure: Directed graph.

may be seen as a relation (A, A, R), where $A = \{1, 2, 3, 4\}$ and

$$R = \{(1,2), (1,3), (2,4), (3,4), (4,1)\}.$$

Functions

Definition

A relation $r = (A, B, R)$ is called a *function* if

$$\forall a \in A, \quad |r < a > | = 1,$$

that is, the relation class with respect to r of every $a \in A$ consists of exactly one element.

In what follows, if $f = (A, B, F)$ is a function, we will mainly use the classical notation for a function, namely $f : A \to B$ or sometimes $A \xrightarrow{f} B$. The unique element of the set $f < a >$ will be denoted by $f(a)$. Then we have

$$(a, b) \in F \Longleftrightarrow f(a) = b.$$

From relations we get the following notions.

Definition

Let $f : A \to B$ be a function. Then A is called the *domain*, B is called the *codomain* and

$$F = \{(a, f(a)) \mid a \in A\}$$

is called the *graph* of the function f.

Definition

Let $f : A \to B$ be a function and let $X \subseteq A$. We call the *image of X by f* the relation class of X with respect to f, that is,

$$f(X) = \{b \in B \mid \exists x \in X : x\, f\, b\} = \{f(x) \mid x \in X\}.$$

We denote $\mathrm{Im} f = f(A)$ and call it the *image of f*.

(*a*) Let A be a set. Then the equality relation (A, A, Δ_A) is a function called the *identity function on* A, denoted by $1_A : A \to A$.

(*b*) Let $A = \{1, 2, 3\}$, $B = \{1, 2\}$ and let $r = (A, B, R)$, $s = (A, B, S)$, $t = (A, B, T)$ be the relations having the graphs

$$R = \{(1,1), (2,1), (3,2)\},$$
$$S = \{(1,2), (3,1)\},$$
$$T = \{(1,1), (1,2), (2,1), (3,2)\}.$$

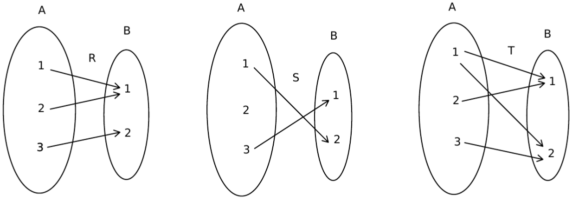

Figure: Diagrams of functions or relations.

Which of them are functions?

Recall that a relation $r = (A, B, R)$ is called *homogeneous* if $A = B$.

Definition

A homogeneous relation $r = (A, A, R)$ on A is called:
(1) *reflexive* (r) if: $\forall x \in A,\ x\, r\, x$.
(2) *transitive* (t) if: $x, y, z \in A,\ x\, r\, y$ and $y\, r\, z \Longrightarrow x\, r\, z$.
(3) *symmetric* (s) if: $x, y \in A,\ x\, r\, y \Longrightarrow y\, r\, x$.
A homogeneous relation $r = (A, A, R)$ is called an *equivalence relation* if r has the properties (r), (t) and (s).

(a) The equality relation δ_A on a set A is an equivalence relation.

(b) The similarity of triangles is an equivalence relation on the set of all triangles.

(c) The inequality relation "$\leq$" on $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{R}$ has (r) and (t), but not (s). Hence it is not an equivalence relation.

(d) Let $n \in \mathbb{N}$ and let ρ_n be the relation defined on $\mathbb{Z}$ by

$$x \, \rho_n \, y \Longleftrightarrow x \equiv y \pmod{n},$$

that is, $n|(x-y)$ or equivalently for $n \neq 0$, x and y give the same remainder when divided by n. Then ρ_n is called the *congruence modulo* n and it is an equivalence relation.
For $n = 0$, we have $x \, \rho_0 \, y \Longleftrightarrow 0|x-y \Longleftrightarrow x = y$, hence $\rho_0 = \delta_{\mathbb{Z}} = (\mathbb{Z}, \mathbb{Z}, \Delta_{\mathbb{Z}})$.
For $n = 1$, we have $x \, \rho_1 \, y \Longleftrightarrow 1|x-y$, which is always true, and thus $\rho_1 = u = (\mathbb{Z}, \mathbb{Z}, \mathbb{Z} \times \mathbb{Z})$.

Partitions

Definition

Let A be a non-empty set. Then a family $(A_i)_{i\in I}$ of non-empty subsets of A is called a *partition* of A if:

(i) The family $(A_i)_{i\in I}$ covers A, that is,

$$\bigcup_{i\in I} A_i = A.$$

(ii) The A_i's are pairwise disjoint, that is,

$$i, j \in I, i \neq j \Longrightarrow A_i \cap A_j = \emptyset.$$

Examples of partitions

(a) Let $A = \{1, 2, 3, 4, 5\}$ and $A_1 = \{1, 2, 3\}$, $A_2 = \{4\}$, $A_3 = \{5\}$. Then $\{A_1, A_2, A_3\}$ is a partition of A.

(b) Let A be a set. Then $\{\{a\} \mid a \in A\}$ and $\{A\}$ are partitions of A.

(c) Let A_1 be the set of even integers and A_2 the set of odd integers. Then $\{A_1, A_2\}$ is a partition of $\mathbb{Z}$.

(d) Consider the intervals

$$A_n = [n, n + 1)$$

for every $n \in \mathbb{Z}$. Then the family $(A_n)_{n \in \mathbb{Z}}$ is a partition of $\mathbb{R}$.

Quotient set

Denote by $E(A)$ the set of all equivalence relations and by $P(A)$ the set of all partitions on a set A.

Definition

Let $r \in E(A)$.
The relation class $r < x >$ of an element $x \in A$ with respect to r is called the *equivalence class of x with respect to r*, while the element x is called a *representative* of $r < x >$.
The set

$$A/r = \{r < x > | \, x \in A\},$$

which is the set of all equivalence classes of elements of A with respect to r, is called the *quotient set of A by r*.

Relation associated to a partition

Definition

Let $\pi = (A_i)_{i \in I} \in P(A)$ and define the relation r_π on A by

$$x \, r_\pi \, y \iff \exists i \in I : x, y \in A_i \, .$$

Then r_π is called the *relation associated to the partition* π.

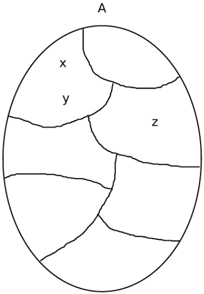

Figure: Relation associated to a partition.

Theorem

(i) Let $r \in E(A)$*. Then* $A/r \in P(A)$*.*
(ii) Let $\pi = (A_i)_{i \in I} \in P(A)$*. Then* $r_\pi \in E(A)$*.*
(iii) Let $F : E(A) \to P(A)$ *be defined by*

$$F(r) = A/r, \quad \forall r \in E(A).$$

Then F *is a bijection, whose inverse is* $G : P(A) \to E(A)$*, defined by*

$$G(\pi) = r_\pi, \quad \forall \pi \in P(A).$$

(a) Let $A = \{1, 2, 3\}$ and let r and s be the homogeneous relations defined on A with the graphs

$$R = \{(1,1),(2,2),(3,3),(1,2),(2,1)\},$$
$$S = \{(1,1),(2,2),(3,3),(1,2),(2,3)\}.$$

Then r is an equivalence relation, but s is not. What is the partition corresponding to r?

(b) Consider the following families of sets:

$$\pi = \{\{1\},\{2,3\},\{4\}\},$$

$$\pi' = \{\{1,2\},\{2,3\},\{4\}\}.$$

Then π is a partition of $A = \{1, 2, 3, 4\}$, but π' is not. What is the equivalence relation corresponding to π?

Illustrations of the theorem II

(c) The congruence relation modulo n is an equivalence relation on $\mathbb{Z}$ and its corresponding partition is

$$\mathbb{Z}/\rho_n = \{\rho_n < x >\mid x \in \mathbb{Z}\} = \{x + n\mathbb{Z} \mid x \in \mathbb{Z}\} = \{\widehat{x} \mid x \in \mathbb{Z}\},$$

where an equivalence class is denoted by $\widehat{x}$. For $n \geq 2$, we denote

$$\mathbb{Z}_n = \mathbb{Z}/\rho_n = \{\widehat{0}, \widehat{1}, \dots, \widehat{n-1}\}.$$

For $n = 0$ and $n = 1$, we have seen that $\rho_0 = \delta_{\mathbb{Z}}$ and $\rho_1 = u$, and we get

$$\mathbb{Z}/\rho_0 = \{\{x\} \mid x \in \mathbb{Z}\} \quad \text{and} \quad \mathbb{Z}/\rho_1 = \{\mathbb{Z}\},$$

that are the two extreme partitions of $\mathbb{Z}$.

Binary relations may be naturally generalized as follows.

Definition

A (finite) tuple

$$r = (A_1, \dots, A_n, R),$$

where $A_1, \dots, A_n$ are sets and

$$R \subseteq A_1 \times \cdots \times A_n = \{(a_1, \dots, a_n) \mid a_1 \in A_1, \dots, a_n \in A_n\},$$

is called an *(n-ary) relation*.
The sets $A_1, \dots, A_n$ are called the *domains* of r, and the set R is called the *graph* of r.
The number n is called the *degree (arity)* of r.
A *relational database* is a (finite) set of relations.

Consider the relation

$$student = (Integer, String, String, Integer, Student),$$

where

$$Student \subseteq Integer \times String \times String \times Integer$$

is given by the following table:

ID (Integer)	**Surname** (String)	**Name** (String)	**Grade** (Integer)
7	Ionescu	Alina	9
11	Ardelean	Cristina	10
23	Ionescu	Dan	7

Some known relational database management systems are:

- Oracle and RDB - Oracle
- SQL Server and Access - Microsoft