Course 4

Generated subspace, linear maps

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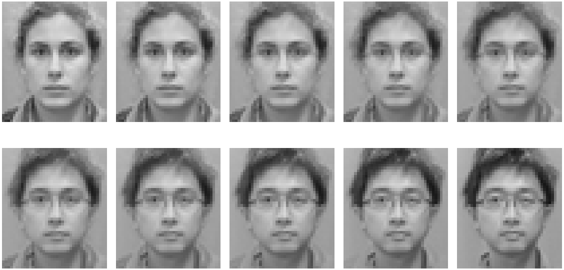
Application: image crossfade

Following [Klein], we describe a way to achieve an image crossfade effect.



Intersection of subspaces

For a vector space V over K, we denote by $S(V)$ the set of all subspaces of V. Sometimes, this set is denoted by $S_K(V)$ if we like to emphasize the field K.

Theorem

Let V be a vector space over K and let $(S_i)_{i\in I}$ be a family of subspaces of V. Then $\bigcap_{i\in I} S_i \in S(V)$.

Proof. For each $i \in I$, we have $S_i \in S(V)$, hence $0 \in S_i$. Then $0 \in \bigcap_{i\in I} S_i \neq \emptyset$. Now let $k_1, k_2 \in K$ and $x, y \in \bigcap_{i\in I} S_i$. Then $x, y \in S_i$, $\forall i \in I$. But $S_i \in S(V)$, $\forall i \in I$. It follows that $k_1x + k_2y \in S_i$, $\forall i \in I$, hence $k_1x + k_2y \in \bigcap_{i\in I} S_i$. Therefore, $\bigcap_{i\in I} S_i \in S(V)$. □

In general, the union of two subspaces of a vector space is not a subspace. For instance, the sets

$$S = \{(x, 0) \mid x \in \mathbb{R}\},$$

$$T = \{(0, y) \mid y \in \mathbb{R}\}$$

are subspaces of the canonical real vector space $\mathbb{R}^2$, but $S \cup T$ is not a subspace of $\mathbb{R}^2$. Indeed, for instance, we have $(1, 0), (0, 1) \in S \cup T$, but $(1, 0) + (0, 1) = (1, 1) \notin S \cup T$.

Generated subspace

Now we are interested in how to "complete" a given subset of a vector space to a subspace in a minimal way.

Definition

Let V be a vector space and let $X \subseteq V$. Then we denote

$$\langle X \rangle = \bigcap \{S \leq V \mid X \subseteq S\}$$

and we call it the *subspace generated by* X or the *subspace spanned by* X. Here X is called the *generating set* of $\langle X \rangle$. If $X = \{v_1, \dots, v_n\}$, we denote $\langle v_1, \dots, v_n \rangle = \langle \{v_1, \dots, v_n\} \rangle$.

(1) $\langle X \rangle$ is the "smallest" (with respect to inclusion) subspace of V containing X.

(2) $\langle \emptyset \rangle = \{0\}$.

(3) If $S \leq V$, then $\langle S \rangle = S$.

Definition

A vector space V over K is called *finitely generated* if $\exists v_1, \dots, v_n \in V$ ($n \in \mathbb{N}$) such that

$$V = \langle v_1, \dots, v_n \rangle.$$

Then the set $\{v_1, \dots, v_n\}$ is called a *system of generators for* V.

Definition

Let V be a vector space over K and $v_1, \dots, v_n \in V$ ($n \in \mathbb{N}$). A finite sum of the form

$$k_1 v_1 + \cdots + k_n v_n \,,$$

where $k_i \in K$ ($i = 1, \dots, n$), is called a (finite) *linear combination* of the vectors $v_1, \dots, v_n$.

Theorem

Let V be a vector space over K and let $\emptyset \neq X \subseteq V$. Then

$$\langle X \rangle = \{k_1 v_1 + \cdots + k_n v_n \mid k_i \in K\,,\ v_i \in X\,, i = 1, \ldots, n\,,\ n \in \mathbb{N}^*\}\,,$$

that is, the set of all finite linear combinations of vectors of X.

Proof. We prove the result in 3 steps, by showing that

$$L = \{k_1 v_1 + \cdots + k_n v_n \mid k_i \in K\,,\ v_i \in X\,, i = 1, \ldots, n\,,\ n \in \mathbb{N}^*\}$$

is the smallest subspace of V containing X.

(i) Let $v \in X$. Then $v = 1 \cdot v \in L$, hence $L \neq \emptyset$. Now let $k, k' \in K$ and $v, v' \in L$. Then $v = \sum_{i=1}^{n} k_i v_i$ and $v' = \sum_{j=1}^{m} k'_j v'_j$ for some $k_1, \dots, k_n, k'_1, \dots, k'_m \in K$ and $v_1, \dots, v_n, v'_1, \dots, v'_m \in X$. Then $kv + k'v' \in L$, because it is a finite linear combination of vectors of X. Hence we have $L \leq V$.

(ii) If $v \in X$, then $v = 1 \cdot v \in L$. Hence $X \subseteq L$.

(iii) Let $S \leq V$ be such that $X \subseteq S$. Let $k_1, \dots, k_n \in K$ and $v_1, \dots, v_n \in X$. Since $X \subseteq S$ and $S \leq V$, $k_1 v_1 + \dots + k_n v_n \in S$. Hence $L \subseteq S$.
Thus, we have $\langle X \rangle = L$ by the initial remark. $\square$

Corollary

Let V be a vector space over K and let $x_1, \dots, x_n \in V$. Then

$$\langle x_1, \dots, x_n \rangle = \{k_1 x_1 + \dots + k_n x_n \mid k_i \in K\,,\ x_i \in X\,, i = 1, \dots, n\}\,.$$

Examples I

(a) Consider the canonical real vector space $\mathbb{R}^3$. Then

$$\begin{aligned}&\langle(1,0,0),(0,1,0),(0,0,1)\rangle\\&=\{k_1(1,0,0)+k_2(0,1,0)+k_3(0,0,1)\mid k_1,k_2,k_3\in\mathbb{R}\}\\&=\{(k_1,0,0)+(0,k_2,0)+(0,0,k_3)\mid k_1,k_2,k_3\in\mathbb{R}\}\\&=\{(k_1,k_2,k_3)\mid k_1,k_2,k_3\in\mathbb{R}\}=\mathbb{R}^3\,.\end{aligned}$$

Hence $\mathbb{R}^3$ is generated by the three vectors $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$, and thus it is finitely generated.

(b) Consider the canonical vector space $\mathbb{Z}_2^3$ over $\mathbb{Z}_2$. Similarly as above, we have:

$$\langle(\widehat{1},\widehat{0},\widehat{0}),(\widehat{0},\widehat{1},\widehat{0})\rangle=\{(k_1,k_2,\widehat{0})\mid k_1,k_2\in\mathbb{Z}_2\}\neq\mathbb{Z}_2^3\,.$$

Examples II

Hence $\mathbb{Z}_2^3$ is not generated by the two vectors $(\widehat{1}, \widehat{0}, \widehat{0})$ and $(\widehat{0}, \widehat{1}, \widehat{0})$. But it is generated by $(\widehat{1}, \widehat{0}, \widehat{0})$, $(\widehat{0}, \widehat{1}, \widehat{0})$ and $(\widehat{0}, \widehat{0}, \widehat{1})$, hence it is finitely generated.

(c) Consider the subspace

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid x - y - z = 0\}$$

of the canonical real vector space $\mathbb{R}^3$. Let us write it as a generated subspace. Expressing $x = y + z$, we have:

$$\begin{aligned} S &= \{(y + z, y, z) \mid y, z \in \mathbb{R}\} = \{(y, y, 0) + (z, 0, z) \mid y, z \in \mathbb{R}\} \\ &= \{y(1, 1, 0) + z(1, 0, 1) \mid y, z \in \mathbb{R}\} = \langle (1, 1, 0), (1, 0, 1) \rangle. \end{aligned}$$

Alternatively, one may express y or z by using the other two components and get other writings of S as a generated subspace, namely $S = \langle (1, 1, 0), (0, -1, 1) \rangle = \langle (1, 0, 1), (0, 1, -1) \rangle$. We see that S is finitely generated.

Sum of subspaces I

In what follows we shall be interested in "decomposing" a vector space into subspaces.

Definition

Let V be a vector space over K and let $S, T \leq V$.
We define the *sum* of the subspaces S and T as the set

$$S + T = \{s + t \mid s \in S,\ t \in T\}.$$

Theorem

Let V be a vector space over K and $S, T \leq V$. Then $S + T = \langle S \cup T \rangle$, hence $S + T \leq V$.

Proof. We prove the equality by double inclusion.

Sum of subspaces II

First, let $v = s + t \in S + T$, for some $s \in S$ and $t \in T$. Then

$$v = 1 \cdot s + 1 \cdot t$$

is a linear combination of the vectors $s, t \in S \cup T$, hence $v \in \langle S \cup T \rangle$. Thus, $S + T \subseteq \langle S \cup T \rangle$.
Now let $v \in \langle S \cup T \rangle$. Then

$$v = \sum_{i=1}^{n} k_i v_i = \sum_{i \in I} k_i v_i + \sum_{j \in J} k_j v_j \, ,$$

where $I = \{i \in \{1, \dots, n\} \mid v_i \in S\}$ and $J = \{j \in \{1, \dots, n\} \mid v_j \in T \setminus S\}$. But the first sum is a linear combination of vectors of S, hence it belongs to S, while the second sum is a linear combination of vectors of T, hence it belongs to T. Thus, $v \in S + T$ and so $\langle S \cup T \rangle \subseteq S + T$. $\square$

Definition

Let V be a vector space over K and let $S, T \leq V$.
If $S \cap T = \{0\}$, then $S + T$ is denoted by $S \oplus T$ and is called the *direct sum* of the subspaces S and T.

Theorem

Let V be a vector space over K and let $S, T \leq V$. Then

$$V = S \oplus T \iff \forall v \in V \,, \exists! s \in S \,, t \in T \ : v = s + t \,.$$

Proof. $\boxed{\Longrightarrow}$ Assume that $V = S \oplus T$. Let $v \in V$. Then $\exists s \in S \,, t \in T$ such that $v = s + t$. Now suppose that $\exists s' \in S \,, t' \in T$ such that $v = s' + t'$. Then $s + t = s' + t'$, whence

$$s - s' = t' - t \in S \cap T = \{0\} \,.$$

Direct sum of subspaces II

Hence $s = s'$ and $t = t'$, that show the uniqueness.

$\boxed{\Longleftarrow}$ Assume that $\forall v \in V$, $\exists! s \in S, t \in T$ such that $v = s + t$. Then $V \subseteq S + T$. Clearly, we have $S + T \subseteq V$ and consequently $V = S + T$. Now suppose that $0 \neq v \in S \cap T$. Then

$$v = v + 0 = 0 + v \in S + T.$$

But this is a contradiction, since we have the uniqueness of writing of v as a sum of an element of S and an element of T. Therefore, $S \cap T = \{0\}$ and thus, $V = S \oplus T$. $\square$

Example

Consider the canonical real vector space $\mathbb{R}^2$. Then $\mathbb{R}^2 = S \oplus T$, where $S = \{(x, 0) \mid x \in \mathbb{R}\}$ and $T = \{(0, y) \mid y \in \mathbb{R}\}$.

Definition

Let V and V' be vector spaces over the same field K. A function $f : V \to V'$ is called:

(1) *(K-)linear map* (or *(vector space) homomorphism* or *linear transformation*) if

$$f(v_1 + v_2) = f(v_1) + f(v_2), \quad \forall v_1, v_2 \in V,$$
$$f(kv) = kf(v), \quad \forall k \in K, \forall v \in V.$$

(2) *isomorphism* if it is a bijective K-linear map.
(3) *endomorphism* if it is a K-linear map and $V = V'$.
(4) *automorphism* if it is a bijective K-linear map and $V = V'$.

Properties of linear maps

If $f : V \to V'$ is a K-linear map, then the first condition from its definition tells us that f is a group homomorphism between the groups $(V, +)$ and $(V', +)$. Then we have $f(0) = 0'$ and $f(-v) = -f(v)$, $\forall v \in V$.

We denote by $V \simeq V'$ the fact that two vector spaces V and V' are isomorphic. We also denote

$$\begin{aligned}
\mathrm{Hom}_K(V, V') &= \{f : V \to V' \mid f \text{ is } K\text{-linear}\}\,,\\
\mathrm{End}_K(V) &= \{f : V \to V \mid f \text{ is } K\text{-linear}\}\,,\\
\mathrm{Aut}_K(V) &= \{f : V \to V \mid f \text{ is bijective } K\text{-linear}\}\,.
\end{aligned}$$

Theorem

Let V and V' be vector spaces over K and $f : V \to V'$. Then f is a K-linear map $\Longleftrightarrow f(k_1 v_1 + k_2 v_2) = k_1 f(v_1) + k_2 f(v_2), \forall k_1, k_2 \in K, \forall v_1, v_2 \in V$.

Proof. $\boxed{\Longrightarrow}$ Let $k_1, k_2 \in K$ and $v_1, v_2 \in V$. Then

$$f(k_1 v_1 + k_2 v_2) = f(k_1 v_1) + f(k_2 v_2) = k_1 f(v_1) + k_2 f(v_2).$$

$\boxed{\Longleftarrow}$ Choose $k_1 = k_2 = 1$ and then $k_2 = 0$ to get the two conditions of a K-linear map. $\square$

(a) Let V and V' be vector spaces over K and let $f : V \to V'$ be defined by $f(v) = 0'$, $\forall v \in V$. Then f is a K-linear map, called the *trivial linear map*.

(b) Let V be a vector space over K. Then the identity map $1_V : V \to V$ is an automorphism of V.

(c) Let V be a vector space and $S \leq V$. Define $i : S \to V$ by $i(v) = v$, $\forall v \in S$. Then i is a K-linear map, called the *inclusion linear map*.

(d) Let V be a vector space over K and $a \in K$. Define $t_a : V \to V$ by $t_a(v) = av$, $\forall v \in V$. Then t_a is an endomorphism of V.

Examples II

(e) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = x + y$. Then f is an $\mathbb{R}$-linear map, because we have

$$\begin{aligned} f(k_1(x_1, y_1) + k_2(x_2, y_2)) &= f(k_1x_1 + k_2x_2, k_1y_1 + k_2y_2) \\ &= (k_1x_1 + k_2x_2) + (k_1y_1 + k_2y_2) \\ &= k_1(x_1 + y_1) + k_2(x_2 + y_2) \\ &= k_1f(x_1, y_1) + k_2f(x_2, y_2) \end{aligned}$$

for every $k_1, k_2 \in K$ and for every $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$.
On the other hand, $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = xy$ is not an $\mathbb{R}$-linear map, because, for instance, we have

$$f((1, 0) + (0, 1)) = f(1, 1) = 1 \neq 0 = f(1, 0) + f(0, 1).$$

(f) Let $\theta \in \mathbb{R}$ and let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$f(x, y) = (x \cos\theta - y \sin\theta, x \sin\theta + y \cos\theta),$$

which is the counterclockwise rotation of angle θ about the origin in the plane. Then f is an $\mathbb{R}$-linear map. In particular, for $\theta = \frac{\pi}{2}$, we have $f(x, y) = (-y, x)$.

(g) For an interval $I = [a, b] \subseteq \mathbb{R}$ we considered the real vector space

$$\mathbb{R}^I = \{f \mid f : I \to \mathbb{R}\}$$

and its subspaces

$$C(I, \mathbb{R}) = \{f \in \mathbb{R}^I \mid f \text{ continuous on } I\},$$

$$D(I, \mathbb{R}) = \{f \in \mathbb{R}^I \mid f \text{ derivable on } I\}.$$

Then

$$F : D(I, \mathbb{R}) \to \mathbb{R}^I, \quad F(f) = f',$$

$$G : C(I, \mathbb{R}) \to \mathbb{R}, \quad G(f) = \int_a^b f(t)\mathrm{d}t,$$

are $\mathbb{R}$-linear maps.

Properties of linear maps

Theorem

(i) Let $f : V \to V'$ be an isomorphism of vector spaces over K. Then $f^{-1} : V' \to V$ is again an isomorphism of vector spaces over K.
(ii) Let $f : V \to V'$ and $g : V' \to V''$ be K-linear maps. Then $g \circ f : V \to V''$ is a K-linear map.

Proof. Homework.

Kernel and image of a linear map

Definition

Let $f : V \to V'$ be a K-linear map. Then the set

$$\operatorname{Ker} f = \{v \in V \mid f(v) = 0'\}$$

is called the *kernel* (or the *null space*) of the K-linear map f and the set

$$\operatorname{Im} f = \{f(v) \mid v \in V\}$$

is called the *image* (or the *range space*) of the K-linear map f.

Kernel and image are subspaces

Theorem

Let $f : V \to V'$ be a K-linear map. Then $\mathrm{Ker} f \leq V$ *and* $\mathrm{Im} f \leq V'$.

Proof. We have $f(0) = 0'$, hence $0 \in \mathrm{Ker}\, f \neq \emptyset$. Let $k_1, k_2 \in K$ and $v_1, v_2 \in \mathrm{Ker}\, f$. We prove that $k_1 v_1 + k_2 v_2 \in \mathrm{Ker}\, f$. We have:

$$f(k_1 v_1 + k_2 v_2) = k_1 f(v_1) + k_2 f(v_2) = k_1 \cdot 0' + k_2 \cdot 0' = 0',$$

and thus $k_1 v_1 + k_2 v_2 \in \mathrm{Ker}\, f$. Hence $\mathrm{Ker}\, f \leq V$.

Now note that $0' = f(0) \in \mathrm{Im}\, f \neq \emptyset$. Let $k_1, k_2 \in K$ and $v_1', v_2' \in \mathrm{Im}\, f$. We prove that $k_1 v_1' + k_2 v_2' \in \mathrm{Im}\, f$. We have $v_1' = f(v_1)$ and $v_2' = f(v_2)$ for some $v_1, v_2 \in V$. Then:

$$k_1 v_1' + k_2 v_2' = k_1 f(v_1) + k_2 f(v_2) = f(k_1 v_1 + k_2 v_2) \in \mathrm{Im}\, f.$$

Hence $\mathrm{Im}\, f \leq V'$. $\square$

When is the kernel minimal?

Theorem

Let $f : V \to V'$ be a K-linear map. Then

$$\operatorname{Ker} f = \{0\} \iff f \text{ is injective}.$$

Proof. $\boxed{\Longrightarrow}$ Assume that $\operatorname{Ker} f = \{0\}$. Let $v_1, v_2 \in V$ be such that $f(v_1) = f(v_2)$. Then $f(v_1 - v_2) = f(v_1) - f(v_2) = 0'$, hence $v_1 - v_2 \in \operatorname{Ker} f = \{0\}$, and thus $v_1 = v_2$. Therefore, f is injective.

$\boxed{\Longleftarrow}$ Assume that f is injective. Clearly, we have $\{0\} \subseteq \operatorname{Ker} f$. Now let $v \in \operatorname{Ker} f$. Then $f(v) = 0' = f(0)$. By the injectivity of f, we deduce that $v = 0$. Thus $\operatorname{Ker} f \subseteq \{0\}$, and consequently, $\operatorname{Ker} f = \{0\}$. $\square$

Theorem

Let $f : V \to V'$ be a K-linear map and let $X \subseteq V$. Then

$$f(\langle X\rangle) = \langle f(X)\rangle\,.$$

Proof. If $X = \emptyset$, then we have:

$$f(\langle\emptyset\rangle) = f(\{0\}) = \{f(0)\} = \{0'\} = \langle\emptyset\rangle = \langle f(\emptyset)\rangle.$$

If $X \neq \emptyset$, use

$$\langle X\rangle = \{k_1v_1 + \cdots + k_nv_n \mid k_i \in K\,,\ v_i \in X\,, i = 1,\dots,n\,,\ n \in \mathbb{N}^*\}.$$

Since f is a K-linear map, it follows that:

$$\begin{aligned} f(\langle X\rangle) &= \{f(k_1v_1 + \cdots + k_nv_n) \mid k_i \in K, v_i \in X, i = 1,\dots,n, n \in \mathbb{N}^*\} \\ &= \{k_1f(v_1) + \cdots + k_nf(v_n) \mid k_i \in K, v_i \in X, i = 1,\dots,n, n \in \mathbb{N}^*\} \\ &= \langle f(X)\rangle\,, \end{aligned}$$

which proves the result. $\square$

Theorem

Let V and V' be vector spaces over K. Consider on $\mathrm{Hom}_K(V, V')$ the operations: $\forall f, g \in \mathrm{Hom}_K(V, V')$ and $\forall k \in K$, $f + g, k \cdot f \in \mathrm{Hom}_K(V, V')$, where

$$(f+g)(v) = f(v) + g(v),$$
$$(kf)(v) = kf(v)$$

$\forall v \in V$. Then $\mathrm{Hom}_K(V, V')$ is a vector space over K.

Corollary

Let V be a vector space over K. Then $\mathrm{End}_K(V)$ is a vector space over K.

Extra: Image crossfade I

A black-and-white image of (say)

$$n = 1024 \times 768$$

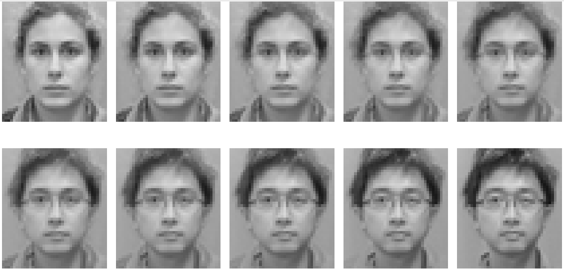
pixels can be viewed as a vector in the real canonical vector space $\mathbb{R}^n$, where each component of the vector is the intensity of the corresponding pixel.

Let us consider two vectors representing images:

$v_1 =$ , $v_2 =$ .

Now consider the following intermediate images:

Extra: Image crossfade II



The vectors corresponding to the above images are the following linear combinations of the vectors v_1 and v_2:

$$v_1, \quad \frac{8}{9}v_1 + \frac{1}{9}v_2, \quad \frac{7}{9}v_1 + \frac{2}{9}v_2, \quad \frac{6}{9}v_1 + \frac{3}{9}v_2, \quad \frac{5}{9}v_1 + \frac{4}{9}v_2,$$
$$\frac{4}{9}v_1 + \frac{5}{9}v_2, \quad \frac{3}{9}v_1 + \frac{6}{9}v_2, \quad \frac{2}{9}v_1 + \frac{7}{9}v_2, \quad \frac{1}{9}v_1 + \frac{8}{9}v_2, \quad v_2.$$

One may use these images as frames in a video in order to get a crossfade effect.