Fractional Matchings and Covers
Observations

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January 30, 2021

This file is for observations about the fractional matchings and packings problem. Any comment or observation, small or large, is good. Unpolished presentations are quite fine. Put each new observation in its own section (using `\section{Descriptive title of observation (XY June 11)}`), with initials and date at the end of the title.

1 Conjecture for K^4's (AB June 11)

As I mentioned, for complete graphs on 4 vertices, i.e. K^4's the easy bound is $\nu \leq \tau \leq 6\nu$. Based on VERY few examples (see the "MATLAB.pdf" file), it seems that the following should true:

Conjecture 1.1. *$\tau \leq 3.5\nu$.*

In the case of K_8, we can have 2 edge disjoint copies of K^4 and need to delete 7 edges. So the above conjecture would be best possible.

As a start, it might be instructive just to look at complete graphs. I said it might be the case that this value is monotonically increasing until $n = 8$, and then starts decreasing. But this is not true (refer to the "Matlab.pdf" file). I was only able to check values of n up till 12. Maybe we can try to do this more efficiently and see what happens for larger values of n.

Here is another approach: For transversals, the problem is this: What is the best way to delete every K^4 from K_n. The answer is the following: partition the n vertices into 3 (roughly) equal size classes, and then delete the edges within each class. In fact, if we want to delete K_t for any $t \geq 2$, then we should do the same with $t - 1$ classes.

The corresponding matching problem is the following: What is the largest number of edge disjoint copies of K_t in K_n. I believe this is an open problem, but for $t = 3$ and 4 we know the answer. See [1]

For the fractional counterparts, we know the following:

$$\tau \leq 6\nu^* \quad \text{and} \quad \tau^* \leq 6\nu.$$

Can we improve this to anything better than 6? The conjecture here would be the same, i.e.

$$\tau \leq 3.5\nu^* \quad \text{and} \quad \tau^* \leq 3.5\nu.$$

For the fractional versions of the conjectures, I do not know if the 3.5 bound would be best possible. So for this, there are two questions:

1. Can we prove anything better than $\tau \leq 3.5\nu^*$ and $\tau^* \leq 3.5\nu$?

2. Is 3.5 best possible for the fractional numbers or could it be improved?

Note that while I talk about K^4's, the same problems could be studied for any K_t, $t \geq 3$. It doesn't make sense to ask the same question for $t = 2$ (why?). The case $t = 3$ has been studied a lot. There are some references in the writeup (and many more). We'll try to understand these along the way to see if similar approaches give anything for our problem. f

2 An approach to the fractional problem (AB June 13)

Problem 2.1. *Can we prove* $\tau^* \leq 3.5\nu$*?*

Here is an approach to a proof: Let G be a graph and let H be the hypergraph of K^4's in G. Let t be a minimum fractional transversal of H, so $\tau^* = |t|$. We want to find a large matching. (of size at least $\tau^*/3.5$)

Presumably one might try to go the other way. Start with a maximum matching ν and construct (small) a fractional transversal from this. The paper of Füredi in the references folder uses this approach for certain types of hypergraphs. Let's spend a few hours (or more) next week reading this paper.

Here is how we might get a handle on this: the idea is the following: We'll look at two cases: (i) $t(v) > 1/6$ for some vertex, or (ii), $t(v) = 1/6$ for every vertex, so $|t| = |V(H)|/6$.

In Case (i), one might look at the vertex v for which $t(v) > 1/6$. Then, presumably there is a *reason* that this vertex got a lot of weight. For example, maybe it is contained in few edges? Conversely, if a vertex gets little weight, it should be in many edges.

I suggest thinking about Case (ii) first. In this case, we should be able to argue that H must be regular (maybe assuming that its a minimal counterexample?) and I'm hoping we can say what these graphs look like. If this is doable, then we can explicitly say what a matching in such a graph should be.

Question 2.2. *So the question to think about is the following: We know that the hypergraph is* 6*-uniform. Can we argue that the hypergraph must be regular? (Maybe look at the minimal counterexample?) What can we say about the degrees of this regular graph?*

If we know that this hypergraph is d-regular, can we say what the original graph should look like? In terms of the original graph, this says that every edge is in the same number of K^4*'s.*

E.g. One way to get regular hypergraphs is for the original graph to be the complete graph. Or disjoint copies of the complete graph. We already "know" (based on checking the first 200 values) that the conjecture is true for complete graphs.

Are there other examples?

3 A proof that no projective planes are realizable as simple graphs of K^rs. (KD June 19)

Definition. An r-uniform *projective plane*, $\mathscr{P}_r$, is a hypergraph which satisfies the following:

(i) any two vertices fall in exactly one common edge,

(ii) any two edges share exactly one vertex,

(iii) any edge contains r vertices,

(iv) there are 4 vertices, no three of which lie in the same edge.

Theorem 3.1. *No $\mathscr{P}_r$ can be represented by a simple graph G of K^rs such that the vertices of $\mathscr{P}_r$ are edges of G and edges of $\mathscr{P}_r$ are K^rs in G.*

Corollary 3.2. *Any hypergraph that contains a $\mathscr{P}_r$ is not representable as a simple graph.* □

Proof of theorem. We argue by contradiction. Suppose there exists such a simple graph G which represents $\mathscr{P}_r$. Clearly, then, $\|G\| = |\mathscr{P}_r| = r(r-1)+1$. We pick any edge of G and one of its two endpoints, v. There are other edges in G, so this edge is incident with a K^r, thus, $d(v) \geqslant r-1$. We now choose any one of the $r-2$ edges incident with v which is not our original edge, say e, and some other edge which is not in our original K^r. By axiom (i), these two edges fall in another K^r, which shares only e with our original K^r. Hence, $d(v) \geqslant 2(r-1)-1$. However, these two K^rs supply only $2\binom{r}{2}-1$ of the $r(r-1)+1$ edges in G, so there is yet another edge, not in either of these K^rs, with which the same process may be repeated on any of the $2(r-1)-3$ edges incident with v not already chosen. Thus, $d(v) \geqslant 2(r-1)$. Since our original edge choice was arbitrary, this must be so for any v.

If $k := |V(G)|$, we recall the handshake lemma to deduce

$$\begin{aligned} 2k(r-1) &\leqslant \sum_{v \in V(G)} d(v) = 2r(r-1)+2 \\ \Rightarrow \quad k &\leqslant \left\lfloor r + \frac{1}{r-1} \right\rfloor \\ &\leqslant r. \end{aligned}$$

But from r vertices, we can construct at most $\binom{r}{2}$ edges, hence,

$$\|G\| \leqslant \binom{r}{2} = \frac{r(r-1)}{2} < r(r-1)+1 = \|G\|,$$

a contradiction. From thence it follows that there cannot exist such a G. □

4 Alternate Proof of Theorem 3.1 (EF June 22)

Proof of theorem. We try to construct a simple graph G which represents $\mathscr{P}_r$. Pick any vertex v in $\mathscr{P}_r$ and denote it as e in G. By definition of projective plane, any two vertices fall in exactly one common edge, and this implies that v is on exactly r edges in $\mathscr{P}_r$: each edge contains r vertices and any two edges share exactly one vertex, namely v. In other words, r distinct K^r intersects only at e in G. Now, we have $r(r-1)+1$ edges total on these distinct K^r's. Since $\mathscr{P}_r$ has $r(r-1)+1$ edges, we would have $r(r-1)+1$ K^r's in G; however, any new edges added would exceed $r(r-1)+1$ number of vertices in $\mathscr{P}_r$, so there exists no simple graph realized by $\mathscr{P}_r$.

□

5 Alternate Proof to Theorem 3.1 (CT June 23)

Proof. We will argue by construction. Let K_1^r be the first K^r set in our simple graph which represents the edge ϵ_1 of $\mathscr{P}_r$. Let v be a vertex of $\mathscr{P}_r$ which does not belong to ϵ_1. There are three ways v can be added to our simple graph:
Case 1. Both ends of the edge v are in K_1^r. This cannot happen since, any two edges fall in exactly one K^r.
Case 2. One end of the edge v falls in K_1^r and the other falls outside i.e v is adjacent to K_1^r. In this case, v is adjacent to at most $(r-1)$ edges of K_1^r. However, K_1^r has exactly $\binom{r}{2}$ edges, each of which v should be adjacent to. Hence, this cannot happen either.
Case 3. Both ends of the edge v lie outside of K_1^r. In this case, we can construct another K^r, call it K_2^r, which uses v and shares exactly one edge with K_1^r. However, this means that there exists another edge $v^\star$ which has one end in K_1^r and the other in v, which lies outside of K_1^r. Hence, we are back in case 2. $\square$

6 Outline of Approach (KD June 25)

Let the hypergraph $\mathscr{H}_r$ be defined from a simple graph G such that $V(\mathscr{H}_r) = E(G)$ and such that the edges of each $K_r \subseteq G$ form a hyperedge in $\mathscr{H}_r$. We then venture to find an upper bound for the fractional transversal number $\tau^*(\mathscr{H}_r) = \tau^*$ in terms of the matching number $\nu(\mathscr{H}_r) = \nu$. To do so, we first observe that we may reduce our search to hypergraphs where

$$|\mathscr{H}_r| \leqslant |V(\mathscr{H}_r)|,$$

since we need no more constraints than variables to find the optimal solution of a linear program. Then, since $\mathscr{H}_r$ is r-uniform,

$$d(\mathscr{H}_r) = \frac{1}{|V(\mathscr{H}_r)|}\sum_{v\in V} d(v) = \frac{r|\mathscr{H}_r|}{|V(\mathscr{H}_r)|} \leqslant r.$$

If we suppose equality holds, then one of the following is true.

(i) $\delta(\mathscr{H}_r) = r = \Delta(\mathscr{H}_r)$, or,
(ii) $\delta(\mathscr{H}_r) < r$.

7 Bounds on τ^* (EF June 25)

In this section, we discuss bounds on τ^* in respect to book of size 2 and 3 with $\upsilon = 1$ in a 6-regular, 6-uniform hypergraph.

Definition. A *book* of size r is a hypergraph with exactly r edges intersecting at exactly one vertex.

Book of size 2 We will start building edges from a book of size 2 and consider the sum of degree of vertices in $X = P_1 \cup P_2$, where P_i is the edge in the book. Notice that the sum of degree in X is equal to the sum of intersections of all edges with $P_1 \cup P_2$, we have:

$$\sum_{u\in X} d(u) = \sum_{e\in\mathscr{H}} |e\cap X| = 6\cdot 2 + \sum_{e\in\mathscr{H}\setminus\{P_1,P_2\}} |e\cap X|$$

Now, observe that only 8 pairs of edges in the simple graph G realized by the book of size 2 can form K^4 such that the K^4's share exactly 2 edges with G. In other words, only 8 edges in $\mathscr{H}$ can intersect P_1 and P_2 exactly 1 times each, and all other K^4 intersects G in at least 3 edges, i.e. a triangle; denote the 8 edges as e_i, $i = 1, ..., 8$, so:

$$\begin{aligned}\sum_{u\in X} d(u) = \sum_{e\in\mathscr{H}} |e\cap X| &= 6\cdot 2 + \sum_{e\in\mathscr{H}\setminus\{P_1,P_2\}} |e\cap X| \\ &= 6\cdot 2 + 8\cdot 2 + \sum_{e\in\mathscr{H}\setminus\{P_1,P_2,e_1,\ldots,e_8\}} |e\cap X| \\ &\geqslant 28 + 3(|\mathscr{H}| - 10)\end{aligned}$$

Since $|X| = 11$, $\sum_{u\in X} d(u) = 6\cdot 11 = 66$, and we get:

$$|\mathscr{H}| \leqslant \frac{38}{3} + 10 < 13 + 10 = 23$$

So $\tau^* \leqslant \dfrac{|\mathscr{H}|}{6} \leqslant \dfrac{22}{6} = 3.67$.

8 Restrictions on Intersections of Edges (KD, CT June 26)

We observe the following claim which is implied by the fact: $|e_i \cap e_j| \in \{0,1,3\}$ for any $i \neq j$.

Claim. $|\bigcap_{j=1}^{k} e_{i_j}| \in \{0,1,3\}$ for any $k \geqslant 2$ and $\{i_1,\ldots,i_k\} \subseteq [n]$.

Proof. Our proof is by induction on k. We suppose $k \geqslant 3$ and that the assertion holds for smaller k. Clearly, since the size of any pairwise intersection is at most three, the aggregate intersection any superset (of k edges, in our case) must also fall on or below three. Furthermore, the sets $e_1,\ldots,e_n$ are discrete, so the size of any intersection of any number of hyperedges is integral. Hence, it is sufficient to show $A_k := |\bigcap_{j=1}^{k} e_{i_j}| \neq 2$ since it is clear that the other possibilities, namely 0, 1, and 3, are attainable.

Let us suppose, then, that $A_k = 2$. It must be so that $S_J := |\bigcap_{j\in J} e_{i_j}| = 3$ for all $J \subset \{e_{i_1},\ldots,e_{i_k}\}$, since $S_J \geqslant 2$ if $A_k \geqslant 2$ and $S_J \neq 2$ by hypothesis. Thus, e_{i_k} intersects every other hyperedge in distinct hypervertices (since 2 hypervertices fall in the aggregate intersection and it must intersect every other edge in 3). Hence 2 hypervertices of e_{i_k} are in a single K^4, and (at least) 2 of the remaining hypervertices fall in distinct K^4s. But this would already produce 5 vertices in the simple graph, which cannot be because e_{i_k} is a collection of edges from a K^4 by definition. We have hence proven that $A_k \neq 2$. $\square$

9 Proof attempt for book size 2 (AB June 27)

Heres an attempt to make sure what I said makes sense.

As always H is a subgraph of the hypergraph of triangles in G with $|H| \leq |V(H)|$.

Suppose the largest size of a book in H is 2. Call the pages p_1 and p_2, with spine s. Now p_1, p_2 and s correspond to a subgraph of G with 11 edges and 6 vertices.

In the case that G has only 6 vertices, G can only be K_6.

Otherwise G has at least one more vertex (i.e., verteices not involved in the book). Each vertex must have degree at least 3, and each of this edges must participate in a K^4 (otherwise deleting this edge doesn't change τ or ν etc).

Suppose G has exactly one more vertex v not in the book. Then this vertex must have degree at least 3 each edge must be in some K^4, and in particular, three edges must be in the same K^4 (call this K). I claim there are only two ways (up to isomorphism) that K can happen. K must contain v, so it can't contain two non-adjacent edges that are in the book. Either the edges are NOT the spine and K contains one vertex from each page; or one of the edges is the spine (in which case) K contains a triangle from the book.

Now suppose that G has more than one vertex not in the book. I claim that again, there should be a vertex v and three edges such that the subgraph looks like one of the two cases described above. Let u, v be the vertices of G not in the book. Again, both contribute at least one K^4 and have degree at least 3. If u and v are adjacent, then they must be in the same K^4 (remember, we're just looking at a subgraph of G AND a subset of K^4's in this subgraph). But that means that u and v must be adjacent to the same two vertices of the book that are adjacent (in the subgraph we're looking at). Then either this new K^4 contains the spine, in which case we have a book of size 3 or we have matching number 2.

So we may assume that u and v are not adjacent. But now pick any one of them. And we're back in the case when G had size 7.

Now, I would like to show that any H which has one of the two subgraphs from the 7 vertex case can't be regular (or that it can't get too big and still have matching number one).

All of this needs to be checked very carefully!

10 Ruling out the three page case (KD July 3)

Theorem 10.1. *Let $\mathscr{H}$ be the hypergraph of K^4s in a simple graph G. If $\mathscr{H}$ is 6-regular, 6-uniform, and has matching number $\nu(\mathscr{H}) = \nu = 1$, then $|G| \neq 8$.*

Proof. Suppose the contrary for the sake of contradiction. Any graph on 8 vertices may be produced by a sequence of edge deletions, $K^8 = G_0 \supseteq \cdots \supseteq G_n = G$, where $G_i = G_{i+1} - e_i$ for some $e_i \in E(G_{i+1})$.

We first observe that every vertex from K^8 must have at least one incident edge deleted. Suppose instead there is a vertex v with $d_G(v) = 7$. Since $\mathscr{H}$ is 6-regular, all 7 edges incident with v fall in a K^4. Any three of these edges form a K^4 (since $\nu = 1$, there cannot be K^4s which are edge disjoint), if we thence pick any three edges, we have found a K^4. But 4 more edges are incident with v, any three of which forming a K^4. Such a K^4 would be edge disjoint with the previous one, thus, $\Delta(G) \leqslant 6$.

We now show that no vertices can have degree smaller than 5, and, moreover, that at least one vertex of degree 5 exists. Suppose there were a vertex v with $d_G(v) \leqslant 4$. Any edge incident with v falls in 6 K^4s, and, furthermore, all such K^4s must include v, and so must be formed with edges incident with v. Hence, we count all K^4s by the number of ways of picking any 2 other edges incident with v (so $d_{K^4}(v) = 3$), namely,

$$\binom{d_G(v) - 1}{2} < 6.$$

Heretofore, $\delta(G) \geqslant 5$. But suppose every vertex in G has degree 6. Also suppose that the edge deleted from each of $N(v)$ is between neighbors of v, because it cannot hurt our argument if this is not so. Let $\{v_1, \ldots, v_6\} = N(v)$ and, without loss of generality, suppose the edges v_1v_2, v_3v_4, and v_5v_6 are deleted. Since each of $N(v)$ has degree 6, these are the only deleted edges among the neighbors of v. We can thus produce two edge-disjoint K^4s on (v, v_1, v_3, v_5) and (v, v_2, v_4, v_6), a contradiction. Therefore there is some vertex v with degree 5.

This vertex v has all incident edges falling in 6 K^4s if and only if $G[N(v) \cup \{v\}] = K^6$, since

$$\binom{d_G(v)-1}{2} = 6,$$

so all neighbors of v must be pairwise adjacent or otherwise not all 6 possible K^4s could be produced. It hence suffices to show that an edge between neighbors of v must be absent. If not, then all the deleted edges incident with neighbors of v must be incident also with vertices not adjacent to v. Since $|G| = 8$ and $d_G(v) = 5$, there are 2 such vertices, u and w. By pigeonhole, one of u or w must have at least 3 more edges deleted, which cannot be since $\delta(G) \geqslant 5$. It then follows that at least one edge ab for $a, b \in N(v)$ is deleted and the theorem is proven. □

11 Possible generalization to n vertices (CT July 2)

Theorem 11.1. *Let e be the number of edges needed to be deleted from a K^n to preserve $\nu = 1$. If $e > n(n-6)/2$ then we cannot have a 6-uniform, 6-regular hypergraph of K^4's on n vertices.*

(**Basit says:** Isn't the following another (easier to understand) way to say this: If $|E(G)| < \binom{n}{2} - n(n-6)/2 = 5n/2$, then the corresponding hypergraph can not be regular?)

Proof. We know from the previous proof that $d_G(v) \geqslant 5$ is a necessary condition for our hypergraph. Also, deleting one edge in the graph reduces the sum of the degrees by two.
Hence:

$$\sum d_{K^n}(v) - 2e \geqslant 5n$$
$$\iff n(n-1) - 2e \geqslant 5n$$
$$\iff e \geqslant n(n-6)/2$$

□

Theorem 11.2. *Let $\mathscr{H}$ be the hypergraph of K^4's in a simple graph G. If $\mathscr{H}$ is 6-regular, 6-uniform, and has matching number $\nu(\mathscr{H}) = 1$, then $|V(G)| \neq 8$.*

Proof. Suppose that the statement of the theorem is false.

We would like to argue that $\Delta(G) \leqslant 6$. I'm still not sure how to do this.

Let v be a vertex with $d_G(v) \leq 4$, and e be an edge incident to v. Note e can be contained in at most $\binom{d(v)-1}{2} < 6$ K^4's, contradicting the assumption of the theorem. It follows that $\delta(G) \geq 5$.

Now, suppose that every vertex in G has degree 6. Consider the complement $\overline{G}$ of G, and note that every vertex has degree 1, i.e., $\overline{G}$ must be a perfect matching. Let $v_1, \ldots, v_6$ be the neighbors of v in G, and, without loss of generality, suppose that the edges v_1v_2, v_3v_4, and v_5v_6 are missing in G. But now the two sets v, v_1, v_3, v_5 and v, v_2, v_4, v_6 induce edge-disjoint K^4's in G.

To sum up, we have shown that $\Delta(G) \leq 6$, $\delta(G) \geq 5$, and there exists a vertex with degree 5. Let v be a vertex of degree 5, and u, w be vertices not adjacent to v. Each edge incident to v must be contained in exactly 6 K^4's, which implies that $G[N(v) \cup \{v\}] = K_6$. Then, since $\Delta(G) \leq 6$, each vertex in $N(v)$ is adjacent to at most one of u, w. But then either $d(u) < 5$ or $d(v) < 5$, which is a contradiction. $\square$

12 Possible Proof of K_6 being the only simple graph that is 6 regular and 6 uniform with $\nu = 1$ (EF July 4)

Let v be a vertex in a graph G and let v_i denotes $N(v)$. Since by assumption, vv_i is 6 regular, it forms K^4 with other vv_j, so there exist edges v_iv_j. We consider the number of K_3's restrict to $N(v)$ such that a given edge v_iv_j is in.

Without lost of generality, we let the edge v_iv_j be v_1v_2 in the following proofs, and use the vertices to indicate K^4.

Lemma 12.1. *v_iv_j cannot be in 6 K_3's on $N(v)$.*

Proof. If v_1v_2 is in 6 K_3's on $N(v)$, then v has at least 8 neighbors, and$\forall v_i, v_j, i, j \in 3, 4, ..., 8$, v_i and v_j are not adjacent, because otherwise vv_1 is in more than 6 K^4's. Now, if vv_3 is in 6 K^4's, it make no use of $vv_i, i = 1, 2, 4, 5, ..., 7$, so $\exists v_9, v_j, j \neq 1, 2, 4, 5, ..., 7$ such that $vv_3v_9v_j$ is a K^4. However, this gives $\nu > 1$ since $vv_1v_2v_4$ is a K^4. So v_1v_2 cannot be in 6 K_3's on $N(v)$. $\square$

Lemma 12.2. *v_iv_j cannot be in 5 K_3's on $N(v)$.*

Proof. By above lemma, if v_1v_2 is in 5 K_3's on $N(v)$, then v has at least 7 neighbors, and there exists at most one $v_iv_j, i, j = 3, 4, 5, 6, 7$. If no such v_iv_j exists, then either (i) $\exists v_8, v_9$ such that $vv_1v_8v_9$ is a K^4. Now, vv_2 cannot use v_8 or v_9 to make a K^4 since otherwise $vv_1v_2v_i, i = 8, 9$ is a K^4 and now v_1v_2 is in 6 K_3's on $N(v)$, contradicts to the above lemma. Or (ii) $\exists v_8$ and $vv_1v_8v_i, i = 3, ..., 7$ forms a K^4. Similarly, vv_2 cannot make use of v_8 to form a K^4, so $\exists v_9$ such that $vv_2v_9v_j, j = 3, ..., 7$ forms a K^4. If v_j is a new vertex, then this contradicts $\nu = 1$. Now, consider vv_3, vv_3 is in at most 3 K^4's if the above $i, j = 3$. Since now vv_1, vv_2 are in 6 K^4's and by assumption no $v_iv_j, i, j = 3, ..., 7$ exists, if vv_3 makes use of v_8, v_9 to form K^4, then $vv_1v_3v_i, i = 8, 9$ is a K^4, and now v_1v_2 is in 6 K_3's on $N(v)$, a contradiction. If vv_3 makes use of two new vertices, then $\nu > 1$.

Now, without lost of generality, assume $v_3v_i, i = 4, ..., 7$ exists. Now, vv_3 is in 3 K^4's, and vv_1, vv_2 are in 6 K^4's. So $\exists v_8, v_9$ such that $vv_3v_8v_9$ is a K^4, but this gives $\nu > 1$ since $vv_1v_2v_4$ is an edge disjoint K^4. So v_iv_j cannot be in 5 K_3's on $N(v)$. $\square$

Lemma 12.3. *v_iv_j cannot be in 4 K_3's on $N(v)$.*

Proof. By above lemma, if v_1v_2 is in 4 K_3's on $N(v)$, then v has at least 6 neighbors, and there exists at most two $v_iv_j, i, j = 3, 4, 5, 6$.
If exists 2 $v_iv_j, i, j = 3, 4, 5, 6$ that share a vertex, then vv_3 is in at most 5 K^4's and vv_1, vv_2 are in 6 K^4's. Assume the two edges to be v_3v_4, v_3v_5 without lost of generality. So $\exists v_7$ such that $vv_3v_7v_i, i \neq 6$ is a K^4, but now $\nu > 1$.
If exists one $v_iv_j, i, j = 3, 4, 5, 6$, without lost of generality, assume it to be $v_3v_i, i = 4, 5, 6$. Now, vv_1, vv_2 are in 5 K^4's. Again, without lost of generality, if $\exists v_7$ such that $vv_1v_6v_7$ is a K^4, then vv_2v_7 cannot be in a new K^4 because otherwise v_1v_2 is in 5 K_3 on $N(v)$, contradicts to the above lemma. If $\exists v_8$ such that $vv_2v_8v_6$ is a K^4, consider vv_3. vv_3 cannot form K^4 with both v_7, v_8 because otherwise $\nu > 1$. If vv_3 forms a K^4 with at most one of v_7, v_8, and since only one v_3v_i exists, there must exist at least one new vertex v_9 such that vv_3v_9 is in a K^4, again this gives $\nu > 1$. Now, if $\exists v_7, v_8$ such that $vv_1v_7v_8$ is a K^4 and by the previous lemma, $\exists v_9$ such that $vv_2v_9v_i, i \neq 7, 8$ is a K^4, but now $\nu > 1$.
If there exists two v_iv_j that do not share a vertex, assume them to be v_3v_4, v_5v_6. Now, in order to let vv_3 be in 6 K^4's, $\exists v_7$ such that $vv_3v_7v_i, i \neq 5, 6$ is a K^4. But now $\nu > 1$.
So we may assume that no $v_iv_j, i, j = 3, 4, 5, 6$ exists. If $\exists v_7$ such that $vv_1v_6v_7$ is a K^4, then by above lemma, vv_2v_7 cannot form a K^4, so $\exists v_8$. If $\exists v_9$ such that $vv_2v_8v_9$ is a K^4, then $\nu > 1$. So we assume that $vv_2v_8v_6$ forms a K^4. Now, vv_1, vv_2 are in 5 K^4's. Since vv_1v_8 cannot be in a K^4 because otherwise we would have v_1v_2 in 5 K_3's on $N(v)$, and a new vertex v_9 such that $vv_1v_7v_9$ is a K^4 gives $\nu > 1$, we may assume that $vv_1v_7v_i, i = 3, 4, 5$ is a K^4, but now $vv_2v_6v_8$ is another edge disjoint K^4.
So v_iv_j cannot be in 4 K_3's on $N(v)$. □

Theorem 12.4. *K_6 is the only simple graph that is 6 regular and 6 uniform with $\nu = 1$.*

Proof. By above lemmas, v_1v_2 is in 3 K_3's on $N(v)$, so v has at least 5 neighbors.
If no $v_iv_j, i, j = 3, 4, 5$ exists, and if $\exists v_6, v_7$ such that $vv_1v_6v_7$ is a K^4, vv_2 must not use v_6, v_7 to form a K^4 because otherwise it contradicts the above lemma. So $\exists v_8$ such that $vv_2v_8v_k, k \neq 6, 7$ is a K^4. But now $vv_1v_6v_7$ and $vv_2v_8v_k, k \neq 6, 7$ are two edge disjoint K^4's. So we can now assume that $vv_1v_6v_5$ forms a K^4 and $vv_2v_7v_5$ forms a K^4 because otherwise we get $\nu > 1$. Now, consider vv_3, in order to keep $\nu > 1$, vv_3 must make use of v_6, v_7 and $vv_3v_6v_i, i = 1, 2$ cannot both exist because now v_1v_2 is in 4 K_3's on $N(v)$. So now vv_3 is in at most 5 K^4's and we assume them to be with v_1, and since no $v_iv_j, i, j = 3, 4, 5$ exists, $\exists v_8$ such that $vv_3v_8v_1$ forms a K^4. We get $\nu > 1$ as $vv_2v_5v_7$ is an edge disjoint K^4.
Without lost of generality, assume only v_3v_4 exists, vv_1, vv_2 needs 2 more K^4 to be 6 regular. If $\exists v_6$ such that $vv_1v_5v_6$ forms a K^4, then v_2v_6 cannot be in a K^4 since otherwise v_1v_2 is in 4 K_3's on $N(v)$, contradicts to the above lemma. So $\exists v_7$ such that $vv_2v_7v_i, i \neq 6$ is a K^4. If $i = 5$, and since vv_2 needs one more K^4, exists $v_j, j \neq 5, 6$ such that $vv_2v_7v_j$ is a K^4. Now, $\nu > 1$ as $vv_1v_6v_5$ is a K^4.
If $\exists v_6, v_7$ such that $vv_1v_6v_7$ is a K^4. Similarly, vv_2 makes no use of v_6 or v_7 to form a K^4, so $\exists v_i, v_j, i, j \neq 6, 7$ such that $vv_2v_iv_j$ is a K^4,again $\nu > 1$.
Now, without lost of generality, if $\exists v_3v_4, v_3v_5$, then vv_1, vv_2 needs one more K^4. If $\exists v_6$ such that $vv_1v_5v_6$ is a K^4, and since vv_2 makes no use of v_6 to form K^4 by above lemma, there exists v_7 such that $vv_2v_7v_i, i \neq 6$ is a K^4.
If $i = 5$, then vv_2 is in 6 K^4's and we consider vv_3. vv_3 is in 5 K^4's now, which are $vv_1v_2v_3, vv_1v_3v_4$, $vv_1v_3v_5, vv_2v_3v_4, vv_2v_3v_5$, and vv_3 makes no use of vv_1, vv_2, vv_6, vv_7 to form K^4 since otherwise $vv_1v_2v_4$ is always an edge disjoint K^4. So $vv_3v_jv_k, j, k \neq 1, ..., 7$ is a new K^4, which gives $\nu > 1$.
So now $i \neq 5$,then $\exists vv_2v_7v_i, i \neq 5$ is a K^4, which is edge disjoint to $vv_1v_5v_6$, a contradiction.
If v_iv_j is in at most 2 K_3's on $N(v)$, then v has 4 neighbors, and consider the following 2 conditions.
If edge v_3v_4 exists, we get a K_5, and if there exists v_5 such that $vv_1v_4v_5$ forms a K^4, edge v_1v_4 is

in 3 K_3's on $N(v)$, and same argument follows as above. If $\exists v_5, v_6$ such that $vv_1v_5v_6$ is a K^4, then $\nu > 1$. So we assume that edge v_3v_4 does not exist. Again similar arguments show that this cannot happen as either we get $\nu > 1$ or some v_iv_j in 3 K_3's on $N(v)$. Thus, the 6 regular and 6 uniform hypergraph is only achieved when all v_3v_4, v_3v_5, v_4v_5 exist, which gives a K_6. □

13 Proof that if $\nu = 1$ and $\delta(\mathscr{H}) \geq 6$, then $G = K_6$. $\nu = 1$(EF July 6)

Theorem 13.1. *Let G be a graph. Suppose that the maximum size of a K^4-packing in G is 1, and that every edge is in at least 6 K^4's. Then $G = K_6$.*

Corollary 13.2. *Let $\mathscr{H}$ be a subgraph of the hypergraph of K^4's in G. Suppose $\mathscr{H}$ is 6-regular, and that $\nu(\mathscr{H}) = \nu(\mathscr{H}_0) = 1$. Then $G = K_6$.*

Proof. Let $\mathscr{H}_0$ be the hypergraph of K^4's in G. By assumption, $\mathscr{H} \subseteq \mathscr{H}_0$. For every vertex v, we have that $d_{\mathscr{H}_0}(v) \geq d_{\mathscr{H}}(v) = 6$. Now, by the theorem applied to $\mathscr{H}_0$, G must be K_6. □

We now prove the theorem.

Proof. Let G be a graph and $v \in V(G)$ be an arbitrary vertex of G. Suppose that $N(v) = \{v_1, v_2, \ldots, v_t\}$. Every edge of the form vv_i is contained in some K^4, and every such K^4 contains an edge of the form vv_j, with $i \neq j$. It follows that there exists at least one edge in $G[N(v)]$. Without loss of generality, suppose $v_1v_2 \in E(G)$. □

Definition. Let $T(e)$ denote the *triangle number* of $e \in E(G[N(v)])$, that is, the number of triangles in $G[N(v)]$ containing e. If A and B are K^4s, let $A \perp B$ indicate that they are edge-disjoint. Let a sequence $v_iv_jv_kv_\ell$ of vertices in G refer to the K^4 which is $G[\{v_i, v_j, v_k, v_\ell\}]$, $a \sim b$ indicate $ab \in E(G)$, and $a \not\sim b$ indicates $ab \notin E(G)$.

Lemma 13.3. $T(v_1v_2) \leqslant 3$.

Proof. Suppose $T(v_1v_2) \geqslant 4$, and that vertices in $X = \{v_3, v_4, \ldots, v_s\}$, $s \geq 6$, are the third vertices of these triangles.

Note that $G[X]$ cannot contain a triangle nor P^k for $k \geqslant 3$. Indeed, if there is a triangle, say v_3, v_4, v_5, then $v_1v_3v_4v_5 \perp vv_1v_2v_6$, that is, $\nu(\mathscr{H}) > 1$, a contradiction. If there is a path of length three, say v_3, v_4, v_5, v_6, then the sets $vv_1v_3v_4 \perp vv_2v_5v_6$, and, again, $\nu > 1$. It follows that $G[X]$ is acyclic, that is, a forest.

If $G[X]$ contains at least one edge, then there is a vertex, say v_3, with $d_{G[X]}(v_3) = 1$. Suppose that $v_3 \sim v_4$. v_3 can be in at most 3 K^4's that include one of v, v_1, v_2, or v_4. Since vv_3 is in at least 6 K^4's, there exists a vertex p such that v, v_3, and p are contained in a K_4. Note that $p \not\sim v_4$, since then $vv_3v_4p \perp vv_1v_2v_5$. Hence, vv_4 can be in at most 5 K^4s, so there must exist a vertex q in a K^4 with v, v_4. Likewise, another vertex r is in a K^4 with v, v_5. Clearly, if vv_3pv_i, vv_4qv_j, and vv_5rv_k are such K^4s, then $v_i, v_j, v_k \in X$. Furthermore, $i = j = k \in \{1, 2\}$ since $i = j$ or else $vv_3pv_i \perp vv_4qv_j$, and v_3 is not adjacent to v_5 nor r, so $k = i$. Let the common value be 1. Then $v_1v_2v_3v_4 \perp vv_5rv_1$ and $\nu > 1$, a contradiction.

If $E(G[X]) = \varnothing$, we pick a vertex in $G[X]$, say v_3. Then vv_3 is in only 1 K^4. Since there are no edges in $G[X]$, there is another vertex, say v_i, which forms a K^4 with vv_3. v_i must be connected to exactly one of v_1, v_2 or else $v_1v_2v_i$ would form another triangle with v_1v_2 and $v_3 \sim v_i$, which contradicts our assusmption on $E(G[X])$. Without loss of generality, let this vertex be v_1. v_3v_i must be in a K^4 other than $v_iv_3vv_1$, say, $v_iv_3v_kv_j$. If neither of v_k, v_j is equal to v or v_1 then

$v_iv_3v_kv_j \perp vv_1v_2v_4$. Assume, then, $v_k = v$. Since $j \neq 1, 2$, $v_iv_3vv_j \perp vv_1v_2v_4$. Similarly for $v_k = v_1$. In any case, $\nu > 1$. This completes the proof.

□

Lemma 13.4. *$T(e) = 3$ for every $e \in E(G[N(v)])$.*

Proof. Since our choice of v_1v_2 was arbitrary, any $e \in E(G[N(v)])$ has $T(e) \leqslant 3$ by Lemma 13.3. We assume $T(e) \leqslant 1$ and do not consider the trivial case, that is, we suppose there is an edge v_1v_2 with $T(v_1v_2) = 1$. We know that there is at least one other K^4 on vv_1 which isn't $vv_1v_2v_3$, call it $vv_1v_jv_k$, where $j, k \neq 2$. Similarly, there exists another K^4 on vv_2 which we denote by $vv_2v_mv_n$ where $m, n \neq 1$. Then $vv_1v_jv_k \perp vv_2v_mv_n$ unless $j = m$. Now, vv_2 is in yet another K^4, say $vv_2v_av_b$, where $a, b \neq \{1, j, k\}$. Then $vv_2v_av_b \perp vv_1v_jv_k$.

Now assume $T(e) \geqslant 2$ for all $e \in E(G[N(v)])$, and, moreover, that there is some edge, v_1v_2, with $T(v_1v_2) = 2$. We label the third vertices of these triangles v_3 and v_4. There is at least one other K^4 on vv_1 which isn't $vv_1v_2v_3$, say $vv_1v_jv_k$. Similarly, there exists another K^4 on vv_2 which we denote by $vv_2v_mv_n$ where $m, n \neq 1$. Then $vv_1v_jv_k \perp vv_2v_mv_n$ unless one of j, k is equal to one of m, n. Suppose, then, $j = m = 3$. But now the edge vv_n is in fewer than 6 K^4s, so there are vertices v_α, v_β and a K^4 $vv_nv_\alpha v_\beta$. If $\alpha, \beta \notin \{2, 3\}$, then clearly $vv_nv_\alpha v_\beta \perp vv_1v_2v_3$, and if at least one of $\alpha, \beta \in \{2, 3\}$, say α, then either $vv_nv_2v_\beta \perp vv_1v_3v_k$ or $vv_nv_3v_\beta \perp vv_1v_2v_4$. In either case, $\nu > 1$. □

Lemma 13.5. *$G[N(v)] = K^5$.*

Proof. By Lemma 13.4, we may assume every edge $e \in E(G[N(v)])$ has $T(e) = 3$. Let $X := \{v_3, v_4, v_5\}$ denote the set of third vertices of the triangles on v_1v_2. If $G[X]$ is complete, then clearly $G[N(v)] = K^5$. Suppose, then, that some pair of vertices are not adjacent in $G[X]$, say v_3 and v_4. $T(v_1v_3) \leqslant 2$, where equality holds if $v_3 \sim v_5$. If indeed $v_3 \sim v_5$, then another vertex, $v_j \in N(v)$ must form a triangle with v_1v_3. If $v_4 \sim v_5$, then $vv_1v_3v_j \perp vv_2v_4v_5$, otherwise, $T(v_2v_5) \leqslant 2$ and so there is a vertex $v_k \in N(v)$ forming a triangle with v_2v_5, and therefore a K^4 $vv_2v_5v_k \perp vv_1v_3v_j$.

If $v_3 \nsim v_5$, then $T(v_1v_3) = 1$ and there is a vertex $v_m \neq v_5$ forming a triangle with v_1v_3. Exactly as above, either $vv_1v_3v_m \perp vv_2v_4v_5$ or $vv_1v_3v_m \perp vv_2v_5v_k$. □

Proof of Theorem. For arbitrary $v \in N(v)$, $G[N(v)] = K^5$ by Lemma 13.5. That $G[N(v)] + v = K^6$ follows immediately. □

14 A map for presentations (AB July 11)

Here's a possible outline of presentations on this problem. For the introductary material, the initial handout (the file `matching-permutations.pdf`) will be a good source.

good introduction is vitally important to a sucessful presentation. When you are presenting your mathematical thoughts to a general audience, you must remember that for the most part the audience will not be as familar with the problem you've been thinking about as you are, and indeed will almost certainly be *much less* familiar with it. That means that it is unreasonable to expect that the majority of your listeners will be able to follow the fine details of proofs that you present. If you can convey clearly what the question is, and give a sense of what progress you've made, you should consider that a successful presentation.

So, to the outline:

- Describe a matching and transversal (or cover) in a graph. Specifically as 0/1 weights on the edges/vertices. Describe a fractional matching and cover in a graph. It'll be good to do a bunch of examples here. E.g. good examples would be complete graphs, cycles, bipartite graphs.

- Describe a hypergraph, and (fractional) matchings transversals in hypergraphs. Again its good to follow the writeup. Again, do some examples (complete hypergraphs, regular hypergraphs, etc).

  Maybe also talk about the greedy algorithm and the bound?

- NOW we can talk about Tuza's conjecture and how it fits into the framework. Talk about the generalization we are looking at. I.e. the question about K_4. Say what we expect to be true and why: So specifically: For general τ and ν we think K_8 is the best possible. So $\tau \leq 3.5\nu$. For the fractional case of $\tau^* \leq 2.5\nu$ we expect K_6 to be tight. We haven't talked about the last case at all $\tau \leq c\nu^*$, so ignore this...

- Talk about our approach to the conjecture: Saying that we can reduce to the case when the average degree must be at most 6, etc... Then we can restrcit into even more specific cases (regular vs not-regular etc)...

- What we have shown so far... VERY rough idea of the proof (don't spend more than 5 minutes on this).

- What we plan to do after this (if anything at all)... So work on the non-regular case for $nu = 1$. Try to generalize to arbitrary nu etc...'

  This should already be MORE than enough... When we meet tomorrow, we can talk about how to split things up...

15 Roadmap for proof of $\tau^* \leq 2.5\nu$ (AB July 13)

Let G be a graph, and suppose that the largest size of a K_4-packing in G is k. Let $\mathcal{F}$ be the hypergraph of K_4's in G. Note that $\nu(\mathcal{F}) = k$. We show that there is a fractional transversal of $\mathcal{F}_4$ of size at most $2.5k$ by induction on k.

By Proposition **??**, it suffices to consider $\mathcal{H} \subseteq \mathcal{F}$, with $|\mathcal{H}| \leq |V(\mathcal{F})|$. This implies that

$$\overline{d}(\mathcal{H}) = \frac{\sum_{v \in V(\mathcal{H})} d(v)}{|V(\mathcal{H})|} = \frac{6|\mathcal{H}|}{V(\mathcal{H})} \leq 6. \tag{1}$$

Case 1: *There exists $x_0 \in V(\mathcal{H})$ with $d(x_0) = k < 6$.*

Let $\mathcal{H}_0 = \{e \in \mathcal{H} : x_0 \in e\} = \{e_1, e_2, \ldots, e_t\}$, and $\mathcal{H}_i = \{e \in \mathcal{H} : e \cap e_i = \varnothing\}$, $0 \leq i \leq t$. That is, $\mathcal{H}_0$ is the set of edges containing x_0, and $\mathcal{H}_i$ is the set of edges that do not intersect the i^{th} edge in $\mathcal{H}_0$. Observe that that $\nu(\mathcal{H}_i) \leq k - 1$.

Question 1: Can we construct a small transversal of h_0? The nice thing here is that everything is centered around x_0, so hopefully should be easy to analyze. Maybe start with thinking what happens when $k = 1$.

If $k = 1$, then every edge intersects all of the e_i's, so all of the $\mathcal{H}_i$'s are empty. Thus, we are done.

Otherwise, if $k > 1$, by the inductive hypothesis, there exists a transversal of t_i of $\mathcal{H}_i$ of size at most $2.5(k-1)$. In this case, the hope is to combine the transversal of $\mathcal{H}_0$ constructed above with the transversals obtained via induction in a *nice* way.

Case 2: $min_{x \in V(\mathcal{H})}\{d(x)\} \geq 6$.

Then by (1), $\mathcal{H}$ is 6-regular, so $|V(\mathcal{H})| = |\mathcal{H}|$. Let e_0 be an arbitrary edge. Let $\mathcal{H}_0 = \{e \in \mathcal{H} : e \cap e_0 \neq \varnothing\}$, and $\mathcal{H}_1 = \mathcal{H} \setminus \mathcal{H}_0$. That is, $\mathcal{H}_0$ is the set of edges that intersect e_0, and $\mathcal{H}_1$ is the set of edges that do not intersect e_1.

Question 2: Can we argue that both $\mathcal{H}_0$ and $\mathcal{H}_1$ are 6-regular? I don't see how to do this in a nice way. But it would imply, by induction, that G is the disjoint union of K_6's.

Otherwise, here's a different approach:

Question 3: What can we say about $\mathcal{H}_0$? Can we argue that $\tau^*(\mathcal{H}_0) \leq 2.5$ or that $|h_0| \leq 15$? Again, when $k = 1$, this was sort of the approach I was suggesting with books etc.

Assuming we can do question 3: We know that $\nu(\mathcal{H}_0) \leq k - 1$, so by the inductive hypothesis, $\tau(h_0) \leq 2.5(k-1)$. This combined with the question above would finish this case.

References Cited

[1] A. E. Brouwer, Optimal Packings of K^4's into a K_n, *Journal of Combinatorial Theory, Series A*, 26(3), pp.278-297.