# 1 Setup

Let $K : \mathbb{T} \to \mathbb{R}$ be an even low-pass kernel of cutoff frequency $f_c$ with $K(0) = 1$, and $t_1, \ldots, t_s \in \mathbb{T}$ with minimum separation $\Delta$. Write $|t_i - t_j|$ for the wraparound distance between $t_i$ and $t_j$, so that $K(t_i - t_j) = K(|t_i - t_j|)$.

Some quantities on $K$ that we will need to estimate: suppose that $\Delta$ is fixed, then we define

$$
\begin{aligned}
M(c) &= \sup_{|t| \geq c\Delta} |K(t)| \\
S_p(t) &= \sup_{\{t_i\}} \sum_{t_j \in T \setminus \{t_0\}} |K(t - t_j)|^p \\
S(t) &= S_1(t) \\
T_p(t) &= \sup_{\{t_i\}} \sum_{t_j \in T \setminus \{t_{-1}, t_0, t_1\}} |K(t - t_j)|^p \\
T(t) &= T_1(t)
\end{aligned}
$$

Write the matrix $\boldsymbol{K} = (K(|t_i - t_j|))_{i,j=1}^s$, then we want to estimate $\boldsymbol{K}^{-1}$.

# 2 Outline

Suppose we are interested in estimating the second column of $\boldsymbol{K}^{-1}$. We have two goals: (1) get good estimates of the first, second, and third entries, and (2) establish that the remaining entries down the column decay.

We expand into a block matrix

$$
\boldsymbol{K} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{B}^\top & \boldsymbol{C} \end{bmatrix}, \tag{1}
$$

with $\boldsymbol{A} \in \mathbb{R}^{3 \times 3}$, $\boldsymbol{B} \in \mathbb{R}^{3 \times (s-3)}$, and $\boldsymbol{C} \in \mathbb{R}^{(s-3) \times (s-3)}$. Inverting the block matrix, we find that the first three columns of the inverse are given by

$$
\boldsymbol{K}^{-1} = \begin{bmatrix} \boldsymbol{A}^{-1} + \boldsymbol{A}^{-1}\boldsymbol{B}(\boldsymbol{K}/\boldsymbol{A})^{-1}\boldsymbol{B}^\top \boldsymbol{A}^{-1} & \cdots \\ -(\boldsymbol{K}/\boldsymbol{A})^{-1}\boldsymbol{B}^\top \boldsymbol{A}^{-1} & \cdots \end{bmatrix}, \tag{2}
$$

where $\boldsymbol{K}/\boldsymbol{A} = \boldsymbol{C} - \boldsymbol{B}^\top \boldsymbol{A}^{-1} \boldsymbol{B}$ is the Schur complement.

# 3 Computing main contribution

We will use the explicit formula for the inverse of $\boldsymbol{A}$,

$$
\boldsymbol{A}^{-1} = \frac{\mathsf{adj}(\boldsymbol{A})}{\det(\boldsymbol{A})},
$$

where (using symmetry and that $\mathsf{diag}(\boldsymbol{A}) = \mathbf{1}$ to simplify)

$$\det(\boldsymbol{A}) = 1 - K(|t_2 - t_3|)^2 - K(|t_1 - t_3|)^2 - K(|t_1 - t_2|)^2 + 2K(|t_1 - t_3|)K(|t_2 - t_3|)K(|t_1 - t_2|)$$

$$\begin{aligned}\mathsf{adj}(\boldsymbol{A}) &= 2\boldsymbol{I}_3 - \boldsymbol{A} + \begin{bmatrix} -K(|t_2 - t_3|)^2 & K(|t_1 - t_3|)K(|t_2 - t_3|) & K(|t_2 - t_3|)K(|t_1 - t_2|) \\ K(|t_1 - t_3|)K(|t_2 - t_3|) & -K(|t_1 - t_3|)^2 & K(|t_1 - t_3|)K(|t_1 - t_2|) \\ K(|t_2 - t_3|)K(|t_1 - t_2|) & K(|t_1 - t_3|)K(|t_1 - t_2|) & -K(|t_1 - t_2|)^2 \end{bmatrix} \\ &= 2\boldsymbol{I}_3 - \boldsymbol{A} + \boldsymbol{D}(\boldsymbol{J}_3 - 2\boldsymbol{I}_3)\boldsymbol{D}, \text{ where} \\ \boldsymbol{D} &= \begin{bmatrix} K(|t_2 - t_3|) & 0 & 0 \\ 0 & K(|t_1 - t_3|) & 0 \\ 0 & 0 & K(|t_1 - t_2|) \end{bmatrix}.\end{aligned}$$

The two terms of the adjugate are $\boldsymbol{A}$ with all of its off-diagonal entries negated. In the second term, the smallest terms of the diagonal matrices are $K(|t_1 - t_3|)$ (by the minimum separation property, assuming $K$ has good decay), thus we expect the entry in $(2, 2)$ to be the smallest, the other entries in row and column 2 to be bigger but still fairly small, and the rest of the entries to be the largest.

**Proposition 1** (Determinant Bound)**.**

$$|\det(\boldsymbol{A}) - 1| \leq 2M(1)^2 + M(2)^2 + 2M(1)^2 M(2).$$

*Proof.* Triangle inequality on the explicit formula. □

**Proposition 2** (Conditioning Bound)**.**

$$\|\boldsymbol{A}^{-1}\| = \lambda_{\min}(\boldsymbol{A})^{-1} \leq \frac{1}{1 - M(1) - M(2)}.$$

*Proof.* Gershgorin circle theorem. □

## 4 Controlling the Schur complement

We compute the entries of $\boldsymbol{K}/\boldsymbol{A}$: for $i, j \geq 4$,

$$\begin{aligned}(\boldsymbol{K}/\boldsymbol{A})_{(i-3)(j-3)} &= \boldsymbol{K}_{ij} - \boldsymbol{K}_{i[1:3]}^\top \boldsymbol{A}^{-1} \boldsymbol{K}_{j[1:3]} \\ &= K(|t_i - t_j|) - 2\boldsymbol{K}_{i[1:3]}^\top \boldsymbol{K}_{j[1:3]} + \boldsymbol{K}_{i[1:3]}^\top \boldsymbol{A} \boldsymbol{K}_{j[1:3]} - \boldsymbol{K}_{i[1:3]}^\top \boldsymbol{D} \mathbf{1}_3 \mathbf{1}_3^\top \boldsymbol{D} \boldsymbol{K}_{j[1:3]} \\ &\quad + 2\boldsymbol{K}_{i[1:3]}^\top \boldsymbol{D}^2 \boldsymbol{K}_{j[1:3]} \\ &= K(|t_i - t_j|) - 2\sum_{k=1}^{3} K(|t_i - t_k|)K(|t_j - t_k|) + \sum_{k=1}^{3}\sum_{\ell=1}^{3} K(|t_i - t_k|)K(|t_j - t_\ell|)K(|t_k - t_\ell|) \\ &\quad - \Big(K(|t_i - t_1|)K(|t_2 - t_3|) + K(|t_i - t_2|)K(|t_1 - t_3|) + K(|t_i - t_3|)K(|t_1 - t_2|)\Big) \times \\ &\quad \Big(K(|t_j - t_1|)K(|t_2 - t_3|) + K(|t_j - t_2|)K(|t_1 - t_3|) + K(|t_j - t_3|)K(|t_1 - t_2|)\Big) \\ &\quad + 2K(|t_i - t_1|)K(|t_j - t_1|)K(|t_2 - t_3|)^2 + 2K(|t_i - t_2|)K(|t_j - t_2|)K(|t_1 - t_3|)^2 + 2K(|t_i - t\end{aligned}$$

We want to control $\|\boldsymbol{I}-(\boldsymbol{K}/\boldsymbol{A})\|_\infty$, which requires bounding the distance of the diagonal from 1 and bounding the remaining entries. For the diagonal, we bound

$$\begin{aligned}
|1-(\boldsymbol{K}/\boldsymbol{A})_{(i-3)(i-3)}| &= |\boldsymbol{K}_{i[1:3]}^\top \boldsymbol{A}^{-1}\boldsymbol{K}_{i[1:3]}| \\
&\le \|\boldsymbol{A}^{-1}\| \cdot \|\boldsymbol{K}_{i[1:3]}\|^2 \\
&= \frac{\sum_{k=1}^3 K(|t_i-t_k|)^2}{\lambda_{\min}(\boldsymbol{A})} \\
&\le \frac{M(i-1)^2+M(i-2)^2+M(i-3)^2}{1-M(1)-M(2)}
\end{aligned}$$

On the off-diagonal, we have the naive bound

$$\begin{aligned}
|(\boldsymbol{K}/\boldsymbol{A})_{(i-3)(j-3)}| &\le |K_{ij}| + \|\boldsymbol{A}^{-1}\| \cdot \|\boldsymbol{K}_{i[1:3]}\|_2 \cdot \|\boldsymbol{K}_{i[1:3]}\|_2 \\
&\le K(|t_i-t_j|) + \frac{1}{2\lambda_{\min}(\boldsymbol{A})}\left(\|\boldsymbol{K}_{i[1:3]}\|_2^2 + \|\boldsymbol{K}_{i[1:3]}\|_2^2\right) \\
&\le K(|t_i-t_j|) + \frac{1}{2(1-M(1)-M(2))}\left(\sum_{k=1}^3 K(|t_i-t_k|)^2 + \sum_{k=1}^3 K(|t_j-t_k|)^2\right)
\end{aligned}$$

# 5 Error in the first three coordinates

# 6 Decay of coefficients