

Super-Resolution of Point Sources Down to the Rayleigh Limit from Multiple Observations

June 12, 2018

1 Outline

1.1 Dual Certificate Construction

The argument by analogy with L^2 minimization gives us a natural construction, so long as we are committed to using the Dirichlet kernel:

$$\begin{aligned} \text{find } q_k(t) &= \sum_{j=1}^s \alpha_{j,k} K(t - t_j) + \sum_{j=1}^s \beta_j V_{j,k} K'(t - t_j) \\ \text{subject to } q_k(t_j) &= V_{j,k} \quad \forall j \in [s], k \in [m] \\ \langle \mathbf{q}'(t_j), \mathbf{v}_j \rangle &= 0 \quad \forall j \in [s] \end{aligned} \tag{INT-FULL}$$

Drawing from [1], it is also plausible that, once we have this, we can swap out the kernel K for any other reasonable kernel and get a sensible construction. If we do this, we lose the L^2 minimization property, but it is not clear how to use that anyway.

Task 1 (*Moderately well-defined.*) *Experiment numerically on whether non-Dirichlet kernels (Fejer, more generally K^{2d} , or even more generally products of Dirichlet kernels with different cutoffs like in [1]) improve (INT-FULL) numerically. Is there any benefit from the kernel being positive?*

1.2 Special Case: Orthogonal Signs

Assuming the \mathbf{v}_j are orthogonal, without loss of generality we may rotate them to the standard basis, and we get the following simplified problem.

$$\begin{aligned} \text{find } q_k(t) &= \sum_{j=1}^s \alpha_{j,k} K(t - t_j) + \beta_k K'(t - t_k) \\ \text{subject to } q_k(t_j) &= \delta_{j,k} \quad \forall j \in [s], k \in [m] \\ q'_k(t_k) &= 0 \quad \forall k \in [m] \end{aligned} \tag{INT-ORTH}$$

Note that the constraints on the dimension q_k are now decoupled; therefore, this problem is just a combination of m independent copies of interpolation problems of the following kind:

$$\begin{aligned} \text{find } q(t) &= \sum_{j=1}^s \alpha_j K(t - t_j) + \beta K'(t - t_1) \\ \text{subject to } q(t_j) &= \delta_{1,j} \quad \forall j \in [s] \\ q'(t_1) &= 0 \end{aligned} \tag{INT-1D}$$

Ultimately, if \mathbf{q} solves (INT-ORTH), we want to control

$$N(t) \stackrel{\text{df}}{=} \|\mathbf{q}(t)\|^2 = \sum_{k=1}^m |q_k(t)|^2.$$

Per [1] and [2], this should break up into an analysis near the t_i and away from the t_i . These two questions each require different estimates on solutions of (INT-1D).

1.2.1 Information From (INT-1D) For “Near” Analysis

Task 2 *Understand the solution of (INT-1D) near t_1 as well as possible.*

- *Is $q''(t_1)$ very negative? Does it stay very negative in a neighborhood?*

Task 3 *Understand the solution of (INT-1D) near t_j where $j \neq 1$ as well as possible.*

- *Is $|q'(t_j)|$ small? Does it stay small in a neighborhood?*
- *Is $|q''(t_j)|$ small? Does it stay small in a neighborhood? (This might not actually be the case—check numerically.)*

This is what we need because of the identities

$$\begin{aligned} N''(t) &= 2\text{Re}(\langle \mathbf{q}(t), \mathbf{q}''(t) \rangle) + 2\|\mathbf{q}'(t)\|^2, \\ N''(t_k) &= 2\text{Re}(q_k''(t_k)) + 2 \sum_{j \neq k} |q_j'(t_k)|^2. \end{aligned}$$

So, the above information would let us argue as follows, for each fixed k :

1. $N''(t_k)$ is very negative.
2. Since $\mathbf{q}(t_k + \epsilon) \approx \mathbf{e}_k$, $q_k''(t_k + \epsilon)$ is negative, and (possibly) $q_j''(t_k + \epsilon)$ is small for $j \neq k$, $N''(t_k + \epsilon)$ will remain negative for small ϵ .
3. For whatever range of ϵ we can ensure this for, we will have $N(t_k + \epsilon) < 1$.

1.2.2 Information From (INT-1D) For “Far” Analysis

Task 4 *Bound the solution $|q(t)|$ of (INT-1D) for $t_j < t < t_{j+1}$ with enough “margin” on either side. This depends on how good of an ϵ we can obtain from the previous part.*

- For $j \notin \{1, s\}$, t lies away from the first interval on either side of t_1 , so $|q(t)|$ should be very small.
- For $j \in \{1, s\}$, t lies in the vicinity of the second lobe of a Dirichlet kernel centered at t_1 , which is roughly how $q(t)$ should look. So, the analysis here might be more delicate (this is also one place where a different kernel may help).

The plan for bounding $N(t)$ is then to simply use the triangle inequality with all of these bounds. Note that for any given t in a “far” region, there will be exactly two contributions from dimensions for which t is in “second lobe range”, and $s - 2$ contributions from the rest.

1.3 Information From (INT-1D) Coefficients

The next question is what information on α and β in (INT-1D) will be necessary to get us the above estimates. In [1], the argument (rewritten in our situation) is that $\alpha \approx e_1$ and $|\beta| \ll 1$. In our case, the former is no longer true for smaller minimum separation, but the latter is still appears to be the case.

In fact, we can eliminate β entirely by solving the constraint $q'(t_1) = 0$. This leaves us with the reduced problem

$$\begin{aligned} \text{find } q(t) &= \sum_{j=1}^s \alpha_j \left(K(t - t_j) - \frac{K'(t_1 - t_j)}{K''(0)} K'(t - t_1) \right) & (\text{INT-1D-2}) \\ \text{subject to } q(t_j) &= \delta_{1,j} \quad \forall j \in [s] \end{aligned}$$

2 Formal Setup

2.1 Single Observation Problem Statement

Suppose we observe a true signal whose representation in the physical domain is

$$x^*(t) = \sum_{j=1}^s a_j \delta_{t_j}(t), \quad (1)$$

for $t \in \mathbb{T}$ where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, which we will usually think of as the unit interval $[0, 1]$ with its endpoints identified, and $T = \{t_j\}_{j=1}^s \subset \mathbb{T}$ a discrete support set. In the Fourier domain, this signal takes the form

$$\widehat{x^*}(k) = \sum_{j=1}^s a_j \exp(-2\pi i k t_j) \quad (2)$$

for $k \in \mathbb{Z}$.

Now, we are interested in recovering this true signal x from an observation that suffers from low resolution, which we represent as convolution of $x(t)$ with a point-spread function (PSF) denoted by $\phi(t)$. We then observe the signal

$$(\phi * x^*)(t) = \sum_{j=1}^s a_j \phi(t - t_j). \quad (3)$$

In the simplest case, ϕ is a Dirichlet kernel with cutoff frequency f_c , in which case the Fourier transform of the above is simply the truncation of the Fourier transform of x^* . Denoting by y the signal we observe in the Fourier domain, we have

$$y(k) = \left(\sum_{j=1}^s a_j \exp(-2\pi i k t_j) \right) \mathbb{1}\{|k| \leq f_c\}. \quad (4)$$

We may then think of the data we observe as simply the $2f_c + 1$ values $y(-f_c), \dots, y(f_c)$. We write \mathcal{F}_n for the sensing operator mapping x^* to these n Fourier coefficients. A popular technique for solving this problem is *total variation minimization*, which attempts to recover x^* by solving the convex problem

$$\begin{aligned} & \text{minimize} && \|x\|_{\text{TV}} \\ & \text{subject to} && \mathcal{F}_n x = y. \end{aligned} \quad (5)$$

2.2 Extension to Multiple Observations

We now consider a generalization of the problem presented in the previous section, where we make *several* observations signals sharing the support $T = \{t_1, \dots, t_s\}$ of x^* , but having varying amplitudes a_j . To formalize this, our true signal is now

$$x_\ell^*(t) = \sum_{j=1}^s a_{\ell,j} \delta_{t_j} \text{ for } \ell \in \{1, \dots, m\}, \quad (6)$$

and our observations are $y_{\ell,k} = (\mathcal{F}_n x_\ell^*)_k$ for $k \in \{-f_c, \dots, f_c\}$ and $\ell \in \{1, \dots, m\}$. We think of the $a_{\ell,j}$ as organized into vectors $\mathbf{a}_j \in \mathbb{R}^m$ for $j \in \{1, \dots, s\}$, and the $y_{\ell,k}$ as organized into a matrix $\mathbf{Y} \in \mathbb{C}^{m \times n}$. The *group total variation minimization* is the natural extension of the previous convex problem to this setting, where we solve

$$\begin{aligned} & \text{minimize} \quad \|x\|_{\mathbf{gTV}} \\ & \text{subject to} \quad \begin{bmatrix} \mathcal{F}_n x_1 & \mathcal{F}_n x_2 & \dots & \mathcal{F}_n x_m \end{bmatrix}^\top = \mathbf{Y}. \end{aligned} \quad (7)$$

2.3 Dual Certificates

In this section, we briefly describe the results of applying Lagrangian duality to TV and \mathbf{gTV} norm minimization, the main tool for theoretical analysis of the performance of these algorithms.

Definition 1 Let $\mu^0 \subset \mathbb{C}$ denote the complex unit circle. A sign pattern on a set is an assignment of points of μ^0 to each point of the set.

Definition 2 For a sign pattern $v \in (\mu^0)^T$, a low-pass trigonometric polynomial $q : \mathbb{T} \rightarrow \mathbb{C}$,

$$q(t) = \sum_{k=-f_c}^{f_c} c_k e^{2\pi i k t}, \quad (8)$$

is a single-observation dual certificate for v if $q(t_j) = v_j$ for $t_j \in T$ and $|q(t)| < 1$ for $t \notin T$.

Proposition 1 (TV Norm Minimization Duality) If a dual certificate for the sign pattern $v_j = a_j/|a_j|$ exists, then x^* is the unique solution of TV minimization for the super-resolution problem.

Therefore, to prove the effectiveness of TV minimization it suffices to show that a dual certificate exists under some conditions on T . The main result of [1] establishes that this is true under a minimum separation condition.

Definition 3 The minimum separation of a set $T \subset \mathbb{T}$ is

$$\Delta(T) = \min_{\substack{s, t \in T \\ s \neq t}} |s - t|, \quad (9)$$

where we measure the wrap-around distance as on a circle.

Theorem 1 (Proposition 2.3 of [1]) If $\Delta(T) \geq 1.26\lambda_c$ and $f_c \geq 10^3$, then a dual certificate exists for any sign pattern on T .

As also observed in [1], the same ideas extend to the multiple observation case in a straightforward fashion.

Definition 4 Let $\mu^{m-1} \subset \mathbb{C}^m$ denote the m -dimensional complex unit sphere. A m -dimensional sign pattern on a set is an assignment of points of μ^{m-1} to each point of the set.

Definition 5 For a sign pattern $\mathbf{v} \in (\mu^{m-1})^T$, a low-pass trigonometric polynomial $q : \mathbb{T} \rightarrow \mathbb{C}^m$ given by $q(t) = (q_1(t), \dots, q_m(t))$ and

$$q_\ell(t) = \sum_{k=-f_c}^{f_c} c_k \exp(2\pi i k t) \text{ for } \ell \in \{1, \dots, m\} \quad (10)$$

is an m -observation dual certificate for \mathbf{v} if $q(t_j) = \mathbf{v}_j$ for $t_j \in T$, and $\|q(t)\|_{\ell^2} < 1$ for $t \notin T$.

Proposition 2 (gTV Norm Minimization Duality) If a dual certificate for the sign pattern $\mathbf{v}_j = \mathbf{a}_j / \|\mathbf{a}_j\|_{\ell^2}$ exists, then x^* is the unique solution of gTV minimization for the super-resolution problem.

However, while numerical experiments in [1] suggest that when the amplitude vectors \mathbf{a}_j are taken randomly then recovery is possible down to a lower critical minimum separation which accumulates at $\frac{1}{2}\lambda_c$ as $m \rightarrow \infty$, the same uniform argument cannot apply—indeed, it is always possible that the amplitudes \mathbf{a}_j are all identical, in which case clearly it cannot be possible to recover x^* any more effectively than in the one observation case. Thus, to show that gTV norm minimization substantially improves on TV norm minimization, it is necessary to make some incoherence assumptions about the data generating process of the \mathbf{a}_j . One simple such assumption that captures the idea of these points being in “general position” is to assume that the signs $\mathbf{v}_j = \mathbf{a}_j / \|\mathbf{a}_j\|_{\ell^2}$ are i.i.d. isotropically random unit vectors.

3 Dual Certificate Construction

Recall that we are interested in building a low-pass trigonometric polynomial $q : \mathbb{T} \rightarrow \mathbb{C}^m$ interpolating the points (t_j, \mathbf{v}_j) while remaining strictly inside the unit sphere in \mathbb{C}^m elsewhere. The general idea of such constructions is usually to choose some *interpolation basis* of polynomials, and build q as a linear combination of them under some constraints. We first describe a simple approach to choosing interpolation bases based on constraints and a variational heuristic, that justifies several existing techniques and motivates our own.

3.1 Equivalence of L^2 Norm Minimization and Interpolation

Minimizing $\|q\|_{L^2([0,1])}$ under some interpolation constraints is equivalent to interpolating with a restricted basis, the basis arising from the nature of the constraints. This general principle can be implemented in specific cases with simple Lagrange multiplier calculations. Consider, for instance, the problem

$$\begin{aligned} & \text{minimize} && \|q\|_{L^2([0,1])} \\ & \text{subject to} && q(t_j) = \mathbf{v}_j \quad \forall j \in [s] \end{aligned} \quad (11)$$

Let us write

$$q_{\mathbf{c}}(t) = \sum_{f=-f_c}^{f_c} c_f \exp(2\pi i f t), \quad (12)$$

and define the matrix $\mathbf{F}_T \in \mathbb{C}^{s \times (2f_c+1)}$ by $(\mathbf{F}_T)_{jf} = \exp(2\pi i f t_j)$, with $-f_c \leq f \leq f_c$. Then, the calculation of the values of q_c on T is a matrix-vector multiplication,

$$\begin{bmatrix} q_c(t_1) \\ \vdots \\ q_c(t_s) \end{bmatrix} = \mathbf{F}_T \mathbf{c}, \quad (13)$$

and $\|q_c\|_{L^2([0,1])} = \|\mathbf{c}\|_2$. Thus, the above problem is equivalently

$$\begin{aligned} & \text{minimize} && \|\mathbf{c}\|_2 \\ & \text{subject to} && \mathbf{F}_T \mathbf{c} = \mathbf{v} \end{aligned} \quad (14)$$

We write a Lagrangian $L(\mathbf{c}, \boldsymbol{\alpha}) = \frac{1}{2} \|\mathbf{c}\|_2^2 - \langle \boldsymbol{\alpha}, \mathbf{F}_T \mathbf{c} - \mathbf{v} \rangle$, and upon differentiating and solving the first-order optimality condition obtain $\mathbf{c} = \mathbf{F}_T^* \boldsymbol{\alpha}$. Generally, we have

$$q_{\mathbf{F}_T^* \boldsymbol{\alpha}}(t) = \sum_{f=-f_c}^{f_c} \left(\sum_{j=1}^s \exp(-2\pi i f t_j) \alpha_j \right) \exp(2\pi i f t) = \sum_{j=1}^s \alpha_j K(t - t_j) \quad (15)$$

where K is the Dirichlet kernel. We therefore have found the formal equivalence:

$$\begin{aligned} & \text{minimize} && \|q\|_{L^2([0,1])} \\ & \text{subject to} && q(t_j) = \mathbf{v}_j \quad \forall j \in [s] \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} & \text{find} && q(t) = \sum_{j=1}^s \alpha_j K(t - t_j) \\ & \text{subject to} && q(t_j) = \mathbf{v}_j \quad \forall j \in [s] \end{aligned} \quad (16)$$

In the same way, one obtains a justification for the popular trick for allowing one to force the derivative of q to be zero at each t_j by adding terms $\beta_j K'(t - t_j)$ to the sum above: an analogous calculation shows

$$\begin{aligned} & \text{minimize} && \|q\|_{L^2([0,1])} \\ & \text{subject to} && q(t_j) = v_j \quad \forall j \in [s] \\ & && q'(t_j) = 0 \quad \forall j \in [s] \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} & \text{find} && q(t) = \sum_{j=1}^s \alpha_j K(t - t_j) + \sum_{j=1}^s \beta_j K'(t - t_j) \\ & \text{subject to} && q(t_j) = v_j \quad \forall j \in [s] \\ & && q'(t_j) = 0 \quad \forall j \in [s] \end{aligned} \quad (17)$$

3.2 Relaxing the Derivative Condition

As mentioned before, in the multidimensional setting, we do not expect setting the derivative to equal zero at each interpolation point will allow us to lower the minimum separation threshold substantially beyond where the analogous one-dimensional dual certificate is feasible. Instead, we propose a new construction where this condition is relaxed. Rather than setting the derivative at each interpolation point to equal zero, we only constrain it to the tangent plane to the complex unit sphere at that interpolation point. [TODO: something about not dividing real from complex for clarity though this helps in numerics]. Following the ideas in the previous section, we propose

as a dual certificate the polynomial solving the following optimization problem:

$$\begin{aligned}
& \text{minimize} && \|q\|_{L^2([0,1])} \\
& \text{subject to} && q(t_j) = \mathbf{v}_j \quad \forall j \in [s] \\
& && \langle q'(t_j), \mathbf{v}_j \rangle = 0 \quad \forall j \in [s]
\end{aligned} \tag{L2-MIN-FULL}$$

Performing the translation to an interpolation problem as before gives the following:

$$\begin{aligned}
& \text{find} && q_k(t) = \sum_{j=1}^s \alpha_{j,k} K(t - t_j) + \sum_{j=1}^s \beta_j V_{j,k} K'(t - t_j) \\
& \text{subject to} && q_k(t_j) = V_{j,k} \quad \forall j \in [s], k \in [m] \\
& && \langle q'(t_j), \mathbf{v}_j \rangle = 0 \quad \forall j \in [s]
\end{aligned} \tag{INT-FULL}$$

where we organize the sign pattern vectors \mathbf{v}_j as the rows of the matrix $\mathbf{V} \in \mathbb{C}^{s \times m}$. Note that if we instead took the constraint $q'(t_j) = \mathbf{0}$, then the second term would involve terms $\beta_{j,k} K'(t - t_j)$ in coordinate k , but here we are only allowed *one* set of coefficients β_j to use across all coordinates. This, along with the randomness of the $V_{j,k}$, essentially ensures that the second term cannot often be large: a single vector $\boldsymbol{\beta}$ cannot align with many weakly dependent random vectors $\mathbf{V}_{\bullet,k}$ at once.

3.3 Explicit Solution

Next, we derive concise formulae solving for the coefficients $\boldsymbol{\alpha}, \boldsymbol{\beta}$ (when this is possible, which will later need some justification). Let $\mathbf{K}_r = (\frac{d^r}{dt^r} K(t_i - t_j))_{i,j=1}^s$, then we may write

$$(q_k(t_j))_{j \in [s], k \in [m]} = \mathbf{K}_0 \boldsymbol{\alpha} + \mathbf{K}_1 \text{diag}(\boldsymbol{\beta}) \mathbf{V} \tag{18}$$

$$(q'_k(t_j))_{j \in [s], k \in [m]} = \mathbf{K}_1 \boldsymbol{\alpha} + \mathbf{K}_2 \text{diag}(\boldsymbol{\beta}) \mathbf{V} \tag{19}$$

Thus, the interpolation equations may be written

$$\mathbf{K}_0 \boldsymbol{\alpha} + \mathbf{K}_1 \text{diag}(\boldsymbol{\beta}) \mathbf{V} = \mathbf{V} \tag{20}$$

$$\text{diag}((\mathbf{K}_1 \boldsymbol{\alpha} + \mathbf{K}_2 \text{diag}(\boldsymbol{\beta}) \mathbf{V}) \mathbf{V}^*) = \mathbf{0} \tag{21}$$

From the first equation we obtain

$$\boldsymbol{\alpha} = \mathbf{K}_0^{-1} (\mathbf{I}_s - \mathbf{K}_1 \text{diag}(\boldsymbol{\beta})) \mathbf{V}, \tag{22}$$

and substituting this into the second obtain

$$\boldsymbol{\beta} = \frac{\text{diag}(\mathbf{V} \mathbf{V}^* \mathbf{K}_0^{-1} \mathbf{K}_1)}{\text{diag}(\mathbf{V} \mathbf{V}^* \mathbf{K}_2) - \text{diag}(\mathbf{V} \mathbf{V}^* \mathbf{K}_1 \mathbf{K}_0^{-1} \mathbf{K}_1)}, \tag{23}$$

where division in the latter is coordinatewise, and these formulae hold provided \mathbf{K}_0 is invertible and $\text{diag}(\mathbf{V} \mathbf{V}^* (\mathbf{K}_2 - \mathbf{K}_1 \mathbf{K}_0^{-1} \mathbf{K}_1))$ is nowhere zero.

4 Special Case: Orthogonal Signs

Consider the special case where the \mathbf{v}_j are orthogonal, and for the sake of simplicity $s = m$, so that they form a complete basis. Then, without loss of generality we further consider the case where the \mathbf{v}_j are rotated to equal the standard basis, so that $\mathbf{V} = \mathbf{I}_s$. One may then simplify the equations from before, but it is even easier to note directly that the equations governing each polynomial coordinate $q_k(t)$ become decoupled: we are now interested in the interpolation problem

$$\begin{aligned} \text{find } q_k(t) &= \sum_{j=1}^s \alpha_{j,k} K(t - t_j) + \beta_k K'(t - t_k) \\ \text{subject to } q_k(t_j) &= \delta_{j,k} \quad \forall j \in [s], k \in [m] \\ q'_k(t_k) &= 0 \quad \forall k \in [m] \end{aligned} \tag{INT-ORTH}$$

Note that this decomposes into s independent interpolation problems, each of the form

$$\begin{aligned} \text{find } q(t) &= \sum_{j=1}^s \alpha_j K(t - t_j) + \beta K'(t - t_1) \\ \text{subject to } q(t_j) &= \delta_{1,j} \quad \forall j \in [s] \\ q'(t_1) &= 0 \end{aligned} \tag{INT-1D}$$

4.1 Explicit Solution

From the second constraint we may directly solve

$$\beta_k = -\frac{1}{K''(0)} \sum_{j=1}^s \alpha_{j,k} K'(t_k - t_j),$$

and therefore

$$q_k(t) = \sum_{j=1}^s \alpha_{j,k} \left(K(t - t_j) - \frac{K'(t_k - t_j)K'(t - t_k)}{K''(0)} \right).$$

We define the matrix corresponding to this linear system, \mathbf{M}_k , by

$$(\mathbf{M}_k)_{ij} = K(t_i - t_j) + \frac{K'(t_k - t_j)K'(t_k - t_i)}{K''(0)},$$

so that, writing $\alpha_k = \alpha_{\bullet,k}$, we find $\alpha_k = \mathbf{M}_k^{-1} \mathbf{e}_k$. We may further decompose

$$\mathbf{M}_k = \mathbf{K}_0 + \Delta_k,$$

where

$$(\Delta_k)_{ij} = \frac{K'(t_k - t_j)K'(t_k - t_i)}{K''(0)}.$$

This matrix is rank one:

$$\Delta_k = \frac{1}{K''(0)} \kappa_{1,k} \kappa_{1,k}^\top \quad (24)$$

$$(\kappa_{1,k})_i = K'(t_k - t_i) \quad (25)$$

Therefore, M_k^{-1} may be computed by the Sherman-Morrison formula:

$$M_k^{-1} = K_0^{-1} - \frac{1}{K''(0) + \kappa_{1,k}^\top K_0^{-1} \kappa_{1,k}} K_0^{-1} \kappa_{1,k} \kappa_{1,k}^\top K_0^{-1},$$

and we obtain the formula

$$\alpha_k = M_k^{-1} e_k = K_0^{-1} e_k - \frac{\kappa_{1,k}^\top K_0^{-1} e_k}{K''(0) + \kappa_{1,k}^\top K_0^{-1} \kappa_{1,k}} K_0^{-1} \kappa_{1,k},$$

expressing α_k as a perturbation of $K_0^{-1} e_k$, which is what we would obtain if we were simply doing Dirichlet kernel interpolation on the sign pattern e_k . Indeed, if only the first term were present, we would simply have $\alpha \approx K_0^{-1}$.

Furthermore, observe that $\langle \kappa_{1,k}, e_k \rangle = K'(0) = 0$, so the perturbation term is very small when K_0^{-1} is close to the identity, which is the case for large minimum separations.

4.2 Three-Point Approximation

We now introduce a means of approximating the solution to (INT-1D). The idea is to solve a truncated version of the same problem in closed form, and hope that the resulting solution is close to the true solution of (INT-1D). The truncated problem is as follows, obtained by simply ignoring all constraints on points other than t_s, t_1, t_2 (the three-point neighborhood surrounding t_1 , where the interpolant must equal 1). Here, it is convenient to assume (without loss of generality) that $t_1 = 0$, and to rename $t_s = -\Delta_-$ and $t_2 = \Delta_+$ where $\Delta_\pm > 0$, for the sake of more symmetrical notation.

$$\begin{aligned} \text{find } & q(t) = \alpha_0 K(t) + \alpha_+ K(t - \Delta_+) + \alpha_- K(t + \Delta_-) + \beta K'(t) \\ \text{subject to } & q(0) = 1 \\ & q(\Delta_+) = q(-\Delta_-) = 0 \\ & q'(0) = 0 \end{aligned} \quad (\text{INT-1D-TRUNC})$$

The coefficients of the solution solve the following linear system of equations in four variables:

$$\begin{bmatrix} K(0) & K(\Delta_+) & K(\Delta_-) & K'(0) \\ K(\Delta_+) & K(0) & K(\Delta_+ + \Delta_-) & K'(\Delta_+) \\ K(\Delta_-) & K(\Delta_+ + \Delta_-) & K(0) & -K'(\Delta_-) \\ K'(0) & K'(\Delta_+) & -K'(\Delta_-) & K''(0) \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_+ \\ \alpha_- \\ \beta \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \quad (26)$$

We may solve this in closed form, obtaining an explicit (if complicated) function that gives the solution:

$$q^{3\text{pt}}(\Delta_+, \Delta_-; t) : [0, 1] \times [0, 1] \times \mathbb{T} \rightarrow \mathbb{R} \quad (27)$$

Let us similarly set some notation for the solution of (INT-1D):

$$q^{1D}(T; t) : [0, 1]^s \times \mathbb{T} \rightarrow \mathbb{R}, \quad (28)$$

then our main hypothesis is that the following approximation is sound: suppose $T = \{0 = t_1, t_2, \dots, t_s\}$ with these points ordered along \mathbb{T} , then

$$q^{1D}(\{0 = t_1, t_2, \dots, t_s\}; \bullet) \approx q^{3pt}(|t_2|, |t_s|; \bullet). \quad (29)$$

5 Further Special Case: Equally Spaced Points

Consider further a special case where the t_j are all equally spaced in \mathbb{T} , so that $\Delta = 1/s$, and every pair of t_j, t_{j+1} achieves this spacing exactly in the wraparound metric. The point of this simplification is to induce some extra structure in the matrix \mathbf{K}_0 to further ease the formal analysis performed in the previous section. If we do this, then if we have $1/s = \Delta = \gamma\lambda_c = \gamma/f_c$, we must have $f_c = \gamma s$ is an integer.

Indeed, in this case we have $t_i - t_j = (i - j)\Delta$ (with equality modulo 1), and therefore

$$(\mathbf{K}_0)_{ij} = K((i - j)\Delta),$$

so \mathbf{K}_0 is circulant. In case K is the Dirichlet kernel, we have $\mathbf{K}_0 = \mathbf{F}\mathbf{F}^*$ where $\mathbf{F} \in \mathbb{C}^{s \times (2f_c+1)}$ having, assuming $t_1 = \Delta$ as we may without loss of generality,

$$F_{jf} = \exp(2\pi i t_j f) = \exp(2\pi i j f \Delta) = \exp\left(2\pi i \frac{jf}{s}\right).$$

That is, \mathbf{F} is nothing but a discrete Fourier transform (DFT) matrix with some redundant frequencies included. In this formulation, we see immediately a simple significance of the threshold $\gamma = \frac{1}{2}$: it is the threshold below which \mathbf{F} is no longer a complete DFT matrix in dimension s .

In this case, the inverse of \mathbf{K}_0 is related to \mathbf{K}_0 itself, which may be seen by computing $\mathbf{F}^*\mathbf{F}$, where we see that the usual orthogonality of Fourier harmonics of order s appears:

$$(\mathbf{F}^*\mathbf{F})_{fg} = \sum_{j=0}^{s-1} \exp\left(2\pi i \frac{j(g-f)}{s}\right) = s[f \equiv g \pmod{s}].$$

We have $(\mathbf{K}_0)_{jj} = 2f_c + 1$, and

$$\begin{aligned} (\mathbf{K}_0)_{jk} &= \sum_{f=-f_c}^{f_c} \exp\left(2\pi i \frac{(k-j)f}{s}\right) \\ &= \frac{\sin((f_c + \frac{1}{2})(k-j))}{\sin(\frac{1}{2}(k-j))} \end{aligned}$$

6 Local Analysis

Proposition 3 *Let $q : [0, 1] \rightarrow \mathbb{C}^m$ be a smooth function. Write $N(t) = \|q(t)\|^2 = \sum_{k=1}^m |q_k(t)|^2$. Then,*

$$N'(t) = 2\text{Re}(\langle q(t), q'(t) \rangle) \quad (30)$$

$$N''(t) = 2\text{Re}(\langle q(t), q''(t) \rangle) + 2\|q'(t)\|^2. \quad (31)$$

Thus, for the local analysis of $N(t)$ near one of the t_j in the special case of orthogonal signs, we have

$$N(t_k) = 1 \quad (32)$$

$$N'(t_k) = 0 \quad (33)$$

$$N''(t_k) = 2\text{Re}(q_k''(t_k)) + 2 \sum_{j \neq k} |q_j'(t_k)|^2 \quad (34)$$

Thus, the local concavity of $N(t)$ near t_k is a battle between, on the one hand, the concavity of q_k near t_k , and on the other hand the total magnitude of the derivatives in all other dimensions. Since $K'''(0) = 0$, we may express

$$q_k''(t_k) = \sum_{j=1}^s \alpha_{jk} K''(t_k - t_j) = \langle \alpha_k, \kappa_{2,k} \rangle,$$

where we substitute in the leading behavior derived in the previous section.

For the derivative terms, we have

$$q_j'(t_k) = \sum_{i=1}^s \alpha_{ij} \left(K'(t_k - t_i) - \frac{K'(t_j - t_i)K''(t_k - t_j)}{K''(0)} \right) \quad (35)$$

$$= \langle \alpha_j, \kappa_{1,k} \rangle - \frac{K''(t_k - t_j)}{K''(0)} \langle \alpha_j, \kappa_{1,j} \rangle \quad (36)$$

The last inner product here may be written fairly simply,

$$\langle \alpha_j, \kappa_{1,j} \rangle = \kappa_{1,j}^\top \mathbf{K}_0^{-1} \mathbf{e}_j \left(1 - \frac{1}{\frac{K''(0)}{\kappa_{1,k}^\top \mathbf{K}_0^{-1} \kappa_{1,k}} + 1} \right).$$

Heuristics If we assume the second term above is negligible, then the “adversary term” for local concavity would be of the form

$$\sum_{j \neq k} |\langle \alpha_j, \kappa_{1,k} \rangle|^2.$$

We may think heuristically of $\kappa_{1,k}$ as having an equal positive and negative weight on entries $k \pm 1$ respectively, whereby this computes the total “gradient” of each α_j between entries $k \pm 1$. Thus, to the extent that \mathbf{K}_0^{-1} , the solution operator of Dirichlet interpolation, operates in a “local” way, the only significant contributions should come from the first few neighbors on either side of dimension

k . (Indeed, away from these neighbors we expect the second correction term to be small as well, since the leading factor will be small according to the decay of K'' away from zero.)

6.1 Steps

Assume orthogonal basis-generating samples (requires $s \leq m$). For a given $\epsilon > 0$ (of choice) and for all t_i , $1 \leq i \leq s$ satisfying minimum separation condition,

1. $N''(t)$ stays negative for $t \in [t_i - \epsilon, t_i + \epsilon]$ Together with the fact that $N'(t) = 2\text{Re}(\langle q(t), q'(t) \rangle) = 0$ due to construction, this ensures $N(t) \leq 1$ for $t \in [t_i - \epsilon, t_i + \epsilon]$.

$$N''(t) = 2\text{Re}\{\langle q(t), (q''(t))^* \rangle\} + 2\|q'(t)\|^2 \quad (37)$$

$$= 2 \sum_{k=1}^m \text{Re}\{q_k(t)(q_k''(t))^*\} + |q_k'(t)|^2 \quad (38)$$

$$= \sum_{k=1}^m 2\text{Re} \left\{ \underbrace{\left(\sum_{j=1}^s \alpha_{j,k} K(t-t_j) + \beta_k K'(t-t_k) \right) \left(\sum_{j=1}^s \alpha_{j,k}^* K''(t-t_j) + \beta_k^* K'''(t-t_k) \right)}_{x_k(t)} \right\} \quad (39)$$

$$+ \underbrace{\left(\sum_{j=1}^s \alpha_{j,k} K'(t-t_j) + \beta_k K''(t-t_k) \right) \left(\sum_{j=1}^s \alpha_{j,k}^* K'(t-t_j) + \beta_k^* K''(t-t_k) \right)}_{y_k(t)} \quad (40)$$

$$x_k(t) = \sum_{j=1}^s \sum_{\ell=1}^s \alpha_{j,k} K(t-t_j) \alpha_{\ell,k}^* K''(t-t_\ell) + \beta_k^* K'''(t-t_k) \sum_{j=1}^s \alpha_{j,k} K(t-t_j) \quad (41)$$

$$+ \beta_k K'(t-t_k) \sum_{j=1}^s \alpha_{j,k}^* K''(t-t_j) + \beta_k K'(t-t_k) \beta_k^* K'''(t-t_k) \quad (42)$$

For $t = t_i + \epsilon$,

$$x_k(t) = \sum_{j=1}^i \sum_{\ell=1}^i \alpha_{j,k} \alpha_{\ell,k}^* K(\delta_{i,j} + \epsilon) K''(\delta_{i,\ell} + \epsilon) + \sum_{j=i+1}^s \sum_{\ell=1}^i \alpha_{j,k} \alpha_{\ell,k}^* K(\delta_{i,j} - \epsilon) K''(\delta_{i,\ell} + \epsilon) \quad (43)$$

$$+ \sum_{j=1}^i \sum_{\ell=i+1}^s \alpha_{j,k} \alpha_{\ell,k}^* K(\delta_{i,j} + \epsilon) K''(\delta_{i,\ell} - \epsilon) + \sum_{j=i+1}^s \sum_{\ell=i+1}^s \alpha_{j,k} \alpha_{\ell,k}^* K(\delta_{i,j} - \epsilon) K''(\delta_{i,\ell} - \epsilon) \quad (44)$$

$$+ \beta_k^* K'''(t_i - t_k + \epsilon) \sum_{j=1}^i \alpha_{j,k} K(\delta_{i,j} + \epsilon) + \beta_k^* K'''(t_i - t_k + \epsilon) \sum_{j=i+1}^s \alpha_{j,k} K(\delta_{i,j} - \epsilon) \quad (45)$$

$$+ \beta_k K'(t_i - t_k + \epsilon) \sum_{j=1}^i \alpha_{j,k}^* K''(\delta_{i,j} + \epsilon) + \beta_k K'(t_i - t_k + \epsilon) \sum_{j=i+1}^s \alpha_{j,k}^* K''(\delta_{i,j} - \epsilon) \quad (46)$$

$$+ |\beta_k|^2 K'(t_i - t_k + \epsilon) K'''(t_i - t_k + \epsilon) \quad (47)$$

fejer kernel, distribution of real alphas. where $\delta_{i,j} = |t_i - t_j|$. We assumed symmetric kernel.

2. $N(t) < 1$ for $t \in [t_i + \epsilon, t_{i+1} - \epsilon]$

3. For non-orthogonal samples Duality question: Is solving the non-orthogonal case and assuming

7 Tools

7.1 Bernstein Polynomial Inequality

The idea of using this comes from [2]. The inequality is as follows.

Lemma 1 *Let $p(z)$ be a complex polynomial of degree N . Then,*

$$\sup_{|z| \leq 1} |p'(z)| \leq N \sup_{|z| \leq 1} |p(z)|.$$

The way this is used in that paper is basically to argue that a well-bounded polynomial must also have a good Lipschitz property over the unit circle:

$$\begin{aligned} |p(e^{ix}) - p(e^{iy})| &\leq |e^{-ix} - e^{iy}| \sup_{|z|=1} |p'(z)| \\ &\leq \left(CN \sup_{|z|=1} |p(z)| \right) |x - y| \end{aligned}$$

There, this is used to show that random polynomials are close to their mean, by showing this on a grid and then applying this Lipschitz property to pass from the grid to the entire interval. It's not clear how this can help us, since there is no natural choice of “comparison” polynomial.

7.2 Nazarov Inequality

Lemma 2 *Let $p(t)$ be a complex trigonometric polynomial of degree n , and $E \subset [0, 1]$. Then,*

$$\max_{t \in [0, 1]} |p(t)| \leq \left(\frac{C}{m(E)} \right)^{n-1} \max_{t \in E} |p(t)|.$$

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