1. Feb 14: Classification of Dynamical Systems

1.1. **Classification is difficult.** Orbit equivalence is an example that describes this difficulty.

Definition 1.1 (Orbit equivalence)**.** All aperiodic systems are equivalent up to measurable change of coordinates.

Definition 1.2 (Rigidity Theorem)**.** Note well define, but if some "nice" condition is met, the system is a "nice" one.

Example 1.3 (Marked Length Spectrum (Otal's theorem))**.** If S is a negatively curved surface and there is a surface S' of constant curvature and a homeomorphism $\phi : S \to S'$ such that for every free homotopy class γ, the length of γ with

$$l_{S'}(\phi(\gamma)) = l_S(\gamma).$$

Then there is an isometry $\phi_0 : S \to S'$.

Importantly, there is some structure on the space being asked, and we wish to find some algebraic structure (for it to be considered "nice"). In the case of Example 1.3, we are looking to build $SL(2, \mathbb{R})$ structure.

Theorem 1.4 (Another Rigidity Theorem)**.** *If $\phi_t : M \to M$ is an Anosov flow on a 3-manifold M, and $h_{top}(\gamma_t) = h_\mu(\phi_t)$ for an invariant volume μ, Then ϕ_t is either:*

- *Conjugate to geodesic flow on T's for a hyperbolic surface S.*
- *Constant-time suspension of an Anosov automorphism of Π^2.*

Let $\mathcal{X}^\infty(M) = \{C^\infty \text{ vector fields on } M\}$, then in the Lie algebraic sense, $[X, Y] \cdot f = XYf - YXf$.

Theorem 1.5 (Frobenius)**.** *If D is a distribution on M, then D is the tangent bundle for some center manifold N ($D = TN$), then $N \iff$ for all $x, y \in \mathcal{X}(M)$ such that $X_p, Y_0 \in D_0$ for all $p \in M$ then $[X, Y]_p \in D_p$ for all $p \in M$.*

Theorem 1.6. *Let M be a C^∞ manifold, and $\gamma \subset \mathcal{X}^\infty(M)$ is a vector subspace such that fro all $x, y \in \gamma$, $[x, y] \in \gamma$ and γ is finite dimensional, then there is a simply connected lie group G such that $Lie(G) = \gamma$ and an action $G \to M$ such that $T_p(G \cdot p) = \gamma_p$ (the operation is almost reversible, that is, it is only locally unique.)*

Big Idea from today's talk: Bring Bracky assumption to the Lie Group level.
We can assume that because γ is finite dimensional and a Lie Algebra, we can reframe this with

$$\phi_t^{(1)}, \dots, \phi_t^{(n)}$$

which are n-many flows where we don't assume regularity to do this solely topologically.

Definition 1.7 (Graev)**.** Let $H_1, \dots, H_n$ be topological groups. Then, up to continuous isomorphism, there exists a unique group $H_1 \star \cdots \star H_n$ such that:

- $H_i \to H_1 \star \cdots \star H_n$ embeds as a homeomorphism onto its image.
- If $\phi_i : H_i \to G$ be any family of continuous homeomorphisms, there exists a unique extension $\Phi : H_1 \star \cdots \star H_n \to G$.

Punishing me for not learning Tickzd. The above can be written as a commutative diagram.

Definition 1.8 (Free Products)**.** Elements of a free product are

- $h_1^{(i)} \star h_2^{(ii)} \star \cdots \star h_m^{(im)}$ where each $h_k^{ik} \in H_{l_k}$.

The only relations that one can use in a free product are the ones that are forced in a group.

Theorem 1.9 (δ domain)**.** *If H_i is a Lie group, $H_i \star \cdots \star H_n$ is a CW-complex topology. It is ∞-dimensional.*

This is a rather weak topology, but it preserves Graev's definition.

Example 1.10 (Word lengths in $\mathbb{R}$)**.**

- At word length 0, there is only the identity element e
- At word length 1, there is a single copy of $\mathbb{R}^2$ intersecting
- At word length 2, there is an intersection of two copies $\mathbb{R}^2$ which represent the combinations of words.
- At word length 3, the structure becomes more complicated, but in the case of $g0g$ for $g \in \mathbb{R}$, it collapses onto a line.

Goal for 2/21: **Show that Stabilizers are normal**.