Exercise 3: Let U be a bounded domain of $\mathbb{R}^n$. We say $u \in C^2(U)$ is subharmonic if $-\Delta u \leq 0$ in U.

(a) Prove for subharmonic v that

$$v(x) \leq \frac{1}{|B(x,r)|}\int_{B(x,r)} v(y)dy \quad \text{for all } \overline{B(x,r)} \subset U.$$

(b) Prove that the weak maximum principle holds for subharmonic $v \in C^2(U) \cap C(\bar{U})$.

(c) Let $\phi : \mathbb{R} \to \mathbb{R}$ be smooth and convex. Assume u is harmonic and $v = \phi(u)$. Prove that v is subharmonic.

(d) Prove $v = |Du|^2$ is subharmonic whenever u is harmonic.

Solution:

(a) It's sufficient to show that $v(x) \leq \frac{1}{|B(x,r)|}\int_{\partial B} v(y)dy$ as

$$\frac{1}{|B(x,r)|}\int_B v(y)dy = \frac{1}{|B(x,r)|}\int_0^r d\rho \int_{\partial B} v(y)dy = \frac{1}{|B(x,r)|}\int_{\partial B} v(y)dy.$$

Let $\phi(r) = \frac{1}{|B(x,r)|}\int_{\partial B} v(y)dy$. Now we see that (using $\int$ for the mean value integral because I'm not sure how to do that in LaTeX.)

$$\phi'(r) = \int_{\partial B(0,1)} Dv(x+rz)\cdot z dS(z)$$

letting $y = x + rz$. Using green's theorem, we see that (using the fact that $\Delta v \geq 0$)

$$\phi'(r) = \int_{\partial B(0,1)} Dv(x+rz)\cdot \frac{y-z}{r} dS(y) = \frac{r}{n}\int_{B(x,r)} \Delta v(y)dy \geq 0.$$

Because $\phi' \geq 0$, we see that it's increasing,

$$\phi(r) = \int_{\partial B(x,r)} v(y)dy \geq v(x).$$

(b) Let $v \in C^2(U) \cap C(\bar{U})$. FSoC, suppose

$$\max_{x\in\bar{U}} v(x) > \max_{x\in\partial\bar{U}} v(x).$$

Now, there is a point $x_0 \in U$ s.t. $v(x_0) > \max_{x\in\partial\bar{U}} v(x)$.
By part (a), for any $r > 0$ for $B(x_0, r) \subset U$,

$$\int_{B(x_0,r)} v(y) - v(x_0)dy \geq 0$$

but $v(y) - v(x_0)$ is nonpositive so $v(y) = v(x_0)$ for all $y \in B(x_0, r)$. Consider $L = \{r > 0 | B(x_0, r) \subset U\}$ and $s = \sup L$. Since $B(x_0, s) = \cup_{r\in L} B(x_0, r) \subset U$, we have that for all $y \in B(x_0, s), v(y) = v(x_0)$.
It's now sufficient to show that $\overline{B(x_0, s)} \cap \partial U \neq \emptyset$.
Proof of Above: Suppose $\overline{B(x_0, s)} \cap \partial U$ is empty. Since $\cap B(x_0, s) \subset \bar{U}$, then $\overline{B(x_0, s)} \cap (\mathbb{R}^n \setminus U)$ is also empty. Since the ball is compact (Heine-Borel) and $\mathbb{R}^n \setminus U$ is closed, then

$$0 < \inf\{|b-a| \; | b \in \overline{B(x_0,s)}, \; a \in (\mathbb{R}^n \setminus U)\} = d.$$

However, this means that $B(x_0, s + d/2) \subset U$. But this means $s + d/2 \in L$ and $s + d/2 > s = \sup L$, which is a contradiction.

(c) ϕ is smooth and convex and u is harmonic. $v = \phi(u)$ so

$$\begin{aligned}
\Delta v &= \nabla\cdot(\nabla v)\\
\nabla v &= \phi'(u)\nabla u\\
\nabla\cdot(\nabla v) &= \nabla\cdot(\phi'(u)\nabla u)\\
&= (\phi''(u)\nabla u)\cdot\nabla u + \phi'(u)\nabla\cdot(\nabla u)\\
&= (\phi''(u)\nabla u)\cdot\nabla u = \phi''(u)|\nabla u|^2 \geq 0
\end{aligned}$$

because ϕ is convex, it has $\phi'' \geq 0$. This implies that $-\Delta v \leq 0$, and thus v is subharmonic.

(d) Since u is harmonic, u_{x_i} for $i = 1, \dots, n$ is harmonic as well. $(u_{x_i})^2$ is subharmonic by (c). Now

$$|Du|^2 = \sum_{i=1}^{n} (u_{x_i})^2$$

which is subharmonic as it's the sum of subharmonic functions.

Exercise 4: If u is weakly harmonic, then u is C^2 and harmonic.

Solution: Because u is weakly harmonic, for test functions $\phi \in C_c^\infty$,

$$\int_U u \Delta \phi dx = 0.$$

Define a mollifier η s.t.

$$\eta(x) := \begin{cases} C \exp\left(\frac{1}{|x|^2 - 1}\right) & \textit{if } |x| < 1 \\ 0 & \textit{if } |x| \geq 1 \end{cases}$$

Where C is chosen such that $\int_{\mathbb{R}^n} \eta dx = 1$. Then, we can define $u_\epsilon = \eta_\epsilon \star u$. Now, for $\epsilon > 0$ set

$$\eta_\epsilon(x) := \frac{1}{\epsilon^n} \eta\left(\frac{x}{\epsilon}\right)$$

Now define the set $U_\epsilon := \{x \in U \ : \ d(x, \partial U) > \epsilon\}$ where d is the distance metric. Because η_ϵ is supported on $B(0, \epsilon)$, we see that $\eta_\epsilon(x - y)$ is compactly supported in U for any $x \in U_\epsilon$.
Fix $x \in U_\epsilon$, and because u is continuous and $\eta_\epsilon(x - y)$ being continuously differentiable,

$$\begin{aligned}
\Delta u_\epsilon &= \Delta_x (\eta_\epsilon \star u) \\
&= \Delta_x \int_{\mathbb{R}^n} \eta_\epsilon(x - y) u(y) dy \\
&= \Delta_x \int_U \eta_\epsilon(x - y) u(y) dy \\
&= \int_U \Delta_x \left[\eta_\epsilon(x - y)\right] u(y) dy \\
&= \int_U \Delta y \left[\eta_\epsilon(x - y)\right] u(y) dy \\
&= \int_U \Delta \phi u dy \\
&= 0
\end{aligned}$$

where $\phi = \eta_\epsilon(x - y)$. We see that Δu_ϵ is harmonic. Now, choose $\epsilon_2 > 0$ and set $u_{\epsilon\epsilon_2} := \eta_{\epsilon_2} \star u_\epsilon$. We see that $\eta_{\epsilon_2}(x - y)$ is compactly supported on U_ϵ for $x \in U_{\epsilon + \epsilon_2}$. Computing in polar coordinates gives

$$\begin{aligned}
u_{\epsilon\epsilon_2} &= \int_{U_\epsilon} \eta_{\epsilon_2}(x - y) u_\epsilon(y) dy \\
&= \frac{1}{\epsilon_2^n} \int_{B(x, \epsilon_2)} \eta\left(\frac{|x - y|}{\epsilon_2}\right) u_\epsilon(y) dy \\
&= \frac{1}{\epsilon_2^n} \int_0^{\epsilon_2} \eta\left(\frac{r}{\epsilon_2}\right) \int_{\partial B(x, r)} u_\epsilon(y) dS(y) dr \\
&= u_\epsilon(x) \int_{B(x, \epsilon)} \eta\left(\frac{x - y}{\epsilon_2}\right) dy \\
&= u_\epsilon(x) \int_{B(0, \epsilon)} \eta\left(y\right) dy \\
&= u_\epsilon(x)
\end{aligned}$$

Because convolution is associative, for $x \in U_{\epsilon+\epsilon_2}$, we have that $u_{\epsilon_2 \epsilon} = u_{\epsilon \epsilon_2} = u_\epsilon \to u(x)$ uniformly. Furthermore,

$$u(x) = u_{\epsilon_2}(x)$$

so we can conclude that for each ϵ, ϵ_2, $u = u_{\epsilon_2} \in C^\infty(U_{\epsilon+\epsilon_2})$ and harmonic.

Exercise 5: Let U be a bounded domain of $\mathbb{R}^n$ and $u \in C^2(U) \cap C(\overline{U})$ which is harmonic in U. Suppose that $u(x_0) = \min_{\overline{U}} u = 0$ at some $x_0 \in \partial U$. Suppose that there exists x_1 so that $B(x_1, r) \in U$ an $\partial(x_1, r) \cap \partial U = \{x_0\}$. Prove that if u is not constant, then

$$\frac{\partial u}{\partial v}(x_0) < 0$$

where v is the outward unit normal to $B(x_1, r)$ at x_0.

Solution: The solution $v = -u$ is still harmonic in the same domain U, so by the Strong maximum principle because $u = -v$ is not constant, $v(x_0) = 0$ is a maximum of v and hence $v < 0 \in U$. Furthermore, $u > 0$ in U.

Exercise 6: Let U be a bounded domain of $\mathbb{R}^n$ with C^2 boundary, in particular it has an interior tangent ball at every boundary point. Use the result of the previous problem to show that any two solutions of the Neumann problem

$$\begin{cases} -\Delta u = f & \text{in} \quad U \\ \frac{\partial u}{\partial v} = h & \text{on} \quad \partial U \end{cases}$$

differ by a constant.

Consider u_1, u_2 solutions to the above. Now consider $\omega = u_2 - u_1$. We see that

$$\begin{cases} -\Delta \omega = 0 & \text{in} \quad U \\ \frac{\partial \omega}{\partial v} = 0 & \text{on} \quad \partial U \end{cases}$$

So ω is harmonic, so using Green's we see that

$$\int_U \omega \Delta \omega dx = \int_{\partial U} \omega \frac{\partial \omega}{\partial v} dS(v) - \int_U |\nabla \omega|^2 dx.$$

Since $\frac{\partial \omega}{\partial v} = \Delta \omega = 0$,

$$\int_U |\nabla \omega|^2 dx = 0 \implies \omega \text{ constant}.$$

Hence $u_2 = u_1 + \text{ constant}$.