Computation 1: Find a number z so that $z^2 = i$. Write it in the form $z = x + iy$.

Solution: We wish to find a number z such that

$$(x + iy)^2 = i.$$

So $x^2 - y^2 = 0$ and $2xy = 1$. Thus we have $x = y = \pm\frac{\sqrt{2}}{2}$ and hence

$$z = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}, \quad z = -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}.$$

Computation 1.1: Shakarchi. Describe geometrically the sets of points in z in the complex plane defined by the list in Sharkarchi.

Solution:

1. This is the line consisting of all points that are equidistant from both z_1, z_2.
2. $z\overline{z} = 1$ is the unit circle.
3. This is the line where all numbers have real part equal 3.
4. The first case is the open half plane with real part greater than c. The second is closed.
5. This is the line which has real part $\mathrm{re}(a)\mathrm{re}(z) - \mathrm{im}(a)\mathrm{im}(z) + \mathrm{re}(b)$.
6. $|z|^2 = (x+1)^2 \implies y^2 = 2x + 1$. This is the complex numbers on this parabola opening to the right.
7. This is a line with imaginary part c for any $c \in \mathbb{R}$.

Computation 9: Shakarchi. Show that in polar coordinates, the Cauchy-Riemann equations take the form

$$\partial_r u = \frac{1}{r}\partial_\theta v \quad \partial_\theta u = -r\partial_r v.$$

Use this to show that $\log(z) = \log(r) + i\theta$ is holomorphic in $\mathbb{C} \setminus (-\infty, 0]$.

Solution:

$$\partial_r u = \partial_x u \partial_r x + \partial_y u \partial_r y \quad \partial_\theta u = \partial_x u \partial_\theta x + \partial_y u \partial_\theta y$$

Using the fact that $x = r\cos\theta, y = r\sin\theta$, we have that

$$u_r = u_x \cos\theta + u_y \sin\theta,\ u_\theta = -u_x r\sin\theta + u_y r\cos\theta$$

Now for u, v, we have that they satisfie Cauchy Riemann So

$$v_r = -u_y\cos\theta + u_x\sin\theta\ v_\theta = u_y r\sin\theta + u_x r\cos\theta$$

It follows directly that

$$ru_r = v_\theta,\ u_\theta = -rv_r.$$

When $z = re^{i\theta}$, $\log z = \log r + i\theta$, so we have that

$$ru_r = r/r = 1 = v_\theta,\ u_\theta = 0 = -r \cdot 0 = -rv_r.$$

So whenever $\theta \neq \pi$, we have holomorphicity.

Computation 16: Shakarchi. Determine the radius of convergence of the series in Shakarchi.

1. $a_n = (\log n)^2$. We want $R = \limsup |a_n|^{1/n}$. Furthermore, we have
$$\lim_{n\to\infty} |\log n^2|^{1/n} = L$$
Taking the log of both sides of the limit, we see that
$$\lim_{n\to\infty} \log|\log n^2|/n \to 0$$
because $O(\log\log n^2) < O(n)$. So we see that $R = 1$.

2. $a_n = n!$. We have
$$|n!|^{1/n}$$
so taking the log on both sides gives $\lim \log n!/n \to \infty$ as $O(\log n!) > O(n)$ given that $n! \approx O(n^n e^{-n})$. So $R = 0$.

3. $a_n = \frac{n^2}{4^n+3n}$. Upon first glance, clearly $a_n \to 0$ so $R = \infty$.

4. $a_n = \frac{(n!)^3}{(3n)!} \approx \frac{c^3 n^{3n+3/2} e^{-3x}}{cn^{3n+1/2}e^{-3x}} = \frac{c^2 n^{3/2}}{n^{1/2}} = c^2 n$. As $n \to \infty$, this goes to ∞ so $R = 0$.

Computation 25: Shakarchi. Cauchy's Theorem preliminaries.

1. Solve
$$\int_C z^n dz$$
for the positive oriented circle containing the origin.

   Solution:
$$\int_C z^n dz = \int_0^{2\pi} (re^{it})^n ire^{it} dt = ir^{n+1}\int_0^{2\pi}(e^{it})^{n+1} dt.$$
We see that when $n \neq -1$, then $\int z^n dz = 0$ because $e^0 = e^{i2\pi}$. Now when $n = -1$, we have that the integral is 2π because it's the definite integral of 1 from 0 to 2π.

2. Now assume the circle does not contain the origin.

   Solution: When $n \geq 0$, the case is trivially 0 as we have the integral above shifted to $|z - z_0|$. We have that when $m < 0$,
$$(1+z)^m = \sum_{k=0}^{\infty}\binom{m}{k} z^k = \sum_{k=0}^{\infty}(-1)^k\binom{m+k-1}{k} z^k.$$
Then, for small enough radius of the contour,
$$\int_0^{2\pi}(2+re^{it})^m ire^{it}dt = c^m ir\int_0^{2\pi}\left(1+\frac{re^{it}}{c}\right)^m e^{it}dt = c^{-m} ir\sum_{k=0}^{\infty}(-1)^k\binom{m+k-1}{k}\frac{r^k}{c^k}$$
Now, we see that $\int_0^{2\pi} e^{i(k+1)t}dt = 0$.

3. Show that if $|a| < r < |b|$, then
$$\int_C \frac{1}{(z-a)(z-b)}dz = \frac{2\pi i}{a-b}.$$
By partial fractions we have
$$\frac{1}{a-b}\left(\int_C \frac{1}{z-a} - \frac{1}{z-b}\right) = \frac{1}{a-b}\int_C \frac{1}{w}dw$$
where $w = z - a$. The integral of $1/(z-b)$ is 0 because $|b| > r$. Furthermore, now we have $a \in C$, so the integral is equal to (by part a)
$$\frac{2\pi i}{a-b}.$$

Exercise 1: Stein Shakarchi 3,4,5,7,23,24

3. With $\omega = se^{i\phi}$, where $s \geq 0$ and $\phi \in \mathbb{R}$, solve the equation $z^n = \omega$ in $\mathbb{C}$ where n is a natural number. How many solutions are there?

   Solution: Write z as $re^{i\theta}$ where $r \geq 0$ and $\theta \in \mathbb{R}$. Now,

$$r^n e^{ni\theta} = se^{i\phi}$$

   so we have that $r = s^{1/n}$ and $\theta = \phi/n$. We see that solutions are the same up to $2n\pi$.

4. Show that it is impossible to define a total ordering on $\mathbb{C}$.

   Solution: Assume such a relation, $\succ$, exists. Now, consider that $i \succ 0$. Furthermore, multiplying by i, we see that $-1 \succ 0$. Now, using (iii) again, multiplying by -1 gives $(-1)(-1) = 1 \succ 0$. This is a contradiction to (i) given that both $1 \succ 0$ and $-1 \succ 0$ hold.

   Solution: Assume such a relation, $\succ$, exists with $i \succ 0$. Using condition (iii), observe that

$$\begin{aligned} i \cdot i \succ 0 \cdot i &\implies -1 \succ 0 \\ -1 \cdot i \succ 0 \cdot i &\implies -i \succ 0 \\ i \cdot (-i) \succ 0 \cdot (-i) &\implies 1 \succ 0. \end{aligned}$$

   This breaks condition (i), so we've arrived at a contradiction, and there is no such relation.

5. Prove that an open set Ω is pathwise connected if and only if Ω is connected.

   (a) Suppose first that Ω is open and pathwise connected, and that it can be written as $\Omega = \Omega_1 \cup \Omega_2$ where Ω_1 and Ω_2 are disjoint non-empty open sets. Choose two points $\omega_1 \in \Omega_1$ and $\omega_2 \in \Omega_2$ and let γ denote a curve in Ω joining ω_1 to ω_2. Consider a parametrization $z : [0,1] \to \Omega$ of this curve with $z(0) = \omega_1$ and $z(1) = \omega_2$, and let

$$t^\star = \sup_{0 \leq t \leq 1} \{t : z(s) \in \Omega_1 \text{ for all } 0 \leq s < t\}.$$

   Arrive at a contradiction by considering the point $z(t^\star)$.

   Solution: 0 is contained in the set $T_1 = \{t : z(s) \in \Omega_1 \text{ for all } 0 \leq s < t\}$. Furthermore, when $t^\star = 1$, $z(t^\star) \in \Omega_2$ so the set $T_2 = \{t : z(s) \in \Omega_2 \text{ for all } t \leq s \leq 1\}$ is non-empty. Now, $T_1 \cup T_2 = [0,1]$, but intervals in $\mathbb{R}$ are connected, so this is a contradiction.

   Define Ω_1, Ω_2, let $z \in \Omega_1$. Since Ω is open and $z \in \Omega$, there exists a ball $B(z,\delta) \subset \Omega$. To show that $B \subset \Omega_1$, let $s \in B$ and consider $f : [0,1] \to \mathbb{C}$ given by $f(t) = st + z(1-t)$. Now, $|f(t) - z| = t|s - z| < \delta$. The image of f is contained in B so we can see that $s \in \Omega_1$. Suppose Ω_2 is not open, then there is some $z \in \Omega_2$ such that every ball around z contains a point of Ω_1. B is one such ball, so consider $s \in \Omega_1 \cap B(z,\delta)$. Similarly, define f such that $|f(t) - z| = t|s - z| < \delta$, so w is path connected to the point s, but that means that $z \in \Omega_1$, which means Ω_2 must be open. Now Ω_1 is non-empty so $\Omega_2 = \emptyset$ by connectedness.

7. Let z, w be two complex numbers such that $\bar{z}w \neq 1$. Prove that

$$\left| \frac{w - z}{1 - \bar{w}z} \right| < 1$$

   if $|z|, |w| < 1$ with equality when $|z|$ or $|w|$ equal 1.
   Then, prove that for a fixed w in the unit disc $\mathbb{D}$, the mapping

$$F : z \mapsto \frac{w - z}{1 - \bar{w}z}$$

   for the given properties.

Solution: Suppose that $|w| < 1$ and $|z| = 1$ then

$$\left|\frac{w-z}{1-\bar{w}z}\right| = \left|\frac{w-z}{\bar{z}-\bar{w}}\right| = 1$$

Because $|w| < 1$, we see that the function $f(z) = (w-z)/(1-\bar{w}z)$ is holomorphic on $\mathbb{D}$. Thus, by the maximum modulus principle, it satisfies $|f(z) < 1$ in $\mathbb{D}$ because it is not constant.
We already have that $F(\mathbb{D}) \subset \mathbb{D}$ by above. Clearly, $F(0) = w$ and $F(w) = 0$. From above, we have $F(\partial\mathbb{D}) \subseteq \partial\mathbb{D}$. F^{-1} equals itself, so it's clearly bijective.

23 Prove that f is infinitely differentiable on $\mathbb{R}$ and $f^{(n)}(0) = 0$ for all $n \geq 1$. Conclude that f does not have a converging power series expansion near the origin.

We see that $f'(x) = -\frac{\exp -1/x^2}{x^3}$, and as $x \to 0$, $e^{-1/x^2} \to 0$ faster than x^3, so $f'(x) \to 0$ as $x \to 0$. This follows for every (n) s.t. $f^{(n)}(x) \approx (-1)^n \frac{e^{-1/x^2}}{x^{2+n}}$ for $n \geq 1$. We see this tends to 0, so f is infinitely differentiable on $\mathbb{R}$. This function can be written using $\omega = 1/x$ so we see that $\exp -\omega^2 = 1 - \omega + \frac{1}{2}\omega^2 - \ldots$, but this actually adopts the Laurent series, as $\omega = x^{-1}$. If there were a polynomial series, then its radius of convergence must be infinite, but the derivatives are all 0 at $x = 0$, so an extension of the summation must be applied.

24 Reversal of curves

We see that

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt = -\int_b^a f(\gamma(t))\gamma'(t)dt = -\int_{-\gamma} f(z)dz.$$

Exercise 2: Prove that if $f : \mathbb{C} \to \mathbb{C}$ is C^2,, and f is holomorphic on $\mathbb{C}$ then f' is holomorphic.

Solution: Because f is holomorphic, it satisfies the cauchy riemann equations with

$$\partial_x u = \partial_y v, \quad \partial_y u = -\partial_x v$$

Additionally, we know that

$$\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2}, \frac{\partial^2 f}{\partial y \partial x}$$

exist (with the last one being equal to its mixed case as it's continuous). Recall that $f'(z) = 2\frac{\partial u}{\partial z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$ let $U = \frac{\partial u}{\partial x}$ and $V = \frac{\partial v}{\partial x}$. Now,

$$\partial_x U = \partial_{xx} u = \partial_{xy} v = \partial_y V, \quad \partial_y U = \partial_{xy} u = \partial_{yx} u = -\partial_{xy} v = -\partial_y V.$$

Hence, f has holomorphic derivative.

Exercise 3: Let γ be a smooth curve in $\mathbb{C}$, when does $\overline{\int_\gamma f(z)dz} = \int_{gamma} \overline{f(z)}dz$ for all f continuous?

WLOG, let $\gamma : [0,1] \to \mathbb{C}$ because γ is smooth, it's sufficiently regular, so we can parametrize it over the contour.

$$\int_C f(z)dz = \int_{[0,1]} f(y(t))y'(t)dt$$

we can take conjugates to yield $\overline{\int_{[0,1]} f(y(t))y'(t)dt} = \int_{[0,1]} \overline{f(y(t))y'(t)}dt = \int_C \overline{f(z)}\overline{dz}$ from the fact that the former is a Riemann integral over $[0,1]$, so the conjugate can be applied over each sum.
Now, we see that there is equality when $\overline{dz} = dz$.

Exercise 4: Suppose that $f : \mathbb{D} \to \mathbb{C}$ is holomorphic and $|f'(z)| \leq M$ then $|f(z) - f(0) \leq M|z|$. Formulate and prove an analogous bound for $|f(z) - f(w)|$ when $f : \Omega \to \mathbb{C}$ where Ω is a general path-connected domain.

Solution: We see that $|f'(z)| = \lim_{w\to z} \frac{|f(z)-f(w)|}{|z-w|} \leq M$. Furthermore, we have that

$$\frac{|f(z) - f(w)|}{|z - w|} \leq M \implies |f(z) - f(w)| \leq M|z - w|.$$

Exercise 5: Show that if f is holomorphic on an open connected set $\Omega \subset \mathbb{C}$ and $f'(z) = 0$ in Ω then f is constant. What if Ω is not connected?

Solution: Because Ω is open and connected, it is also path connected. Let $z_0 \in \Omega$, and any $w \in \Omega$. There is a curve $\gamma : [0, 1] \to \Omega$ such that $\gamma(0) = z_0$ and $\gamma(1) = w$. We then have

$$f(w) - f(z_0) = \int_\gamma f'(z)dz = \int_\gamma 0dz = 0$$

Hence $f(z_0) = f(w)$ for any $w \in \Omega$, so $f(w)$ is constant in Ω. However, if Ω is disconnected, consider $A, B \subset \mathbb{C}$ be disjoint, open sets. Then $A \cup B$ is disconnected. Let $f(A) = 1$ and $f(B) = 0$, then $f'(z) = 0$, but f is not constant.

Exercise 6: Show that there is a holomorphic function on $\mathbb{C} \setminus (-\infty, 0]$ which is equal to $z^{1/2}$ on the positive reals.

$\log(r) + i\theta$ is holomorphic on $r > 0$ and $-\pi < \theta < \pi$. The function $e^{z/2}$ is also holomorphic. A composition of Holomorphic functions is still harmonic so let

$$f(r, \theta) = e^{\log(r)/2+i\theta/2} = r^{1/2}e^{i\theta/2} \implies f(z) = z^{1/2}.$$

So along $z \in \mathbb{R}^+$, we see that $f(z) = \sqrt{z}$. Holomorphicity comes from the branch cut on $(-\infty, 0]$, or when $\theta = \pm\pi$. which is not in the domain.