Exercise 16: The Kelvin transform $\mathcal{K}u = \bar{u}$ of a function $u : \mathbb{R}^n \to \mathbb{R}$ is

$$\bar{u} : u(x/|x|^2)|x|^{2-n}$$

with $\bar{x} = x/|x|^2$. Show that if u is harmonic, then so is $\bar{u}$.

First, compute $D_x\bar{x}$

$$D_x\bar{x} = \frac{I|x|^2 - 2xx^T}{|x|^4}$$

Now compute $D_x\bar{x}(D_x\bar{x})^T$

$$\begin{aligned} D_x\bar{x}(D_x\bar{x})^T &= \frac{1}{|x|^8}(I|x|^2 - 2xx^T)(I|x|^2 - 2xx^T)^T \\ &= \frac{1}{|x|^8}(I|x|^2 - 2xx^T)(I|x|^2 - 2xx^T) \\ &= \frac{1}{|x|^8}(I|x|^4 - 4|x|^2xx^T + 4|x|^2xx^T) = I/|x|^4 = |\bar{x}|^4 I \end{aligned}$$

as $\left|\frac{x}{|x|}\right| = \left|\frac{1}{|x|}\right|$.
Now, we compute as $\Delta\bar{x} = n(2-n)\frac{x}{|x|^4}$. Furthermore, we can compute $\Delta u(\bar{x})$ with

$$\Delta(u(\bar{x})) = \sum_{i=1}^{n} \partial_{x_i}\bar{x}^T D^2u(\bar{x})\partial_{x_i}\bar{x} + Du(\bar{x})^T\Delta\bar{x} = \mathrm{Tr}((D\bar{x})^T D^2 u D\bar{x}) + Du \cdot \Delta\bar{x}.$$

We see that $\mathrm{Tr}((D\bar{x})^T D^2 u D\bar{x}) = |\bar{x}|^4\Delta u$ from the hint above. Now, we compute $\Delta\bar{u}$ with chain rule,

$$\Delta\bar{u} = |\bar{x}|^{n-2}|\bar{x}|^4(\Delta u) + u(\bar{x})\Delta(|\bar{x}|^{n-2}) + 2DuD\bar{x}(D|\bar{x}|^{n-2})^T + |\bar{x}|^{n-2}Du \cdot \Delta\bar{x}.$$

The first two terms are equivalently 0 because $\Delta|\bar{x}|^{2-n} = \sum_{j=1}^{n}(n-2)\left(1 - \frac{nx_j^2}{|x|^2}\right)$ and u is harmonic.
Now we see that

$$2DuD\bar{x}(D|\bar{x}|^{n-2})^T = 2Du\left(|x|^{-2}\left(I - 2\frac{xx^T}{|x|^2}\right)\right)(2-n)|x|^{-n}x^T = 2(n-2)|x|^{-2-n}Du \cdot x^T = -|\bar{x}|^{n-2}Du \cdot \Delta\bar{x}.$$

This term allows $\Delta\bar{u} = 0$ so the Kelvin transform of u is also harmonic.