Exercise 17: Let U be a bounded domain in $\mathbb{R}^n$ and $\alpha \in (0,1)$. Suppose that $u \in C^2(U) \cap C(\overline{U})$ solves

$$\begin{cases} -\Delta u = |u|^\alpha & \text{in } U \\ u(x) = 0 & \text{on } \partial U. \end{cases}$$

Show that

$$\sup_U |u| \leq C$$

where C depends only on n, α, and $\mathrm{diam}(U)$.

Solution. By Evans 2.6,

$$\max_{\bar{U}} |u| \leq \frac{\max_{\overline{U}} |x|^2}{2n} \max_{\overline{U}} |u|^\alpha$$

□

Exercise 19:

Solution. Let u be a solution, then it has

$$u(x) = \int_{\partial B(0,r)} \frac{r^2 - |x|^2}{n\alpha(n)r} \frac{u(y)}{|x-y|^n} dS(y)$$

□

Exercise 20:

Solution. Let u be a strict subsolution so $-\Delta u + |Du|^2 < 0$ with $u < v$ on ∂U. Then, we see that

$$u - v$$

□

Exercise 21:

Solution. For $n \geq 3$, let $v(x) = \frac{1}{n(n-2)\alpha(n)} |x|^{2-n}$. □