Exercise 18: Consider the stationary Schrodinger operators.

Solution.

1. We can let $V(x) = -\pi$ so then in $\mathbb{R}^1$, it solves the ODE

$$u'' = -\pi u$$

which has solution $\sin(\pi x)$, so let this be on the interval $(0,1) \subset \mathbb{R}$. This violates maximum principle as $u(x)$ larger than 0 on its domain.

2. By problem 2.6 in Evans, we see that

$$\max_{\overline{U}} |u| \leq \frac{\operatorname{diam}(U)^2}{2n} \max_{\overline{U}}(|V(x)u|) \leq \frac{M\operatorname{diam}(U)^2}{2n} \max_{\overline{U}}(|u|)$$

Now, let $\delta = \sqrt{\frac{n}{M}}$, and we see that whenever $\operatorname{diam}(U) \leq \delta$

$$\max_{\overline{U}} |u| \leq \frac{1}{2} \max_{\overline{U}} |u|$$

which implies that $u = 0$

3. Again, using problem 2.6 in Evans, this time fixing $x_2, \dots x_n$ and taking the maximum over x_1,

$$\max_{x_1 \in \overline{U}} |u| \leq \frac{\operatorname{diam}(U)^2}{2n} \max_{\overline{U}}(|V(x)u|) \leq \frac{M\operatorname{diam}(U)^2}{2n} \max_{x_1 \in \overline{U}}(|u|)$$

We see that because $x_2, \dots, x_n$ are fixed, $U \subseteq \{0 < x_1 < \operatorname{diam}(U)\}$. Now, letting $\{0 < x_1 < \delta\}$ for $\delta = \sqrt{\frac{n}{M}}$, we see that

$$\max_{x_1 \in \overline{U}} |u| \leq \frac{1}{2} \max_{x_1 \in \overline{U}} |u|$$

Hence, $u(x) = 0$ for any x_1, but this choice of $x_2, \dots x_n$ was arbitrary so $u = 0$ for all $x \in U$ such that $U \subset \{0 < x_1 < \delta\}$.

□