c) $L = \frac{1}{2} I_0 (\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2) + \frac{1}{2} I_z (\dot{\psi} + \dot{\phi}\cos\theta)^2 - mgd\cos\theta$

$$\frac{\partial L}{\partial \dot{\phi}} = I_0 \dot{\phi} \sin^2\theta + \frac{1}{2} I_z \frac{\partial}{\partial \dot{\phi}} (\dot{\psi}^2 + 2\dot{\phi}\dot{\psi}\cos\theta + \dot{\phi}^2\cos^2\theta)$$

$$= I_0 \dot{\phi} \sin^2\theta + I_z \dot{\psi}\cos\theta + I_z \dot{\phi}\cos^2\theta$$

$$p_\phi = \dot{\phi}(I_0 \sin^2\theta + I_z \cos^2\theta) + I_z \dot{\psi}\cos\theta$$

$$\frac{\partial L}{\partial \dot{\psi}} = \frac{1}{2} I_z \frac{\partial}{\partial \dot{\psi}} (\dot{\psi}^2 + 2\dot{\phi}\dot{\psi}\cos\theta + \dot{\phi}^2\cos^2\theta)$$

$$p_\psi = I_z \dot{\psi} + I_z \dot{\phi}\cos\theta$$

$$\frac{\partial L}{\partial \dot{\theta}} = \frac{1}{2} I_0 \cdot 2\sin\theta\cos\theta\,\dot{\phi}^2 + \frac{1}{2} I_z (-2\dot{\phi}\dot{\psi}\sin\theta$$

$$+ \dot{\phi}^2 2(\cos\theta\sin\theta) + mgd\sin\theta$$

$$\frac{\partial L}{\partial \theta} = \dot{\phi}^2 \sin\theta\cos\theta (I_0 - I_z) - \dot{\phi}\dot{\psi}\sin\theta + mgd\sin\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{\partial L}{\partial \theta} \qquad \frac{\partial L}{\partial \dot{\theta}} = I_0 \dot{\theta}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = I_0 \ddot{\theta}$$

$$I_0 \ddot{\theta} = \dot{\phi}^2 \sin\theta\cos\theta (I_0 - I_z) - \dot{\phi}\dot{\psi}\sin\theta + mgd\sin\theta$$

b)

$x = r\cos\theta \qquad y = r\sin\theta$

$m \propto A \qquad m = \rho \pi R^2$

$dm \propto A \qquad dA = 2\pi r\, dr$

$I = \int r^2 dm \qquad m = \rho A$ ctes

$$I = \int_0^r r^2 \rho\, 2\pi r\, dr = \rho 2\pi \int_0^r r^3 dr$$

$$I = \rho \frac{\pi r^4}{2} = = \rho \pi r^2 \cdot \frac{r^2}{2} = \frac{1}{2} m r^2$$

a) Usando ejes paralelos tenemos que:

$$I_0 = I_{cm} + md^2$$

Donde I_{cm} es el momento de inercia del disco bajo el otro eje de rotación.

Que en el caso del disco es $I_{cm} = \frac{1}{4} mr^2$

$$I_0 = \frac{1}{4} mr^2 + md^2$$