$$\Omega = \{(t_{n-2}, f_{n-2}), (t_{n-1}, f_{n-1}), (t_n, f_n)\}$$

$$y_{n+1} = y_n + \frac{h}{12}(23f_n - 16f_{n-1} + 5f_{n-2})$$

$$y_{n+1} = y_n + \frac{h}{24}(55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3}).$$

$$y_{n+1} = y_n + \sum_{i=0}^{k} B_i f_i \qquad P_k(x) = \sum_{i=0}^{k} f(x_i) l_i(x)$$

$$l_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{k} \frac{x - x_j}{x_i - x_j}$$

$$P_3(t) = \sum_{i=0}^{4} f(x_i)\, l_i(x)$$

$$l_1 = \left(\frac{t - t_{n-1}}{t_n - t_{n-1}}\right)\left(\frac{t - t_{n-2}}{t_n - t_{n-2}}\right) = \frac{(t - t_{n-1})}{h}\frac{(t - t_{n-2})}{2h} = \frac{1}{2h^2}\left[t^2 - t t_{n-2} - t t_{n-1} + (t_{n-1} t_{n-2})\right] \quad -2h t_n$$

$$l_2 = \left(\frac{t - t_n}{t_{n-1} - t_n}\right)\left(\frac{t - t_{n-2}}{t_{n-1} - t_{n-2}}\right) = \frac{(t - t_n)(t - t_{n-2})}{-h \cdot h} = -\frac{1}{h^2}\left[t^2 - t t_{n-2} - t t_n + (t_n t_{n-2})\right]$$

$$l_3 = \left(\frac{t - t_n}{t_{n-2} - t_n}\right)\left(\frac{t - t_{n-1}}{t_{n-2} - t_{n-1}}\right) = \frac{(t - t_n)(t - t_{n-1})}{2h \cdot -h} = \frac{1}{2h^2}\left[t^2 - t t_{n-1} - t t_n + (t_n t_{n-1})\right]$$

$$y_{n+1} = y_n + \left[\frac{f_n}{2h^2}\int (t^2 - t t_{n-2} - t t_{n-1} + (t_n t_{n-1}))\,dt + \frac{f_{n-1}}{h^2}\int (t^2 - t t_{n-2} - t h + (t_n t_{n-2}))\,dt + \frac{f_{n-2}}{2h^2}\int (t^2 - t t_{n-1} \ldots)\right]$$

$-\frac{t_n^2 h}{2}$ $\quad h) - t_n^3$

Para f_{n-1}

$$\left(\frac{t_n - t_n}{t_{n-1} - t_n}\right)\left(\frac{t - t_{n-2}}{t_{n-1} - t_{n-2}}\right) = \frac{(t - t_n)}{-h}\frac{(t - t_n + 2h)}{h} = -\frac{1}{h^2}\left[t^2 - t t_n + 2ht - t_n t + t_n^2 - 2h t_n\right]$$

$$= -\frac{1}{h^2}(t^2 - 2t t_n + 2ht + t_n^2 - 2h t_n) \} Q$$

$$\int_{t_n}^{t_{n+1}} Q = -\frac{1}{h^2}\left[\frac{(t_n+h)^3}{3} - \frac{t_n^3}{3} - 2(t_n+h)t_n + 2t_n^2 + \frac{2h(t_n+h)^2}{2} - \frac{2h t_n^2}{2}\right.$$

$$\left. t_n^2(t_n+h) - t_n^3 - 2h t_n(t_n+h) + 2h t_n^2 \right]$$

$$= -\frac{1}{h^2}\left[\frac{(t_n^2 + 2h t_n + h^2)(t_n+h)}{3} - \frac{t_n^3}{3} - 2t_n^2 - 2t_n h + 2t_n^2 + h(t_n^2 + 2h + h^2) - h t_n^2\right.$$

$$\left. t_n^3 + t_n^2 h - t_n^3 - 2h t_n^2 - 2h^2 t_n + 2h t_n^2 \right]$$

$$= -\frac{1}{h^2}\left[\frac{t_n^3}{3} + \frac{2h t_n^2}{3} + \frac{h^2 t_n}{3} + \frac{t_n^2 h}{3} + \frac{2h^2 t_n}{3} + \frac{h^3}{3} - \frac{t_n^3}{3} - 2t_n h + h t_n^2 + 2h^2 + h^3 - h t_n^2 + t_n^2 h - 2h^2 t_n\right]$$

$$= -\frac{1}{h^2}\left[h t_n^2 + h^2 t_n + \frac{h^3}{3} - 2t_n h + 2h^2 + h^3 + t_n^2 h - 2h^2 t_n\right]$$

$$= -\frac{1}{h^2}\left[\frac{4h^3}{3}\right] = -\frac{4}{3}h$$

Para f_{n-2}

$$\int \left(\frac{t - t_n}{t_{n-2} - t_n}\right)\left(\frac{t - t_{n-1}}{t_{n-2} - t_{n-1}}\right) = \frac{(t - t_n)}{-2h}\frac{(t - t_n + h)}{-h} = \frac{1}{2h^2}\left[t^2 - t t_n + t h - t t_n + t_n^2 - t_n h\right]$$

$$= \int_{t_n}^{t_{n+1}} \frac{1}{h^2}\left[t^2 - 2t t_n + t h + t_n^2 - t_n h\right]$$

$$\frac{1}{2h^2}\left[\frac{(t_n+h)^3}{3} - \frac{t_n^3}{3} - \frac{2t_n(t_n+h)^2}{2} + \frac{2t_n^3}{2} + \frac{(t_n+h)^2 h}{2} - \frac{t_n^2 h}{2} + t_n^2(t_n+h) - t_n^3 - t_n h(t_n+h)\right.$$

$$\left. + t_n h(t_n) \right] \quad -2h t_n$$

$$= \frac{1}{2h^2}\left[\frac{(t_n^2+2ht_n+h^2)(t_{n+1})}{3} - \frac{t_n^3}{3} - t_n(t_n^2+2ht_n+h^2) + t_n^3 + (t_n^2+h2t_n+h^2)\frac{h}{2} - \frac{t_n^2 h}{2}\right.$$

$$\left. t_n^3 + t_n^2 h - t_n^3 - t_n^2 h - t_n h^2 + t_n^2 h \right\}$$

$$= \frac{1}{2h^2}\left[\frac{t_n^3}{3} + \frac{2ht_n^2}{3} + \frac{h^2 t_n}{3} + \frac{t^2 h}{3} + \frac{2h^2 t_n}{3} + \frac{h^3}{3} - \frac{t_n^3}{3} - t_n^3 - 2ht_n^2 - h^2 t_n + t_n^3 + \frac{t_n^2 h}{2} + h^2 t_n + \frac{h^3}{2} - \frac{t_n^2 h}{2}\right.$$

$$\left. + t_n^2 h - t_n h^2 \right]$$

$$= \frac{1}{2h^2}\left[ht_n^2 + h^2 t_n + \frac{h^3}{3} - 2ht_n^2 - h^2 t_n + h^2 t_n + \frac{h^3}{2} + t_n^2 h - t_n h^2\right]$$

$$= \frac{1}{2h^2}\left[\frac{5h^3}{6}\right] = \frac{5}{12}h$$

Para f_n.

$$\left(\frac{t-t_{n-1}}{t_n-t_{n-1}}\right)\left(\frac{t-t_{n-2}}{t_n-t_{n-1}}\right) = \left(\frac{t-t_n+h}{h}\right)\left(\frac{t-t_n+2h}{2h}\right) = \frac{1}{2h^2}\left[t^2 - 2t_n + 2ht - t_n t + t_n^2 - 2ht_n + ht - t_n h + 2h^3\right]$$

$$= \int_{t_n}^{t_{n+1}} \frac{1}{2h^2}\left[t^2 - 2tt_n + 3th + t_n^2 - 3ht_n + 2h^3\right]$$

$$= \frac{1}{2h^2}\left[\frac{(t_n+h)^3}{3} - \frac{t_n^3}{3} - 2\frac{(t_n+h)^2 t_n}{2} + \frac{t_n^3}{2} + 3h\frac{(t_n+h)^2}{2} - \frac{3h}{2}t_n^2 + t_n^2(t_n+h) - t_n^3\right.$$

$$\left. - 3ht_n(t_n+h) + 3ht_n^2 + 2h^3(t_n+h) - 2h^3 t_n \right.$$

$$= \frac{1}{2h^2}\left[\frac{(t_n^2+2ht_nh^2)(t_{n+1})}{3} - \frac{t_n^3}{3} - (t_n^2+2ht_n+h^2)t_n) + \frac{t_n^3}{2} + \frac{3h}{2}(t_n^2+2ht_n+h^2) - \frac{3h}{2}t_n^2\right.$$

$$\left. t_n^3 + t_n^2 h - t_n^3 - 3ht_n^2 - 3h^2 t_n + 3ht_n^2 + 2h^3 t_n + 2h^4 t_n - 2h^3 t_n \right\}$$

$$= \frac{1}{2h^2}\left[\frac{23h^3}{6}\right] = \frac{23h}{12}$$

Entonces tenemos

$$\cancel{y_{n+1} = y_n + \frac{23hf_{n+1}}{12} - \frac{8hf_n}{12}}$$

$$y_{n+1} = y_n + \frac{23hf_n}{12} - \frac{4}{3}hf_{n-1} + \frac{5}{12}hf_{n-2}.$$

$$= y_n + \frac{h}{12}\left[23f_n - 16f_{n-1} + 5hf_{n-2}\right].$$

Para el de 3 orden tomamos $R = \{(t_{n-2}, f_{n-2}), (t_{n-1}), f_{n-1}), (t_n, f_n), (t_{n-3}, f_{n-3})$

$$= y_n + \frac{h}{12}\left[23f_n - 16f_{n-1} + 5\right.$$

Para el de 3 orden tomamos $R = \{(t$

del mismo procedimiento del anterior punto

Para f_{n-3}

$$\int_{t_n}^{t_{n+1}} \left(\frac{t-t_{n-1}}{t_{n-3}-t_{n-1}}\right)\left(\frac{t-t_{n-2}}{t_{n-3}-t_{n-2}}\right)\left(\frac{t-t_n}{t_{n-3}-t_n}\right) = ($$

$$= \int \frac{1}{6h^3}\left[(t-t_n+h)(t-t_n+2h)(t-t_n)\right.$$

$$= -\int \frac{1}{6h^3}\left[(t^2 - tt_n + 2ht - tt_n + t_n^2 -\right.$$

$$= \int \frac{1}{6h^3}\left[t^3 - t^2 t_n + 2h^2 - t^2 t\right.$$

$$+ 2htt_n - tt$$

$h f_{n-2}]$.

$\ldots, f_{n-2}), (t_{n-1}, f_{n-1}), (t_n, f_n), (t_{n-3}, f_{n-3})$.

tenemos.

$\frac{(t-t_{n-1})}{-2h} \frac{(t-t_{n-2})}{-h} \left(\frac{t-t_n}{-3h}\right)$

$]$.

$2ht_n + ht - ht_n + 2h^2)(t-t_n)]$

$t_n^2 t - 2ht_n t + ht^2 - ht_n t + 2h^2 t - t_n t^2 - t t_n^2$.

$t_n^2 + t_n^3 - 2ht_n^2 + ht t_n - ht_n^2 + 2h^2 t_n]$.

$$-\frac{1}{6h^3}\left[\frac{t^4}{4} - \frac{t^3}{3}t_n + 2h^2 t - \frac{t^3}{3}t_n + \frac{t_n^2 t^2}{2} - \frac{2ht_n t^2}{2} + \frac{ht^3}{3} - \frac{ht_n t^2}{2} + \frac{2h^2 t^2}{2} - \frac{t_n t^3}{3} - \frac{t^2}{2}t_n^2 + \frac{2ht^2}{2}t_n - \frac{t^2}{2}t_n^2 + t_n^3 t - 2ht_n^2 t + \frac{ht^2}{2}t_n - ht_n^2 t + 2h^2 t_n\right]\Bigg|_{t_n}^{t_{n+1}}$$

$$= \cancel{\ldots} = \cancel{\ldots} = \frac{1}{6h^3}\left[-\frac{9h^4}{4}\right] = -\frac{9h}{24}$$

Para f_{n-2}.

$$\int_{t_n}^{t_{n+1}} \left(\frac{t-t_{n-3}}{t_{n-2}-t_{n-3}}\right)\left(\frac{t-t_{n-1}}{t_{n-2}-t_{n-1}}\right)\left(\frac{t-t_n}{t_{n-2}-t_n}\right) = \left(\frac{t-t_n+3h}{h}\right)\left(\frac{t-t_n+h}{-h}\right)\left(\frac{t-t_n}{-2h}\right)$$

$$= \int \frac{1}{2h^3}\left[(t^2 - tt_n + th + tt_n^2 - tt_n - ht_n + 3ht - 3ht_n + 3h^2)(t-t_n)\right]$$

$$= \int \frac{1}{2h^3}\Big(t^3 - t^2 t_n + t^2 h + t_n^2 t - t^2 t_n - ht_n t + 3ht^2 - 3ht_n t + 3h^2 t$$
$$- t^2 t_n + tt_n^2 + tht_n - t_n^3 + tt_n^2 + ht_n^2 - 3htt_n + 3ht_n^2 - 3h^2 t_n\Big]$$

$$= \frac{1}{2h^3}\Big[\frac{t^4}{4} - \frac{t^3}{3}t_n + \frac{t^3}{3}h + t_n^2\frac{t^2}{2} - \frac{t^3}{3}t_n - \frac{ht^2}{2}t_n + \frac{3ht^3}{3} - 3h\frac{t^2 t_n}{2} + 3h^2\frac{t^2}{2}$$
$$- \frac{t^3}{3}t_n + \frac{t^2}{2}t_n^2 - \frac{t^2}{2}ht_n - t_n^3 t + \frac{t^2}{2}t_n^2 + ht_n^2 t - 3h\frac{t^2}{2}t_n + 3ht_n^2 t - 3h^2 t_n t\Big)\Big|_{t_n}^{t_{n+1}}$$

$$= \frac{1}{2h^3}\left[\frac{37h^4}{12}\right] = \frac{37h}{24}$$

Para f_{n-1}

$$\int_{t_n}^{t_{n+1}} \left(\frac{t-t_{n-3}}{t_{n-1}-t_{n-3}}\right)\left(\frac{t-t_{n-2}}{t_{n-1}-t_{n-2}}\right)\left(\frac{t-t_n}{t_{n-1}-t_n}\right) = \int \frac{(t-t_{n-3})}{2h}\frac{(t-t_{n-2})}{h}\frac{(t-t_n)}{-h} = \int \frac{-1}{2h^3}$$

$$(t-t_{n-3})(t-t_{n-2})(t-t_n) = \int_{t_n}^{t_{n+1}} (t-t_n+3h)(t-t_n+2h)(t-t_n) \cdot \frac{-1}{2h^3} = \frac{-1}{2h^3}\left(\frac{h^4 59}{12}\right)$$

$$= \frac{-59}{24}h$$

Para f_n tenemos

$$\int_{t_n}^{t_{n+1}} \left(\frac{t-t_{n-3}}{t_n-t_{n-3}}\right)\left(\frac{t-t_{n-2}}{t_n-t_{n-2}}\right)\left(\frac{t-t_{n-1}}{t_n-t_{n-1}}\right) = \left(\frac{t-t_n+3h}{3h}\right)\left(\frac{t-t_n+2h}{2h}\right)\left(\frac{t-t_n+h}{h}\right)$$

$$= \int_{t_n}^{t_{n+1}} \frac{1}{6h^3}\left[(t-t_n+3h)(t-t_n+2h)(t-t_n+h)\right] = \frac{1}{6h^3}\left[\frac{55h^4}{4}\right] = \frac{55h}{24}.$$

En conjunto tenemos

$$y_{n+1} = y_n + \frac{h55}{24}f_n - \frac{59}{24}hf_{n-1} + \frac{37}{24}hf_{n-2} - \frac{9}{24}hf_{n-3}.$$

$$= y_n + \frac{h}{24}(55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3}).$$

3

$= y_n + \frac{h}{24}(55 f_n - 59 f_{n-1} + 37 f_{n-2} - 9 f_{n-3})$.

Adams Moulton . . .

De 3 puntos $\Omega = \{(t_{n-1}, f_{n-1}), (t_n, f_n), (t_{n+1}, f_{n+1})\}$.

Vamos Para f_{n-1} $\int_{t_n}^{t_{n+1}} \frac{(t-t_n)}{(t_{n-1}-t_n)} \frac{(t-t_{n+1})}{(t_{n-1}-t_{n+1})} = \int \frac{(t-t_n)}{-h} \frac{(t-t_n-h)}{-2h} = \int \frac{1}{2h^2}(t-t_n)(t-t_n-$

$= \int \frac{1}{2h^2}\left[\frac{2h^3}{8}\right] = -\frac{8h^2}{[illegible]} = -h^2$

Para f_n

$\int_{t_n}^{t_{n+1}} \left(\frac{t-t_{n-1}}{t_n-t_{n-1}}\right)\left(\frac{t-t_{n+1}}{t_n-t_{n+1}}\right) = \int_{t_n}^{t_{n+1}} \frac{(t-t_n+h)}{h} \frac{(t-t_n-h)}{-h} = \frac{-1}{h^2}\left[\frac{-h^3 8}{12}\right] = \frac{h8}{12} = \frac{8h}{12}$.

$y_{n+1} = y_n + \left[\frac{f_n}{2h^2}\right]_{t_{n-1}}$

Para f_{n+1}

$\int_{t_n}^{t_{n+1}} \left(\frac{t-t_n}{t_{n+1}-t_n}\right)\left(\frac{t-t_{n-1}}{t_{n+1}-t_{n-1}}\right) = \int \frac{(t-t_n)}{h} \frac{(t-t_n+h)}{2h} = \frac{1}{2h^2}\left[\frac{5}{6}h^3\right] = \frac{5h}{12}$

de modo que en total tenemos

$y_{n+1} = y_n + \frac{f_{n+1} 5h}{12} + \frac{f_n 8h}{12} - \frac{f_{n-1} h}{12}$

$y_{n+1} = y_n + \frac{h}{12}[5f_{n+1} + 8f_n - f_{n-1}]$.

$= \frac{5h}{12}$

Para el 4 orden tomamos el conjunto de soporte. $\Omega = \{(t_{n+1}, f_{n+1})(t_n, f_n)(t_{n-1}, f_{n-1})$
$(t_{n-2}, f_{n-2})\}$.

Para f_{n-2}

$$\int_{t_n}^{t_{n+1}} \left(\frac{t-t_n}{t_{n-2}-t_n}\right)\left(\frac{t-t_{n-1}}{t_{n-2}-t_{n-1}}\right)\left(\frac{t-t_{n+1}}{t_{n-2}-t_{n+1}}\right) = \int \frac{(t-t_n)}{-2h} \frac{(t-t_n+h)}{-h} \frac{(t-t_n-h)}{-3h} = \int -\frac{1}{6h^3}\left[-\frac{h^4}{4}\right]$$

$$= \frac{h}{24}$$

Para f_{n-1}

$$\int_{t_n}^{t_{n+1}} \left(\frac{t-t_n}{t_{n-1}-t_n}\right)\left(\frac{t-t_{n-2}}{t_{n-1}-t_{n-2}}\right)\left(\frac{t-t_{n+1}}{t_{n-1}-t_{n+1}}\right) = \int \frac{(t-t_n)}{-h} \frac{(t-t_n+2h)}{h} \frac{(t-t_n-h)}{-2h} = \frac{1}{2h^3}\left[-\frac{5h^4}{12}\right]$$

$$= -\frac{5h}{24}$$

Para f_n

$$\int_{t_n}^{t_{n+1}} \left(\frac{t-t_{n-1}}{t_n-t_{n-1}}\right)\left(\frac{t-t_{n-2}}{t_n-t_{n-2}}\right)\left(\frac{t-t_{n+1}}{t_n-t_{n+1}}\right) = \int \frac{(t-t_n+h)}{h} \frac{(t-t_n+2h)}{2h} \frac{(t-t_n-h)}{-h} = -\frac{1}{2h^3}\left[\frac{-h^4 19}{12}\right]$$

$$= \frac{19h}{24}$$

Para f_{n+1}

$$\int_{t_n}^{t_{n+1}} \left(\frac{t-t_n}{t_{n+1}-t_n}\right)\left(\frac{t-t_{n-1}}{t_{n+1}-t_{n-1}}\right)\left(\frac{t-t_{n-2}}{t_{n+1}-t_{n-2}}\right) = \int \frac{(t-t_n)}{h} \frac{(t-t_n+h)}{2h} \frac{(t-t_n+2h)}{3h} = \frac{1}{6h^3}\left[\frac{9h^4}{4}\right]$$

$$= \frac{9h}{24}$$

Entonces $y_{n+1} = y_n + \frac{9 f_{n+1} h}{24} + \frac{19 f_n h}{24} - \frac{5 f_{n-1}}{24} + \frac{f_{n-2}}{24}$

$$y_{n+1} = y_n + \frac{h}{24}\left(9f_{n+1} + 19f_n - 5f_{n-1} + f_{n-2}\right)$$

1

Verlet

$r_{n+1} = 2r_n - r_{n-1} + h^2 a_n$

$v_n = (r_{n+1} - r_{n-1})/(2h)$

9a $\quad r_{n+1} = r_n + hv_n + h^2 \sum_{p=1}^{q-1} b_p a_{n-q+1}$ $\quad b_p, c_p$ y d_p se obtienen expandiendo en Taylor.

9b $\quad r_{n+1} = r_n + hv_n + h^2 \sum_{p=1}^{q-1} c_p a_{n-q+2}$

9c $\quad hv_{n+1} = r_{n+1} - r_n + h^2 \sum_{p=1}^{q-1} d_p a_{n-q+2}$.

For $q = 3$

$r_{n+1} = r_n + hv_n + (h^2/6)(4a_n - a_{n-1}) + (1/8)h^4 r_n^{(4)}$

$r_{n+1} = r_n + hv_n + (h^2/6)(a_{n+1} + 2a_n) - (1/12)h^4 r_n^{(4)}$

$hv_{n+1} = r_{n+1} - r_n + (h^2/6)(2a_{n+1} + a_n) - (1/24)h^4 r_n^{(4)}$.

Expandiendo en series de Taylor tenemos.

$r_{n+1}(t_n) + r_{n+1}'(t_n)(r_n - t_n) + \frac{r_{n+1}''(t_n)}{2!}(r_n - t_n)^2 + \frac{r_{n+1}'''(t_n)}{3!}(r_n - t_n)^3$

$+ \frac{r_{n+1}''''(t_n)}{4!}(r_n - t_n)^4 + \dots$

Tomando la primera expresión

$$r_{n+1} = r_n + hv_n + \frac{h^2}{6}(4a_n - a_{n-1})$$

la expansión en serie de Taylor alrededor de t_n. Usando la fórmula $f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3} \cdots$

Iniciando todo términos).

y la expresión para r_{n+1} es

$$r_{n+1} = r_n + hv_n + h^2 \sum_{p=1}^{q-1} b_p a_{n-q+1}.$$

$$f(a) = r_n + hv_n + h^2 \sum_{p=1}^{q-1} b_p a_{n-q+1}.$$

$$f'(a) = v_n + ha_n + h^2 \sum_{p=1}^{q-1} b_p a'_{n-q+1}$$

$$f''(a) = a_n + ha'_n + h^2 \sum_{p=1}^{q-1} b_p a''_{n-q+1}$$

$$f'''(a) = a'_n + ha''_n + h^2 \sum_{q=1}^{q-1} b_p a'''_{n-q}$$

$\vdots$

$$r_{n+1} = r_n + hv_n + h^2 \sum_{q=1}^{q-1} b_p a_{n-q+1}.$$

q=1 ⟶ el error

$$r_{n+1} = r_n + hv_p + h^2 b_p (a_n + a_{n-1})$$

expando

$$r_{n+1} = r_n + hv_p + h^2 b_p (a_n + a_{n-1}) + v_n + ha_n + h^2 b_p (a'_n + a'_{n-1})$$

$$+ \frac{a_n}{2} + \frac{ha'_n}{2} + \frac{h^2 b_p}{2}(a'_n + a'_{n-1}) \cdots = r(t_n + h)$$

$$r_{n+1} = r_n + hv_p + h^2 b_p (a_n + a_{n-1}) + v_n (t - t + h) + ha_n (t - t + h)$$
$$+ h^2 b_p (a'_n + a'_{n-1})(t + t - t) + \frac{a_n}{2}(t - t + h)^2 + h\frac{a'_n}{2} \cdots$$

$$= r_n + hv_p + h^2 b_p (a_n - a_{n-1}) + v_n h + h^2 a_n + h^3 b_p (a'_n - a'_{n-1})$$

$$= r_n + 2hv_n + h^2 (b_p a_n - b_p a_{n-1} + a_n) + \cdots$$

$$= r_n + 2hv_n + h^2 (6a_n - 6a_{n-1}) - hv_n + h^2 a_n + h^3 \cdots$$

$$= \cancel{r_n + 2hv_n +} = r_n + hv_n + h^2 4a_n - a_{n-1}$$

$$= r_n + hv_n + 4h^2 6a_n - 6a_{n-1}.$$

$$r_{n+1} = r_n + hv_n + h^2 \sum_{q=1}^{q-1} c_p a_{n-q+1}.$$

$$r_{n+1} = r_n + hv_n + h^2 \left(c_p (a_{n-1} + a_n \right) + v_n h + h^2 a_n + h^3 c_p \left(a'_{n-1} + a'_n \right)$$

$$+ a_n h^2 + h^3 a'_n + h^4 c_p \left(a''_{n-1} + a''_n \right)$$

$$= r_n + hv_r + h^2 c_p a_{n-1} + h^2 a_n + v_n h + h^2 a_n + h^3 c_p \left(a'_{n-1} + a'_n \right).$$

$$+ a_n h^2 + h^3 a'_n + h^4 c_r \left(a''_{n-} + a''_n \right)$$

$$=. r_n + hv_r + h^2 c_p (a_{n-1} + a_n).$$

$$h v_{n+1} = r_{n+1} - r_n + h^2 d \left(a_{n-1} + a_n \right) + v_{n+1} h - v_n h + h^3 d \left(a'_{n-1} + a_n \right.$$

$$+ a_{n+1} h^2 - a_n h + h^4 d \left(a''_{n-1} - a''_n \right).$$