# UTS TRU

## Muhammad Ya'mal Amilun (10222070)

$$ds^2 = -f(r)c^2dt^2 + g(r)dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

Tinjaulah sebuah ruang waktu lengkung yang memiliki metrik seperti persamaan di atas dengan koordinat bola (ct, r, $\theta$, $\phi$).

**A. Hitunglah semua symbol Christoffel untuk metrik di atas.**

❖ *Penurunan Levi-Civita Connection Coefficient*

Christoffel simbol sering juga disebut sebagai *connection coefficient* pada suatu covariant derivative (atau yang biasanya disebut sebagai connection). Apabila suatu connection memenuhi properti *torsion free* dan *metric compatibility,* maka disebut dengan levi-civita connection. Kedua properti tersebut menyebabkan simbol christoffel memiliki persamaan seperti yang akan diturunkan di bawah.

$$Torsion\ Free \rightarrow \nabla_{\vec{e_i}}\vec{e_j} = \nabla_{\vec{e_j}}\vec{e_i} \rightarrow \Gamma^k_{ij} = \Gamma^k_{ji}$$

$$Metric\ Compativility \rightarrow \nabla_{\vec{w}}(\vec{u}\cdot\vec{v}) = (\nabla_{\vec{w}}\vec{u})\cdot\vec{v} + (\nabla_{\vec{w}}\vec{v})\cdot\vec{u} \rightarrow \partial_{\mathrm{k}}g_{ij} = \Gamma^l_{ki}g_{lj} + \Gamma^l_{kj}g_{li}$$

$$Dari\ kedua\ properti\ di\ atas, didapatkanlah\ Levi - Civita\ Connection\ Coefficient$$

$$\Gamma^c_{ab} = \frac{1}{2}g^{cd}(\partial_a g_{bd} + \partial_b g_{ad} - \partial d_d g_{ab})$$

❖ *Menentukan komponen-komponen simbol Christoffel*

Dari persamaan di atas, untuk menentukan semua simbol Christoffelnya dibutuhkan $metric\ g^{ij}\ dan\ turunan\ metric\ g_{ij}$ terhadap semua sumbu koordinatnya $(ct, r, \theta, \phi)$. Metric $g_{ij}$ merupakan hasil dari dot product setiap basisnya $\vec{e_i}\cdot\vec{e_j}$. Dari persamaan di atas $ds^2 = -f(r)c^2dt^2 + g(r)dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$, kita dapat mengetahui bahwa semua basisnya saling orthogonal dengan metric $g_{00} = -f(r), g_{11} = g(r), g_{22} = r^2, g_{33} = \sin^2\theta$. Selanjutnya, metric $g^{ij}$ didapatkan dengan menginverskan metric $g_{ij}$ sehingga memenuhi $g^{ij}g_{jk} = \delta^i_k$.

$$g_{ij} = \begin{vmatrix} -f(r) & 0 & 0 & 0 \\ 0 & g(r) & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & \sin^2\theta \end{vmatrix}$$

$$g^{ij} = \begin{vmatrix} -\dfrac{1}{f(r)} & 0 & 0 & 0 \\ 0 & \dfrac{1}{g(r)} & 0 & 0 \\ 0 & 0 & \dfrac{1}{r^2} & 0 \\ 0 & 0 & 0 & \dfrac{1}{\sin^2\theta} \end{vmatrix}$$

Untuk mempermudah perhitungan, kita buat tabel berisikan semua penurunan metric $g_{ij}$ terhadap semua basis koordinatnya $(ct, r, \theta, \phi) = (x^0, x^1, x^2, x^3)$ $[Notasi\ Einstein]$.

| | $-f(r)$ | $g(r)$ | $r^2$ | $\sin^2\theta$ |
|---|---|---|---|---|
| $\partial_{ct}$ | $0$ | $0$ | $0$ | $0$ |
| $\partial_r$ | $-\dfrac{\partial f(r)}{\partial r}$ | $-\dfrac{\partial g(r)}{\partial r}$ | $2r$ | $0$ |
| $\partial_\theta$ | $0$ | $0$ | $0$ | $0$ |
| $\partial_\phi$ | $0$ | $0$ | $0$ | $2\sin\theta\cos\theta$ |

- *Perhitungan Simbol Christoffel (Levi-Civita Connection)*

$$Levi - Civita\ Connection\ Coefficient \rightarrow \Gamma^c_{ab} = \frac{1}{2} g^{cd}(\partial_a g_{bd} + \partial_b g_{ad} - \partial d_d g_{ab})$$

Perhatikan nilai $g^{cd}$ akan bernilai 0 apabila $c \neq d$, sehingga $c$ harus sama dengan $d$ agar $\Gamma^c_{\mathrm{ab}} \neq 0$. Perhatikan juga nilai $g_{ab}$ akan bernilai 0 apabila $a \neq b$, sehingga $\Gamma^c_{ab} \neq 0$ hanya jika $a = b$ atau $(a\ or\ b = d = c)$.

- Untuk $\Gamma^0_{ab}$

$$\Gamma^0_{00} = \Gamma^0_{11} = \Gamma^0_{22} = \Gamma^0_{33} = 0$$
$$\Gamma^0_{01} = \Gamma^0_{10} = -\frac{1}{2f}\frac{\partial(-f)}{\partial r} = \frac{f'}{2f}$$
$$\Gamma^0_{02} = \Gamma^0_{20} = 0$$
$$\Gamma^0_{03} = \Gamma_{30}$$

- Untuk $\Gamma^1_{ab}$

$$\Gamma^1_{00} = -\frac{1}{2g}\frac{\partial(-f)}{\partial r} = \frac{f'}{2g}$$
$$\Gamma^1_{11} = \frac{1}{2g}\frac{\partial g}{\partial r} = \frac{g'}{2g}$$
$$\Gamma^1_{22} = -\frac{1}{2g}\frac{\partial r^2}{\partial r} = -\frac{r}{g}$$
$$\Gamma^1_{33} = -\frac{1}{2g}\frac{\partial r^2 \sin^2\theta}{\partial r} = -\frac{r\sin^2\theta}{g}$$
$$\Gamma^1_{10} = \Gamma^1_{01} = 0$$
$$\Gamma^1_{12} = \Gamma^1_{21} = 0$$
$$\Gamma^1_{13} = \Gamma^1_{31} = 0$$

- Untuk $\Gamma^2_{ab}$

$$\Gamma^2_{00} = \Gamma^2_{11} = \Gamma^2_{22} = 0$$
$$\Gamma^2_{33} = -\frac{1}{2r^2}\frac{\partial r^2 \sin^2\theta}{\partial\theta} = -sin\theta cos\theta$$
$$\Gamma^2_{20} = \Gamma^2_{02} = 0$$
$$\Gamma^2_{21} = \Gamma^2_{12} = \frac{1}{2r^2}\frac{\partial r^2}{\partial r} = \frac{1}{r}$$
$$\Gamma^2_{23} = \Gamma^2_{32} = 0$$

- Untuk $\Gamma^3_{ab}$

$$\Gamma^3_{00} = \Gamma^3_{11} = \Gamma^3_{22} = \Gamma^3_{33} = 0$$
$$\Gamma^3_{30} = \Gamma^3_{03} = 0$$

$$\Gamma_{31}^{3} = \Gamma_{13}^{3} = \frac{1}{2r^2\sin^2\theta}\frac{\partial r^2 \sin^2\theta}{\partial r} = \frac{1}{r}$$

$$\Gamma_{32}^{3} = \Gamma_{23}^{3} = \frac{1}{2r^2\sin^2\theta}\frac{\partial r^2 \sin^2\theta}{\partial \theta} = \frac{cos\theta}{sin\theta}$$

Sehingga, semua simbol Christoffel tersebut dalam bentuk matriks adalah sebagai berikut:

$$\left[\begin{bmatrix} 0 & \frac{\frac{d}{dr}f(r)}{2f(r)} & 0 & 0 \\ \frac{\frac{d}{dr}f(r)}{2f(r)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\frac{d}{dr}f(r)}{2g(r)} & 0 & 0 & 0 \\ 0 & \frac{\frac{d}{dr}g(r)}{2g(r)} & 0 & 0 \\ 0 & 0 & -\frac{r}{g(r)} & 0 \\ 0 & 0 & 0 & -\frac{r\sin^2(\theta)}{g(r)} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{r} & 0 \\ 0 & \frac{1}{r} & 0 & 0 \\ 0 & 0 & 0 & -\sin(\theta)\cos(\theta) \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{r} \\ 0 & 0 & 0 & \frac{\cos(\theta)}{\sin(\theta)} \\ 0 & \frac{1}{r} & \frac{\cos(\theta)}{\sin(\theta)} & 0 \end{bmatrix}\right]$$

## B. Tensor Ricci

Tensor Ricci diturunkan dari Tensor Riemann, Tensor Riemann sendiri adalah tensor yang dapat menentukan apakah suatu ruang waktu *flat* atau *curve.*

$$R(\overrightarrow{e_a},\overrightarrow{e_b})\overrightarrow{e_c} = \nabla_{\overrightarrow{\mathrm{e_a}}}\nabla_{\overrightarrow{e_b}}\overrightarrow{e_c} - \nabla_{\overrightarrow{e_b}}\nabla_{\overrightarrow{e_a}}\overrightarrow{e_c} - \nabla_{[\overrightarrow{e_a},\overrightarrow{e_b}]}\overrightarrow{e_c}$$

Pada ruang waktu koordinat bola *lie bracket* dari basis-basisnya selalu nol $[\overrightarrow{e_a},\overrightarrow{e_b}] = 0$

$$R(\overrightarrow{e_a},\overrightarrow{e_b})\overrightarrow{e_c} = \nabla_{\overrightarrow{\mathrm{e_a}}}(\Gamma_{bc}^{i}\overrightarrow{e_\iota}) - \nabla_{\overrightarrow{e_b}}(\Gamma_{ac}^{j}\overrightarrow{e_J})$$

$$R(\overrightarrow{e_a},\overrightarrow{e_b})\overrightarrow{e_c} = \partial_a\Gamma_{bc}^{i}\overrightarrow{e_\iota} + \Gamma_{bc}^{i}\nabla_{\overrightarrow{\mathrm{e_a}}}\overrightarrow{e_\iota} - \partial_b\Gamma_{ac}^{j}\overrightarrow{e_J} - \Gamma_{ac}^{j}\nabla_{\overrightarrow{e_b}}\overrightarrow{e_J}$$

$$R(\overrightarrow{e_a},\overrightarrow{e_b})\overrightarrow{e_c} = \partial_a\Gamma_{bc}^{i}\overrightarrow{e_\iota} - \partial_b\Gamma_{ac}^{j}\overrightarrow{e_J} + \Gamma_{bc}^{i}\Gamma_{ai}^{d}\overrightarrow{e_d} - \Gamma_{ac}^{j}\Gamma_{bj}^{f}\overrightarrow{e_f}$$

$$Ubah\ basisnya\ jadi\ d \rightarrow\ R(\overrightarrow{e_a},\overrightarrow{e_b})\overrightarrow{e_c} = [R_{cab}^{d}]\overrightarrow{e_d} = [\partial_a\Gamma_{bc}^{d} - \partial_b\Gamma_{ac}^{d} + \Gamma_{bc}^{i}\Gamma_{ai}^{d} - \Gamma_{ac}^{j}\Gamma_{bj}^{d}]\overrightarrow{e_d}$$

$$[R_{cab}^{d}]\ disebut\ sebagai\ komponen\ Riemann\ tensor$$

Ricci Tensor sendiri memiliki fungsi untuk *track* volume sepanjang geodesic, dengan rumus:

$$R_{cb} = R_{ckb}^{k} = g^{ka}(g_{kd}R_{cab}^{d}) = R_{bc}$$

$$R_{cb} = \partial_k\Gamma_{bc}^{k} - \partial_b\Gamma_{kc}^{k} + \Gamma_{bc}^{i}\Gamma_{ki}^{k} - \Gamma_{kc}^{j}\Gamma_{bj}^{k}$$

$$R_{00} = \partial_k\Gamma_{00}^{k} - \partial_0\Gamma_{k0}^{k} + \Gamma_{00}^{i}\Gamma_{ki}^{k} - \Gamma_{k0}^{j}\Gamma_{0j}^{k} = \frac{f''}{2g} - \frac{f'g'}{4g^2} - \frac{f'^2}{4fg} + \frac{f'}{rg}$$

$$R_{01} = R_{10} = R_{02} = R_{20} = R_{03} = R_{30} = 0$$

$$R_{11} = \partial_k\Gamma_{11}^{k} - \partial_1\Gamma_{k1}^{k} + \Gamma_{11}^{i}\Gamma_{ki}^{k} - \Gamma_{k1}^{j}\Gamma_{1j}^{k} = \frac{f'^2}{4f^2} - \frac{f''}{2f} + \frac{f'g'}{4fg} + \frac{g'}{rg}$$

$$R_{12} = R_{21} = R_{13} = R_{31} = 0$$

$$R_{22} = \partial_k\Gamma_{22}^{k} - \partial_2\Gamma_{k2}^{k} + \Gamma_{22}^{i}\Gamma_{ki}^{k} - \Gamma_{k2}^{j}\Gamma_{2j}^{k} = \frac{rg'}{2g^2} - \frac{rf'}{2fg} - \frac{1}{g} + 1$$

$$R_{23} = R_{32} = 0$$

$$R_{33} = \partial_k \Gamma^k_{33} - \partial_3 \Gamma^k_{k3} + \Gamma^i_{33}\Gamma^k_{ki} - \Gamma^j_{k3}\Gamma^k_{3j} = \frac{[rfg' - rgf' + 2(g-1)fg]\sin^2\theta}{2fg^2}$$

Sehingga, diperoleh tensor Ricci dengan $R_{ab} \neq 0$ secara diagonal $R_{00}, R_{11}, R_{22}, R_{33}$.

Kita ketahui bahwa fungsi $f\ dan\ g$ sedemikian sehingga tensor Riccinya sama dengan nol

$$R_{ab} = R_{00} + R_{11} + R_{22} + R_{33} = 0$$

$$R_{ab} = \left(\frac{f''}{2g} - \frac{f'g'}{4g^2} - \frac{f'^2}{4fg} + \frac{f'}{rg}\right) + \left(\frac{f'^2}{4f^2} - \frac{f''}{2f} + \frac{f'g'}{4fg} + \frac{g'}{rg}\right) + \left(\frac{rg'}{2g^2} - \frac{rf'}{2fg} - \frac{1}{g} + 1\right) + \left(\frac{[rfg' - rgf' + 2(g-1)fg]\sin^2\theta}{2fg^2}\right) = 0$$

Apabila kita perhatikan terdapat hubungan antara $R_{22}\ dengan\ R_{33} \rightarrow R_{33} = \sin^2\theta\, R_{22}$

$$R_{ab} = \left(\frac{f''}{2g} - \frac{f'g'}{4g^2} - \frac{f'^2}{4fg} + \frac{f'}{rg}\right) + \left(\frac{f'^2}{4f^2} - \frac{f''}{2f} + \frac{f'g'}{4fg} + \frac{g'}{rg}\right) + R_{22}(1 + \sin^2\theta) = 0$$

$$R_{ab} = R_{00} + R_{11} + R_{22}(1 + \sin^2\theta) = 0$$

Salah satu cara untuk menyelesaikannya dapat dengan menggunakan properti *Contracted Bianchi Identity* atau sering juga disebut *Einstein Equation*.

$$R_{mn;n} - \frac{1}{2} g_{mn} R_{;n} = 0$$

Untuk menggunakan persamaan tersebut ( Einstein Equation ) dibutuhkan Skalar Ricci yang hasilnya ada pada bagian ***C. Skalar Ricci*** . Apabila kita substitusi semua semua tensor Riccinya maka didapatkan sebagai berikut.

$$R_{00} - \tfrac{1}{2} g_{00} R = R_{00} + \tfrac{f}{2} R = -\tfrac{1}{r}\tfrac{g'}{g^2} - \tfrac{1}{r^2}\left(1 - \tfrac{1}{g}\right) = 0$$

$$R_{11} - \tfrac{1}{2} g_{11} R = R_{11} - \tfrac{g}{2} R = \tfrac{f'}{rfg} - \tfrac{1}{r^2}\left(1 - \tfrac{1}{g}\right) = 0$$

$$R_{22} - \frac{1}{2} g_{22} R = R_{22} - \frac{r^2}{2} R = \frac{f}{f'} - \frac{g'}{g} + \frac{rf''}{f} - \frac{rf'}{2}\frac{g'}{fg} - \frac{rf'^2}{2f^2} = 0$$

$$R_{33} - \frac{1}{2} g_{33} R = -\sin^2\theta\, R_{22} - \frac{r^2\sin^2\theta}{2} R = -R_{22} - \frac{r^2}{2} R = 0$$

Perhatikan, bahwa $R_{00}$ merupakan fungsi dari $g$ saja sehingga dapat kita selesaikan sebagai berikut

$$R_{00} - \frac{1}{2} g_{00} R = 0$$

$$-\frac{g'}{g} - \frac{1}{r}(g-1) = 0$$

$$\frac{dg}{g(g-1)} = -\frac{dr}{r}$$

Integralkan kedua sisi dengan aturan integral $\int \frac{dx}{ax+bx^2} = -\frac{1}{a}\ln\left(\frac{a+bx}{x}\right)$

$$\ln\left(\frac{g-1}{g}\right) = \ln r + C$$

$$\frac{C}{r} = \frac{g-1}{g}$$

$$g = \frac{1}{1-\frac{c}{r}}$$

Selanjutnya, masukan solusi tersebut pada $R_{11} - \frac{1}{2}g_{11}R = 0$, sehingga didapatkan

$$\frac{f'}{f}\left(\frac{(r-c)}{r}\right) - \frac{1}{r}\left(1-\left(1-\frac{c}{r}\right)\right) = 0$$

$$\frac{f'}{f} = \frac{c}{r^2 - cr}$$

$$\frac{df}{f} = \frac{c\,dr}{r^2 - cr}$$

$$\ln(f) = \frac{C}{C}\ln\left(\frac{r-c}{r}\right)$$

$$f = 1 - \frac{C}{r}$$

Apabila kita perhatikan, fungsi ini $(f, g)$ sangat mirip dengan metric pada ruang waktu Schwarzschild. Sehingga dapat kita simpulkan fungsi $f\ dan\ g$ sebagai berikut:

$$f = 1 - \frac{C}{r} \quad dan \quad g = \frac{1}{f} = \frac{1}{1-\frac{C}{r}}\ dengan\ C = \frac{2GM}{c^2}$$

$$ds^2 = -(1 - \frac{2GM}{c^2 r})c^2 dt^2 + \frac{dr^2}{1 - \frac{2GM}{c^2 r}} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

## C. Skalar Ricci

Ricci skalar memiliki makna fisis untuk men-*track* bagaimana ukuran suatu objek berubah dari ukuran standardnya pada ruangwaktu flat. Ricci skalar merupakan tensor ricci dengan menaikan salah satu indeksnya seperti berikut:

$$R = R_a^a = g^{ab}R_{ab}$$

Dalam bagian sebelumnya, kita hanya memiliki Ricci tensor pada bagian diagonalnya sehingga pada kasus ini Ricci skalarnya bernilai $R = g^{00}R_{00} + g^{11}R_{11} + g^{22}R_{22} + g^{33}R_{33}$

$$g^{00}R_{00} = -\frac{1}{f}\left(\frac{f''}{2g} - \frac{f'g'}{4g^2} - \frac{f'^2}{4fg} + \frac{f'}{rg}\right)$$

$$g^{11}R_{11} = \frac{1}{g}\left(\frac{f'^2}{4f^2} - \frac{f''}{2f} + \frac{f'g'}{4fg} + \frac{g'}{rg}\right)$$

$$g^{22}R_{22} = \frac{1}{r^2}\left(\frac{rg'}{2g^2} - \frac{rf'}{2fg} - \frac{1}{g} + 1\right)$$

$$g^{33}R_{33} = \frac{1}{r^2 \sin^2\theta}\left(\frac{[rfg' - rgf' + 2(g-1)fg]\sin^2\theta}{2fg^2}\right)$$

$$R = -\frac{f''}{fg} + \frac{f'}{2}\frac{g'}{fg^2} + \frac{f'^2}{2f^2g} - \frac{2}{r}\frac{f'}{fg} + \frac{2}{r}\frac{g'}{g^2} + \frac{2}{r^2}\left(1 - \frac{1}{g}\right)$$

$[R\ dibutuhkan\ pada\ bagian\ b\ untuk\ menyelesaikan\ Einstein\ Equations]$

Pada persamaan *Einstein Equation atau Contracted Bianchi Identity,* apabilka kita mengatur agar skalar Riccinya sama dengan nol maka:

$$R_{mn;n} - \frac{1}{2}g_{mn}R_{;n} = 0$$

$$R_{mn;n} = R_{00} = R_{11} = R_{22} = R_{33} = 0$$

Apabila kita asumsikan $f = \frac{1}{g}\ dan\ f' = \frac{g'}{g^2}$, kemudian kita substitusikan pada persamaan $R_{22} = 0$, maka akan didapatkan fungsi $f\ dan\ g\ sebagai\ berikut$:

$$R_{22} = \frac{rg'}{2g^2} - \frac{rf'}{2fg} - \frac{1}{g} + 1 = 0$$

$$R_{22} = \frac{rg'}{2g^2} - \frac{rg'}{2g^2} - \frac{1}{g} + 1 = 0$$

$$\frac{1}{g} = 1 \rightarrow g = 1\ dan\ f = 1$$