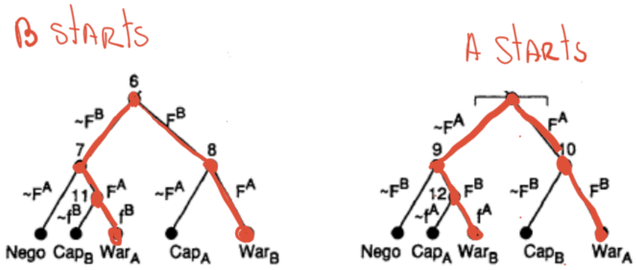
The International Interaction Game

Finding the Probability of Two Counties Engaging into War

We will consider uniquely the subgame crisis where we only have Force (F) or Not Force ($\tilde{F}$) as actions.



$\Pr(\text{War}_B\,,\ \text{War}_A) = F_B\,F_A\,\tilde{F}_B\ \big|\ S\rangle\ +\ F_A\,F_B\ \big|\ S\rangle\ \big|^2$

(* defining the superposition state S according to B's basis*)

In[286]:= $\text{SB} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} * \dfrac{1}{\sqrt{2}};$

In[189]:= `MatrixForm[SB];`

(* Define the Projector Operator ~F1B and F1B and basis FB *)

In[290]:= $\text{F1}_{\bar{B}} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix};\ \text{F1}_B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix};\ \text{F}_B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix};$

(* Define the Rotation Operator for Player A using Force *)

In[291]:= $\text{R} = \begin{pmatrix} \text{Cos}[\text{Re}[\phi]] & -\text{Sin}[\text{Re}[\phi]] \\ \text{Sin}[\text{Re}[\phi]] & \text{Cos}[\text{Re}[\phi]] \end{pmatrix};$

In[292]:= $\text{F1}_A = \text{R}.\begin{pmatrix} 1 \\ 0 \end{pmatrix};$

In[293]:= $\text{MatrixForm}[\text{F1}_A];$

(* Define the Rotation Operator for Player B *)

In[294]:= $\text{R2} = \begin{pmatrix} \text{Cos}[\text{Re}[\theta]] & -\text{Sin}[\text{Re}[\theta]] \\ \text{Sin}[\text{Re}[\theta]] & \text{Cos}[\text{Re}[\theta]] \end{pmatrix};$

(* Rotating from player A's basis back to player B's basis*)

In[242]:= $\text{F1}_{\text{RAtoB}} = \text{FullSimplify}[\text{Inverse}[\text{F}_B].\text{R2}.\text{F}_B.\text{F1}_A]$

Out[242]= {{Cos[Re[θ + ϕ]]}, {Sin[Re[θ + ϕ]]}}

(* Rotating from player B's basis to a new Basis where B attacks *)

In[264]:= $\text{Rfb} = \text{FullSimplify}\left[\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.\text{F1}_{\text{RAtoB}}\right]$

Out[264]= {{Cos[Re[θ + ϕ]] + Sin[Re[θ + ϕ]]}, {Cos[Re[θ + ϕ]] − Sin[Re[θ + ϕ]]}}

In[310]:= $fb = Rfb \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Out[310]= {{Cos[Re[θ + ϕ]] + Sin[Re[θ + ϕ]]}, {0}}

```
(* Probability of war given that B started *)
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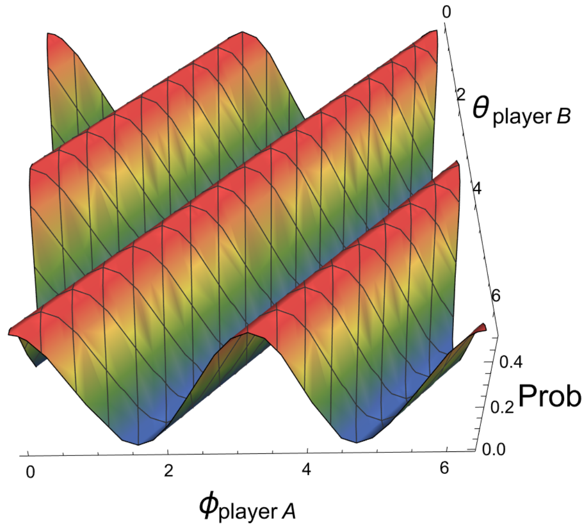
In[313]:= probBwarAwar = FullSimplify[Norm[fb $F1_{RAtoB}$ $F1_{\bar{B}}$. SB + $F1_{RAtoB}$ $F1_{B}$. SB]^2]

Out[313]= $\frac{1}{2}$ Cos[Re[θ + ϕ]]^2

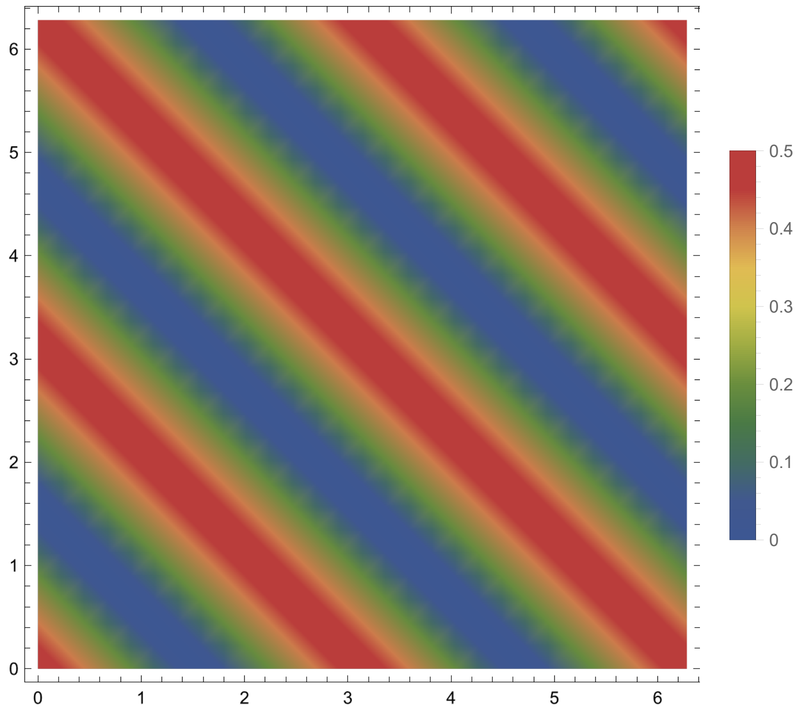
In[346]:= Plot3D[probBwarAwar, {θ, 0, 2 π}, {ϕ, 0, 2 π}, Boxed → False,
AxesLabel → {Style["$θ_{\text{player B}}$", 16], Style["$ϕ_{\text{player A}}$", 16], Style["Prob", 16]},
ColorFunction → (ColorData["DarkRainbow"][#3] &)]

Out[346]=

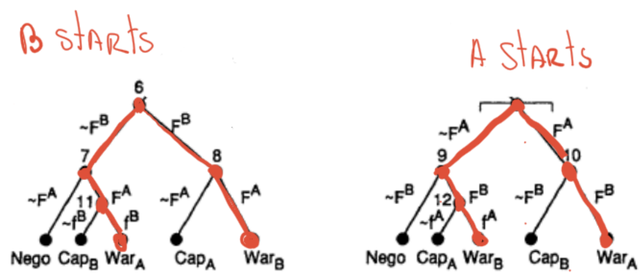


In[347]:=
```
DensityPlot[probBwarAwar, {θ, 0, 2 π}, {ϕ, 0, 2 π},
 Ticks → {{0, Pi / 2, Pi, 3 Pi / 2, 2 Pi}, {0, Pi / 2, Pi, 3 Pi / 2, 2 Pi}},
 ColorFunction → "DarkRainbow", PlotLegends → Automatic]
```

Out[347]=



Computing Probabilities when A starts




```
(* We need to define the superposition state according to A's basis *)
```

In[316]:= $SA = \text{FullSimplify}\left[\text{Inverse}[R].\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.R.SB\right];$

In[317]:=
```
MatrixForm[SA]
```

Out[317]//MatrixForm=

$$\begin{pmatrix} \frac{\text{Cos}[2\,\text{Re}[\phi]]+\text{Sin}[2\,\text{Re}[\phi]]}{\sqrt{2}} \\ \frac{\text{Cos}[2\,\text{Re}[\phi]]-\text{Sin}[2\,\text{Re}[\phi]]}{\sqrt{2}} \end{pmatrix}$$

In[327]:= $F2_A = \begin{pmatrix} \text{Cos}[\text{Re}[\phi]] \\ \text{Sin}[\text{Re}[\phi]] \end{pmatrix};\ F2_{\bar{A}} = \begin{pmatrix} -\text{Sin}[\text{Re}[\phi]] \\ \text{Cos}[\text{Re}[\phi]] \end{pmatrix};$ (* basis of A *)

In[325]:= $\text{Norm}[F2_A]\,\hat{}\,2$

Out[325]= $\text{Cos}[\text{Re}[\phi]]^2 + \text{Sin}[\text{Re}[\phi]]^2$

```
(* Then, we define Player's B roation according to A *)
```

In[323]:= $F2_B = R2 . \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Out[323]= {{Cos[Re[θ]]}, {Sin[Re[θ]]}}

In[331]:= MatrixForm[$F2_A$ $F2_B$ $F2_{\bar{A}}$ SA]

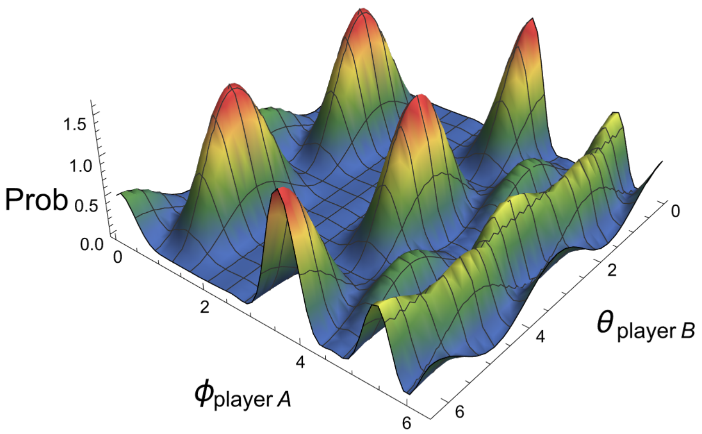
Out[331]//MatrixForm=

$$\begin{pmatrix} -\frac{\text{Cos}[\text{Re}[\theta]]\,\text{Cos}[\text{Re}[\phi]]\,\text{Sin}[\text{Re}[\phi]]\,(\text{Cos}[2\,\text{Re}[\phi]]+\text{Sin}[2\,\text{Re}[\phi]])}{\sqrt{2}} \\ \frac{\text{Cos}[\text{Re}[\phi]]\,\text{Sin}[\text{Re}[\theta]]\,\text{Sin}[\text{Re}[\phi]]\,(\text{Cos}[2\,\text{Re}[\phi]]-\text{Sin}[2\,\text{Re}[\phi]])}{\sqrt{2}} \end{pmatrix}$$

In[332]:= probAwarBwar = FullSimplify[Norm[$F2_A$ $F2_B$ $F2_{\bar{A}}$ SA + $F2_B$ $F2_A$ SA]^2]

Out[332]=
$$\frac{1}{2}\left(-(1+\text{Cos}[\text{Re}[\phi]])^2\,\text{Sin}[\text{Re}[\theta]]^2\,\text{Sin}[\text{Re}[\phi]]^2\,(-1+\text{Sin}[4\,\text{Re}[\phi]])+\text{Cos}[\text{Re}[\theta]]^2\,\text{Cos}[\text{Re}[\phi]]^2\,(-1+\text{Sin}[\text{Re}[\phi]])^2\,(1+\text{Sin}[4\,\text{Re}[\phi]])\right)$$

In[348]:= Plot3D[probAwarBwar, {θ, 0, 2 π}, {ϕ, 0, 2 π}, Boxed → False,
AxesLabel → {Style["$\theta_{\text{player B}}$", 16], Style["$\phi_{\text{player A}}$", 16], Style["Prob", 16]},
ColorFunction → (ColorData["DarkRainbow"][#3] &)]

Out[348]=


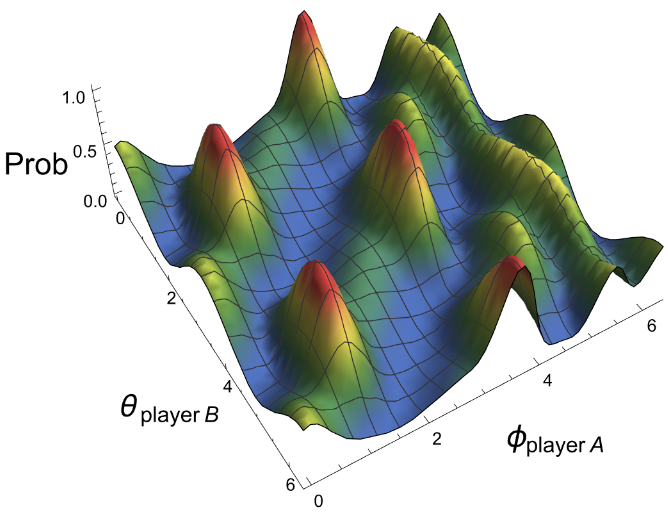
Final Probability of WAR: Pr(WarB, WarA) + Pr(WarA, WarB)

In[345]:= final = FullSimplify[(probBwarAwar + probAwarBwar) / 2]

Out[345]=
$$\frac{1}{4}\left(\text{Cos}[\text{Re}[\theta+\phi]]^2-(1+\text{Cos}[\text{Re}[\phi]])^2\,\text{Sin}[\text{Re}[\theta]]^2\,\text{Sin}[\text{Re}[\phi]]^2\,(-1+\text{Sin}[4\,\text{Re}[\phi]])+\text{Cos}[\text{Re}[\theta]]^2\,\text{Cos}[\text{Re}[\phi]]^2\,(-1+\text{Sin}[\text{Re}[\phi]])^2\,(1+\text{Sin}[4\,\text{Re}[\phi]])\right)$$

In[349]:=
```
Plot3D[final, {θ, 0, 2 π}, {ϕ, 0, 2 π}, Boxed → False,
 AxesLabel → {Style["θ player B", 16], Style["ϕplayer A", 16], Style["Prob", 16]},
 ColorFunction → (ColorData["DarkRainbow"][#3] &)]
```

Out[349]=



In[351]:=
```
DensityPlot[final, {θ, 0, 2 π}, {ϕ, 0, 2 π},
 Ticks → {{0, Pi / 2, Pi, 3 Pi / 2, 2 Pi}, {0, Pi / 2, Pi, 3 Pi / 2, 2 Pi}},
 ColorFunction → "DarkRainbow", PlotLegends → Automatic]
```

Out[351]=