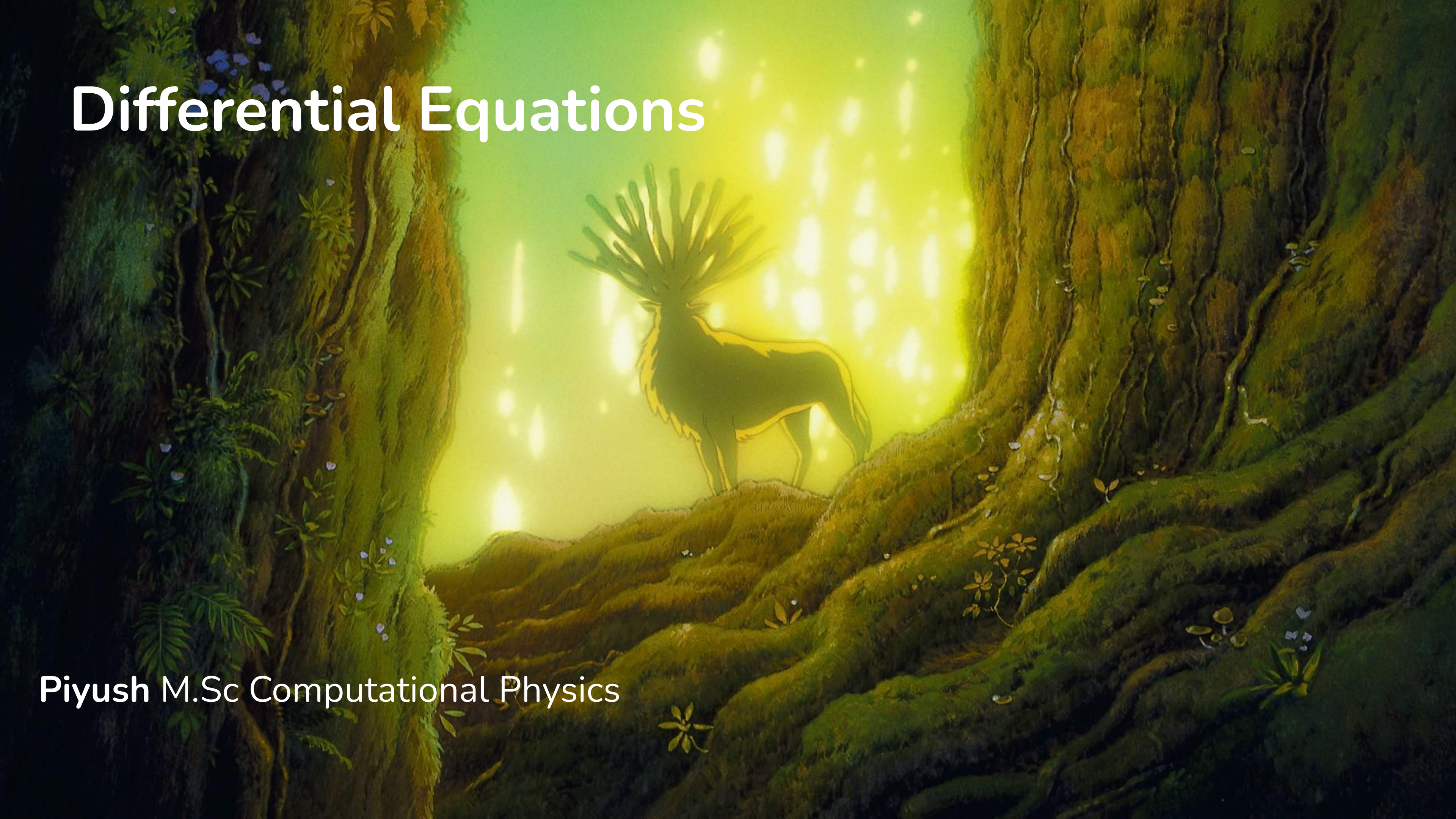


Differential Equations

Piyush M.Sc Computational Physics

Differential Equations

A differential equation is a mathematical equation that relates a function to its derivatives.

$$\frac{1}{\eta^2}\frac{d}{d\eta}\left(\eta^2\frac{d\varphi}{d\eta}\right)+(\varphi^2-C)^{3/2}=0$$

Chandrasekhar White Dwarf equation

$$\left.\frac{\mathrm{d}[\mathrm{H}::]}{\mathrm{d}t}\right|_{\mathrm{form}}=\alpha(T)\,v_{\mathrm{th}}\,\langle n\sigma_{\mathrm{gr}}\rangle\,[\mathrm{H}]\,\exp\left(-\frac{T_1}{T}\right)\left(1-\frac{[\mathrm{H}::]}{[\mathrm{H}::]_{\mathrm{max}}}\right)$$

Eley-Rideal Mechanism

$$\frac{d}{dt}\{X\}=\{A\}+\{X\}^2\{Y\}-\{B\}\{X\}-\{X\}$$
$$\frac{d}{dt}\{Y\}=\{B\}\{X\}-\{X\}^2\{Y\}$$

Autocatalytic Reactions

$$I=C_m\frac{\mathrm{d}V_m}{\mathrm{d}t}+g_K(V_m-V_K)+g_{Na}(V_m-V_{Na})+g_l(V_m-V_l)$$

Hodgkin–Huxley model

$$\frac{\mathrm{d}B}{\mathrm{d}t}=-\alpha A \quad \frac{\mathrm{d}B}{\mathrm{d}t}=-\alpha A$$

Lanchester's laws

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2S^2\frac{\partial^2V}{\partial S^2}+rS\frac{\partial V}{\partial S}-rV=0$$

Black–Scholes equation

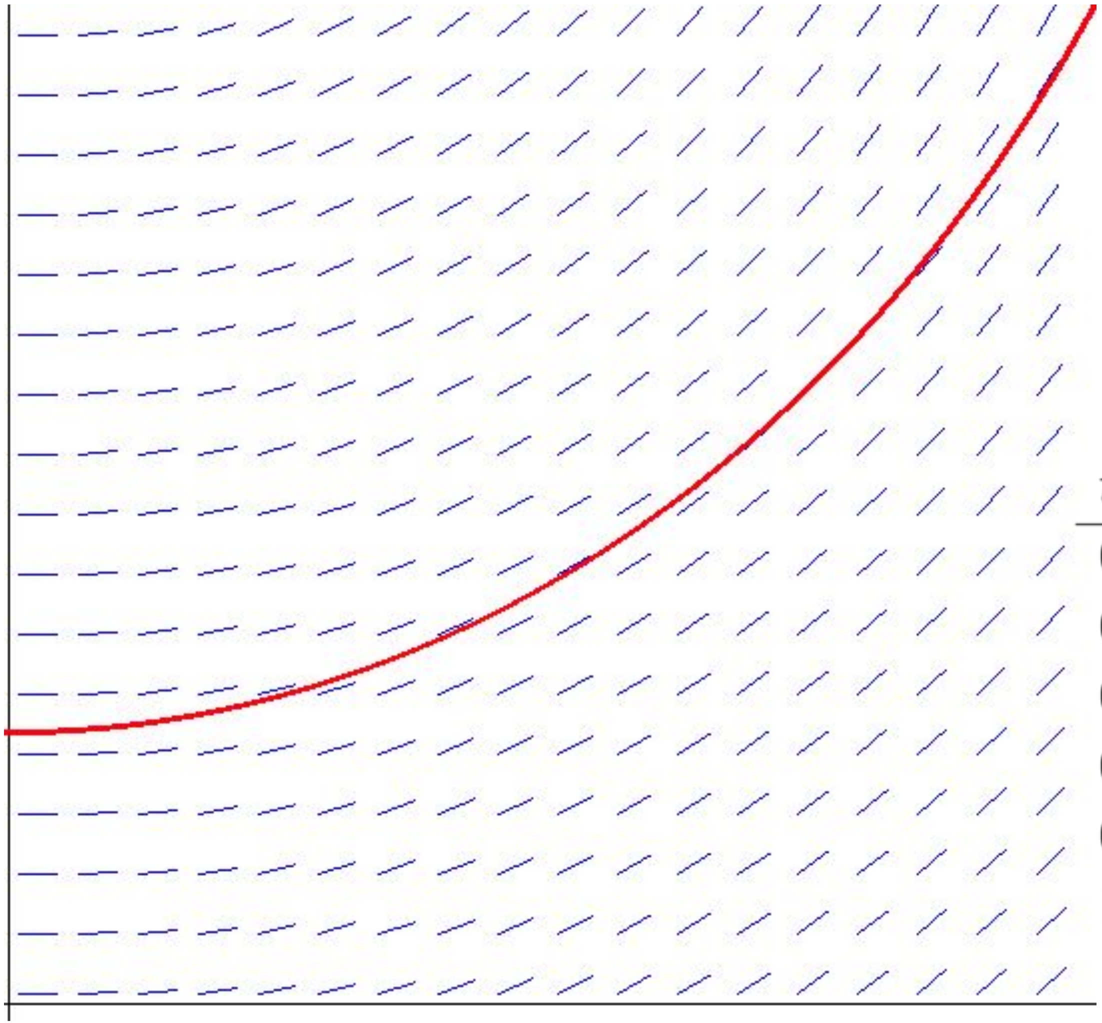
Ordinary Differential Equation: Euler Method

$$y(x+h) \approx y(x) + hy'(x) + \frac{h^2 y''(x)}{2!} + \frac{h^3 y'''(x)}{3!} + \cdots$$

$$y(x+h) \approx y(x) + hy'(x)$$

The integral can be thought of as the differential equation

$$y = \int f(x)dx \qquad y' = f(x) \qquad y' \equiv \frac{dy}{dx}$$



$$y' = ty^2 + t$$ Differential Equation

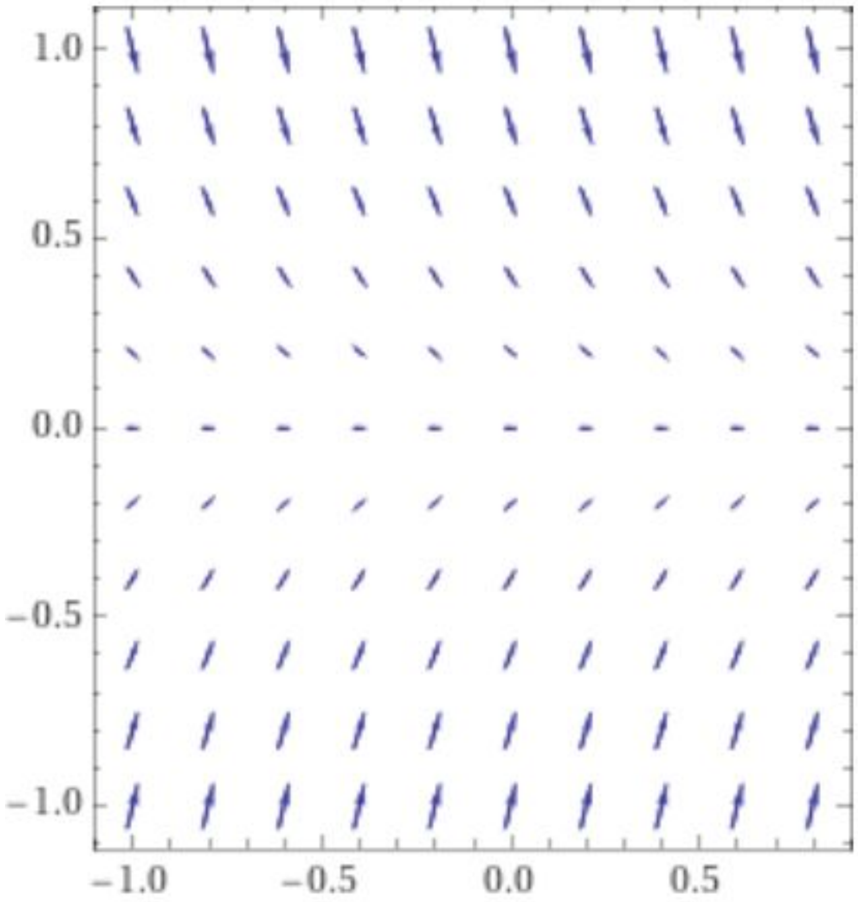
$$y(0) = .25$$ Boundary Condition

$$t \in [0, 1].$$ Range

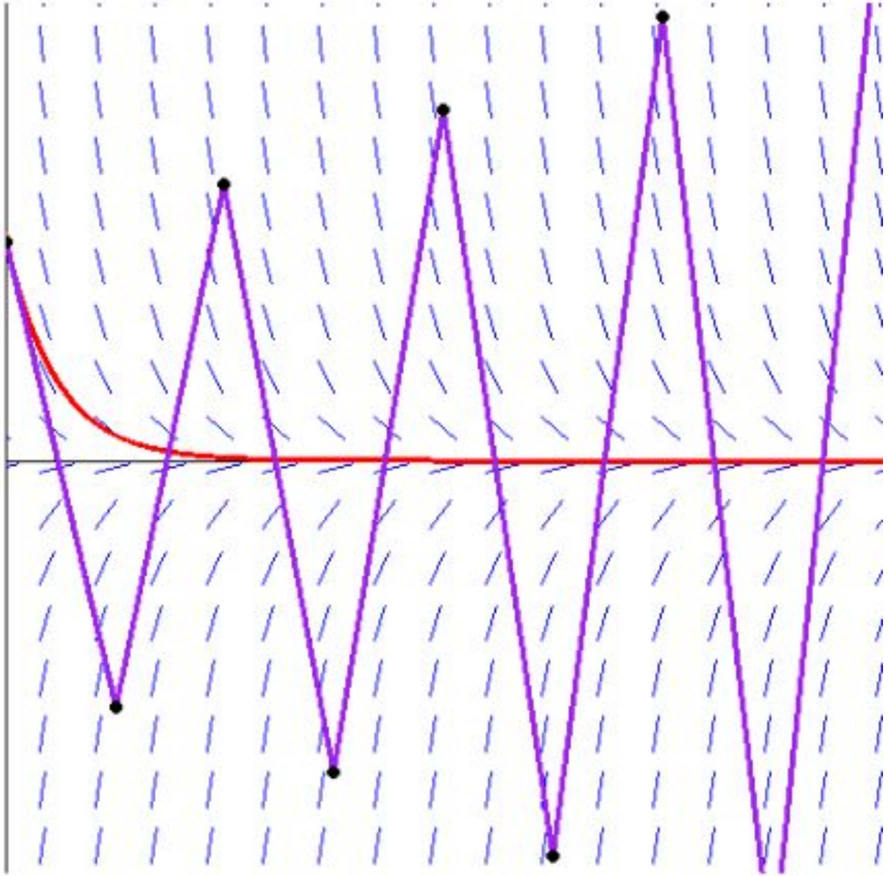
t	y
0.0	0.25
0.2	$0.25 + .2(0 \cdot .25^2 + 0) = .25$
0.4	$0.25 + .2(.2 \cdot .25^2 + .2) = .29$
0.6	$0.29 + .2(.4 \cdot .29^2 + .4) = .38$
0.8	$0.38 + .2(.6 \cdot .38^2 + .6) = .51$
1.0	$0.51 + .2(.8 \cdot .51^2 + .8) = .72$

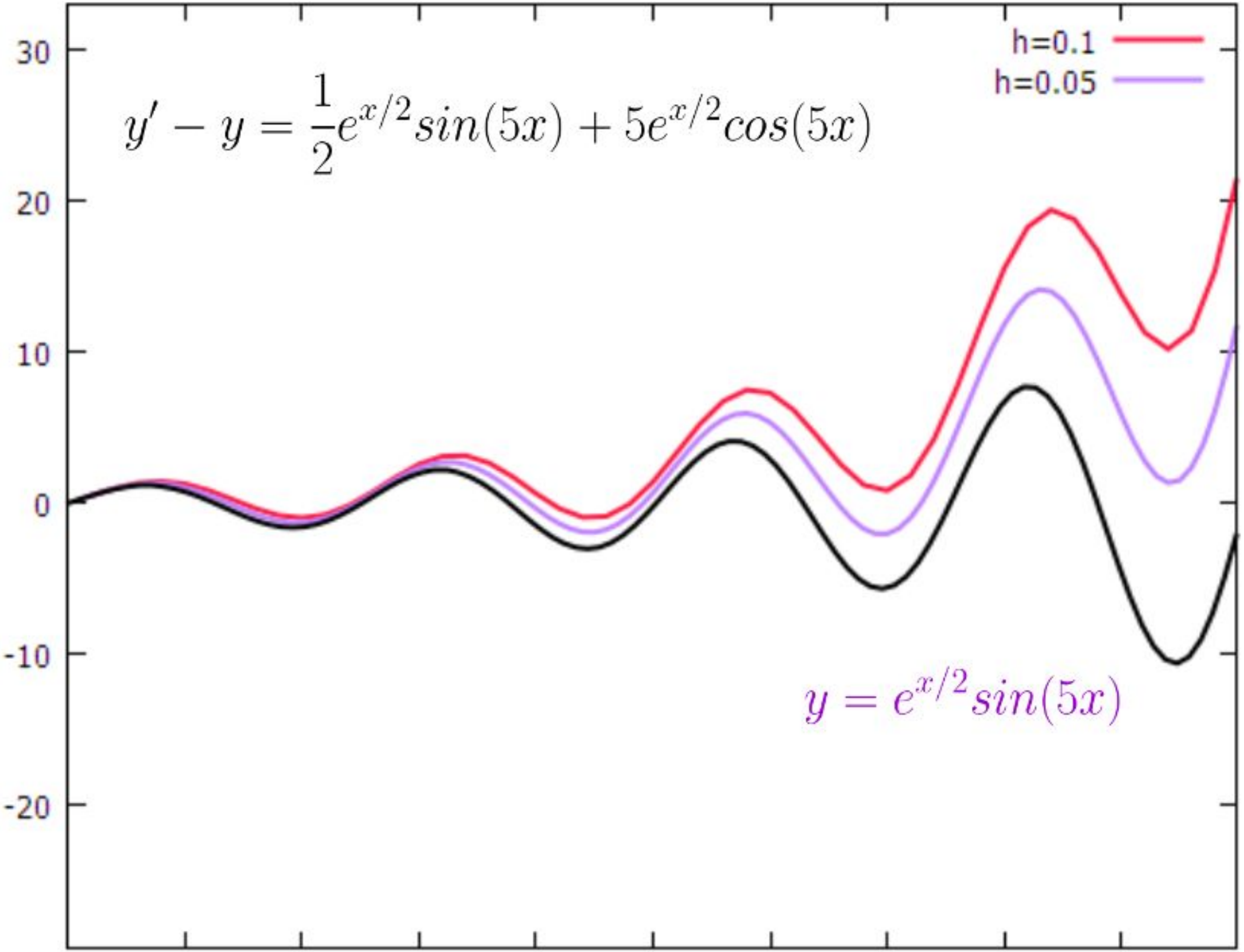
Slope Fields

$$y' = -4.25y$$



$$y = e^{-4.25t}$$

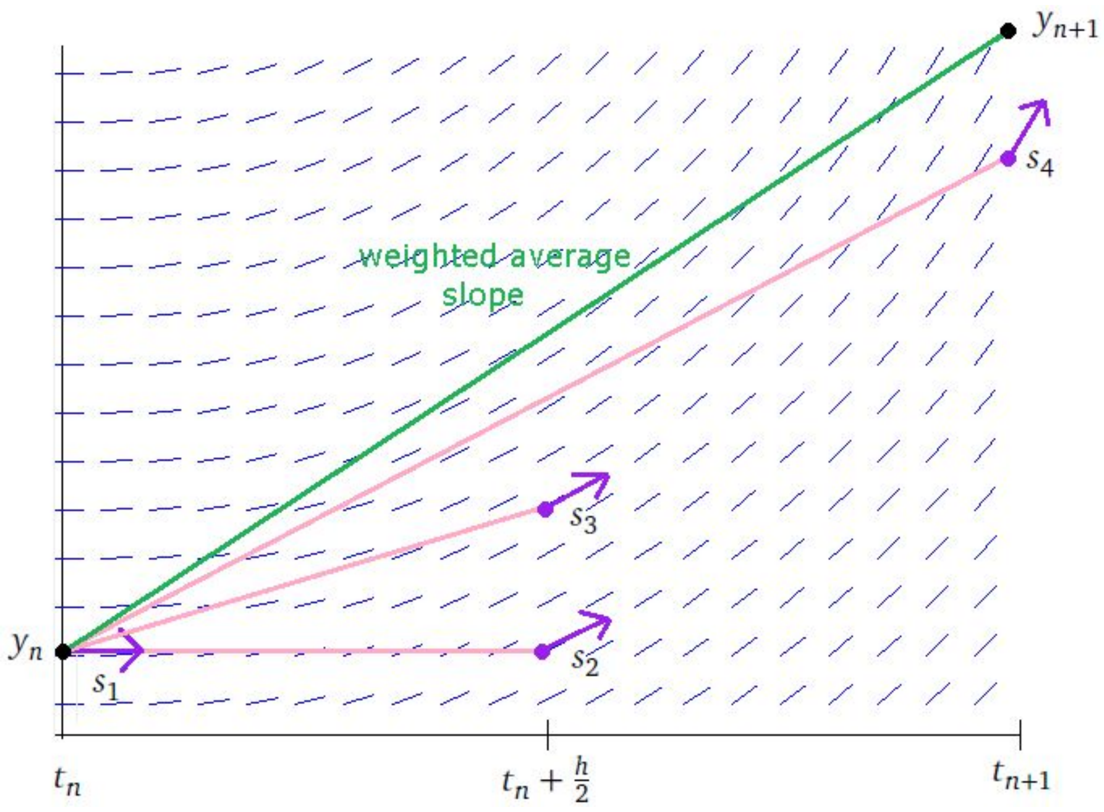




$$\frac{Ch}{L}\left(e^{L(t_i-a)} - 1\right)$$

Ordinary Differential Equation: Runge Kutta-4

$$y_{n+1} = y_n + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$



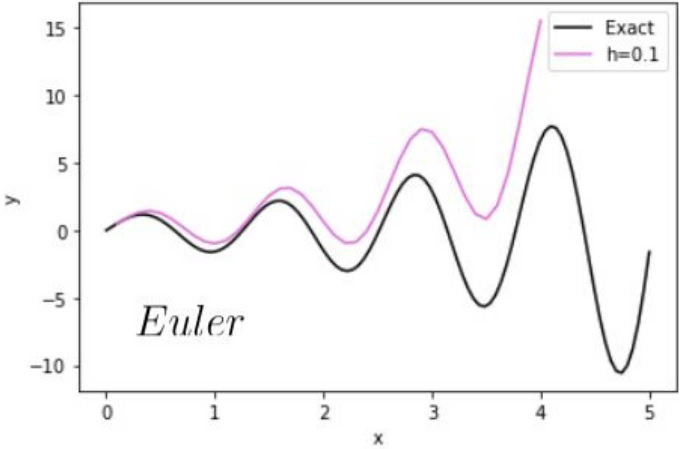
The RK4 method builds off of the RK2 methods (trapezoid, midpoint, etc.) by using four slopes instead of two.

- s_1 — Euler's slope at the left endpoint
- s_2 — Slope at midpoint from following Euler's slope
- s_3 — Improved slope at midpoint, from following s_2 instead of s_1
- s_4 — Slope at right endpoint, from following s_3

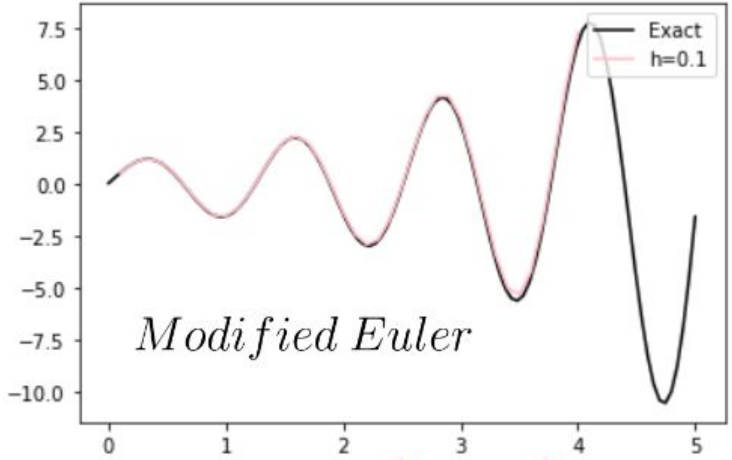
The new y_{n+1} is computed as a weighted average those four slopes.

$$\boxed{y' - y = \frac{1}{2}e^{x/2}sin(5x) + 5e^{x/2}cos(5x)}$$

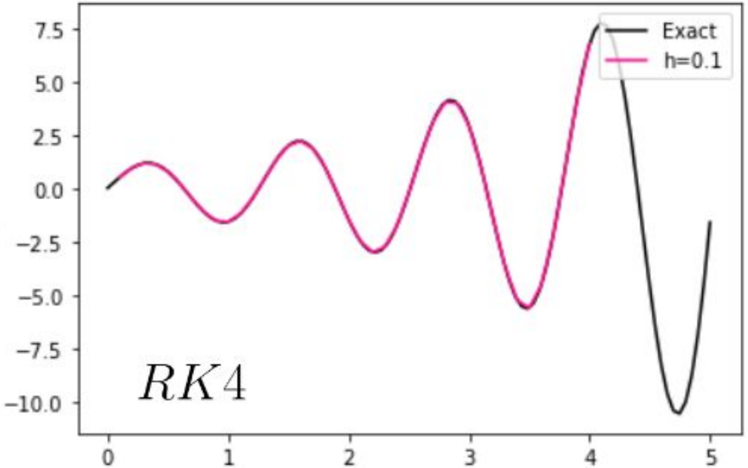
$$y_{n+1} = y_n + hf(x_n, y_n)$$



$$y_{n+1} = y_n + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$



$$y_{n+1} = y_n + hf(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hf(x_n, y_n))$$



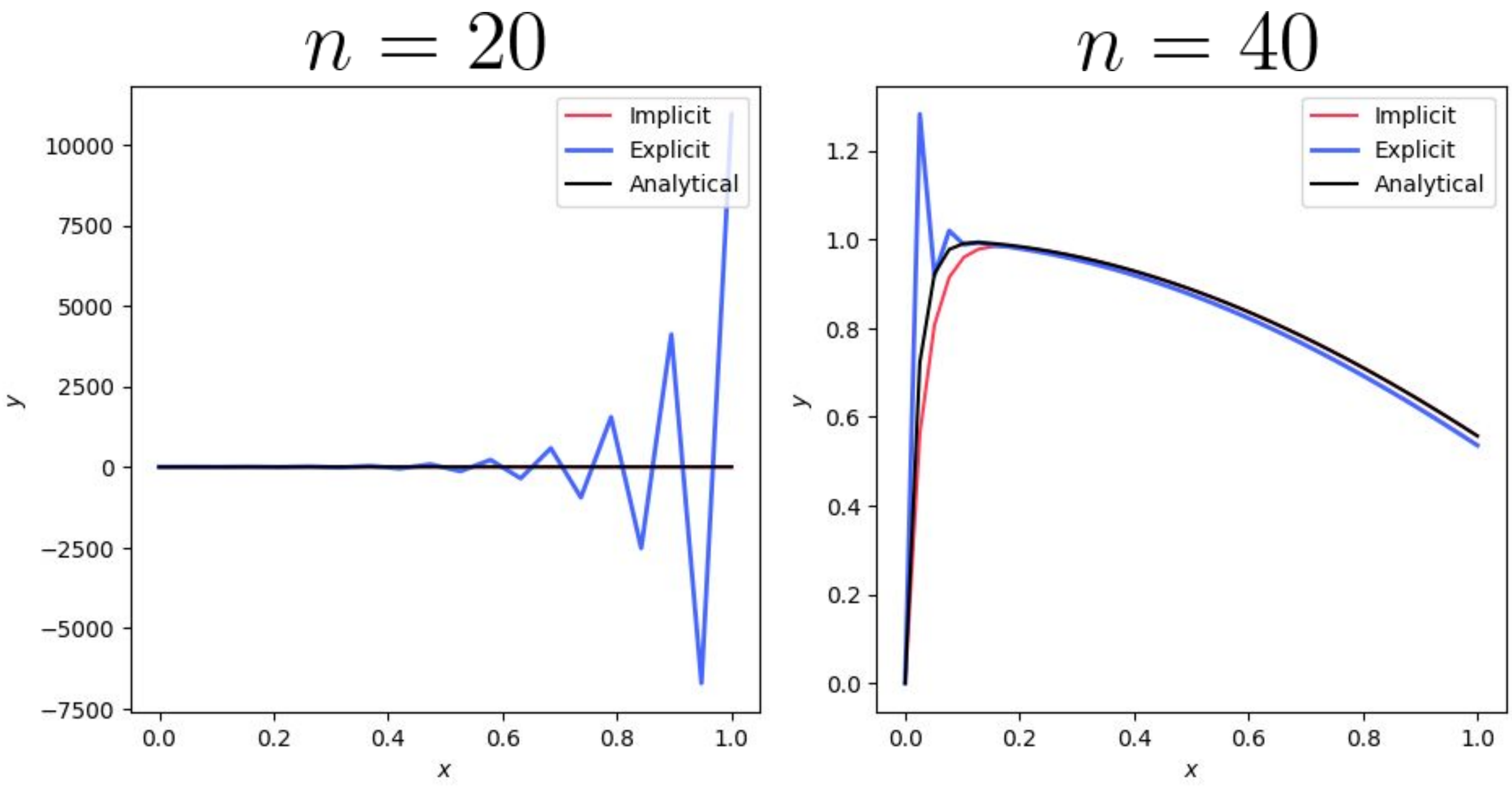
Slaying the Dragon : Stiff differential equations

$$
\begin{aligned}
y' &= -50(y - cos(t)) \\
&= -50y + 50cos(t) = 50\left(cos(t) - y\right)
\end{aligned}
$$

$$y(i+1) = y(i) + 50[cos(t(i)) - y(i)]dt$$ **Explicit**

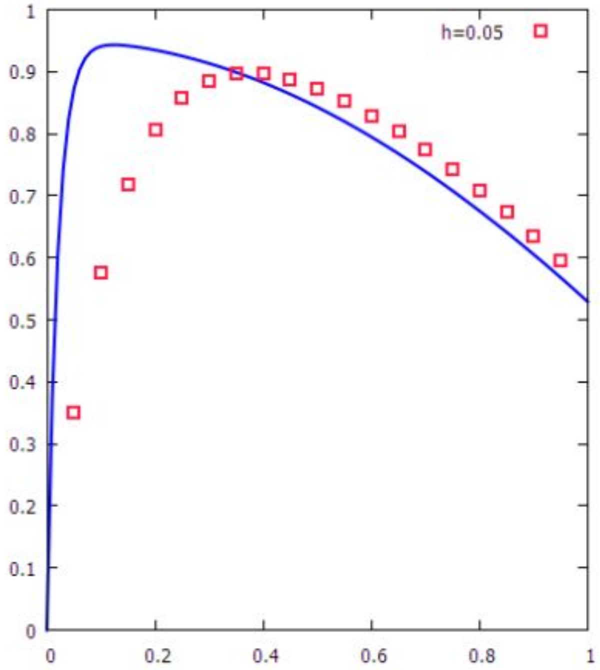
$$y(i+1) = y(i) + 50[cos(t(i+1)) - y(i+1)]dt$$ **Implicit**

$$y(i+1) = \frac{[y(i) + 50cos(t(i+1))]dt}{1 + 50dt}$$

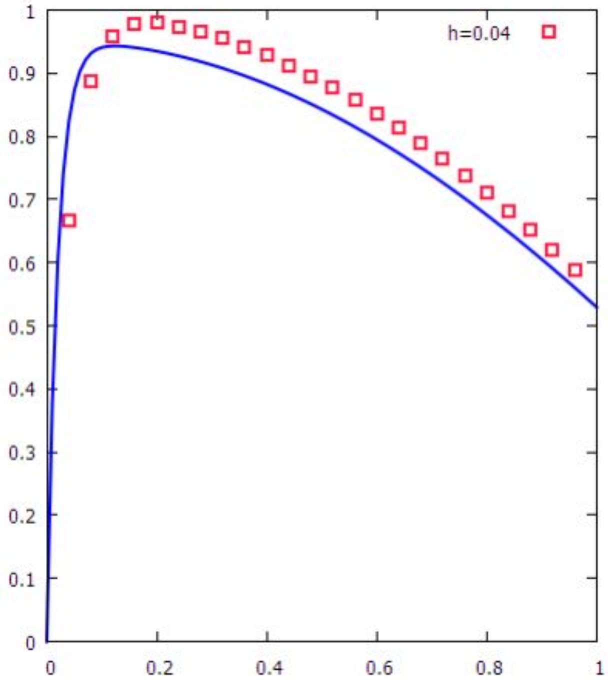


$n = 20$
$n = 40$
Implicit
Explicit
Analytical
Implicit
Explicit
Analytical
10000
7500
5000
2500
0
−2500
−5000
−7500
1.2
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.2
0.4
0.6
0.8
1.0
0.0
0.2
0.4
0.6
0.8
1.0
y
y
x
x

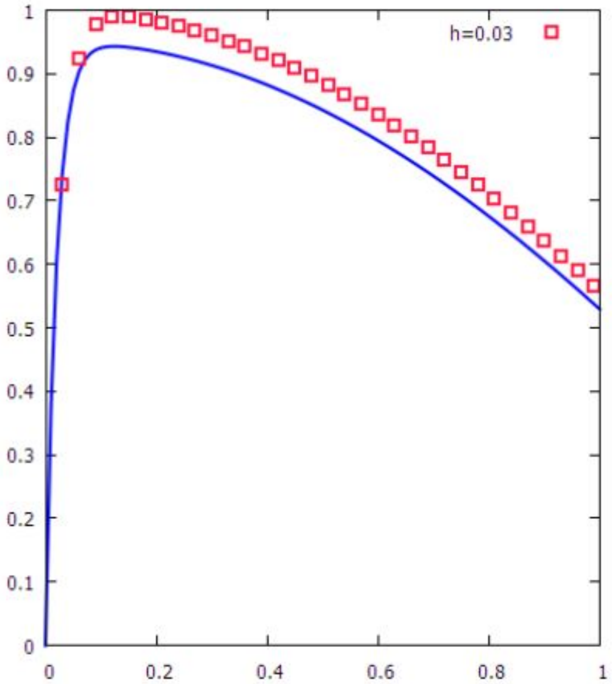
$$y(t) = \frac{50}{2501}\left(sin(t) + 50cos(t)\right) - \frac{2500}{2501}e^{-50x}$$



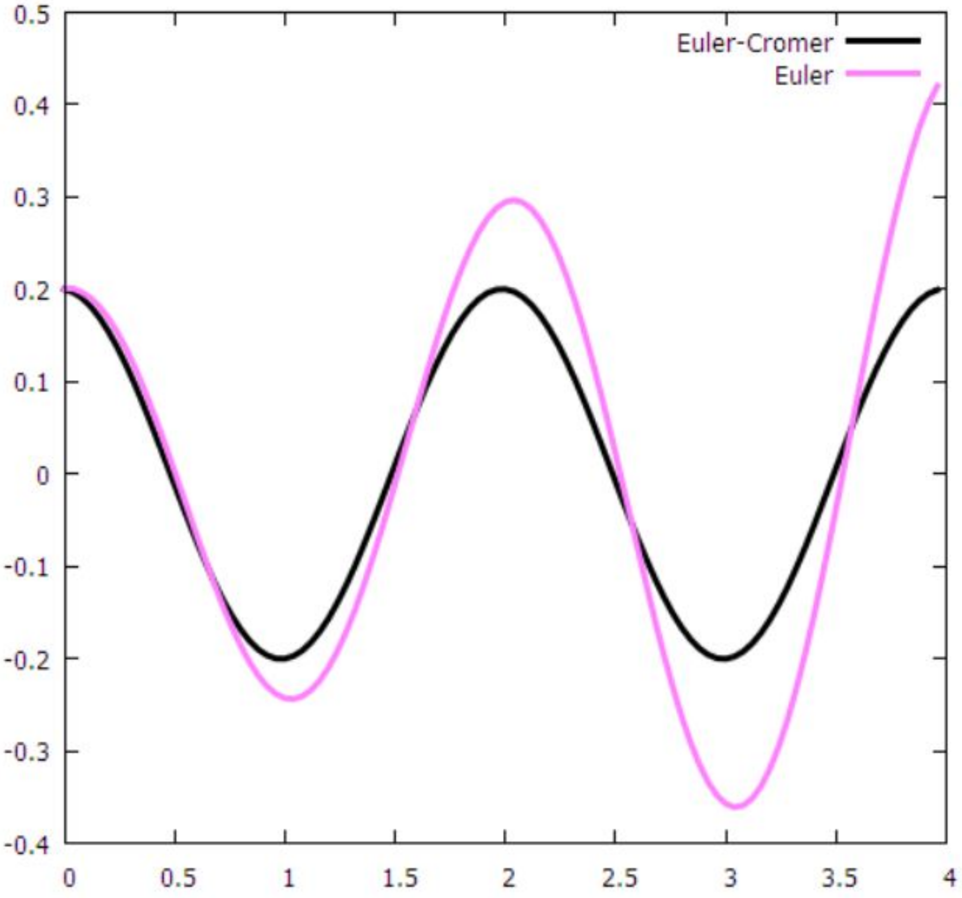
h = 0.05



h = 0.04



h = 0.03



Simple Pendulum

$$\theta' = \omega$$

$$\omega' = -\frac{g}{L}\theta$$

$$\theta_k = \theta_{k-1} + \omega_{k-1}\delta t$$

$$\omega_k = \omega_{k-1} - \frac{g}{L}\sin\theta_{k-1}\delta t$$

Equations of Stellar Structure

$$\frac{\mathrm{d}m(r)}{\mathrm{d}r} = 4\pi r^2 \rho(r).$$

mass continuity

$$\rho = \frac{\overline{m}}{k}\frac{P}{T}$$

$$\frac{\mathrm{d}P(r)}{\mathrm{d}r} = -\frac{G\,m(r)\,\rho(r)}{r^2}$$

hydrostatic equilibrium

$$\frac{\mathrm{d}L(r)}{\mathrm{d}r} = 4\pi r^2\,\varepsilon(r)$$

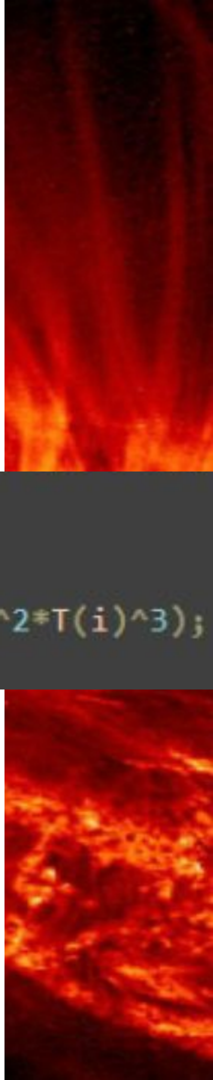
energy generation

$$\frac{\mathrm{d}T(r)}{\mathrm{d}r} = -\frac{3\kappa(r)\,\rho(r)\,L(r)}{(4\pi r^2)(16\sigma)\,T^3(r)}$$

radiative diffusion

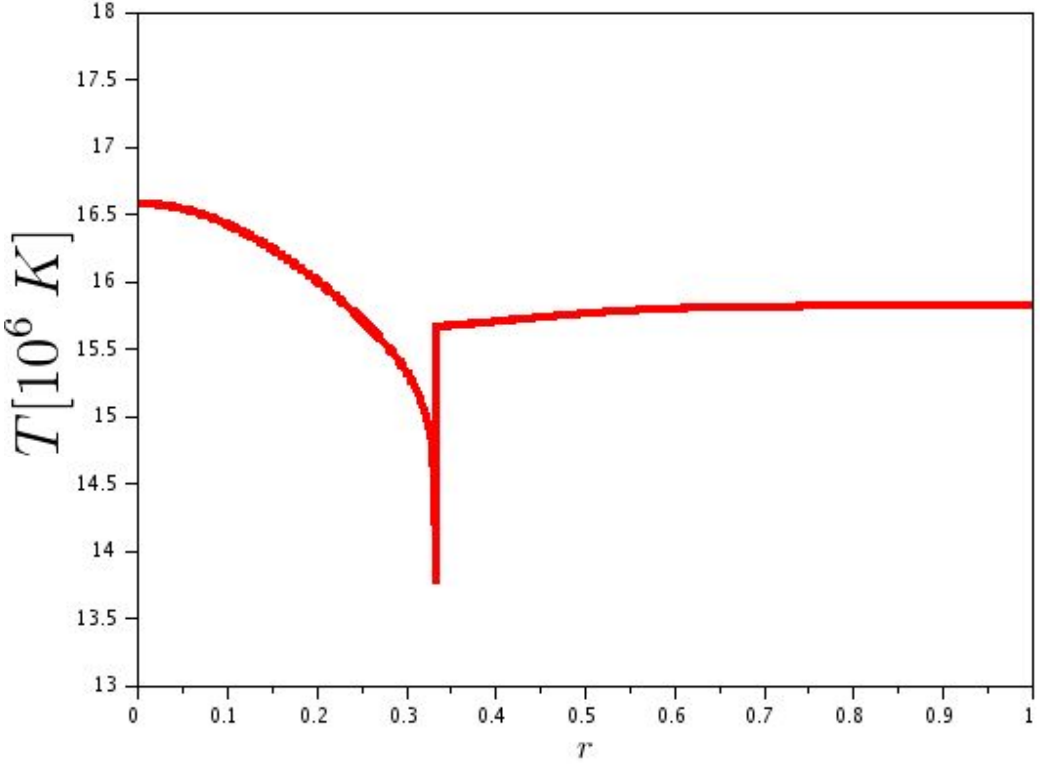
Euler Method

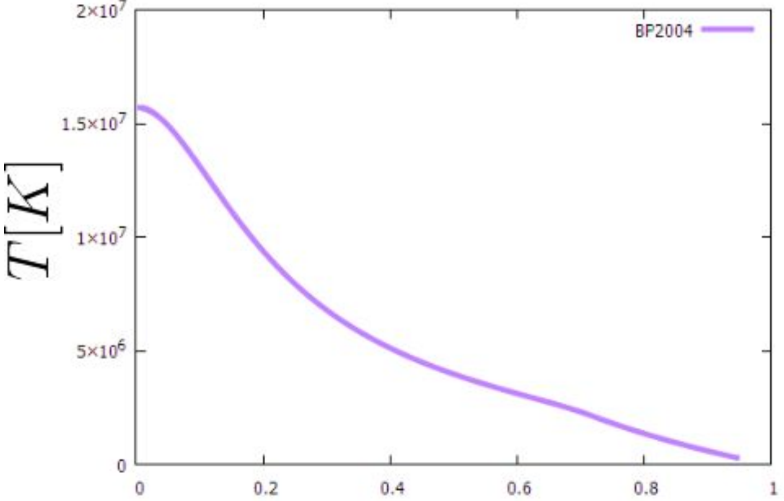
```
r(i+1) = r(i) + h;
P(i+1) = P(i) - h*G*M(i)*rho(i)/(r(i)^2);
M(i+1) = M(i) + h*4*pi*r(i)^2*rho(i);
T(i+1) = T(i) - h*3*opa(i)*rho(i)*L(i)/(64*pi*sig*r(i)^2*T(i)^3);
L(i+1) = L(i) + h*4*pi*r(i)^2*rho(i)*con*T(i)^4;
```

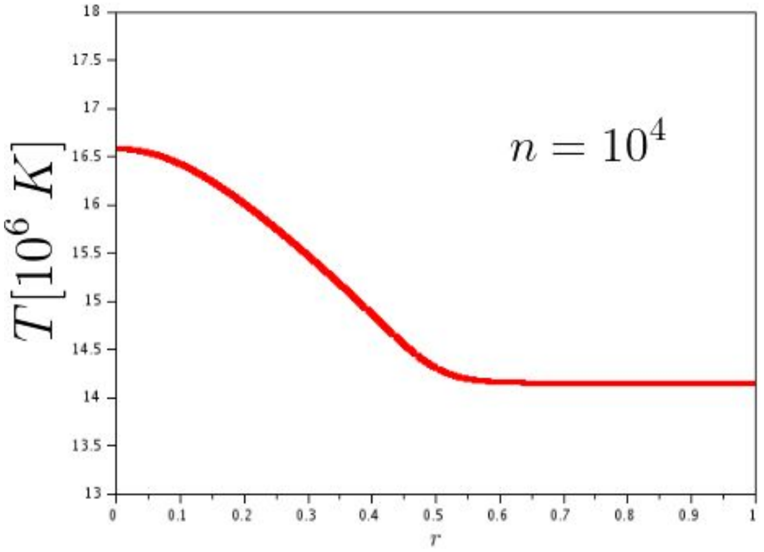


$$\frac{\mathrm{d}T(r)}{\mathrm{d}r} = -\frac{3\kappa(r)\,\rho(r)\,L(r)}{(4\pi r^2)(16\sigma)\,T^3(r)}$$

radiative diffusion



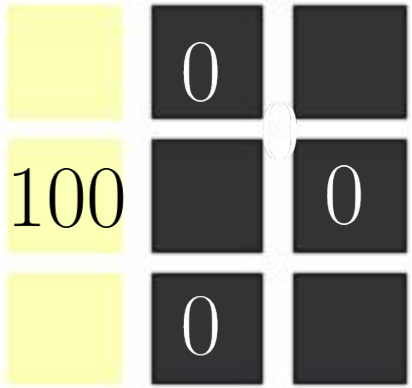


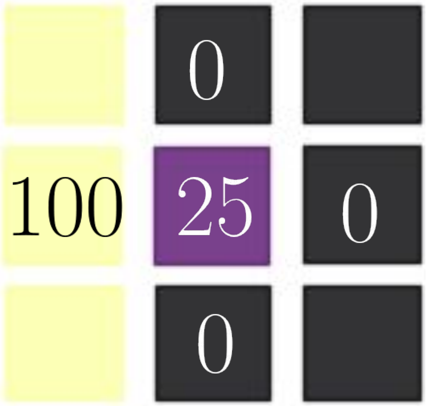


Partial Differential Equation: Heat Equation

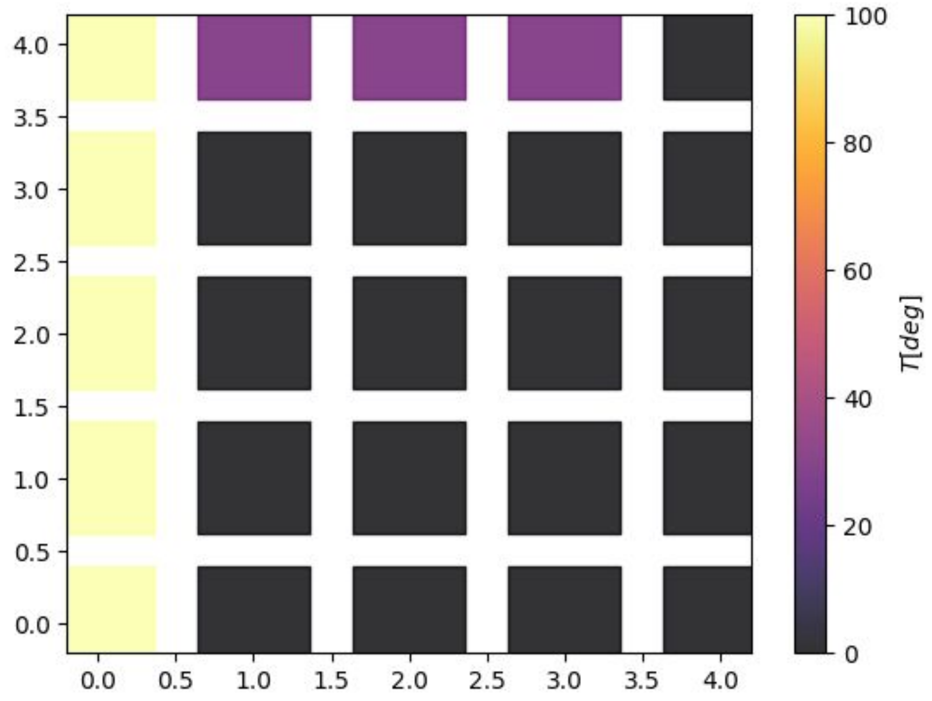
$$\nabla^2 f(x,y) \approx \frac{f(x-h,y)+f(x+h,y)+f(x,y-h)+f(x,y+h)-4f(x,y)}{h^2}.$$

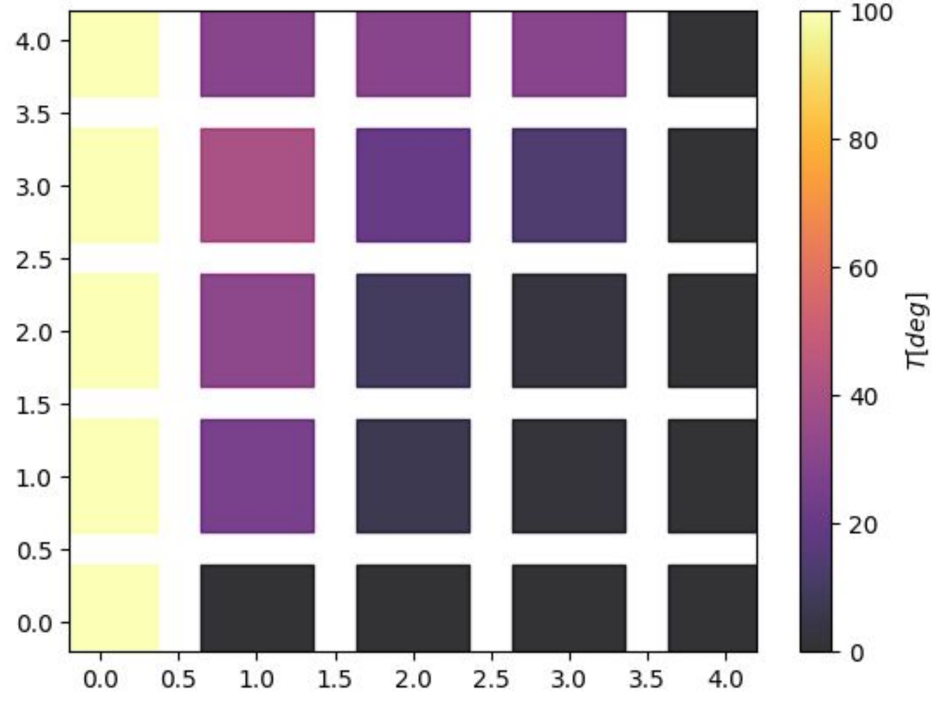
$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \qquad T_{i,j} = \frac{1}{4}\left(T_{i+1,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1}\right)$$





$$T_{i,j} = \frac{1}{4}(0 + 100 + 0 + 0) = 25$$

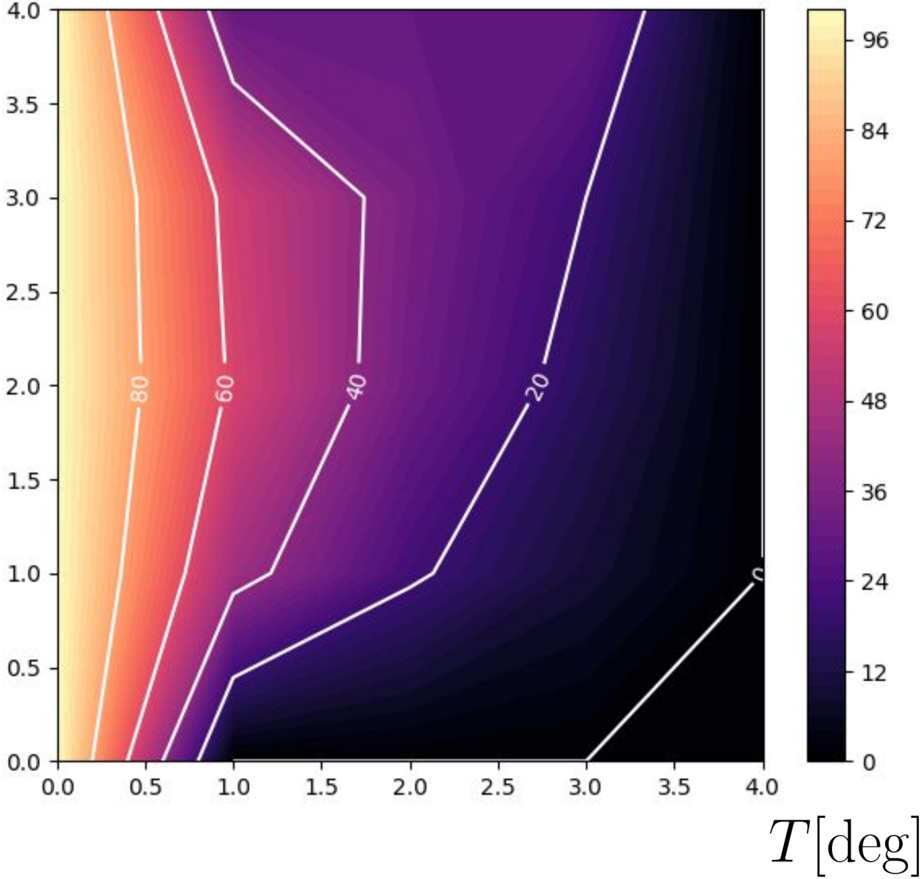




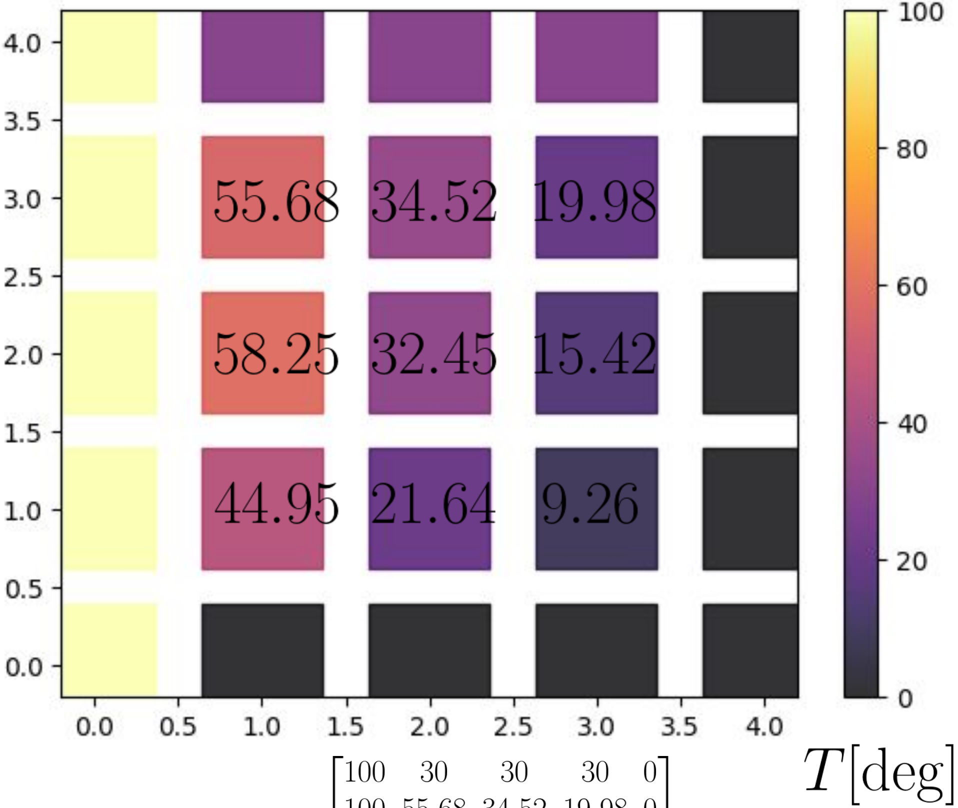
$$N = 0 \quad \begin{bmatrix} 100 & 30 & 30 & 30 & 0 \\ 100 & 0 & 0 & 0 & 0 \\ 100 & 0 & 0 & 0 & 0 \\ 100 & 0 & 0 & 0 & 0 \\ 100 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The code is evaluating from the bottom row so the red row will be calculated first and then the other row s afterwards it

$$N = 1 \quad \begin{bmatrix} 100 & 30 & 30 & 30 & 0 \\ 100 & 40.31 & 19.92 & 13.16 & 0 \\ 100 & 31.25 & 9.37 & 2.73 & 0 \\ 100 & 25 & 6.25 & 1.56 & 0 \\ 100 & 0 & 0 & 0 & 0 \end{bmatrix}$$



$N = 10$



$$\begin{bmatrix} 100 & 30 & 30 & 30 & 0 \\ 100 & 55.68 & 34.52 & 19.98 & 0 \\ 100 & 58.25 & 32.45 & 15.42 & 0 \\ 100 & 44.95 & 21.64 & 9.26 & 0 \\ 100 & 0 & 0 & 0 & 0 \end{bmatrix}$$