Computational Methods

Differential Equations :)

$$\textbf{Taylor Series :} y(x+h) = y(x) + hy'(x) + \frac{h^2y''(x)}{2!} + \frac{h^3y'''(x)}{3!} + \cdots \tag{1}$$

$$\textbf{Euler } y_{n+1} = y_n + hf(x_n, y_n) \quad \textbf{Modified Euler } y_{n+1} = y_n + hf(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hf(x_n, y_n)) \tag{2}$$

$$\textbf{RK4} \quad y_{n+1} = y_n + \frac{h}{6}(k_1 + k_2 + 2k_3 + k_4) \tag{3}$$

where,

$$k_1 = f(t_n, y_n) \quad k_2 = f(t_n + \frac{h}{2}, y_n + h\frac{k_1}{2}) \quad k_3 = f(t_n + \frac{h}{2}, y_n + h\frac{k_2}{2}) \quad k_4 = f(t_n + h, y_n + hk_3)$$

Adam Bashforth-Moulton

$$\textbf{Predictor} \quad y_{n+1} = y_n + \frac{3}{2}hf(t_{n+1}, y_{n+1}) - \frac{1}{2}hf(t_n, y_n) \quad \textbf{Corrector} \quad y_{n+1} = y_n + \frac{1}{2}(hf(t_{n+1}, y_{n+1}) + \frac{1}{2}hf(t_n, y_n)) \tag{4}$$

Integral Equations

$$\textbf{Trapezoidal} \int_a^b f(x)dx \approx \frac{h}{2}\left(f(x_0) + 2\left(\sum_{i=1}^{n-1} f(x_i)\right) + f(x_n)\right) \tag{5}$$

$$\textbf{Simpson} \int_a^b f(x)dx \approx \frac{h}{3}\left[f(x_0) + 4\left(\sum_{i=1, i \text{ odd}}^{n-1} f(x_i)\right) + 2\left(\sum_{i=2, i \text{ even}}^{n-2} f(x_i)\right) + f(x_n)\right] \tag{6}$$

$$\textbf{Gauss Quadrature} \int_a^b f(x)dx \approx p[c_1 f(px_1 + q) + c_2 f(px_2 + q) + \cdots + c_n f(px_n + q)] \quad p = \frac{b-a}{2} \quad q = \frac{b+a}{2} \tag{7}$$

Weights and Abscissae: $n = 2$ ($x_i = -\sqrt{1}/3, \sqrt{1}/3$ $c_i = 1{,}1$) $\quad n = 3$ ($x_i = -\sqrt{3/5}, 0, \sqrt{3/5}$ $\quad c_i = 5/9{,}8/9{,}5/9$)

Finite Difference Methods

$$\textbf{Central } f'(x_j) = \frac{f(x_{j+1}) - f(x_{j-1})}{2h} \quad \textbf{Forward} f'(x_j) = \frac{f(x_{j+1}) - f(x_j)}{h} \quad \textbf{Backward } f'(x_j) = \frac{f(x_j) - f(x_{j-1})}{h} \tag{8}$$

Matrix Methods

For a matrix equation $\mathbf{A}\vec{x} = \vec{b}$ with an initial guess $\vec{x}^0$

$$\textbf{Gauss Seidel} \qquad x_i^k = \left(-\sum_{j=1}^{i-1} a_{ij}x_j^k - \sum_{j=i+1}^{n} a_{ij}x_j^{k-1} + b_i\right) \Big/ a_{ii} \tag{9}$$

Convergence Criteria : $\rho(\mathbf{A}) = \max|\lambda_i| < 1$ [*Spectral radius*], $|a_{ii}| \geq \sum_{j=1}^n |a_{ij}|$ [*Diagonal Dominance*]

$$\textbf{Succesive Over Relaxation } (D + \omega L)x_{n+1} = -((1-\omega)L + U)x_n + b \quad \omega = 0[Jacobi] \quad \omega = 1[GaussSeidel] \tag{10}$$

Partial Differential Equations

$$\nabla^2 f(x,y) \approx \frac{f(x-h,y) + f(x+h,y) + f(x,y+h) - 4f(x,y))}{h^2} \;; T_{i,j} = \frac{1}{4}\left(T_{i+1,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1}\right) \tag{11}$$

# f eval	2	3	4	5≤ n ≤7	8≤ n ≤ 9	$n \geq 10$
Error	$O(h^2)$	$O(h^3)$	$O(h^4)$	$O(h^{n-1})$	$O(h^{n-2})$	$O(h^{n-3})$

Table 1: Accuracy of ODE solvers