Answer key :) July 6 , 2023

The following equation is the mathematical definition of the derivative of a function. Evaluate the derivative of *f*(*x*) = *x*^2 at *x* = 3. Take *h* = .0001

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Answer: 6.0001

Which finite difference formula is this?

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}.$$

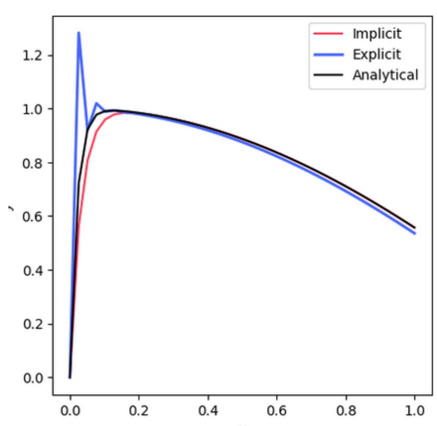
Answer: Centered

The *degree of precision* of an integration rule is the highest degree polynomial that the rule will integrate exactly.
Which of these methods has the highest degree of precision?

Rule	Error	Degree of precision
Simple rectangles	$\frac{b-a}{n}\lvert\max f'\rvert$	0
Midpoint rule	$\frac{(b-a)^3}{24n^2}\lvert\max f''\rvert$	1
Trapezoid rule	$\frac{(b-a)^3}{12n^2}\lvert\max f''\rvert$	1
Simpson's rule	$\frac{(b-a)^5}{2880n^4}\lvert\max f^{(4)}\rvert$	3

Answer: Simpson

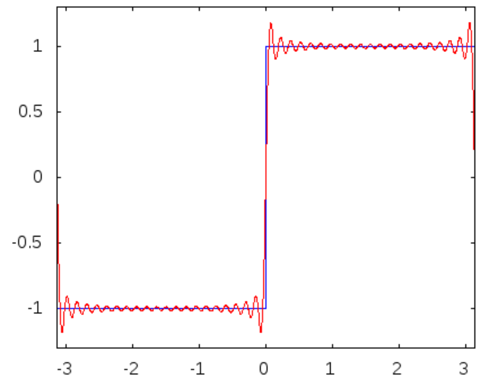
Which method is working better here?



Answer : Implicit

The Fourier series given below represents which kind of wave?
You can use a graphing calculator.

$$\sin t + \frac{1}{3}\sin 3t + \frac{1}{5}\sin 5t + \frac{1}{7}\sin 7t + \ldots.$$



Answer: Square Wave

Using the formula for the Newton-Raphson Method. Evaluate the value of the root after the first iteration for the equation x10-1=0 starting at x_0=0.5**

Example 4.6.6 Discuss the Newton-Raphson method to find the root of the equation $x^{10} - 1 = 0$ starting with $x_0 = 0.5$.

Solution. The real roots of this equation are ± 1.
Here $f(x) = x^{10} - 1$.
Therefore,

$$x_{n+1} = x_n - \frac{x_n^{10} - 1}{10x_n^9} = \frac{9x_n^{10} + 1}{10x_n^9}.$$

When $x_0 = 0.5$ then $x_1 = \dfrac{9 \times (0.5)^{10} + 1}{10 \times (0.5)^9} = 51.65$, which is far away from the root 1.
This is because 0.5 was not close enough to the root $x = 1$.
But the sequence $\{x_n\}$ will converge to the root 1, although very slowly.
The initial root $x_0 = 0.9$ gives the first approximate root $x_1 = 1.068$, which is close to the root 1.
This example points out the role of initial approximation in Newton-Raphson method.

The 5 point stencil is used to solve which equation?

$$T_{i,j} = \frac{1}{4}\left(T_{i+1,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1}\right)$$

Answer : Partial Differential Equation

How can we classify this iteration scheme for solving an ODE ?

$$y(i+1) = y(i) + 50[cos(t(i+1)) - y(i+1)]dt$$

Answer : Implicit Scheme

Explicit scheme:

$$y_{n+1} = y_n + hf'(x_n, y_n)$$

Implicit Scheme:

$$y_{n+1} = y_n + hf'(x_{n+1}, y_{n+1})$$

Is this matrix diagonally dominant?

$$B = \begin{bmatrix} -2 & 2 & 1 \\ 1 & 3 & 2 \\ 1 & -2 & 0 \end{bmatrix}$$

Answer : No

is *not* diagonally dominant because

$$|b_{11}| < |b_{12}| + |b_{13}| \quad \text{since} \quad |-2| < |+2| + |+1|$$
$$|b_{22}| \geq |b_{21}| + |b_{23}| \quad \text{since} \quad |+3| \geq |+1| + |+2|$$
$$|b_{33}| < |b_{31}| + |b_{32}| \quad \text{since} \quad |+0| < |+1| + |-2|\,.$$

That is, the first and third rows fail to satisfy the diagonal dominance condition.

Find the value of x1 after the first iteration taking the initial guess (x1,x2,x3,x4)=(0,0,0,0).
Use the Gauss-Seidel method.

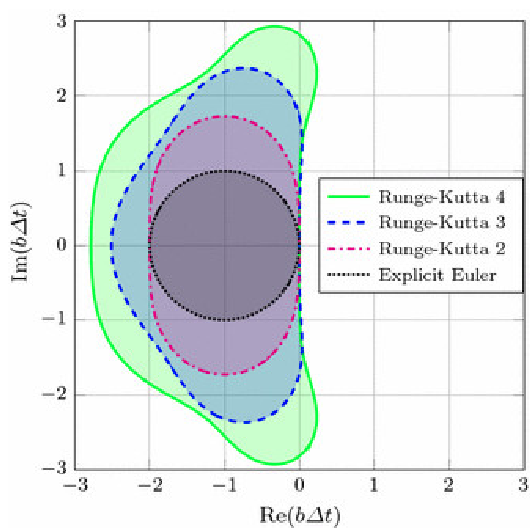
$$
\begin{array}{rrrrl}
10x_1 & -x_2 & +2x_3 & & = 6, \\
-x_1 & +11x_2 & -x_3 & +3x_4 & = 25, \\
2x_1 & -x_2 & +10x_3 & -x_4 & = -11, \\
 & 3x_2 & -x_3 & +8x_4 & = 15.
\end{array}
$$

Answer: 0.6

$$x_1 = x_2/10 - x_3/5 + 3/5,$$

$$x_1 = 3/5 = 0.6$$

The stability diagram of the different ODE solvers depicts what?



Answer: RK4 method is more stable than Explicit Euler

Stability of the Euler method is given by the following formula.If lambda=-200 what will be the value of *h* restricted to ?

$$-2 \leq h\lambda \leq 0$$

Answer: 0.01

Example We study how Euler's method behaves for the stable model problem above, i.e., in the case $\lambda \leq 0$. Since $f(t, y) = \lambda y(t)$ Euler's method states that

$$\begin{aligned} y_{n+1} &= y_n + hf(t_n, y_n) \\ &= y_n + h\lambda y_n \\ &= (1 + \lambda h) y_n. \end{aligned}$$

Therefore, by induction,

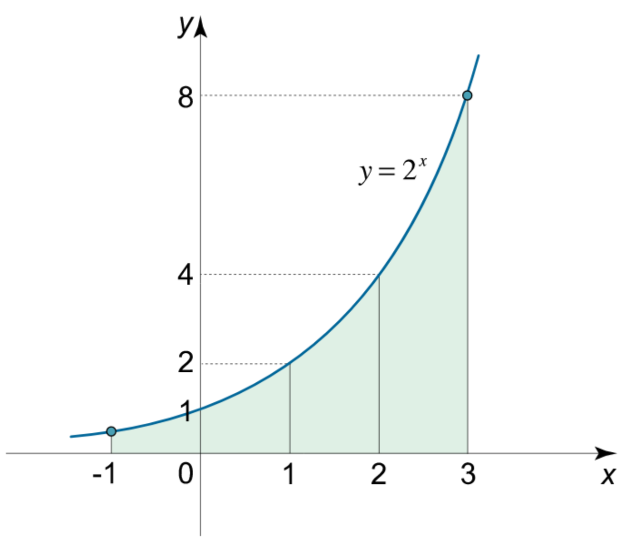
$$y_n = (1 + \lambda h)^n y_0.$$

Since the exact problem has an exponentially decaying solution for $\lambda < 0$, a stable numerical method should exhibit the same behavior. Therefore, in order to ensure stability of Euler's method we need that the so-called *growth factor* $|1 + \lambda h| < 1$. For real $\lambda < 0$ this is equivalent to

$$-2 < h\lambda < 0 \iff h < \frac{-2}{\lambda}.$$

Thus, Euler's method is only *conditionally stable*, i.e., the step size has to be chosen sufficiently small to ensure stability.

Approximate the area under y = 2x from x=[-1,3] using n=5 points. *Use the Trapezoid rule***

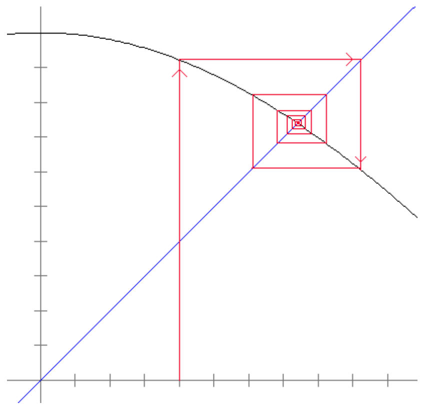


Solution:

$$I_{trap} = \frac{\Delta x}{2} \left[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4) \right].$$

$$A \approx I_{Trap} = \frac{1}{2} \left[\frac{1}{2} + 2 \cdot 1 + 2 \cdot 2 + 2 \cdot 4 + 8 \right] = \frac{1}{2} \cdot 22\frac{1}{2} = 11\frac{1}{4} \approx 11.25$$

Find the fixed point of cos(x).
Start from x=2



We can visualize the fact that the fixed point of cos x is attracting by using something called a cobweb diagram.

```
from math import cos
x = 2
for i in range(20):
    x = cos(x)
    print(x)
```

Answer: 0.739

The roots of the Legendre polynomials give the abscissas for Gauss Legendre Quadrature. Determine the roots of these equations by solving for x.
Do the second equation.

$$\frac{1}{2}\left(5x^3 - 3x\right)$$

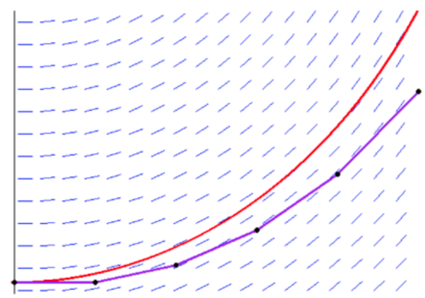
This is the Legendre Polynomial for n=3. Equating to zero you get the abscissae for 3 point gauss quadrature.

<table>
<tr><td rowspan="2">3</td><td colspan="2">0</td></tr>
<tr><td>$\pm\sqrt{\frac{3}{5}}$</td><td>±0.774597...</td></tr>
</table>

Solve this differential equation using the Explicit Euler Method. Take stepsize as *h=0.2*. Find the value of *y* at *t=0.4*. *Note the initial value of y.*

$$\begin{cases} y' = ty^2 + t \\ y(0) = .25 \\ t \in [0,1]. \end{cases}$$

t	y
0.0	0.25
0.2	$0.25 + .2(0 \cdot .25^2 + 0) = .25$
0.4	$0.25 + .2(.2 \cdot .25^2 + .2) = .29$
0.6	$0.29 + .2(.4 \cdot .29^2 + .4) = .38$
0.8	$0.38 + .2(.6 \cdot .38^2 + .6) = .51$
1.0	$0.51 + .2(.8 \cdot .51^2 + .8) = .72$



Answer: 0.29

Find the value of this integral by 2 point Gauss-Legendre quadrature.

$$\int_0^1 \frac{1}{1+x^2} dx$$

Answer:

We need to change the limits of integration so that we can apply the general gauss quadrature

Example 7.15.1 Find the value of $\int_0^1 \frac{1}{1+x^2}dx$ by Gauss's formula for $n = 2, 4, 6$. Also, calculate the absolute errors.

Solution. To apply the Gauss's formula, the limits are transferred to $-1, 1$ by substituting $x = \frac{1}{2}u(1-0) + \frac{1}{2}(1+0) = \frac{1}{2}(u+1)$.

Then $I = \int_0^1 \frac{1}{1+x^2}dx = \int_{-1}^1 \frac{2du}{(u+1)^2+4} = 2\sum_{i=1}^{n} w_i f(u_i)$,

where $f(x_i) = \frac{1}{(x_i+1)^2+4}$.

For the two-point formula ($n = 2$)

$x_1 = -0.57735027, x_2 = 0.57735027, w_1 = w_2 = 1$.
Then $I = 2[1 \times 0.23931272 + 1 \times 0.15412990] = 0.78688524$.

Answer : 0.786

Determine the values of the difference table using Newton-Forward interpolation.
Put the value of the first second order difference as your answer.

x	:	0.61	0.62	0.63	0.64	0.65
y	:	1.840431	1.858928	1.877610	1.896481	1.915541

Answer: 0.000185

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0.61	1.840431			
		0.018497		
0.62	1.858928		0.000185	
		0.018682		0.000004
0.63	1.877610		0.000189	
		0.018871		0.0
0.64	1.896481		0.000189	
		0.019060		
0.65	1.915541			