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A class of correlated weighted Poisson processes

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ABSTRACT

In this paper, we propose new classes of correlated Poisson processes and correlated weighted Poisson processes on the interval [0,1], which generalize the class of weighted Poisson processes defined by Balakrishnan and Kozubowski (2008), by incorporating a dependence structure between the standard uniform variables used in the construction. In this manner, we obtain another process that we refer to as correlated weighted Poisson process. Various properties of this process such as marginal and joint distributions, stationarity of the increments, moments, and the covariance function, are studied. The results are then illustrated through some examples, which include processes with length-biased Poisson, exponentially weighted Poisson, negative binomial, and COM-Poisson distributions.

1. Introduction

In a recent paper, Balakrishnan and Kozubowski (2008) introduced a class of weighted Poisson processes (WPP) on the interval [0,1]. This family assumes that the sequence of standard uniform variables are independent and the unobservable count variable follows a weighted Poisson distribution with parameter λ and weight function $w(\cdot)$; see, for example, Castillo and Pérez-Casany (1998, 2005). The probability mass function (PMF) of the WPP with intensity λ, $\{N^w(t), t \in [0,1]\}$, is given by

$$\mathbb{P}[N^w(t)=k]=\frac{e^{-\lambda t}(\lambda t)^k}{k!}\frac{w_t(k)}{e^{\lambda(1-t)}\mathbb{E}_\lambda[w(N)]},\quad k=0,1,\ldots,\lambda>0,\ t\in[0,1],$$

where $w_t(k)=\sum_{n=0}^{\infty}([(1-t)\lambda]^n/n!)w(n+k)$.

Following their approach, we generalize in this paper the above WPP by considering a dependence structure between the standard uniform variables. In this manner, we obtain a new class of processes that we refer to as correlated weighted Poisson process (CWPP) with intensity λ and correlation ρ, $\{N_\rho^w(t), t \in [0,1]\}$, whose PMF is

$$\mathbb{P}[N_\rho^w(t)=k]=\begin{cases}\rho(1-t)+\rho t\mathbb{G}_{N^w}(0)+(1-\rho)\mathbb{G}_{N^w}(1-t), & k=0,\\ \rho t\mathbb{P}[N^w=k]+(1-\rho)\dfrac{e^{-\lambda t}(\lambda t)^k}{k!}\dfrac{w_t(k)}{e^{\lambda(1-t)}\mathbb{E}_\lambda[w(N;\phi)]}, & k=1,2,\ldots\end{cases}$$

for $\lambda>0$, $t,\rho\in[0,1]$, where N^w is a weighted Poisson variable with parameter λ and weight function $w(\cdot)$, $\mathbb{G}_{N^w}(\cdot)$ is the probability generating function of N^w, and $w_t(k)=\sum_{n=0}^{\infty}([(1-t)\lambda]^n/n!)w(n+k)$. Particular choices of $w(\cdot)$ lead to some

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important special cases, including CWPP with geometric as well as negative binomial marginal distributions, which frequently arise in applications; see for example, Kozubowski and Podgórski (2009) and the references therein. This new class of processes will enable one to tackle the problem of overdispersion and underdispersion inherent in the analysis of count data (see Hinde and Demetrio, 1998) that may arise due to the presence of some correlation between the events. For example, in models of carcinogenesis (see Yakovlev and Tsodikov, 1996), it is usually assumed that cells in a tissue (modeled by a homogeneous Poisson process with intensity λ) can give rise to tumors *independently* of each other, i.e., they are biologically independent as far as carcinogenesis is concerned. Then, statistical independence is used to model biological independence. However, the biological independence assumption may not be true when the dynamics of the cell population of a normal tissue is considered. It is therefore desirable to construct mathematically tractable models of carcinogenesis that can adequately incorporate biological dependence, and this is what motivated the present research work.

The rest of this paper is organized as follows. In Section 2, we introduce the correlated Poisson process before formulating the class of correlated weighted Poisson processes in Section 3. In Section 4, the theoretical results are illustrated through some examples. Finally, some concluding remarks are made in Section 5.

2. The correlated Poisson process

Let N have a Poisson distribution with parameter $\lambda > 0$, and let $U_1, U_2, \ldots, U_N$ be a sequence of independent standard uniform variables, independently of N. Then, the random sum $N(t) = \sum_{j=1}^{N} I_{[0,t]}(U_j)$, where I_A is an indicator of the set A, is a classical Poisson process on the interval $[0,1]$, $\{N(t), t \in [0,1]\}$, with intensity λ. We generalize this Poisson process model by relaxing the assumption of mutual independence between the standard uniform variables. To introduce dependence, we assume that the variables are assigned with equal correlation between them. In this process, we obtain another process on the interval $[0,1]$, $\{N_\rho(t), t \in [0,1]\}$, with intensity λ and correlation ρ, which we refer to as correlated Poisson process (CPP). Thus, for the correlated Poisson process, we have

$$N_\rho(t) = \sum_{j=1}^{N} I_{[0,t]}(U_j) \quad \text{for } t \in [0,1], \tag{1}$$

where U_j's are now equicorrelated standard uniform variables, independently of N; i.e.,

$$\mathbb{C}orr(U_i, U_j) = \rho, \quad i \neq j,\ i,j = 1,2,\ldots,N = n. \tag{2}$$

Remark 1. This model is strongly connected with the concept of exchangeability which restricts the possible correlation structure for the sequences of random variables (r.v.'s) with finite first two moments (Aldous et al., 1985). If $(U_1, U_2, \ldots, U_n)$ is a finite sequence of exchangeable r.v.'s, then

$$\mathbb{C}orr(U_i, U_j) = \rho \in \left[-\frac{1}{n-1}, 1\right], \quad i \neq j. \tag{3}$$

Conversely, every ρ satisfying (3) serves as a correlation in some exchangeable sequence. To see this, let $(X_1, \ldots, X_n)$ be a sequence of independent r.v.'s with $\mathbb{E}(X_i) = 0$, $\mathbb{V}(X_i) = 1$, and define

$$U_i = X_i + d \sum_{k=1}^{n} X_k, \quad i = 1,2,\ldots,n,$$

for a constant $d \geq -1/n$. Thus, $(U_1, \ldots, U_n)$ is an exchangeable sequence and

$$\rho = \mathbb{E}(U_i U_j) = 1 - \frac{1}{nd^2 + 2d + 1}, \quad i \neq j.$$

Varying d, it is possible to obtain all values of $\rho \in [-1/(n-1), 1]$. For example, $\rho = -1/(n-1)$ when $d = -1/n$, and $\rho = 0$ when $d = 0$. For infinite exchangeable sequences, we of course have $\rho \in [0,1]$.

Now, let us note that distributionally, the variables $Z_j = I_{[0,t]}(U_j)$ have the same Bernoulli distribution with parameter $t = \mathbb{P}[U_j < t] \in [0,1]$ and corresponding probability generating function (PGF) as $\mathbb{G}_Z(s) = (1-t) + ts$, $s \in [0,1]$. However, now $U_1, \ldots, U_n$ are equicorrelated ($\mathbb{C}orr(U_i, U_j) = \rho$ for $i \neq j$) with identical marginal distributions. Therefore, $Z_1, Z_2, \ldots, Z_n$ are exchangeable Bernoulli variables with $\mathbb{C}orr(Z_i, Z_j) = \rho_1$, for some $\rho_1 \in [-1/(n-1), 1]$ and $i \neq j$, as noted above in Remark 1. The following lemma gives a necessary and sufficient condition for $\rho_1 = \rho$.

Lemma 1. *Let $\mathbb{C}orr(U_i, U_j) = \rho$ for $i \neq j$, where*

$$\rho \in \left[\max\left\{-\frac{t}{1-t}, -\frac{1-t}{t}, -\frac{1}{n-1}\right\}, 1\right].$$

Then, $\mathbb{C}orr(Z_i,Z_j)=\rho$ *iff*

$$\mathbb{P}[Z_i=1|Z_j=1]=t+(1-t)\rho \quad \textit{and} \quad \mathbb{P}[Z_i=0|Z_j=0]=(1-t)+t\rho. \tag{4}$$

Proof. The r.v.'s Z_i ($i=1,2,\ldots$) are identically distributed as Bernoulli with parameter t, and so $\mathbb{E}(Z)=t=P(U_i<t)$ and $\mathbb{V}ar(Z)=t(1-t)$. Then, $\mathbb{C}orr(Z_i,Z_j)=\rho$ for $i\neq j$ iff $\mathbb{C}ov(Z_i,Z_j)=\rho\ \mathbb{V}ar(Z)$, i.e., when $\mathbb{E}(Z_iZ_j)-t^2=t(1-t)\rho$. On the other hand, we simply have

$$\mathbb{E}(Z_iZ_j)=\mathbb{P}[Z_i=1,Z_j=1].$$

As a result, we have $\mathbb{P}[Z_i=1,Z_j=1]=t^2+t(1-t)\rho$, and by the conditional probability formula, we then obtain $\mathbb{P}[Z_i=1|Z_j=1]=t+(1-t)\rho$.

The second relation in (4) follows from the fact that

$$Z_i^*=1-Z_i, \quad i=1,2,\ldots$$

are also Bernoulli r.v.'s with $\mathbb{P}[Z_i^*=0]=1-(1-t)=t$, $\mathbb{P}[Z_i^*=1]=1-t$, and consequently

$$\mathbb{C}orr(Z_i^*,Z_j^*)=\frac{\mathbb{E}[(1-Z_i^*)(1-Z_j^*)]-(1-t)^2}{t(1-t)}=\rho.$$

The conditional probabilities should be non-negative, and therefore we have the restriction that

$$\rho\in\left[\max\left\{-\frac{t}{1-t},-\frac{1-t}{t},-\frac{1}{n-1}\right\},1\right]. \qquad \square$$

The indicator r.v.'s Z_i are equicorrelated, being exchangeable. Then, the conditional probabilities in (4) show when the existing correlation structure between the original r.v.'s U_i can be preserved by the r.v.'s Z_i given the value of the correlation coefficient ρ. Hence, the process in (1) is a sum of equicorrelated Bernoulli variables (i.e., $\mathbb{C}orr(Z_i,Z_j)=\rho$ for $i\neq j$).

2.1. Properties of the CPP

Here, we establish some basic properties of the correlated Poisson process in (1). Let us take N to be a Poisson variable with PMF

$$p_n=\mathbb{P}[N=n]=\frac{e^{-\lambda}\lambda^n}{n!}, \quad \lambda>0 \text{ and } n=0,1,2,\ldots\ . \tag{5}$$

The mean and variance of this model are known to be $\mathbb{E}(N)=\mathbb{V}ar(N)=\lambda$.

2.1.1. Marginal distributions

Given $N=n$ and for each $t\in[0,1]$, the process $N_\rho(t)=\sum_{j=1}^{n}I_{[0,t]}(U_j)$ is the sum of equicorrelated Bernoulli variables with probability of success t and an equal correlation coefficient ρ. Then, Tallis (1962) showed that the conditional PGF takes on the form

$$\mathbb{G}_{N_\rho(t)|(N=n)}(z)=\mathbb{E}(z^{N_\rho(t)|(N=n)})=\rho(1-t+tz^n)+(1-\rho)(1-t+tz)^n. \tag{6}$$

The corresponding distribution has been referred to in the literature as *correlated binomial* with parameters n, t and ρ (Luceño, 1995), denoted by $CB(n,t,\rho)$, i.e., $N_\rho(t)|(N=n)\sim CB(n,t,\rho)$. The PMF, the mean and the variance of $N_\rho(t)|(N=n)$ (Diniz et al., 2010) are given, respectively, by

$$\mathbb{P}[N_\rho(t)=k|n,t,\rho]=(1-\rho)\binom{n}{k}t^k(1-t)^{n-k}I_{A_1}(k)+\rho t^{k/n}(1-t)^{1-(k/n)}I_{A_2}(k), \tag{7}$$

where $A_1=\{0,1,\ldots,n\}$, $A_2=\{0,n\}$, $k=0,\ldots,n$, and $\rho\in[0,1]$, and

$$\mathbb{E}(N_\rho(t)=k|n,t,\rho)=nt \quad \text{and} \quad \mathbb{V}ar(N_\rho(t)|n,t,\rho)=t(1-t)\{n+\rho n(n-1)\}. \tag{8}$$

Note that the $CB(n,t,\rho)$ model is equivalent to the binomial model when $\rho=0$. Now, by unconditioning with respect to N, we can readily derive the marginal PGF, PMF, mean and variance of $N_\rho(t)$. For example, we obtain the PGF as

$$\begin{aligned}\mathbb{G}_{N_\rho(t)}(z)&=\mathbb{E}(z^{N_\rho(t)})=\mathbb{E}(\mathbb{E}(z^{N_\rho(t)}|N))=\sum_{n=0}^{\infty}\mathbb{G}_{N_\rho(t)|n}(z)\mathbb{P}[N=n]=\sum_{n=0}^{\infty}\{\rho(1-t+tz^n)+(1-\rho)(1-t+tz)^n\}\frac{\lambda^n e^{-\lambda}}{n!}\\&=\rho(1-t)+t\rho e^{-\lambda(1-z)}+(1-\rho)e^{-\lambda t(1-z)}.\end{aligned} \tag{9}$$

Next, we obtain the PMF of $N_\rho(t)$ in a similar manner as

$$g_{N_\rho(t)}(k)=\mathbb{P}[N_\rho(t)=k]=\sum_{n=k}^{\infty}\mathbb{P}[N_\rho(t)=k|n,t,\rho]\mathbb{P}[N=n]=\begin{cases}\rho(1-t)+\rho te^{-\lambda}+(1-\rho)e^{-\lambda t}, & k=0,\\ \dfrac{\rho t\lambda^k e^{-\lambda}}{k!}+\dfrac{(1-\rho)(t\lambda)^k e^{-\lambda t}}{k!}, & k=1,2,\ldots .\end{cases} \tag{10}$$

Furthermore, we also find

$$\mathbb{E}(N_\rho(t))=\mathbb{E}(\mathbb{E}(N_\rho(t)|N))=\mathbb{E}(NT)=\lambda t \tag{11}$$

and

$$\mathbb{V}ar(N_\rho(t))=\mathbb{E}(\mathbb{V}ar(N_\rho(t)|N))+\mathbb{V}ar(\mathbb{E}(N_\rho(t)|N))=t(1-t)(\lambda+\lambda^2\rho)+t^2\lambda. \tag{12}$$

Remark 2. From the expressions of mean and variance of $N_\rho(t)$ in (11) and (12), respectively, we readily find the Fisher index of $N_\rho(t)$ to be

$$\frac{\mathbb{V}ar(N_\rho(t))}{\mathbb{E}(N_\rho(t))}=1+\lambda\rho(1-t)\quad \text{for } t\in[0,1],$$

which reveals that the proposed correlated Poisson process accommodates overdispersion naturally.

2.1.2. Distributions of the increments

Consider the increment variable $N_\rho(t)-N_\rho(s)$ for $0\le s<t\le 1$. The computation of the PGF of $N_\rho(t)-N_\rho(s)$ is analogous to that of $N_\rho(t)$ presented above. In this case, we have

$$\mathbb{G}_{N_\rho(t)-N_\rho(s)}(t)=\mathbb{E}(z^{N_\rho(t)-N_\rho(s)})=\sum_{n=0}^{\infty}\mathbb{E}(z^{\sum_{j=1}^{n}(I_{[0,t]}(U_j)-I_{[0,s]}(U_j))})\mathbb{P}[N=n]. \tag{13}$$

Since this time the quantities $I_{[0,t]}(U_j)-I_{[0,s]}(U_j)$ are equicorrelated Bernoulli variables with probability $t-s$ and correlation coefficient $\rho\in[0,1]$, then, given $N=n$, the increment variable

$$N_\rho(t)-N_\rho(s)=\sum_{j=1}^{n}(I_{[0,t]}(U_j)-I_{[0,s]}(U_j))\sim CB(n,t-s,\rho).$$

Hence,

$$\mathbb{G}_{N_\rho(t)-N_\rho(s)}(t)=\rho\{1-(t-s)\}+(t-s)\rho e^{-\lambda(1-z)}+(1-\rho)e^{-\lambda(t-s)(1-z)}. \tag{14}$$

Interestingly, we observe that the distribution of the increment is stationary, depending only on the difference $t-s$.

2.1.3. Joint distributions

We now consider the joint distribution of variables $N_\rho(t_1),\ldots,N_\rho(t_k)$ for $0\le t_1<\cdots<t_k\le 1$. For simplicity, we present here the expression for the case $k=2$. We consider in this case the vector $(N_\rho(s),N_\rho(t))$ for $0\le s<t\le 1$, and derive the PMF of this vector. For any $0\le k\le n$, we have

$$\mathbb{P}[N_\rho(s)=k,N_\rho(t)=n]=\mathbb{P}[N_\rho(s)=k|N_\rho(t)=n]\mathbb{P}[N_\rho(t)=n].$$

Upon noting that, conditioned on $N_\rho(t)=n$, the variable $N_\rho(s)\sim CB(n,s/t,\rho)$, we readily find

$$\mathbb{P}[N_\rho(s)=k,N_\rho(t)=n]=\left\{(1-\rho)\binom{n}{k}\left(\frac{s}{t}\right)^k\left(1-\frac{s}{t}\right)^{n-k}I_{A_1}(k)+\rho\left(\frac{s}{t}\right)^{k/n}\left(1-\frac{s}{t}\right)^{1-(k/n)}I_{A_2}(k)\right\}\mathbb{P}[N_\rho(t)=n], \tag{15}$$

where $A_1=\{0,1,\ldots,n\}$, $A_2=\{0,n\}$, $k=0,\ldots,n$, $\rho\in[0,1]$, and $P[N_\rho(t)=n]$ is as given earlier in (10).

Next, we derive the characteristic function (ChF) of the vector $(N_\rho(s),N_\rho(t))$, for $0\le s<t\le 1$, as follows:

$$\begin{aligned}\mathbb{E}(e^{iuN_\rho(s)+ivN_\rho(t)})&=\sum_{n=0}^{\infty}\sum_{k=0}^{n}e^{iuk+ivn}\mathbb{P}[N_\rho(s)=k,N_\rho(t)=n]=\sum_{n=0}^{\infty}e^{ivn}\mathbb{P}[N_\rho(t)=n]\underbrace{\sum_{k=0}^{n}e^{iuk}\mathbb{P}[N_\rho(s)=k|N_\rho(t)=n]}_{\mathbb{G}_{N_\rho(s/t)|n}(e^{iu})}\\&=\rho\left(1-\frac{s}{t}\right)\mathbb{G}_{N_\rho(t)}(e^{iv})+\rho\frac{s}{t}\mathbb{G}_{N_\rho(t)}(e^{i(u+v)})+(1-\rho)\mathbb{G}_{N_\rho(t)}\left(\left(1-\frac{s}{t}+\frac{s}{t}e^{iu}\right)e^{iv}\right).\end{aligned} \tag{16}$$

This expression reduces to the ChF of $N_\rho(s)$ when $v=0$, and to the ChF of $N_\rho(t)$ when $u=0$.

2.1.4. Covariance function

We now derive the covariance of $N_\rho(s)$ and $N_\rho(t)$, as well as that of two consecutive increments, viz., $N_\rho(s)$ and $N_\rho(t)-N_\rho(s)$ for any $0\le s<t\le 1$. First, we observe

$$\mathbb{V}ar(N_\rho(t)-N_\rho(s))=\mathbb{V}ar(N_\rho(t))+\mathbb{V}ar(N_\rho(s))-2\mathbb{C}ov(N_\rho(s),N_\rho(t)),$$

and consequently,

$$\mathbb{C}ov(N_\rho(s),N_\rho(t)) = \tfrac{1}{2}[\mathbb{V}ar(N_\rho(t))+\mathbb{V}ar(N_\rho(s))-\mathbb{V}ar(N_\rho(t)-N_\rho(s))].$$

Since, $\mathbb{V}ar(N_\rho(t)-N_\rho(s)) = \mathbb{V}ar(N_\rho(t-s))$ as shown earlier in Section 2 and $\mathbb{V}ar(N_\rho(\cdot))$ is as given in (12), we get

$$\mathbb{C}ov(N_\rho(s),N_\rho(t)) = \lambda s\{1-\lambda\rho(1-t)\}. \tag{17}$$

This leads to an expression for the covariance between two consecutive increments, viz., $N_\rho(s)-N_\rho(0)\equiv N_\rho(s)$ and $N_\rho(t)-N_\rho(s)$, as

$$\mathbb{C}ov(N_\rho(s),N_\rho(t)-N_\rho(s)) = \mathbb{C}ov(N_\rho(s),N_\rho(t))-\mathbb{V}ar(N_\rho(s)) = \lambda s(1-\lambda\rho(1-t))-s(1-s)(\lambda+\lambda^2\rho)-s^2\lambda = \lambda^2 s\rho(t+s-2). \tag{18}$$

The expression in (18) reveals that two consecutive increments are always negatively correlated.

3. The correlated weighted Poisson process

Let N^w be an unobservable weighted Poisson random variable with PMF

$$p^w(n;\lambda,\phi) = \mathbb{P}[N^w = n;\lambda,\phi] = \frac{w(n;\phi)p(n;\lambda)}{\mathbb{E}_\lambda[w(N;\phi)]}, \quad n=0,1,2,\ldots, \tag{19}$$

where $w(\cdot;\phi)$ is a non-negative weight function with parameter $\phi>0$, $p(\cdot;\lambda)$ is the PMF of a Poisson distribution with parameter $\lambda>0$, and $\mathbb{E}_\lambda[\cdot]$ indicates that the expectation is taken with respect to the variable N following a Poisson distribution with mean parameter λ. We shall denote (19) by $WP_\lambda(w)$, which stands for the weighted Poisson distribution with parameter λ and weight function $w(\cdot;\phi)$. This concept was first introduced by Fisher (1934), but it was Rao (1965) who studied the weighted distributions in a unified way. Rao (1965) pointed out that in many situations the recorded observations cannot be considered as a random sample from the original distribution for many reasons including non-observability of some events, damage caused to original observations, and adoption of unequal probability sampling. Many weighted distributions have been used in practice and, for example, the weighted distribution with identity weight function is called the length-biased distribution, and it has found many important applications in environmetrics.

Let $U_1,U_2,\ldots,U_{N^w}$ be a sequence of equicorrelated standard uniform variables, independently of N^w, i.e., $\mathbb{C}orr(U_i,U_j)=\rho$, $\forall i\neq j$. Then, the random sum

$$N_\rho^w(t) = \sum_{j=1}^{N^w} I_{[0,t]}(U_j) \quad \text{for } t\in[0,1] \tag{20}$$

is said to be a correlated weighted Poisson process on the interval [0,1], $\{N_c^w(t), t\in[0,1]\}$, with intensity λ and correlation ρ. Thus, we can define a class of correlated weighted Poisson processes (CWPP) on the interval [0,1] indexed by $w\in\mathcal{W}$, where $\mathcal{W}$ is the set of all non-negative weight functions on $\mathbb{Z}_+$, as follows.

Definition 1. For any $w\in\mathcal{W}$, the process in (20) is said to be a correlated weighted Poisson process with parameter $\lambda>0$, correlation ρ, and weight function w, denoted by $CWPP_{[\lambda,\rho]}(w)$.

Remark 3. When $\rho=0$, the process in (20) simply becomes the weighted Poisson process proposed by Balakrishnan and Kozubowski (2008) and if $w(N;\phi)=constant$, the process in (20) becomes the correlated Poisson process, introduced earlier in Section 2.

Since $U_1,U_2,\ldots,U_{N^w}$ are equicorrelated with an identical distribution, $I_{[0,t]}(U_1),\ldots,I_{[0,t]}(U_{N^w})$ become exchangeable Bernoulli variables, with $\mathbb{C}orr(I_{[0,t]}(U_i),I_{[0,t]}(U_j))=\rho_1$, $i\neq j$, for some $\rho_1\in[-1/(n-1),1]$. Lemma 1 shows that when the existing correlation structure between the original r.v's U_i can be preserved by the r.v's $Z_i = I_{[0,t]}(U_i)$ for a given value of the correlation coefficient ρ.

Therefore, given $N^w=n$ and for each $t\in[0,1]$, the process in (20) is the sum of equicorrelated Bernoulli variables with probability of success t and an equal correlation coefficient ρ. Thus, by using the expression (6), we readily derive the marginal PGF of $N_\rho^w(t)$ as

$$\begin{aligned}\mathbb{G}_{N_\rho^w(t)}(z) = \mathbb{E}(z^{N_\rho^w(t)}) = \mathbb{E}(\mathbb{E}(z^{N_\rho^w(t)}\,|\,N^w)) = \sum_{n=0}^{\infty}\mathbb{G}_{N_\rho^w(t)|n}(z)\mathbb{P}[N^w=n] = \sum_{n=0}^{\infty}\{\rho(1-t+tz^n)+(1-\rho)(1-t+tz)^n\}\mathbb{P}[N^w=n]\\ = \rho(1-t)\underbrace{\sum_{n=0}^{\infty}\mathbb{P}[N^w=n]}_{1}+\rho t\underbrace{\sum_{n=0}^{\infty}z^n\mathbb{P}[N^w=n]}_{\mathbb{G}_{N^w}(z)}+(1-\rho)\underbrace{\sum_{n=0}^{\infty}(1-t+tz)^n\mathbb{P}[N^w=n]}_{\mathbb{G}_{N^w}(1-t+tz)}\\ = \rho(1-t)+\rho t\mathbb{G}_{N^w}(z)+(1-\rho)\mathbb{G}_{N^w}(1-t+tz),\end{aligned} \tag{21}$$

where $\mathbb{G}_{N^w}(\cdot)$ is the probability generating function of the weighted Poisson random variable N^w. In the following lemma (see Rodrigues et al., 2009), we present the probability generating function of the variable N^w.

Lemma 2. *Let $\phi \geq 0$ and $\theta \in \Theta \subset \mathbb{R}$, and the PMF of the discrete variable N^w be of the form*

$$p^w(n;\theta,\phi) = \varphi(n;\phi)\exp\{\theta n - K(\theta,\phi)\}, \quad n = 0,1,\ldots .$$

Then, the corresponding PGF is given by

$$\mathbb{G}_{N^w}(z) = \exp\{-\lambda(1-z)\}\frac{\mathbb{E}_{\lambda z}[w(N;\phi)]}{\mathbb{E}_{\lambda}[w(N;\phi)]},$$

where $\lambda = \exp\{\theta\}$ and $w(n;\phi) = n!\varphi(n;\phi)$.

Thus, by using Lemma 2, we can express $\mathbb{G}_{N^w_\rho(t)}(z)$ as

$$\mathbb{G}_{N^w_\rho(t)}(z) = \rho(1-t) + \rho t\exp\{-\lambda(1-z)\}\frac{\mathbb{E}_{\lambda z}[w(N;\phi)]}{\mathbb{E}_{\lambda}[w(N;\phi)]} + (1-\rho)\exp\{-\lambda t(1-z)\}\frac{\mathbb{E}_{\lambda(1-t+tz)}[w(N;\phi)]}{\mathbb{E}_{\lambda}[w(N;\phi)]}. \tag{22}$$

3.1. Properties of the CWPP

In this section, we derive some basic properties of the correlated weighted Poisson processes in (20), such as the marginal and joint distributions, stationarity of the increments, moments, and the covariance function.

3.1.1. Marginal distributions

Given the PGF $\mathbb{G}_{N^w_\rho(t)}(z)$ in (21), we obtain the PMF of the variable $N^w_\rho(t)$, by taking derivatives of $\mathbb{G}_{N^w_\rho(t)}(z)$, as

$$g_{N^w_\rho(t)}(k) = \mathbb{P}[N^w_\rho(t) = k] = \begin{cases} \mathbb{G}_{N^w_\rho(t)}(0), & k = 0, \\ \mathbb{G}^{(k)}_{N^w_\rho(t)}(0), & k = 1,2,\ldots, \end{cases}$$

where

$$\mathbb{G}^{(k)}_{N^w_\rho(t)} = \frac{\partial^k}{\partial z^k}\mathbb{G}_{N^w_\rho(t)}(z)\bigg|_{z=0} = \rho t k!\mathbb{P}[N^w = k] + (1-\rho)\sum_{n=k}^{\infty}\mathbb{P}[N^w = n]k!\binom{n}{k}t^k(1-t)^{n-k}.$$

Thus, we obtain

$$g_{N^w_\rho(t)}(k) = \begin{cases} \rho(1-t) + \rho t\mathbb{G}_{N^w}(0) + (1-\rho)\mathbb{G}_{N^w}(1-t), & k = 0, \\ \rho t\mathbb{P}[N^w = k] + (1-\rho)\dfrac{e^{-\lambda t}(\lambda t)^k}{k!}\dfrac{w_t(k;\phi)}{e^{\lambda(1-t)}\mathbb{E}_\lambda[w(N;\phi)]}, & k = 1,2,\ldots, \end{cases} \tag{23}$$

where $w_t(k;\lambda,\phi) = \sum_{n=0}^{\infty}([(1-t)\lambda]^n/n!)w(n+k;\phi)$ and $\mathbb{P}[N^w = k]$ is as in (19). Furthermore, by using the expression in (8), we also find

$$\mathbb{E}(N^w_\rho(t)) = \mathbb{E}(\mathbb{E}(N^w_\rho(t)|N^w)) = t\mathbb{E}(N^w) \tag{24}$$

and

$$\mathbb{V}ar(N^w_\rho(t)) = \mathbb{E}(\mathbb{V}ar(N^w_\rho(t)|N^w)) + \mathbb{V}ar(\mathbb{E}(N^w_\rho(t)|N^w)) = t(1-t)[(1-\rho)\mathbb{E}(N^w) + \rho\mathbb{E}(N^{w^2})] + t^2\ \mathbb{V}ar[N^w]. \tag{25}$$

Remark 4. From the expressions of mean and variance of $N^w_\rho(t)$ in (24) and (25), respectively, we readily find the Fisher index of $N^w_\rho(t)$ to be

$$\frac{\mathbb{V}ar(N^w_\rho(t))}{\mathbb{E}(N^w_\rho(t))} = 1 + [t + \rho(1-t)]\left\{\frac{\mathbb{V}ar(N^w)}{\mathbb{E}(N^w)} - 1\right\} + \rho(1-t)\mathbb{E}(N^w) \quad \text{for } t \in [0,1],$$

which reveals that the proposed correlated weighted Poisson process is underdispersed for each $t \in [0,1]$ if and only if $\mathbb{E}(N^w) < t + \rho(1-t)/\rho(1-t)$ and $\mathbb{V}ar(N^w)/\mathbb{E}(N^w) < 1 - \rho(1-t)/t + \rho(1-t)\mathbb{E}(N^w)$. Otherwise, the process is overdispersed.

3.1.2. Distributions of the increments

Consider the increment variable $N^w_\rho(t) - N^w_\rho(s)$ for $0 \leq s < t \leq 1$. The derivation of the PGF of $N^w_\rho(t) - N^w_\rho(s)$ is analogous to that of $N^w_\rho(t)$ presented above. In this case, we have

$$\mathbb{G}_{N^w_\rho(t)-N^w_\rho(s)}(z) = \mathbb{E}(z^{N^w_\rho(t)-N^w_\rho(s)}) = \sum_{n=0}^{\infty}\mathbb{E}(z^{\sum_{j=1}^{n}(I_{[0,t]}(U_j)-I_{[0,s]}(U_j))})\mathbb{P}[N^w = n]. \tag{26}$$

Now, since the quantities $I_{[0,t]}(U_j) - I_{[0,s]}(U_j)$ are equicorrelated Bernoulli variables with probability $t-s$ and correlation coefficient $\rho \in [0,1]$, then, given $N^w = n$, the increment variable $N^w_\rho(t) - N^w_\rho(s) = \sum_{j=1}^{n}(I_{[0,t]}(U_j) - I_{[0,s]}(U_j)) \sim CB(n, t-s, \rho)$, and consequently

$$\mathbb{G}_{N^w_\rho(t)-N^w_\rho(s)}(z) = \rho\{1-(t-s)\} + (t-s)\rho e^{-\lambda(1-z)} + (1-\rho)e^{-\lambda(t-s)(1-z)}. \tag{27}$$

Interestingly, we observe one again that the distribution of the increment is stationary, depending only on the difference $t-s$.

3.1.3. Joint distributions

Here, we consider the joint distribution of variables $N_\rho^w(t_1),\ldots,N_\rho^w(t_k)$ for $0\le t_1<\cdots<t_k\le 1$. For simplicity, we derive the expression for the case $k=2$, and the result for general k can be derived in an analogous manner. We consider in this case the vector $(N_\rho^w(s),N_\rho^w(t))$ for $0\le s<t\le 1$, and derive the joint PMF of this vector. For any $0\le k\le n$, we have

$$\mathbb{P}[N_\rho^w(s)=k,N_\rho^w(t)=n]=\mathbb{P}[N_\rho^w(s)=k|N_\rho^w(t)=n]\mathbb{P}[N_\rho^w(t)=n].$$

Upon noting, by conditioning on $N_\rho^w(t)=n$, that the variable $N_\rho^w(s)\sim CB(n,s/t,\rho)$, we readily find

$$\mathbb{P}[N_\rho^w(s)=k,N_\rho^w(t)=n]=\left\{(1-\rho)\binom{n}{k}\left(\frac{s}{t}\right)^k\left(1-\frac{s}{t}\right)^{n-k}I_{A_1}(k)+\rho\left(\frac{s}{t}\right)^{k/n}\left(1-\frac{s}{t}\right)^{1-(k/n)}I_{A_2}(k)\right\}\mathbb{P}[N_\rho^w(t)=n], \tag{28}$$

where $A_1=\{0,1,\ldots,n\}$, $A_2=\{0,n\}$, $k=0,\ldots,n$, $\rho\in[0,1]$, and $\mathbb{P}[N_\rho^w(t)=n]$ is as given earlier in (23).

Next, we derive the joint ChF of the vector $(N_\rho^w(s),N_\rho^w(t))$, for $0\le s<t\le 1$, as follows:

$$\begin{aligned}\mathbb{E}(e^{iuN_\rho^w(s)+ivN_\rho^w(t)})&=\sum_{n=0}^{\infty}\sum_{k=0}^{n}e^{iuk+ivn}\mathbb{P}[N_\rho^w(s)=k,N_\rho^w(t)=n]\\&=\sum_{n=0}^{\infty}e^{ivn}\mathbb{P}[N_\rho^w(t)=n]\underbrace{\sum_{k=0}^{n}e^{iuk}\mathbb{P}[N_\rho^w(s)=k|N_\rho^w(t)=n]}_{\mathbb{G}_{N_\rho^w(s/t)|n}(e^{iu})}\\&=\rho\left(1-\frac{s}{t}\right)\mathbb{G}_{N_\rho^w(t)}(e^{iv})+\rho\frac{s}{t}\mathbb{G}_{N_\rho^w(t)}(e^{i(u+v)})+1-\rho)\mathbb{G}_{N_\rho^w(t)}\left(\left(1-\frac{s}{t}+\frac{s}{t}e^{iu}\right)e^{iv}\right).\end{aligned} \tag{29}$$

Naturally, the expression in (29) reduces to the ChF of $N_\rho^w(s)$ when $v=0$, and to the ChF of $N_\rho^w(t)$ when $u=0$.

3.1.4. Covariance function

Here, we derive the covariance of $N_\rho^w(s)$ and $N_\rho^w(t)$, as well as that of two consecutive increments, viz., $N_\rho^w(s)$ and $N_\rho^w(t)-N_\rho^w(s)$ for any $0\le s<t\le 1$. First, we observe

$$\mathbb{V}ar(N_\rho^w(t)-N_\rho^w(s))=\mathbb{V}ar(N_\rho^w(t))+\mathbb{V}ar(N_\rho^w(s))-2\,\mathbb{C}ov(N_\rho^w(s),N_\rho^w(t)),$$

and consequently,

$$\mathbb{C}ov(N_\rho^w(s),N_\rho^w(t))=\tfrac{1}{2}[\mathbb{V}ar(N_\rho^w(t))+\mathbb{V}ar(N_\rho^w(s))-\mathbb{V}ar(N_\rho^w(t)-N_\rho^w(s))].$$

Since $\mathbb{V}ar(N_\rho^w(t)-N_\rho^w(s))=\mathbb{V}ar(N_\rho^w(t-s))$ as shown earlier, and $\mathbb{V}ar(N_\rho^w(\cdot))$ is as given in (25), we obtain

$$\mathbb{C}ov(N_\rho^w(s),N_\rho^w(t))=s\{(1-t)[(1-\rho)\mathbb{E}[N^w]+\rho\mathbb{E}(N^{w^2})]+t\mathbb{V}ar(N^w)\}. \tag{30}$$

This leads to an expression for the covariance of two consecutive increments, viz., $N_\rho^w(s)-N_\rho^w(0)\equiv N_\rho^w(s)$ and $N_\rho^w(t)-N_\rho^w(s)$, as

$$\mathbb{C}ov(N_\rho^w(s),N_\rho^w(t)-N_\rho^w(s))=\mathbb{C}ov(N_\rho^w(s),N_\rho^w(t))-\mathbb{V}ar(N_\rho^w(s))=s(t-s)\{\mathbb{V}ar[N^w]-[(1-\rho)\mathbb{E}[N^w]+\rho\mathbb{E}[N^{w^2}]]\}. \tag{31}$$

The expression in (31) readily reveals that two consecutive increments are positively correlated if only if the variable N^w is overdispersed and $\mathbb{V}ar[N^w]>(1-\rho)\mathbb{E}[N^w]+\rho\mathbb{E}[N^{w^2}]$. Otherwise, the increments are negatively correlated.

4. Some illustrative examples

In this section, we present a few specific processes that arise from our general formulation. In Table 1, we present the probability generating function, mean and variance corresponding to these processes.

4.1. Length-biased Poisson process

When the weight function of the r.v. N^w is simply $w(n;\phi)=n$, then the weight function $w_t(\cdot)$ has an explicit form given by $w_t(k;\lambda,t)=k+\lambda(1-t)$. Recall that, in this case, N^w has a Poisson distribution with parameter λ shifted up by one, and is underdispersed, since $\mathbb{V}ar[N^w]<\mathbb{E}[N^w]$. Also, note that the compound Poisson distribution of $N_\rho^w(t)$ is not shifted-Poisson, since its PMF is of the form

$$\mathbb{P}[N_\rho^w(t)=k]=\begin{cases}\rho(1-t)+\rho te^{-\lambda}+(1-\rho)e^{-\lambda t}, & k=0,\\ \rho t\dfrac{\lambda^{k-1}e^{-\lambda}}{(k-1)!}+(1-\rho)\dfrac{e^{-\lambda t}(\lambda t)^k}{k!}\left(1-t+\dfrac{k}{\lambda}\right), & k=1,2,\ldots\ .\end{cases}$$

Table 1
Probability generating function ($G_{N_\rho^w(t)}(\cdot)$), mean ($\mathbb{E}[N_\rho^w(\cdot)]$), and variance ($\mathbb{V}ar[N_\rho^w(\cdot)]$) for different processes.

Correlated process	$\mathbb{G}_{N_\rho^w(t)}(z)$	$\mathbb{E}[N_\rho^w(t)]$	$\mathbb{V}ar[N_\rho^w(t)]$
Length-biased Poisson	$\rho(1-t)+\rho t\exp\{-\lambda(1-z)\}+(1-\rho)(1-t(1-z))\exp\{-\lambda t(1-z)\}$	$t(1-\lambda)$	$t(1-t)[1+\lambda(1+2\rho+\rho\lambda)]+t^2\lambda$
Exponentially weighted Poisson	$\rho(1-t)+\rho t\exp\{-\lambda e^{2\phi}(1-z)\}+(1-\rho)\exp\{-\lambda t(1-z)e^{2\phi}\}$	$t\lambda\exp\{\phi+2\lambda e^{\phi}\}$	$t(1-t)\lambda[\exp\{\phi+2\lambda e^{\phi}\}+\rho\lambda\exp\{2(\phi+\lambda e^{\phi})\}]+t^2\lambda[\lambda\exp\{2(\phi+\lambda e^{\phi})\}$ $+\exp\{\phi+2\lambda e^{\phi}\}-\lambda\exp\{2(\phi+2\lambda e^{\phi})\}]$
Negative binomial	$\rho(1-t)+\rho t(1+\phi\lambda(1-z))^{-1/\phi}+(1-\rho)(1+\phi\lambda t(1-z))^{-1/\phi}$	$t\lambda$	$t(1-t)\lambda[1+\rho\lambda(1+\phi)]+t^2\lambda(1+\phi\lambda)$
COM-Poisson	$\rho(1-t)+\rho t\dfrac{Z(\lambda z,\phi)}{Z(\lambda,\phi)}+(1-\rho)\dfrac{Z(\lambda(1-t(1-z)),\phi)}{Z(\lambda,\phi)}$	$t\sum_{j=0}^{\infty}\dfrac{j\lambda^j}{(j!)^{\phi}Z(\lambda,\phi)}$	$t(1-t)\left[\sum_{j=0}^{\infty}\dfrac{j\lambda^j}{(j!)^{\phi}Z(\lambda,\phi)}\times(1-\rho(1-j))\right]+t^2\left[\sum_{j=0}^{\infty}\dfrac{j^2\lambda^j}{(j!)^{\phi}Z(\lambda,\phi)}-\left(\sum_{j=0}^{\infty}\dfrac{j\lambda^j}{(j!)^{\phi}Z(\lambda,\phi)}\right)^2\right]$

4.2. Exponentially weighted Poisson process

When the weight function of the r.v. N^w is of exponential form, i.e., $w(n;\phi)=e^{\phi m}$, $\phi\in\mathbb{R}$, then the weight function $w_t(\cdot)$ has an explicit form given by $w_t(k;\lambda,\phi,t)=\exp\{\phi k-\lambda(1-t)(1-e^{\phi})\}$. It is easy to see that N^w has a Poisson distribution with parameter λe^{ϕ} and that the variable $N_\rho^w(t)$ has PMF given by

$$\mathbb{P}[N_\rho^w(t)=k]=\begin{cases}\rho(1-t)+\rho t e^{-\lambda e^{2\phi}}+(1-\rho)\exp\{-\lambda t e^{2\phi}\}, & k=0,\\ \rho t\dfrac{\lambda^k\exp\{k\phi+\lambda e^{\phi}\}}{k!}+(1-\rho)\dfrac{(\lambda t)^k\exp\{k\phi-\lambda(1-2e^{\phi})+\lambda t(1-e^{\phi})\}}{k!}, & k=1,2,\ldots\ .\end{cases}$$

4.3. Negative binomial process

Let us take the unobservable count variable, N^w, as a variable with a negative binomial distribution with parameters $\phi>0$ and $\lambda>0$ (see Piegorsch, 1990; Saha and Paul, 2005), with PMF

$$\mathbb{P}[N^w=n;\lambda,\phi]=\frac{\Gamma(\phi^{-1}+n)}{\Gamma(\phi^{-1})n!}\left(\frac{\phi\lambda}{1+\phi\lambda}\right)^n(1+\phi\lambda)^{-1/\phi},\quad n=0,1,2,\ldots\ . \tag{32}$$

Note that $\phi=1$ leads to the geometric distribution with parameter $1/(1+\lambda)$. Comparing (32) with (19), we realize that (32) is a weighted Poisson distribution with parameter $\phi\lambda/(1+\phi\lambda)$ and weight function $w(n;\phi)=\Gamma(\phi^{-1}+n)$. So, we arrive at a closed-form expression for the weight function $w_t(k;\lambda,\phi,t)$ in this case as

$$w_t(k;\lambda,\phi,t)=\exp\left\{\frac{-\phi\lambda(1-t)}{1+\phi\lambda}\right\}\left[\frac{1}{1-\dfrac{\phi\lambda(1-t)}{1+\phi\lambda}}\right]^k\frac{\Gamma(\phi^{-1}+k)}{\left[1-\dfrac{\phi\lambda(1-t)}{1+\phi\lambda}\right]^{1/\phi}}.$$

Thus, the variable $N_\rho^w(t)$ in this case has PMF as

$$\mathbb{P}[N_\rho^w(t)=k]=\begin{cases}\rho(1-t)+\rho t(1+\phi\lambda)^{-1/\phi}+(1-\rho)(1+\phi\lambda t)^{-1/\phi}, & k=0,\\ \rho t\dfrac{\Gamma(\phi^{-1}+k)}{\Gamma(\phi^{-1})k!}\left(\dfrac{\phi\lambda}{1+\phi\lambda}\right)^k(1+\phi\lambda)^{-1/\phi}+(1-\rho)\dfrac{\Gamma(\phi^{-1}+1)}{\Gamma(\phi^{-1})k!}\left(\dfrac{\phi\lambda t}{1+\phi\lambda t}\right)^k(1+\phi\lambda t)^{-1/\phi}, & k=1,2,\ldots\ .\end{cases}$$

4.4. COM-Poisson process

Let us take the unobservable count variable N^w as a variable with a COM-Poisson distribution with parameters $\lambda>0$ and $\phi>0$ (see Shmueli et al., 2005), with PMF

$$\mathbb{P}[N^w=n;\lambda,\phi]=\frac{1}{Z(\lambda,\phi)}\frac{\lambda^n}{(n!)^{\phi}},\quad m=0,1,2,\ldots, \tag{33}$$

where $Z(\lambda,\phi)=\sum_{j=0}^{\infty}\lambda^j/(j!)^{\phi}$. In particular, when $\phi=0$ and $\lambda<1$, the COM-Poisson distribution reduces to the geometric distribution with parameter $1-\lambda$. The distribution in (33) may also be viewed as a weighted Poisson distribution with weight function $w(n;\phi)=(n!)^{1-\phi}$; see Kokonendji et al. (2008). Therefore, for integer ϕ, the weight function $w_t(\cdot)$ of the variable N_ρ^w is given by

$$\begin{aligned}w_t(k;\lambda,\phi,t)&=\exp\{-\lambda(1-t)\}(k!)^{1-\phi}\sum_{n=0}^{\infty}[(k+1)(k+2)\times\cdots\times(k+n)]^{1-\phi}\frac{[\lambda(1-t)]^n}{n!}\\&=\exp\{-\lambda(1-t)\}(k!)^{1-\phi}{}_1\mathcal{F}_{\phi}(k+1;k+1,\ldots,k+1;\lambda(1-t)),\end{aligned} \tag{34}$$

where the generalized hypergeometric function is defined by

$${}_u\mathcal{F}_v(a_1,\ldots,a_u;b_1,\ldots,b_v;x)=\sum_{n=0}^{\infty}\frac{(a_1)_n(a_2)_n\times\cdots\times(a_u)_nx^n}{(b_1)_n(b_2)_n\times\cdots\times(b_v)_nn!},$$

with $(a)_n=a(a+1)\times\cdots\times(a+n-1)$; see Gradshteyn and Ryzhik (2000). Taking $k=0$ and $t=0$ in (33), we obtain from (22) a closed-form expression for the COM-Poisson process as

$$\mathbb{G}_{N_\rho^w(t)}(z)=\rho(1-t)+\rho t\frac{{}_1\mathcal{F}_{\phi}(1;1,\ldots,1;\lambda z)}{{}_1\mathcal{F}_{\phi}(1;1,\ldots,1;\lambda)}+(1-\rho)\frac{{}_1\mathcal{F}_{\phi}(1;1,\ldots,1;\lambda(1-t+tz))}{{}_1\mathcal{F}_{\phi}(1;1,\ldots,1;\lambda)}.$$

In this case, the distribution of the variable N^w is

$$\mathbb{P}[N_\rho^w(t)=k]=\begin{cases}\rho(1-t)+\rho t\dfrac{1}{Z(\lambda,\phi)}+(1-\rho)\dfrac{{}_1\mathcal{F}_\phi(1;1,\ldots,1;\lambda(1-t))}{{}_1\mathcal{F}_\phi(1;1,\ldots,1;\lambda)}, & k=0,\\ \rho t\dfrac{1}{Z(\lambda,\phi)}\dfrac{\lambda^k}{(k!)^\phi}+(1-\rho)\dfrac{(\lambda t)^k}{(k!)^\phi}\dfrac{{}_1\mathcal{F}_\phi(1;1,\ldots,1;\lambda(1-t))}{{}_1\mathcal{F}_\phi(1;1,\ldots,1;\lambda)}, & k=1,2,\ldots\,.\end{cases}$$

5. Concluding remarks

In this paper, we have introduced correlated Poisson processes and correlated weighted Poisson processes as generalizations of the class of weighted Poisson processes on the interval [0;1] proposed by Balakrishnan and Kozubowski (2008) by incorporating a dependence structure between the standard uniform random variables used in the construction. The dependence is measured by the correlation coefficient which is a natural measure for the underlying exchangeable sequence. We have then studied some basic properties of these processes such as the marginal and joint distributions, stationarity of the increments, moments, and the covariance function. These results are then illustrated through some examples, which include processes with length-biased Poisson, exponentially weighted Poisson, negative binomial, and COM-Poisson distributions. Since this class of processes naturally accommodates underdispersion and overdispersion, it will be useful when dealing with data exhibiting greater variability than that allowed by the Poisson process. It will therefore be of great interest to develop inferential methods for the correlated Poisson processes and the correlated weighted Poisson processes. Work in this direction is currently under progress and we hope to report these findings in a future paper.

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