CSC321 Tutorial 10: Policy Gradient

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University of Toronto

Overview

Determinant of an upper triangular matrix

Change-of-Variables Formula

Policy Gradient iPython Notebook

Determinant of an upper triangular matrix

Upper and Lower Triangular Matrix

Upper Triangular Matrices:

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n} \\ 0 & a_{2,2} & a_{2,3} & \cdots & a_{2,n} \\ 0 & 0 & a_{3,3} & \cdots & a_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n,n} \end{pmatrix} \tag{1}$$

Lower Triangular Matrices:

$$A = \begin{pmatrix} a_{1,1} & 0 & 0 & \cdots & 0 \\ a_{2,1} & a_{2,2} & 0 & \cdots & 0 \\ a_{3,1} & a_{3,2} & a_{3,3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,n} \end{pmatrix} \tag{2}$$

Determinant of an upper triangule matrix

Let A be an upper triangular matrix (or, a lower triangular matrix). Then, $\det(A)$ is the product of the diagonal elements of A, namely

$$\det(A) = \prod_{i=1}^{n} a_{ii} \tag{3}$$

For example:

$$A = \begin{pmatrix} 1 & 5 & 8 & 10 \\ 0 & 2 & 6 & 9 \\ 0 & 0 & 3 & 7 \\ 0 & 0 & 0 & 4 \end{pmatrix}, \det(A) = 1 \times 2 \times 3 \times 4 = 24 \tag{4}$$

Determinant of a 2x2 Matrix

In the case of 2x2 matrix:

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \tag{5}$$

So for an upper triangular 2x2 matrix, the determinant is:

$$|A| = \begin{vmatrix} a & b \\ 0 & d \end{vmatrix} = ad - b(0) = ad \tag{6}$$

Which is the product of its diagonal entries.

Determinant of a 3x3 Matrix

In the case of 3x3 matrix:

$$|A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - d \begin{vmatrix} b & c \\ h & i \end{vmatrix} + g \begin{vmatrix} b & c \\ e & f \end{vmatrix} \tag{7}$$

If the matrix is upper triangular:

$$|A| = \begin{vmatrix} a & b & c \\ 0 & e & f \\ 0 & 0 & i \end{vmatrix} = a \begin{vmatrix} e & f \\ 0 & i \end{vmatrix} - 0 \begin{vmatrix} b & c \\ 0 & i \end{vmatrix} + 0 \begin{vmatrix} b & c \\ e & f \end{vmatrix} = aei \tag{8}$$

Determinant of a Matrix (General)

Let A be a $n \times n$ matrix. Then the *determinant* of A is defined by the following. If $n = 1$, then $\det(A) = a_{1,1}$. Otherwise:

$$\det(A) = \sum_{i=1}^{n} a_{i,1} A_{i,1} \tag{9}$$

Where $A_{i,j}$ is the (i, j) *-cofactor* associated with A:

$$A_{i,j} = (-1)^{i+j} \det(M_{i,j}) \tag{10}$$

Where $M_{i,j}$ is $(n-1)(n-1)$ matrix obtained from A by removing the $i-th$ row and $j-th$ column. $M_{i,j}$'s are called the *minors* of A

Proof for Determinant of Upper Triangular Matrix

Use proof by induction on n, the number of rows in the matrix A_n

Basis for induction: When $n = 1$, the determinant is $a_{1,1}$, which is the diagonal element

Induction Hypothesis: When $n \in \mathcal{N}$, we hypothesize that $\det(A_n) = \prod_{i=1}^{n} a_{ii}$

Induction Step: Let A_{n+1} be an upper triangular matrix of order $n + 1$. Apply the definition of determinant, expanding across the $n + 1$-th row:

$$\det(A_{n+1}) = \sum_{i=1}^{n+1} a_{n+1,i} A_{n+1,i} \tag{11}$$

Proof for Determinant of Upper Triangular Matrix

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n} & a_{1,n+1} \\ 0 & a_{2,2} & a_{2,3} & \cdots & a_{2,n} & a_{2,n+1} \\ 0 & 0 & a_{3,3} & \cdots & a_{3,n} & a_{3,n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n,n} & a_{n,n+1} \\ 0 & 0 & 0 & \cdots & 0 & a_{n+1,n+1} \end{pmatrix} \tag{12}$$

Induction Step: Notice that $a_{n+1,i} = 0$ when $i < n+1$.
Therefore:

$$\det(A_{n+1}) = a_{n+1,n+1} A_{n+1,n+1} \tag{13}$$

Where $A_{n+1,n+1} = (-1)^{n+1+n+1} \det(M_{n+1,n+1}) = \det(A_n)$

Proof for Determinant of Upper Triangular Matrix

Substituting expression for $A_{n+1,n+1}$:

$$\det(A_{n+1}) = a_{n+1,n+1}A_{n+1,n+1} \tag{14}$$

$$\det(A_{n+1}) = a_{n+1,n+1}\det(A_n) \tag{15}$$

$$= a_{n+1,n+1}\prod_{i=1}^{n} a_{ii} \tag{16}$$

$$\det(A_{n+1}) = \prod_{i=1}^{n+1} a_{ii} \tag{17}$$

As desired ■.

Change-of-Variables Formula

Change-of-Variables Formula

Let f denote a differentiable, bijective mapping from space $\mathcal{Z}$ to space $\mathcal{X}$ (1-to-1 mapping).

If $\mathbf{x} = f(\mathbf{z})$, then

$$p_X(\mathbf{x}) = p_Z(z) \left| \det \left(\frac{\partial \mathbf{x}}{\partial z} \right) \right|^{-1} \tag{18}$$

Intuition

If the mapping f is 1-to-1, then the total area (or volume) should not change after the transformation from x to z.

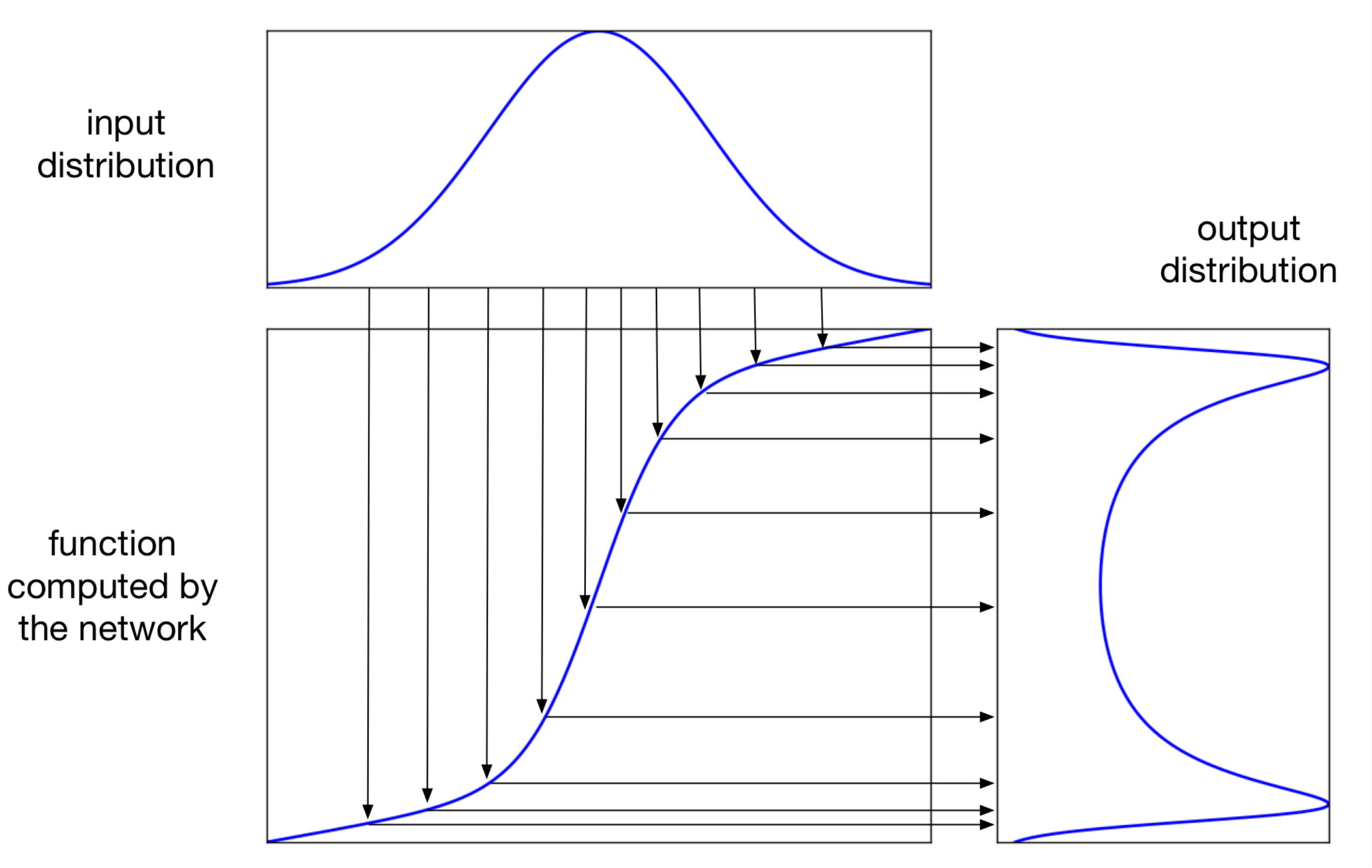

Figure 1: Mapping from one probability density to another. Source: Lecture 19 notes

Example

Qs: Let $p_X(x) = 2x$, for $0 \leq x \leq 1$. Let $f(x) = \sqrt{x} = z$.
What is $p_Z(z)$?

Ans:

$$\sqrt{x} = z \Leftrightarrow x = z^2 \tag{19}$$

$$\frac{\partial x}{\partial z} = 2z \tag{20}$$

$$p_X(\mathbf{x}) = p_Z(z)\left| \det\left(\frac{\partial \mathbf{x}}{\partial z}\right)\right|^{-1} \tag{21}$$

$$p_Z(z) = p_X(\mathbf{x})\left| \det\left(\frac{\partial \mathbf{x}}{\partial z}\right)\right| \tag{22}$$

$$p_Z(z) = (2x)|2z| \tag{23}$$

$$p_Z(z) = (2(z^2))2z \tag{24}$$

$$p_Z(z) = 4z^3 \tag{25}$$

Example

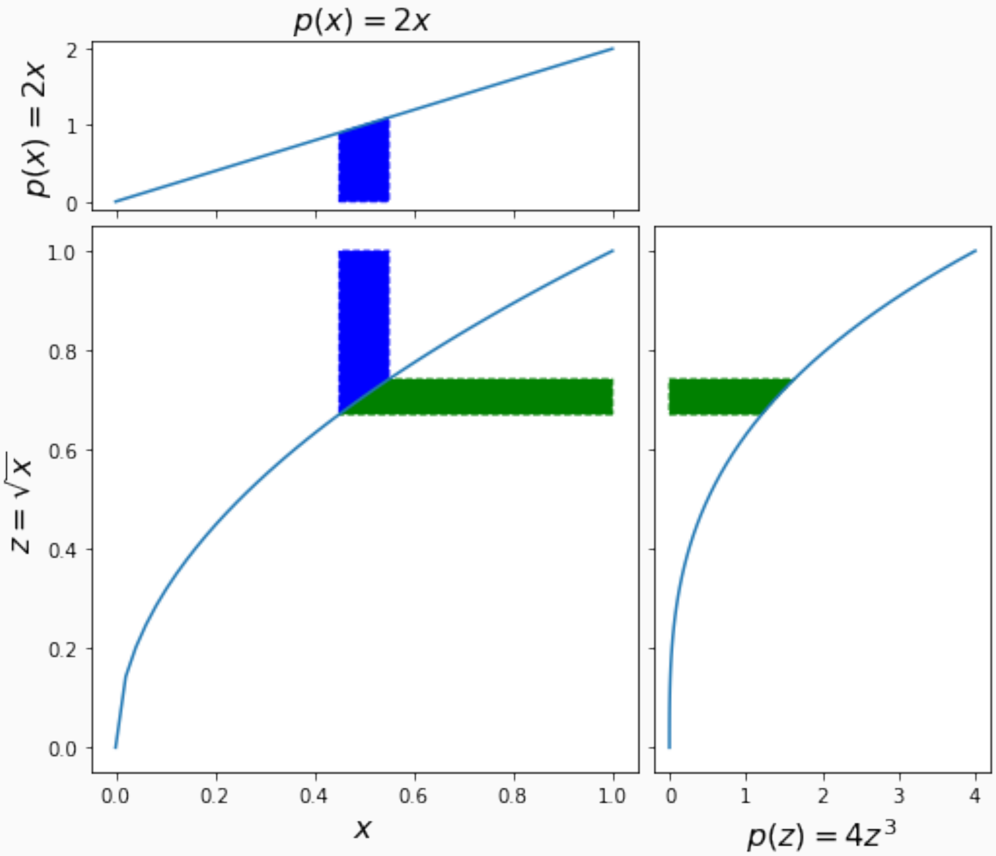

Figure 2: Mapping from $p(x)$ to $p(z)$

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See Demo