ME4010 Modelling assignment

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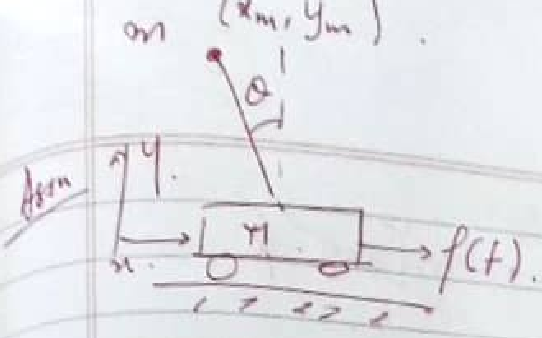

Equations of Motion

$x_m = x - l\sin\theta \qquad y_m = l\cos\theta$.

$\dot{x}_m = \dot{x} - l\dot{\theta}\cos\theta \qquad \dot{y}_m = -l\dot{\theta}\sin\theta$.

Lagrangian method to write eqn of motion:

$PE = V = mg\, l\cos\theta$

$T = KE = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left(\dot{x}_m^2 + \dot{y}_m^2\right)$

$= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left(\cancel{a}\,\dot{x}^2 + l^2\dot{\theta}^2\cancel{\cos^2\theta} - 2\dot{x}l\dot{\theta}\cos\theta + l^2\dot{\theta}^2\sin^2\theta\right)$

$l^2\dot{\theta}^2$

$= \frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2}m l^2\dot{\theta}^2 - m l\dot{\theta}\dot{x}\cos\theta$

Lagrange's eqn: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i$

Let $L = T - V$ (Lagrangian)

$L = \frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2}m l^2\dot{\theta}^2 - m l\dot{\theta}\dot{x}\cos\theta - mg\,l\cos\theta$.

$\cancel{\frac{\partial L}{\partial x} = \frac{1}{2}(M+m)\left(2\dot{x}\frac{\partial \dot{x}}{\partial x}\right)}$

Eqns of Motion:

$\cancel{x: (M+m)\ddot{x} -}$

$\frac{\partial L}{\partial \dot{x}} = \frac{1}{2}(M+m)2\dot{x} + 0 - m l\dot{\theta}\cos\theta - 0$.

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = (M+m)\ddot{x} - \left(m l\ddot{\theta}\cos\theta + m l\dot{\theta}(-\sin\theta)\dot{\theta}\right)$

$\frac{\partial L}{\partial x} = 0$.

$\Rightarrow (M+m)\ddot{x} - m l\ddot{\theta}\cos\theta + m l\dot{\theta}^2\sin\theta = F(t)$

θ:

$$\cancel{\frac{\partial L}{\partial \dot\theta} = 0 + \frac{1}{2} m l^2 \cdot 2\dot\theta - ml\cos\theta \cdot \dot x}$$

$$\cancel{\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot\theta}\right) = ml^2\ddot\theta + ml\sin\theta\,\dot\theta}$$

$$\cancel{\frac{\partial L}{\partial \theta} = -ml\dot\theta(-\sin\theta) - mgl(-\sin\theta)}$$

$$\cancel{\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot\theta}\right) - \frac{\partial L}{\partial \theta} = 0}$$

$$\cancel{ml^2\ddot\theta + ml\dot\theta\sin\theta - ml\dot\theta\sin\theta - mgl\sin\theta = 0}$$

$$l\ddot\theta \quad - g\sin\theta = 0$$

$$L = \frac{1}{2}(M+m)\dot x^2 + \frac{1}{2} ml^2\dot\theta^2 - ml\dot\theta\dot x\cos\theta - mgl\cos\theta$$

$$\frac{\partial L}{\partial \dot\theta} = 0 + \frac{1}{2} ml^2\dot\theta - ml\dot x\cos\theta - 0$$

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot\theta}\right) = ml^2\ddot\theta - \left(ml\ddot x\cos\theta + ml\dot x(-\sin\theta)\dot\theta\right)$$

$$= ml^2\ddot\theta - ml\ddot x\cos\theta + ml\dot x(\sin\theta)\dot\theta$$

$$\frac{\partial L}{\partial \theta} = +ml\dot\theta\dot x(+\sin\theta) + mgl(+\sin\theta)$$

$$= ml\dot\theta\dot x\sin\theta + mgl\sin\theta$$

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot\theta}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$ml^2\ddot\theta - ml\ddot x\cos\theta + \cancel{ml\dot x\dot\theta\sin\theta}$$
$$\cancel{-ml\dot\theta\dot x\sin\theta} - mgl\sin\theta = 0$$

$$l\ddot\theta - \ddot x\cos\theta - g\sin\theta = 0$$

State eqn

$(M+m)\ddot{x} - ml\ddot{\theta}\cos\theta + ml\dot{\theta}^2\sin\theta = F(t)$.

$l\ddot{\theta} - \ddot{x}\cos\theta - g\sin\theta = 0$.

$\begin{bmatrix} \ddot{\theta} \\ \dot{\theta} \end{bmatrix} =$ $\cancel{\ddot{x}}$ $\begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$ $\quad \ddot{x}\ \dot{x}\ \theta$ $\quad \dot{\theta}\ \ddot{\theta}$ $\quad F(t)$

$\cancel{\dot{x}}$ $\dot{\theta}$

$\begin{bmatrix} \ddot{x} \\ \dot{x} \\ \ddot{\theta} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ x \\ \dot{\theta} \\ \theta \end{bmatrix}$

Input (U) = F(t)

Output (Y) = θ

$\ddot{x} = f_1(\dot{x}, \dot{\theta}, \theta)$.

$= 1/(M+m)$

linearizing abt $\theta = \dot{\theta} \to 0$.

$\sin\theta \to \theta \quad \cos\theta \to 1$.

$\dot{\theta}^2 = 0$.

$(M+m)\ddot{x} - ml\ddot{\theta} = F(t)$.

$l\ddot{\theta} - \ddot{x} - g\theta = 0$.

$l\ddot{\theta} = \ddot{x} + g\theta$.

$(M+m)\ddot{x} - ml\left(\frac{\ddot{x}}{l} + \frac{g\theta}{l}\right) = F(t)$.

$\ddot{x}(M+m-m) - mg\theta = F(t)$.

$\ddot{x} = \frac{1}{M}(mg\theta + F(t))$.

$\ddot{x} = l\ddot{\theta} - g\theta$.

$(M+m)(l\ddot{\theta} - g\theta) - ml\ddot{\theta} = F(t)$

$l\ddot{\theta}(M+m-\cancel{m}) - g\theta(M+m) = F(t)$.

$\ddot{\theta} = \frac{F(t) + g\theta(M+m)}{lM}$

2 2nd order eqn $\Rightarrow$ 4 state vars.

~~$\ddot{x}, \dot{x}, \dot{\theta}, \ddot{\theta}$~~ $x, \dot{x}, \theta, \dot{\theta}$.

State eqn abt vertically up (θ=0) fixed pt:

$$\ddot{x} = \frac{mg\theta}{M} + \frac{F}{M} \qquad \ddot{\theta} = \frac{g\theta(M+m)}{Ml} + \frac{F}{Ml}$$

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{mg}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{g(M+m)}{Ml} & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 1/M \\ 0 \\ 1/Ml \end{bmatrix} F$$

Jacobian

$$(M+m)\ddot{x} - ml\ddot{\theta}\cos\theta + ml\dot{\theta}^2\sin\theta = F(t) \quad -(1)$$

$$l\ddot{\theta} - \ddot{x}\cos\theta - g\sin\theta = 0 \quad -(2)$$

System variables: $x, \dot{x}, \theta, \dot{\theta}$

From (2), $l\ddot{\theta} = \ddot{x}\cos\theta + g\sin\theta$.
– Subst in (1),

$$(M+m)\ddot{x} - m\cos\theta(\ddot{x}\cos\theta + g\sin\theta) + ml\dot{\theta}^2\sin\theta = F(t).$$

$$\ddot{x}(M+m - m\cos^2\theta) - mg\cos\theta\sin\theta + ml\dot{\theta}^2\sin\theta = F(t) \quad -(A)$$

$m\sin^2\theta$

$$\ddot{x} = \frac{l\ddot{\theta} - g\sin\theta}{\cos\theta}.$$

Subst in (1),

$$(M+m)\frac{(l\ddot{\theta} - g\sin\theta)}{\cos\theta} - ml\ddot{\theta}\cos\theta + ml\dot{\theta}^2\sin\theta = F(t)$$

$$l\ddot{\theta}\left(\frac{M+m}{\cos\theta} - m\cos\theta\right) - g\tan\theta(M+m) + ml\dot{\theta}^2\sin\theta = F(t) \quad -(B)$$

$$\dot{x} = \dot{x} \quad -(C)$$

$$\dot{\theta} = \dot{\theta}. \quad -(D)$$

from (A), (B), (C), (D), the jacobian is calculated in the matlab cod.

Stat eqn abt $\theta = \pi$:

$$\lim_{\theta \to \pi} (\cos\theta) = -1$$
$$\lim_{x \to \pi} (\sin\theta) = \pi - \theta$$
$$\dot{\theta}^2 = 0$$

Non linearised eqns:

$$(M+m)\ddot{x} - ml\ddot{\theta}\cos\theta + ml\dot{\theta}^2\sin\theta = F(t)$$
$$l\ddot{\theta} - \ddot{x}\cos\theta - g\sin\theta = 0.$$

Linearised:

$$(M+m)\ddot{x} + ml\ddot{\theta} = F(t) \quad -(1)$$
$$l\ddot{\theta} + \ddot{x} + g(\theta) = \pi g \quad -(2)$$

from (2),
$$l\ddot{\theta} = \pi g - \ddot{x} - g\theta$$
subst in (1):

$$(M+m)\ddot{x} + m(\pi g - \ddot{x} - g\theta) = F.$$
$$M\ddot{x} + m\pi g - mg\theta = F. \quad -(3)$$
$$\ddot{x} = \tfrac{1}{M}(F - m\pi g + mg\theta)$$

from (2), $\ddot{x} = \pi g - l\ddot{\theta} - g\theta$

Subst in (1)

$$(M+m)(\pi g - l\ddot{\theta} - g\theta) + ml\ddot{\theta} = F$$
$$l\ddot{\theta}(-M - m + m) + (M+m)(\pi g - g\theta) = F.$$
$$-Ml\ddot{\theta} - g(M+m)\theta + \pi g(M+m) = F \quad -(4)$$

State eqn:

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{mg}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{g(M+m)}{M} & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} - \frac{m\pi g}{FM} \\ 0 \\ -\frac{1}{M} + \frac{\pi g(M+m)}{FM} \end{bmatrix} F$$

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```
syms x1 x2 x3 x4 m g l M F;
D=M+m*(sin(x3))^2;
x1_=[x2];
x2_=[(.5*m*g*sin(2*x3))/D+F/D-(m*l*sin(x3)*(x4)^2)/D];
x3_=[x4]
x4_=[(M+m)*g*sin(x3)/D+F*cos(x3)/(D*l)-(.5*m*sin(2*x3)*(x4)^2)/D]
X=[x1_;x2_;x3_;x4_]
v=[x1 ;x2; x3;x4 ]

J_v=jacobian(X,v)
```

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