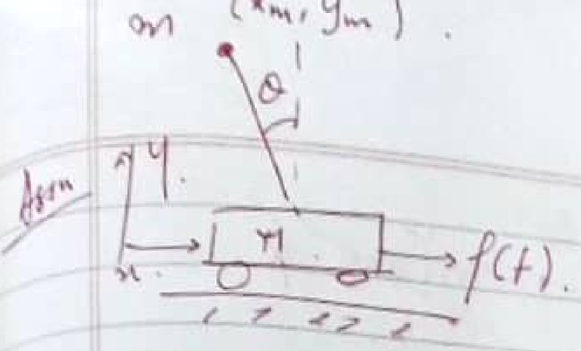


Equations of Motion

$x_m = x - l\sin\theta \qquad y_m = l\cos\theta.$

$\dot{x}_m = \dot{x} - l\dot{\theta}\cos\theta \qquad \dot{y}_m = -l\dot{\theta}\sin\theta.$

Lagrangian method to write eqn of motion;

$PE = V = mg.l\cos\theta$

$T = KE = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x}_m^2 + \dot{y}_m^2)$

$= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x}^2 + l^2\dot{\theta}^2\cos^2\theta - 2\dot{x}l\dot{\theta}\cos\theta + l^2\dot{\theta}^2\sin^2\theta)$

$= \frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2}m l^2\dot{\theta}^2 - ml\dot{\theta}\dot{x}\cos\theta$

Lagrange's eqn: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i$

Let $L = T - V$ (Lagrangian)

$L = \frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2}ml^2\dot{\theta}^2 - ml\dot{\theta}\dot{x}\cos\theta - mgl\cos\theta.$

~~$\frac{\partial L}{\partial x} = \frac{1}{2}(M+m)\left(2\dot{x}\frac{\partial \dot{x}}{\partial x}\right)$~~

~~Eqns of Motion:~~

~~$x: (M+m)\ddot{x} -$~~

$\frac{\partial L}{\partial \dot{x}} = \frac{1}{2}(M+m)2\dot{x} + 0 - ml\dot{\theta}\cos\theta - 0.$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = (M+m)\ddot{x} - (ml\ddot{\theta}\cos\theta + ml\dot{\theta}(-\sin\theta)\dot{\theta})$

$\frac{\partial L}{\partial x} = 0.$

$\Rightarrow (M+m)\ddot{x} - ml\ddot{\theta}\cos\theta + ml\dot{\theta}^2\sin\theta = F(t).$

$\theta$:

$\frac{\partial L}{\partial \dot\theta} = 0 + \cancel{\frac{1}{2}} ml^2 \cdot \cancel{2}\dot\theta - ml\cos\theta \cdot \dot x$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) = ml^2\ddot\theta + ml\sin\theta\,\dot\theta$

$\frac{\partial L}{\partial \theta} = -ml\dot\theta(-\sin\theta) - mgl(-\sin\theta)$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) - \frac{\partial L}{\partial \theta} = 0$

$ml^2\ddot\theta + \cancel{ml\dot\theta\sin\theta} - \cancel{ml\dot\theta\sin\theta} - mgl\sin\theta = 0$

$l\ddot\theta \quad - g\sin\theta = 0$

$L = \frac{1}{2}(M+m)\dot x^2 + \frac{1}{2} ml\dot\theta^2 - ml\dot\theta\dot x\cos\theta - mgl\cos\theta$

$\frac{\partial L}{\partial \dot\theta} = 0 + \cancel{\frac{1}{2}} ml^2\dot\theta - ml\dot x\cos\theta - 0$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) = ml^2\ddot\theta - \left(ml\ddot x\cos\theta + ml\dot x(-\sin\theta)\dot\theta\right)$

$= ml^2\ddot\theta - ml\ddot x\cos\theta + ml\dot x(\sin\theta)\dot\theta$

$\frac{\partial L}{\partial \theta} = +ml\dot\theta\dot x(+\sin\theta) + mgl(+\sin\theta)$

$= ml\dot\theta\dot x\sin\theta + mgl\sin\theta$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) - \frac{\partial L}{\partial \theta} = 0$

$ml^2\ddot\theta - ml\ddot x\cos\theta + \cancel{ml\dot x\dot\theta\sin\theta}$

$-\cancel{ml\dot\theta\dot x\sin\theta} - mgl\sin\theta = 0$

$l\ddot\theta - \ddot x\cos\theta - g\sin\theta = 0$ //

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State eqn

$(M+m)\ddot{x} - ml\ddot{\theta}\cos\theta + ml\dot{\theta}^2\sin\theta = F(t)$.

$l\ddot{\theta} - \ddot{x}\cos\theta - g\sin\theta = 0$.

$F(t)$

$\begin{bmatrix} \ddot{\theta} \\ \dot{\theta} \end{bmatrix} =$ $\cancel{x}\begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$ $\ddot{x}\,\ddot{\theta}\,/\,\theta$ $\dot{\theta}\dot{\theta}$

$\cancel{x}$

$\dot{\theta}$

$\begin{bmatrix} \ddot{x} \\ \dot{x} \\ \ddot{\theta} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \cancel{\ddot{x}} \\ \cancel{\dot{x}} \\ \dot{\theta} \\ \theta \end{bmatrix}$

Input (U) = F(t)
Output (Y) = $\theta$

$\ddot{x} = f_1(\dot{x}, \dot{\theta}, \theta)$.

$= 1/(M+m)$

linearizing abt $\theta = \dot{\theta} \to 0$.
$\sin\theta \to 0 \quad \cos\theta \to 1$.
$\dot{\theta}^2 = 0$.

$(M+m)\ddot{x} - ml\ddot{\theta} = F(t)$.

$l\ddot{\theta} - \ddot{x} - g\theta = 0$.

$l\ddot{\theta} = \ddot{x} + g\theta$.

$(M+m)\ddot{x} - ml\left(\frac{\ddot{x}}{l} + \frac{g\theta}{l}\right) = F(t)$.

$\ddot{x}(M+m-m) - mg\theta = F(t)$.

$\ddot{x} = \frac{1}{M}(mg\theta + F(t))$.

$\ddot{x} = l\ddot{\theta} - g\theta$.

$(M+m)(l\ddot{\theta} - g\theta) - ml\ddot{\theta} = F(t)$

$l\ddot{\theta}(M+\cancel{m}-\cancel{m}) - g\theta(M+m) = F(t)$.

$\ddot{\theta} = \frac{F(t) + g\theta(M+m)}{lM}$

2 $2^{nd}$ order eqn $\Rightarrow$ 4 state vars.
$\cancel{x, \dot{x}, \theta, \dot{\theta}}$ $x, \dot{x}, \theta, \dot{\theta}$.

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State eqn abt vertically ~~down~~ up $(\theta = 0)$ fixed pt:

$$\ddot{x} = \frac{mg\theta}{M} + \frac{F}{M} \qquad \ddot{\theta} = \frac{g\theta(M+m)}{Ml} + \frac{F}{Ml}$$

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{mg}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{g(M+m)}{Ml} & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 1/M \\ 0 \\ 1/Ml \end{bmatrix} F$$

$$\cancel{\begin{bmatrix} \ddot{x} \\ \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix} \begin{bmatrix} \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} \quad \end{bmatrix} F}$$

Jacobian

$$(M+m)\ddot{x} - ml\ddot{\theta}\cos\theta + ml\dot{\theta}^2\sin\theta = F(t) \quad -(1)$$

$$l\ddot{\theta} - \ddot{x}\cos\theta - g\sin\theta = 0 \quad -(2)$$

System variables: $x, \dot{x}, \theta, \dot{\theta}$

From (2), $\quad l\ddot{\theta} = \ddot{x}\cos\theta + g\sin\theta$.
- Subst in (1),

$$(M+m)\ddot{x} - m\cos\theta(\ddot{x}\cos\theta + g\sin\theta) + ml\dot{\theta}^2\sin\theta = F(t).$$

$$\ddot{x}(M+m\sin^2\theta) - mg\cos\theta\sin\theta + ml\dot{\theta}^2\sin\theta = F(t) \quad -(A)$$

$$\ddot{x} = \frac{l\ddot{\theta} - g\sin\theta}{\cos\theta}.$$

Subst in (1),

$$(M+m)\frac{(l\ddot{\theta} - g\sin\theta)}{\cos\theta} - ml\ddot{\theta}\cos\theta + ml\dot{\theta}^2\sin\theta = F(t)$$

$$l\ddot{\theta}\left(\frac{M+m}{\cos\theta} - m\cos\theta\right) - g\tan\theta(M+m) + ml\dot{\theta}^2\sin\theta = F(t) \quad -(B)$$

$$\dot{x} = \dot{x} \quad -(C)$$

$$\dot{\theta} = \dot{\theta}. \quad -(D)$$

from (A), (B), (C), (D), the Jacobian is calculated in the matlab code.

State eqn abt $\theta = \pi$:

$\lim_{\theta \to \pi} (\cos\theta) = -1$

$\lim_{\theta \to \pi} (\sin\theta) = \pi - \theta$

$\dot{\theta}^2 = 0$.

Non linearised eqns:

$(M+m)\ddot{x} - ml\dot{\theta}\cos\theta + ml\dot{\theta}^2\sin\theta = F(t)$

$l\ddot{\theta} - \ddot{x}\cos\theta - g\sin\theta = 0$.

Linearised:

$(M+m)\ddot{x} + ml\ddot{\theta} = F(t)$ — (1)

$l\ddot{\theta} + \ddot{x} + g(\theta) = \pi g$ — (2)

from (2),

$l\ddot{\theta} = \pi g - \ddot{x} - g\theta$

subst in (1):

$(M+m)\ddot{x} + m(\pi g - \ddot{x} - g\theta) = F$.

$M\ddot{x} + m\pi g - mg\theta = F$. — (3).

$\ddot{x} = \frac{1}{M}(F - m\pi g + mg\theta)$

from (2), $\ddot{x} = \pi g - l\ddot{\theta} - g\theta$

subst in (1)

$(M+m)(\pi g - l\ddot{\theta} - g\theta) + ml\ddot{\theta} = F$

$l\ddot{\theta}(-M - m + m) + (M+m)(\pi g - g\theta) = F$.

$-Ml\ddot{\theta} - g(M+m)\theta + \pi g(M+m) = F$ — (4).

State eqn:

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{mg}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{g(M+m)}{M} & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} - \frac{m\pi g}{FM} \\ 0 \\ -\frac{1}{M} + \frac{\pi g(M+m)}{FM} \end{bmatrix} F$$