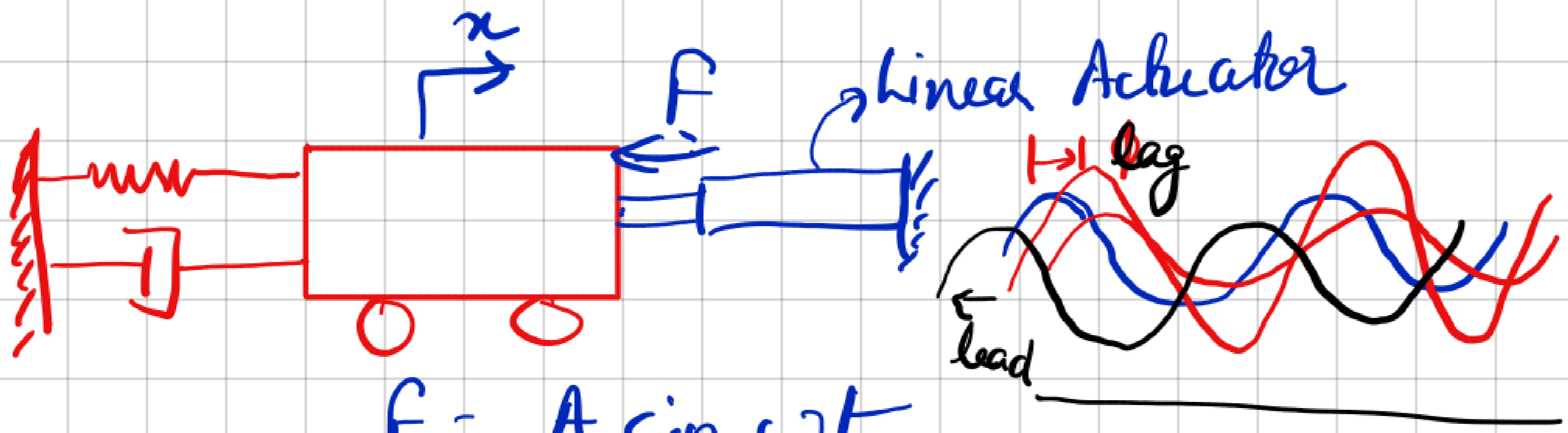
Frequency Response

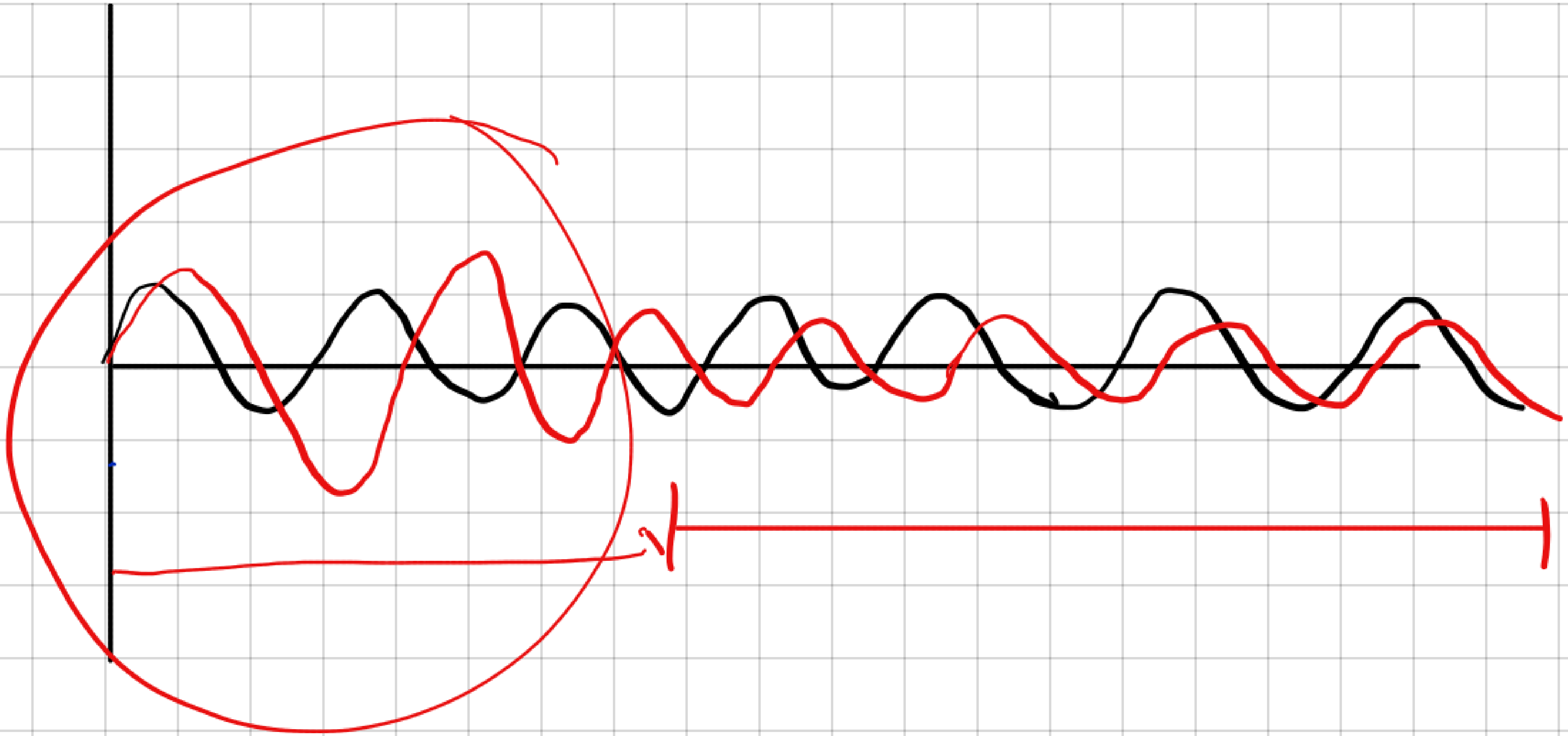
Frequency response is the amplitude & phase relationship of the o/p of a sys to a generalized sinusoidal i/p to the system.



$$F = A \sin \omega t$$

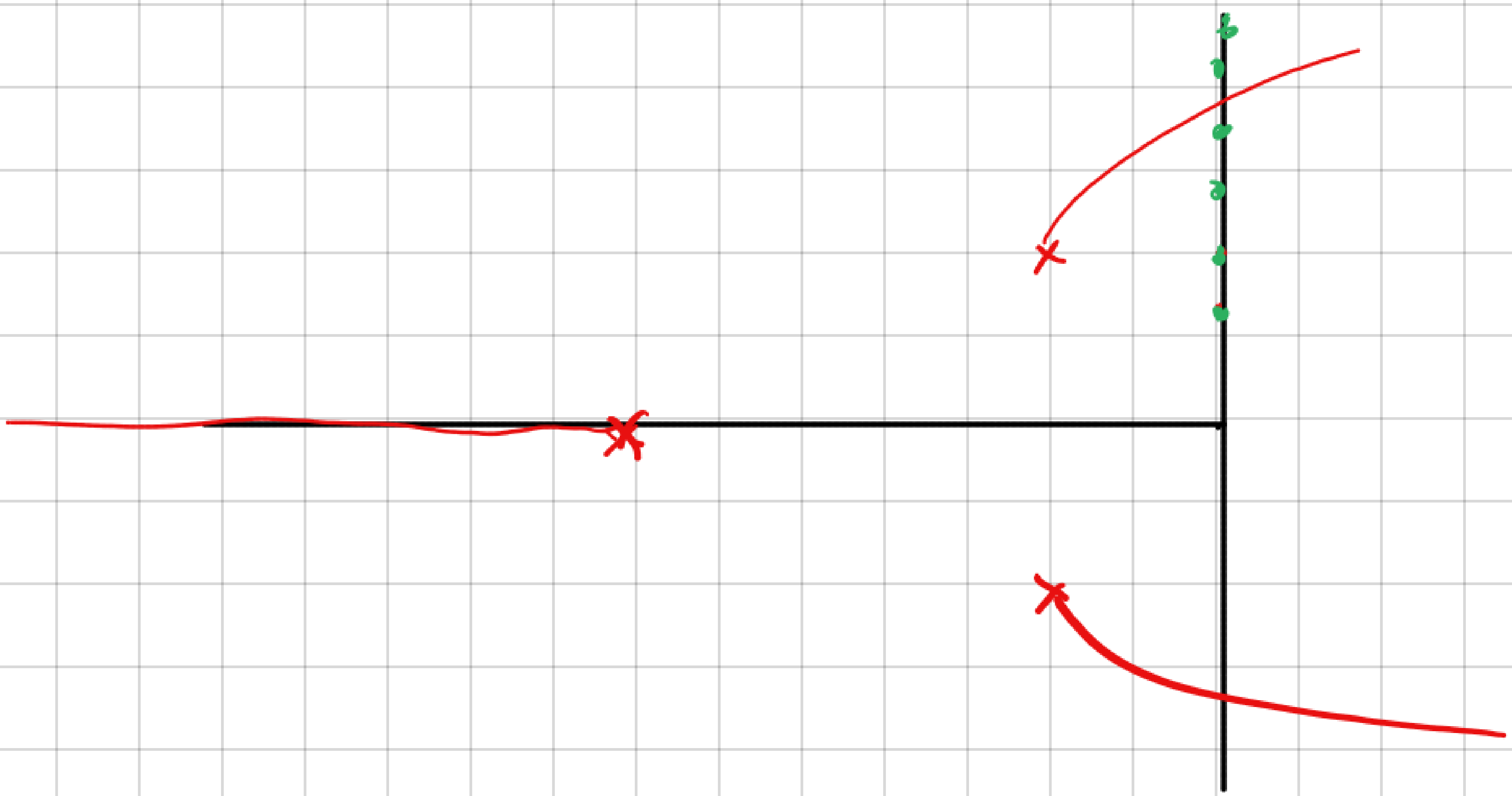
$$x = B \sin(\omega t + \phi)$$

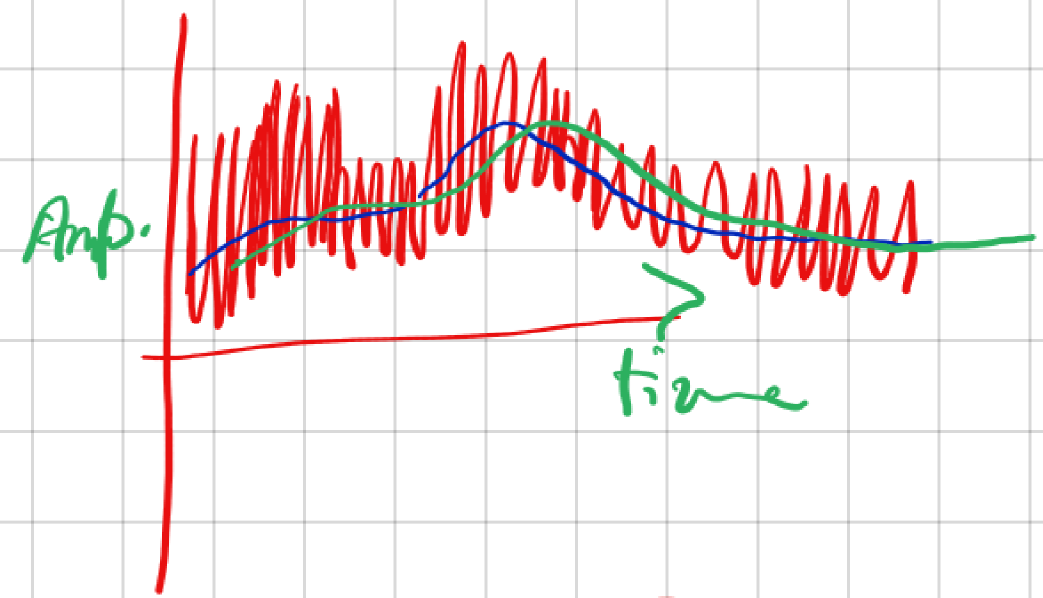
$$\left[\frac{B}{A}\right] \qquad \phi \rightarrow f(\omega)$$

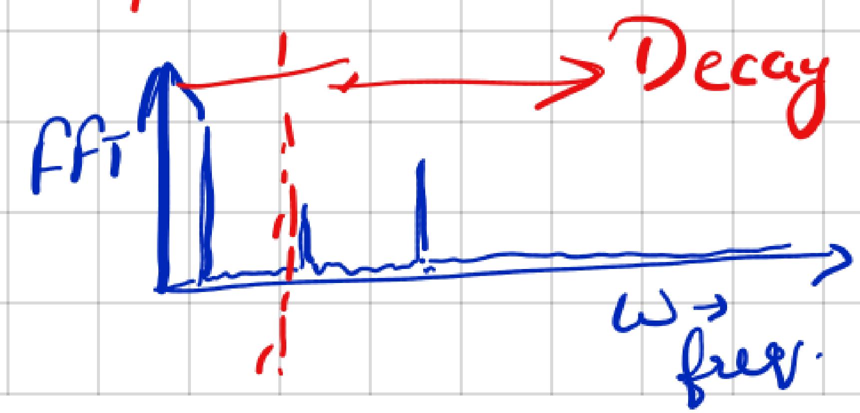


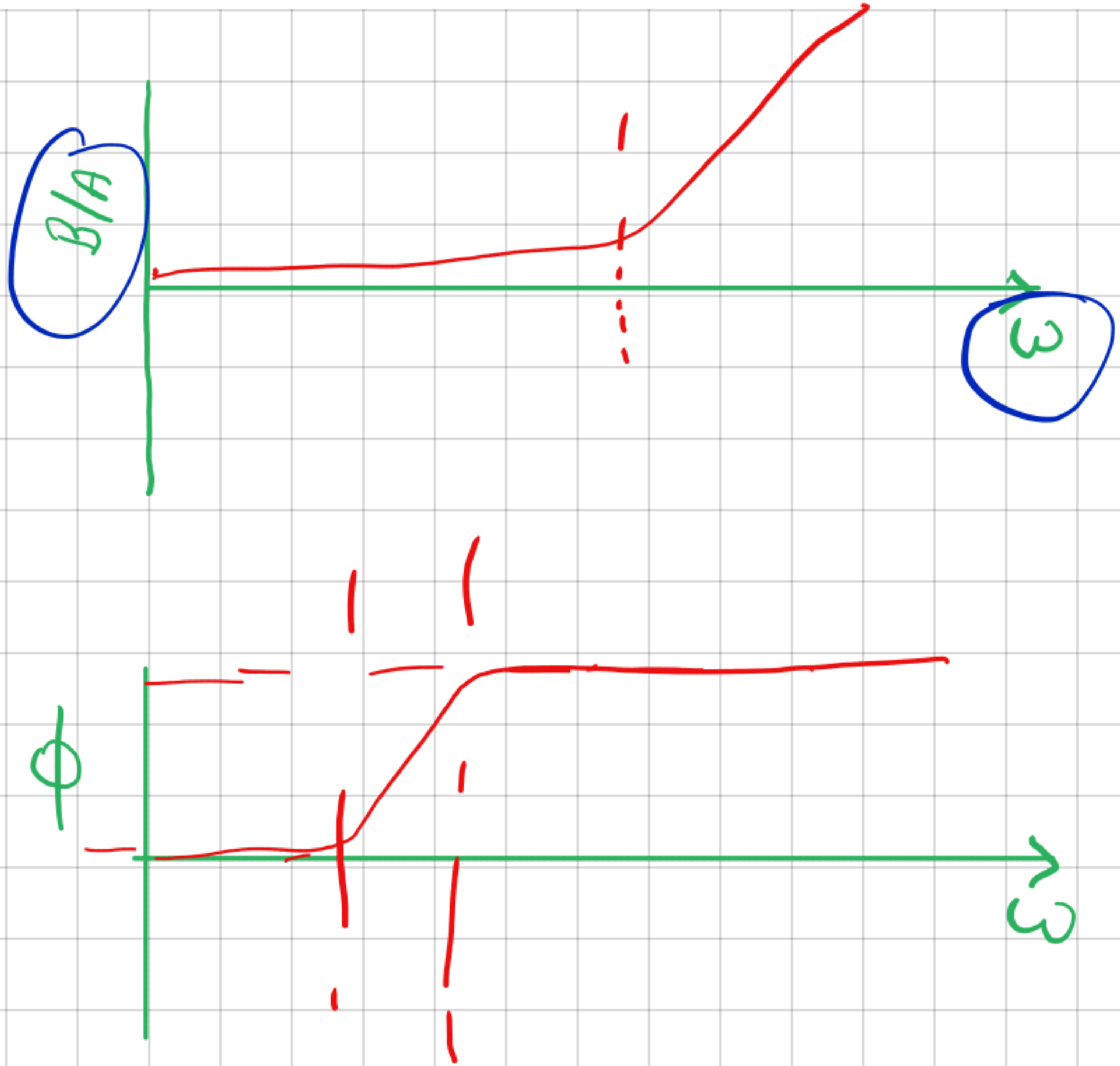
Phasors → Representation of complex numbers

$$\rightarrow \underline{M\angle\phi} = M\cos(\omega t + \phi)$$









BODE plot → Defined later on (8)

→ Plots representing frequency response of a system across a broad spectrum of input frequencies.

$$R(s) \longrightarrow \boxed{G(s)} \longrightarrow C(s)$$

$$r(t) = A\cos(\omega t) + B\sin(\omega t)$$

$$R(s) = \frac{As + B\omega}{s^2 + \omega^2}$$

$$C(s) = \frac{As + B\omega}{s^2 + \omega^2}\, G(s)$$

$$= \left[\frac{K_1}{s + j\omega} + \frac{K_2}{s - j\omega}\right] + \text{Partial frac. from } G(s)$$

$$K_1 = \frac{As + B\omega}{(s - j\omega)}\, G(s)\Bigg|_{s \to -j\omega}$$

$$K_2 = \frac{As + B\omega}{s + j\omega}\, G(s)\Bigg|_{s \to j\omega}$$

$$K_1 = \frac{M_i M_G}{2} e^{-j(\phi_i + \phi_G)}$$

$$K_2 = \frac{M_i M_G}{2} e^{j(\phi_i + \phi_G)}$$

$$= \overline{K_1} \text{ (complex conjugate of } K_1\text{)}$$

$$M_i = \sqrt{A^2 + B^2}$$

$$\phi_i = -\tan^{-1}(B/A)$$

M_G = Mag. of the sys. $G(s)$

ϕ_G = phase of the sys. $G(s)$

$$|G(j\omega)|$$

steady state → $C_{ss} = \frac{K_1}{(s+j\omega)} + \frac{K_2}{(s-j\omega)} + G(s)$

$G(s) \to \sim 0$

eg. $G(s) = \frac{1}{s^2 + 3s + 2}$

$$G(j\omega) = \frac{1}{(j\omega)^2 + 3j\omega + 2}$$

$$C_{ss} = \frac{M_i M_G}{2}\left[\frac{e^{-j(\phi_i+\phi_G)}}{(s+j\omega)} + \frac{e^{j(\phi_i+\phi_G)}}{(s-j\omega)}\right]$$

$$\underbrace{c(t)_{ss} = M_i M_G \cos(\omega t + \phi_i + \phi_G)}_{\underline{(M_i \angle \phi_i)}\ \underline{(M_G \angle \phi_G)}}$$

$\rightarrow$ Frequency response function.

Ploting Frequency Response

Decibel scale for magnitude $\rightarrow$ $20 \log_{10}(M)$

Phase angle ϕ

vs.

$\log \omega$

$\boxed{\text{Bode Plots}}$

Can be simplified using asymptotic approximations.

$$G(s) = \frac{K(s+z_1)(s+z_2)\ldots(s+z_k)}{s^m(s+P_1)(s+P_2)\ldots(s+P_n)}$$

$$|G(j\omega)| = \frac{|K|\,|s+z_1|\,|s+z_2|\ldots|s+z_k|}{|s^m|\,|s+P_1|\,|s+P_2|\ldots|s+P_n|}\Bigg|_{\substack{t\to\infty \\ s\to j\omega}}$$

$\to$ Dominant pole at steady state i.e. $t\to\infty$

$$20\log|G(j\omega)| = 20\log|K| + 20\log|s+z_1| + \ldots - 20\log|s^m| - 20\log|s+P_1| - 20\log|s+P_2| \ldots 20\log|s+P_n|$$

at $\omega = a$ 3 dB amplification in $(s+a)$

$G = (S+a) \quad | \quad S \to j\omega$

$= (j\omega + a) = a\left(j\frac{\omega}{a} + 1\right)$

$.01a \to |G| = |a(.01j + 1)|$

$= a\sqrt{1.0001}$

$\sim a \xrightarrow{dB} 20\log a$

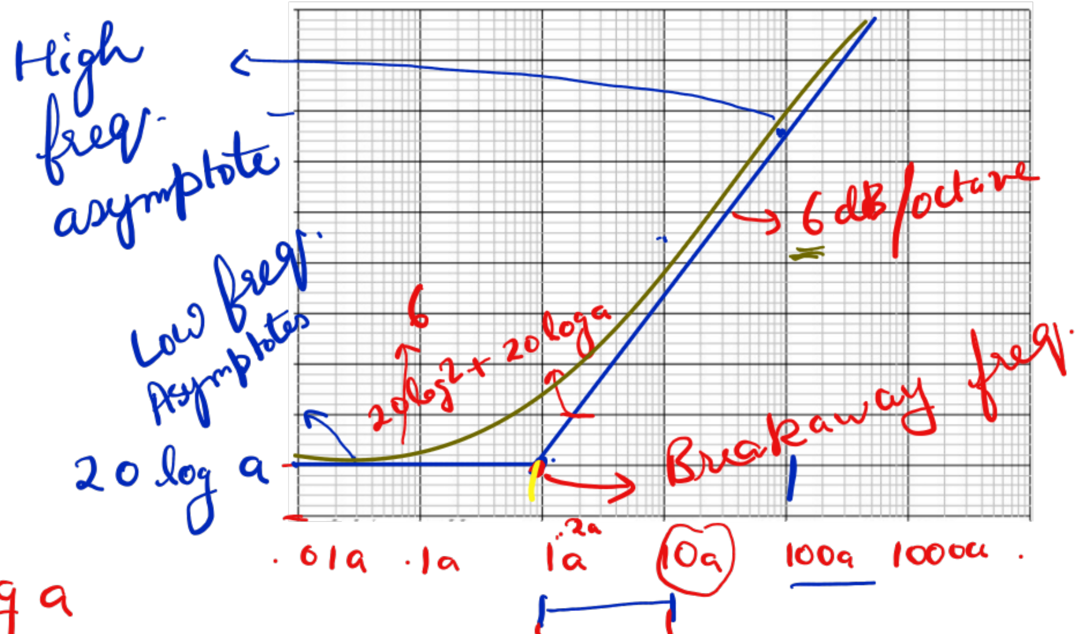
$.001a \to \sim a \xrightarrow{dB} 20 \log a$

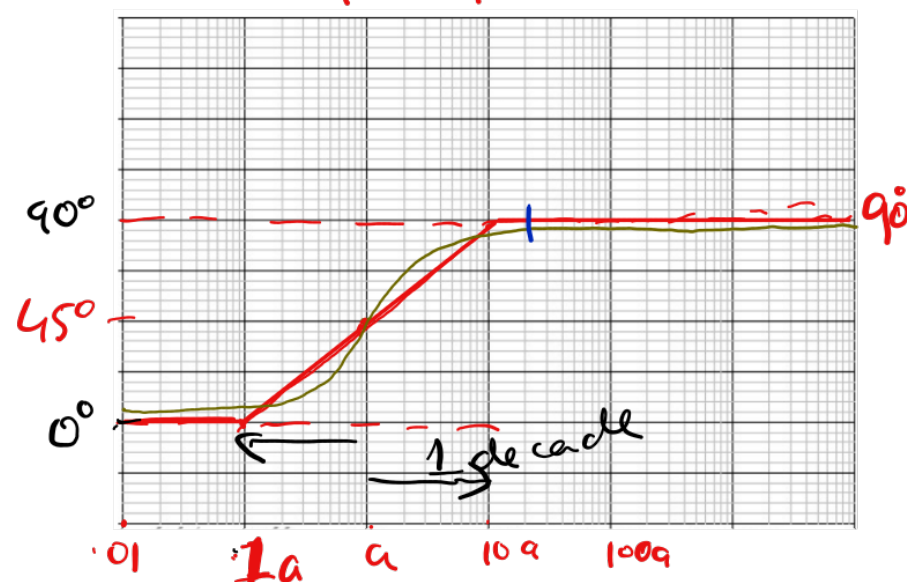
$\omega \gg a$

$|G| = |j\omega| = \omega$

$dB|G| = 20\log\omega$ slope is 20 dB/decade

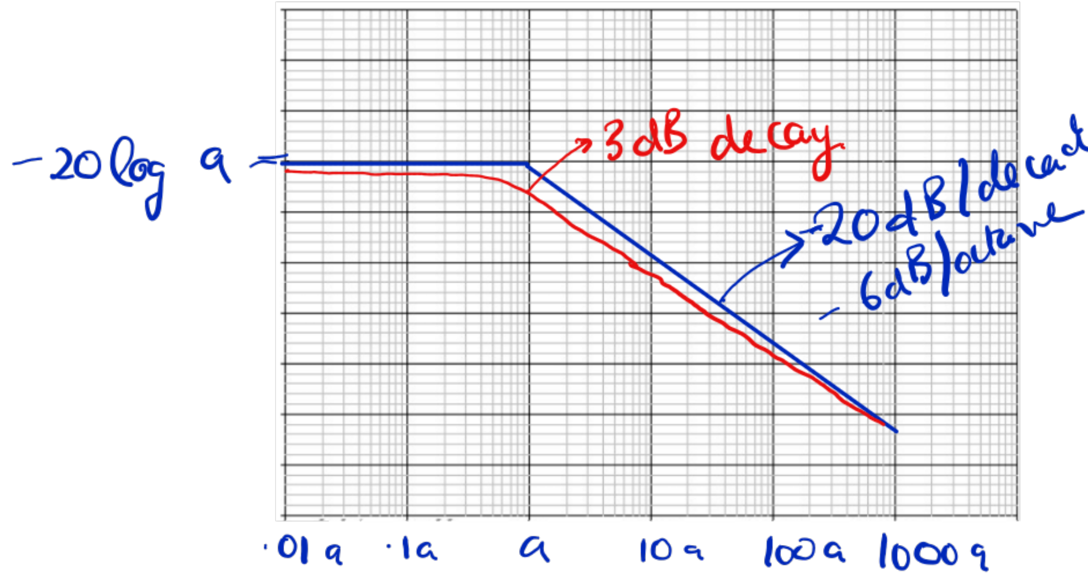
or 6dB/octave → freq. doubl

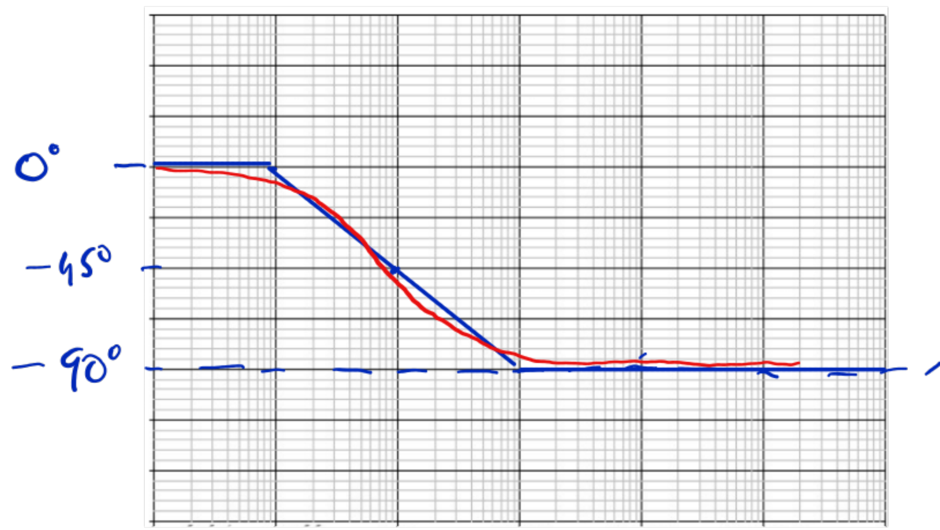




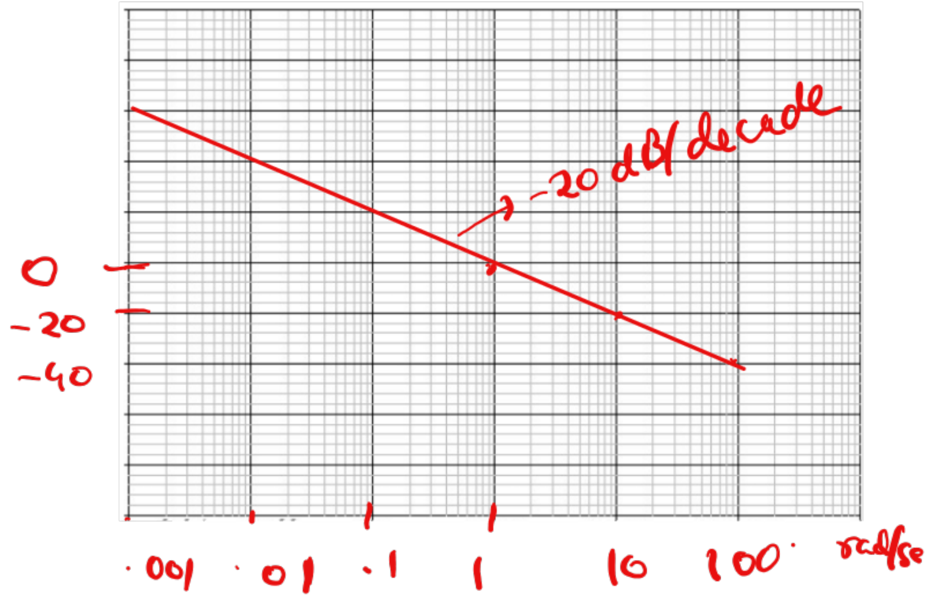
$$G(s) = \frac{1}{(s+a)}\Bigg|_{s\to j\omega}$$

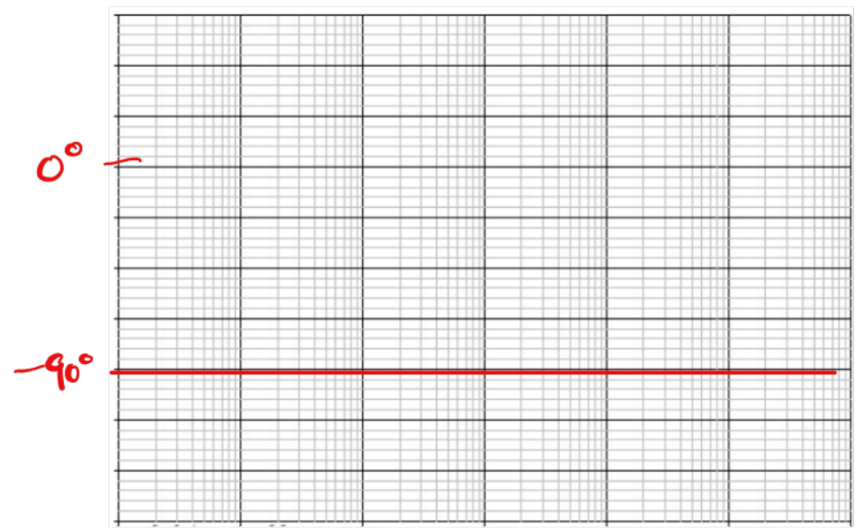
$$\frac{1}{j\omega + a} = \boxed{\frac{1}{a\left(\frac{\omega j}{a}+1\right)}}$$



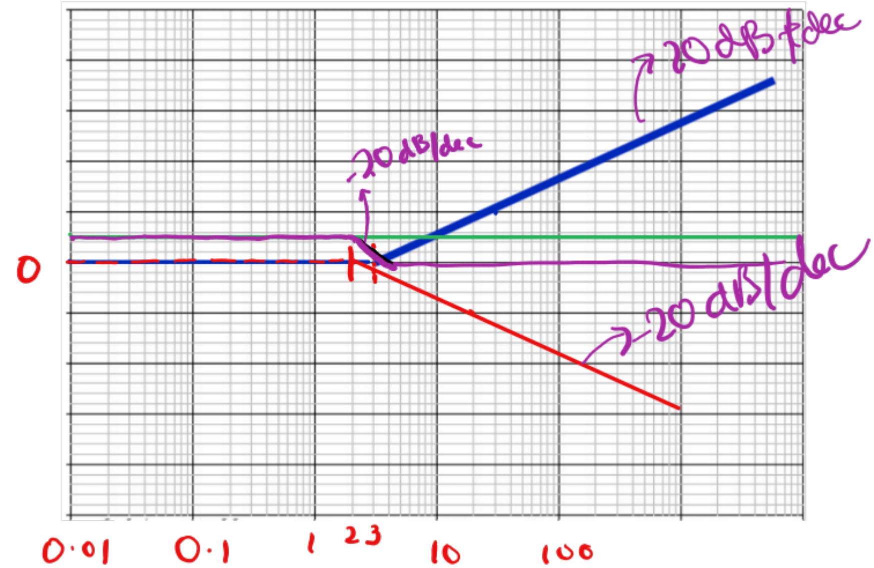


$G = \frac{1}{s}$

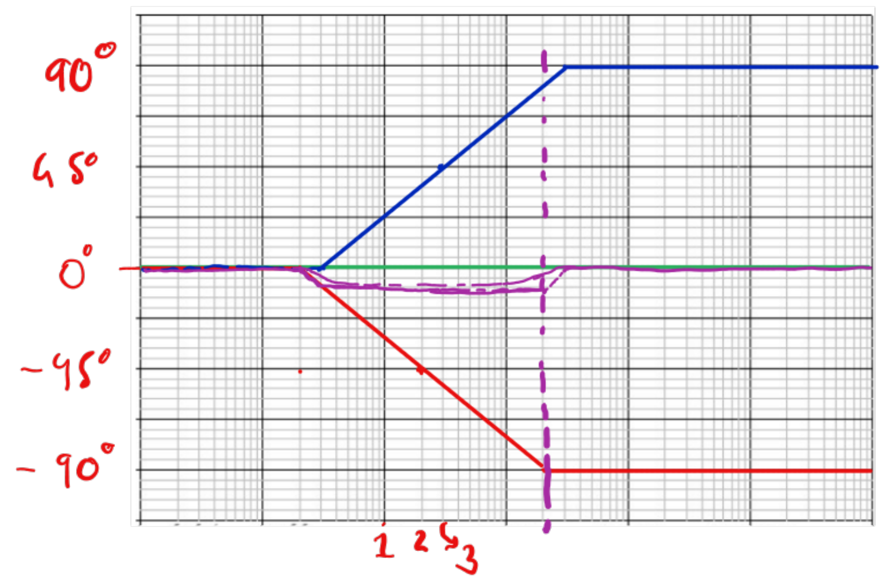




$$G(s) = \frac{(s+3)}{(s+2)} = \frac{\frac{3}{2}\left(\frac{s}{3}+1\right)}{\left(\frac{s}{2}+1\right)}$$

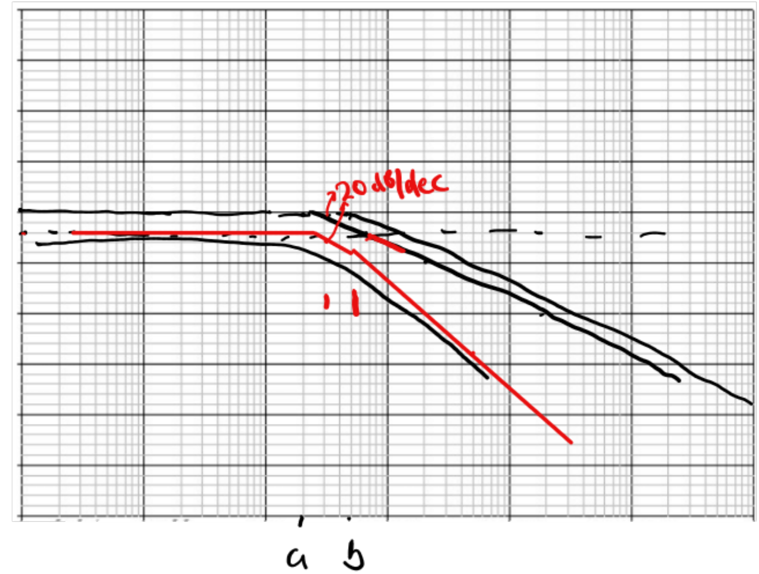


$$20\log 1.5 + 20\log\frac{\omega}{3} - 20\log\frac{\omega}{2}$$

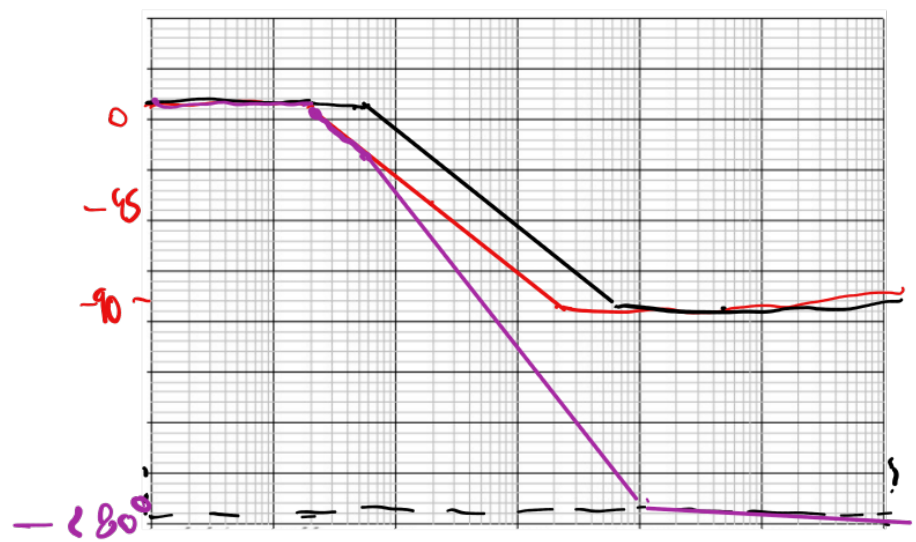


$$G(s) = \frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Overdamped $\rightarrow$ $\dfrac{1}{(s+a)(s+b)}$



$$\frac{1}{ab\left(\frac{s}{a}+1\right)\left(\frac{s}{b}+1\right)}$$



$$G(s)_{s\to j\omega} = G(j\omega) = \frac{\omega_n^2}{(j\omega_n)^2 + 2j\omega\omega_n + \omega_n^2} = \frac{1}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right] + 2\xi\frac{\omega}{\omega_n}j}$$

Log magnitude

$$\underline{\underline{20\log}}\,|G(j\omega)| = -20\log\left(\sqrt{\left(1-\cancel{\frac{\omega^2}{\omega_n^2}}\right)^2 + \cancel{\left(2\xi\frac{\omega}{\omega_n}\right)^2}}\right) = M$$

$\omega \ll \omega_n$

$M = 0$

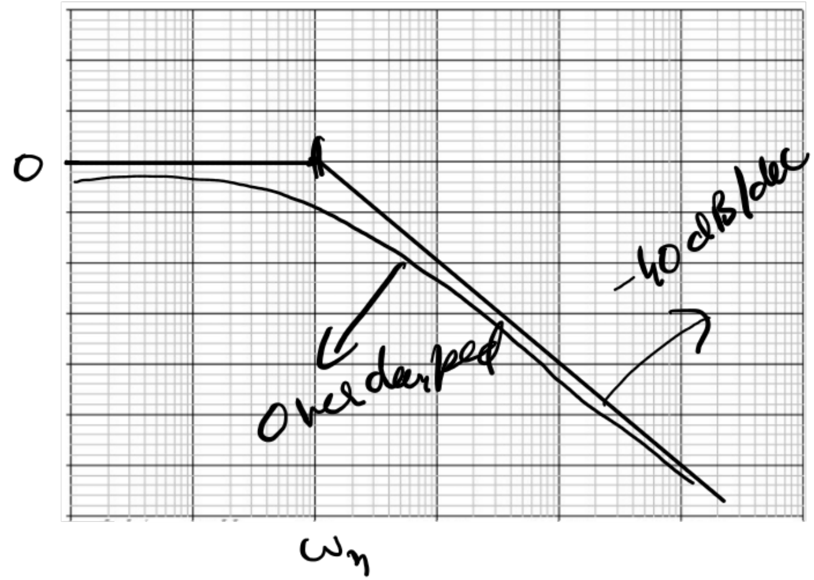
$\omega \gg \omega_n$

$$M = -20\log\frac{\omega^2}{\omega_n^2} = -40\log\frac{\omega}{\omega_n}$$

@ $\omega = \omega_n$ $M = 0$

Underdamped Sys.

Resonant freq. → Resonant peak M_r close to ω_n



$$\left|G(j\omega)\right| = \frac{1}{\sqrt{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\xi\frac{\omega}{\omega_n}\right)^2}} \rightarrow$$

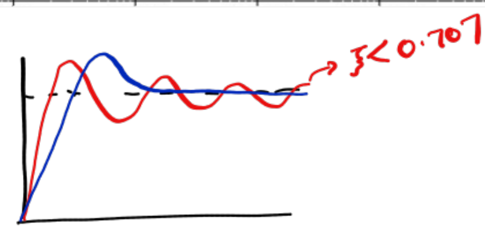
~~where is the magnitude the highest~~

resonance will occur when denom. is min

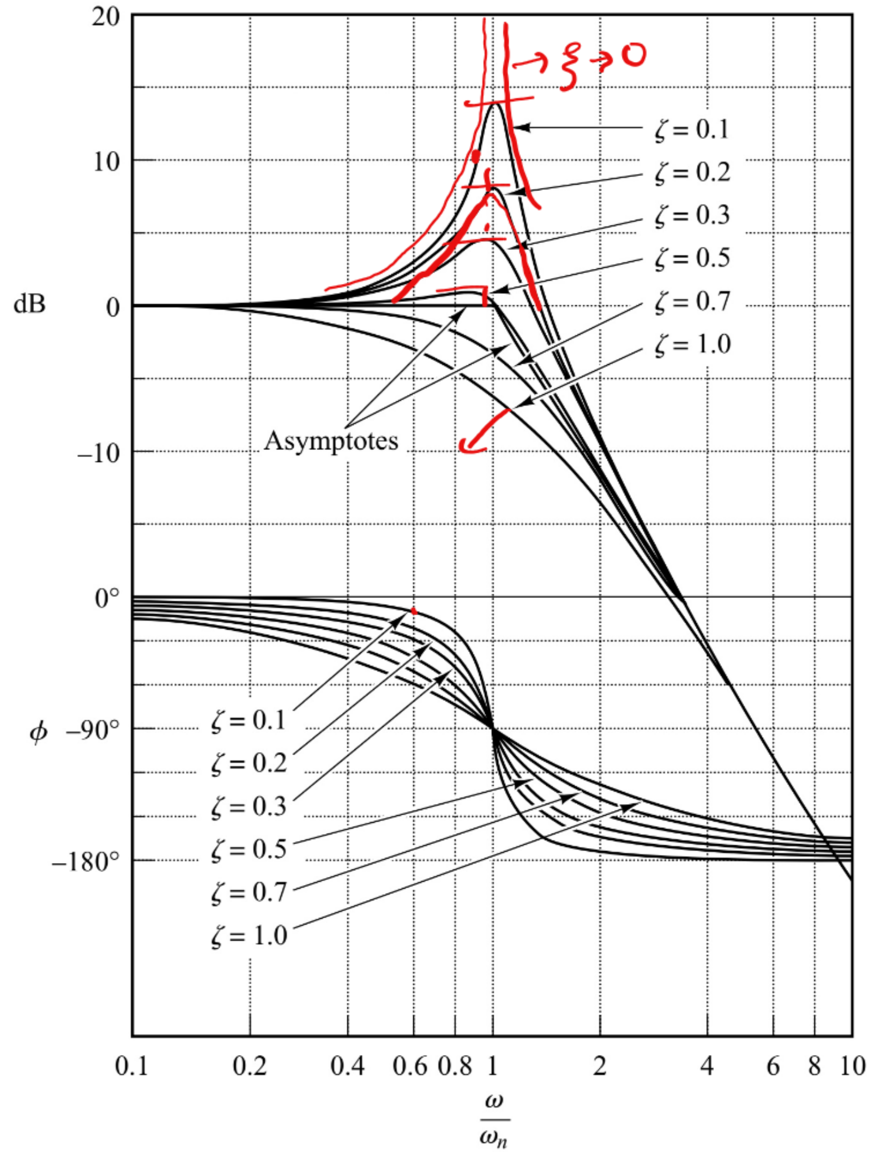
$$g(\omega) = \left(1-\frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\xi\frac{\omega}{\omega_n}\right)^2$$

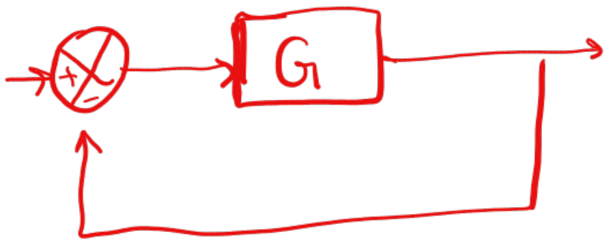
↳ min. occurs at

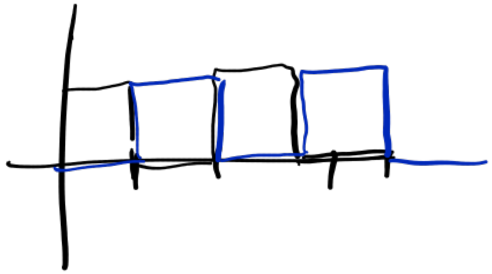
$$\boxed{\omega = \omega_n\sqrt{1-2\xi^2}} \quad 0 \le \xi \le 0.707$$

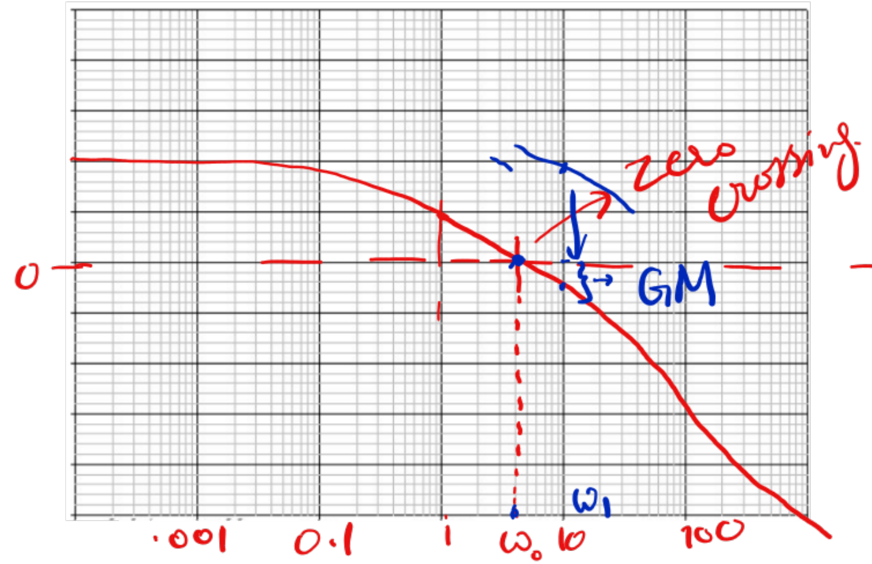


as $\xi \rightarrow 0$ $\omega = \omega_n$


20
10
0
–10
dB
0°
–90°
–180°
ϕ
$\zeta = 0.1$
$\zeta = 0.2$
$\zeta = 0.3$
$\zeta = 0.5$
$\zeta = 0.7$
$\zeta = 1.0$
Asymptotes
$\zeta = 0.1$
$\zeta = 0.2$
$\zeta = 0.3$
$\zeta = 0.5$
$\zeta = 0.7$
$\zeta = 1.0$
$\rightarrow \xi \rightarrow 0$
0.1
0.2
0.4
0.6
0.8
1
2
4
6
8
10
$\frac{\omega}{\omega_n}$

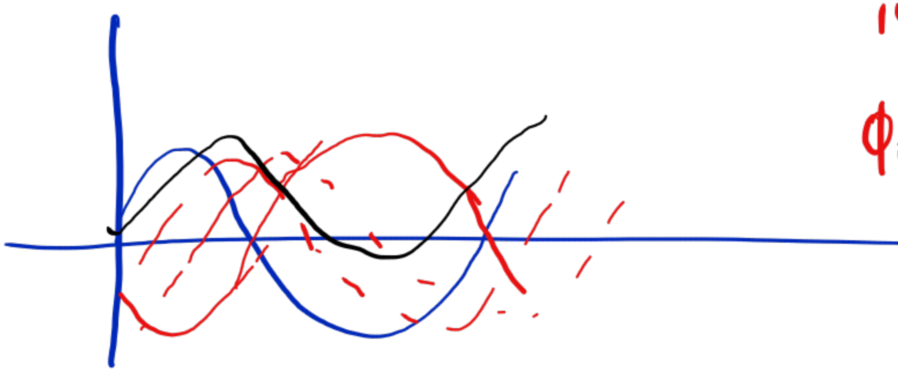


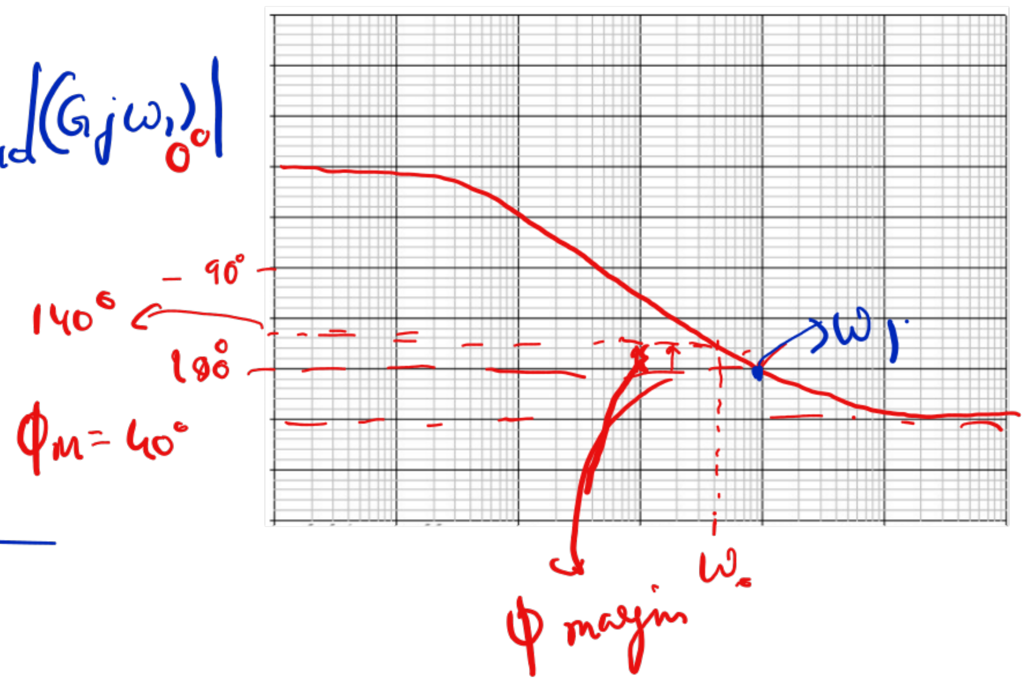


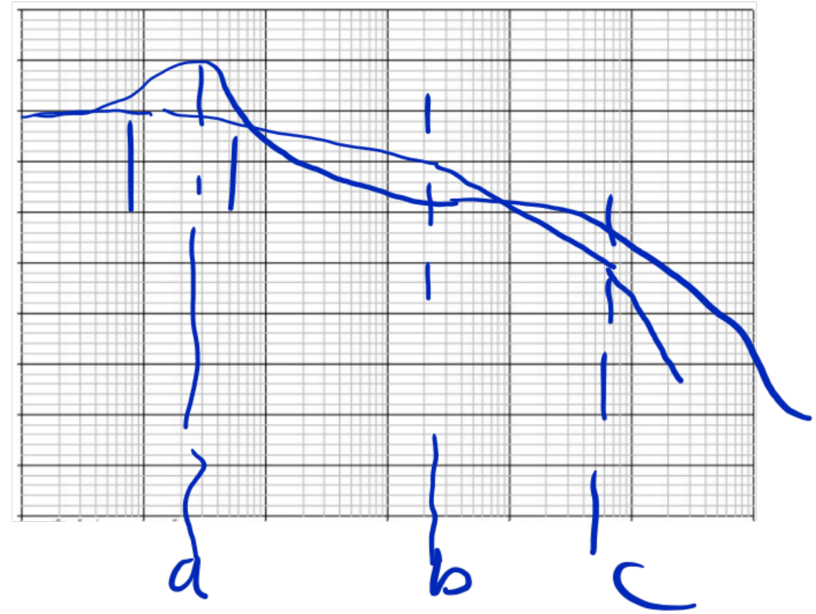


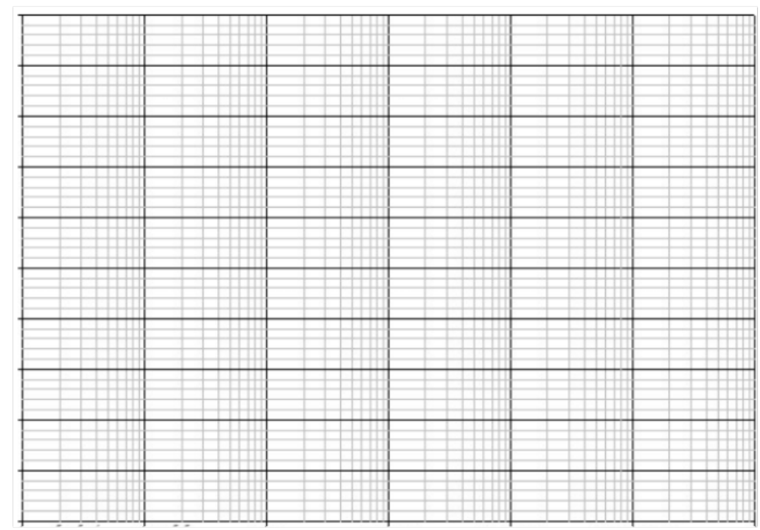
Phase margin $= \angle G(j\omega_0) + 180^\circ$

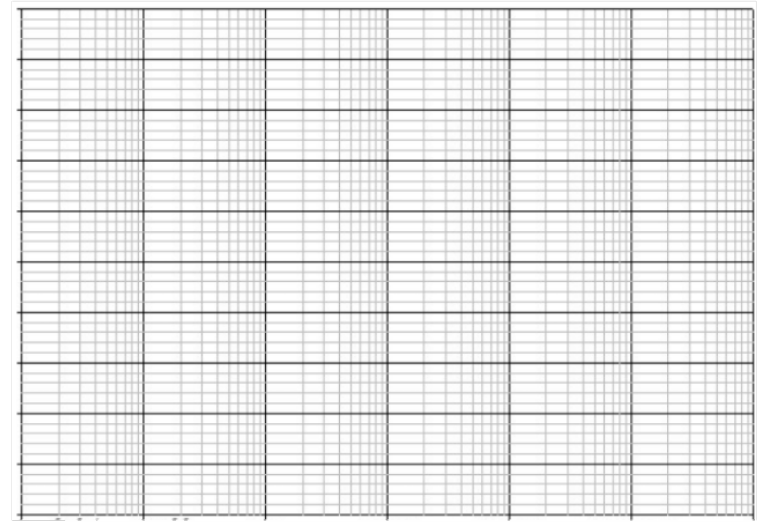
Gain margin $= 0 - 20\log_{10}|G(j\omega_1)|$

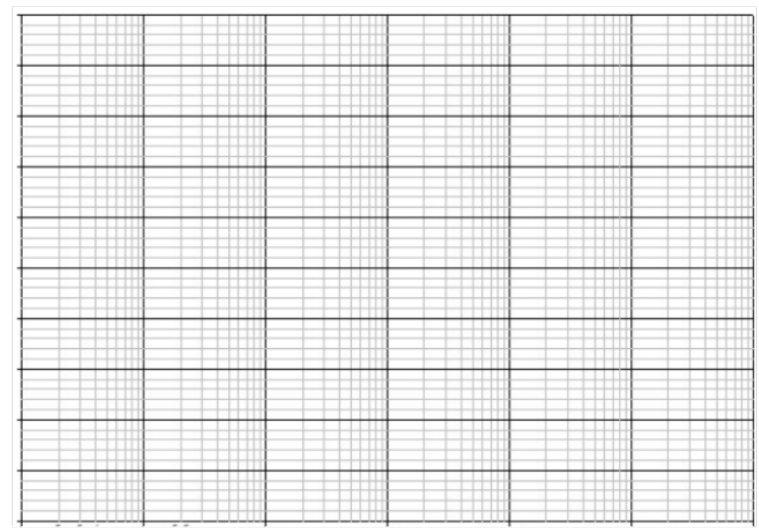


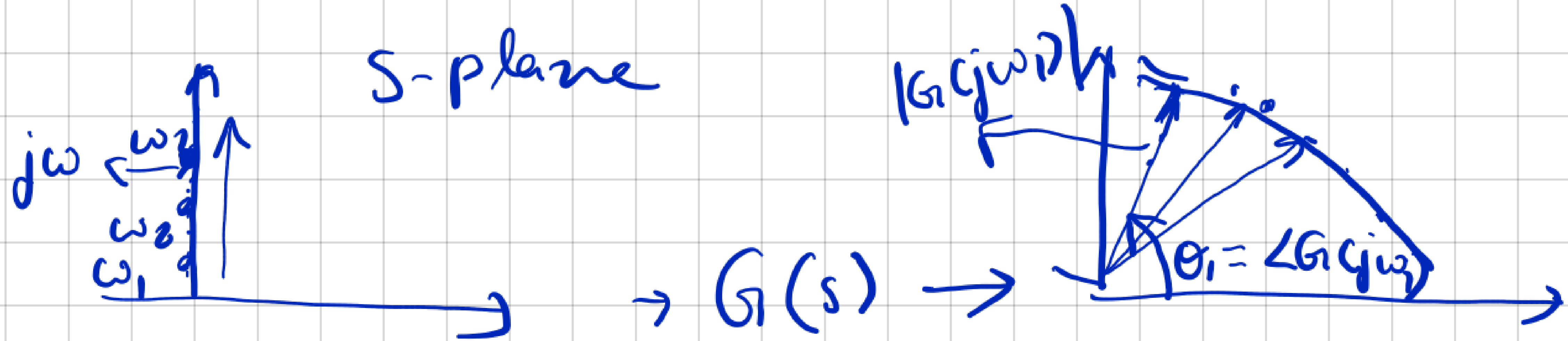



a
b
c









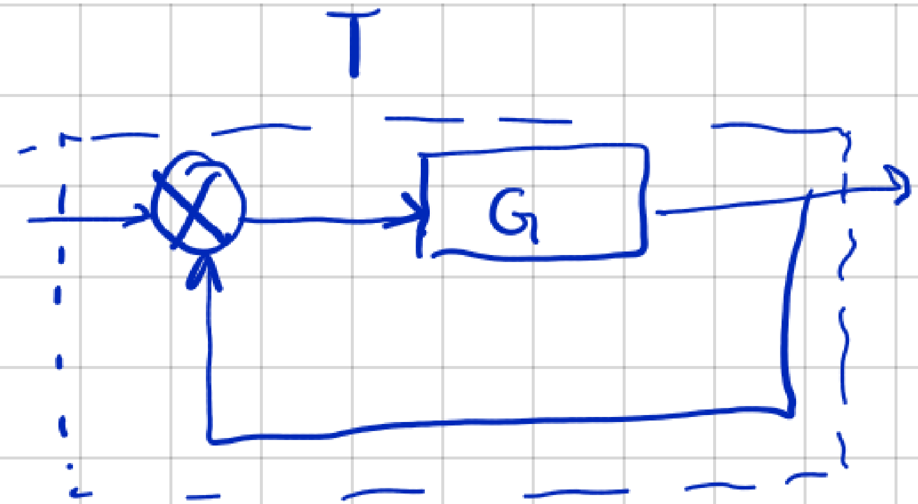
S-plane
jω
ω2
ω1
G(s)
|G(jω1)|
θ1 = ∠G(jω1)

↳ open loop sys.

Freq. Response in BODE plot

↓

Poles in RHP make this sys unstable.



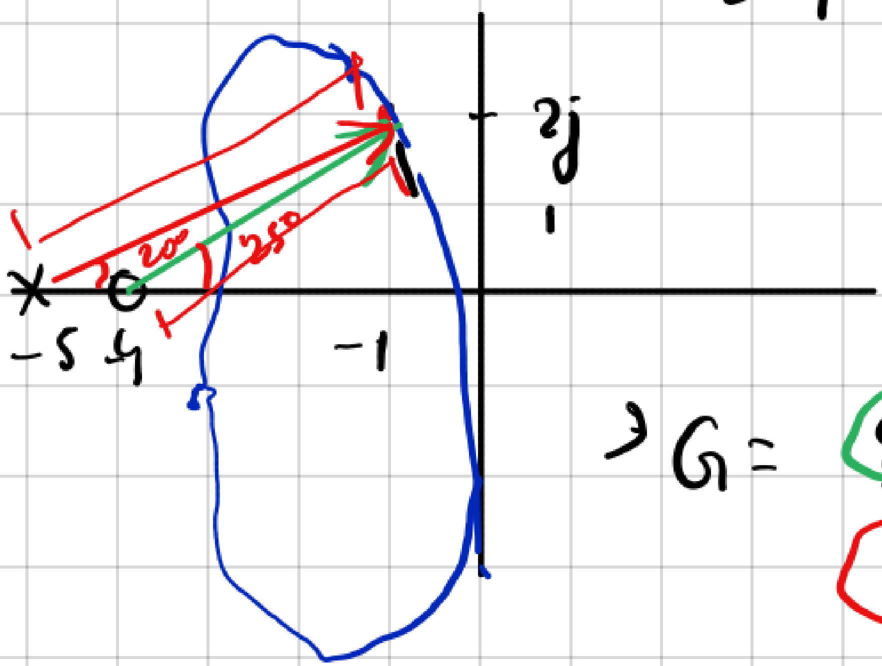
$$T = \frac{G}{1+G}$$

Poles of T = zeros of [1+G]

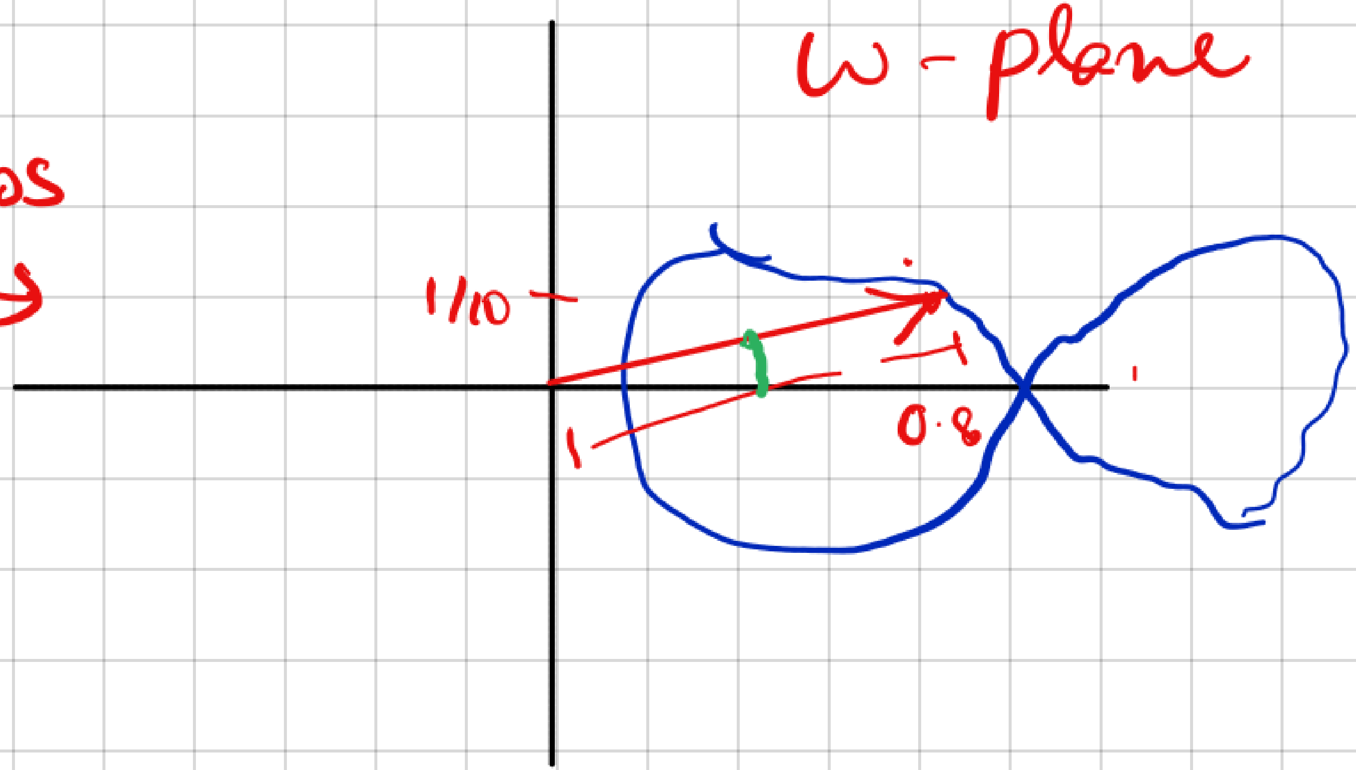
If freq. response of 1+G is known, we understand the closed loop stability & characteristics.

We want to understand stability through zeros of $(1+G)$.

Cauchy's Argument Principle



Maps



→ Hypothetical mapping

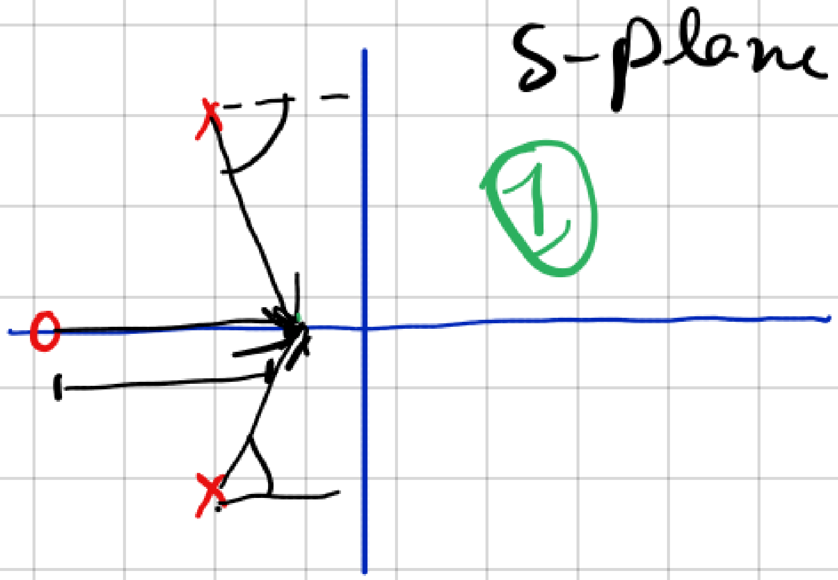
$\rightarrow G = \frac{s+4}{s+5}$

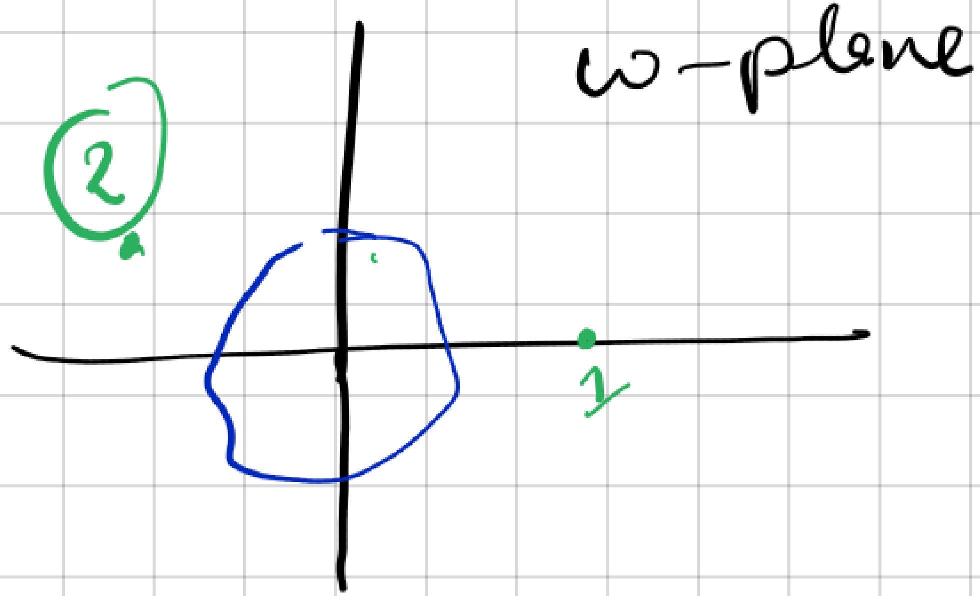
$\frac{|s+4|}{|s+5|}$ $\quad \angle = +5°$

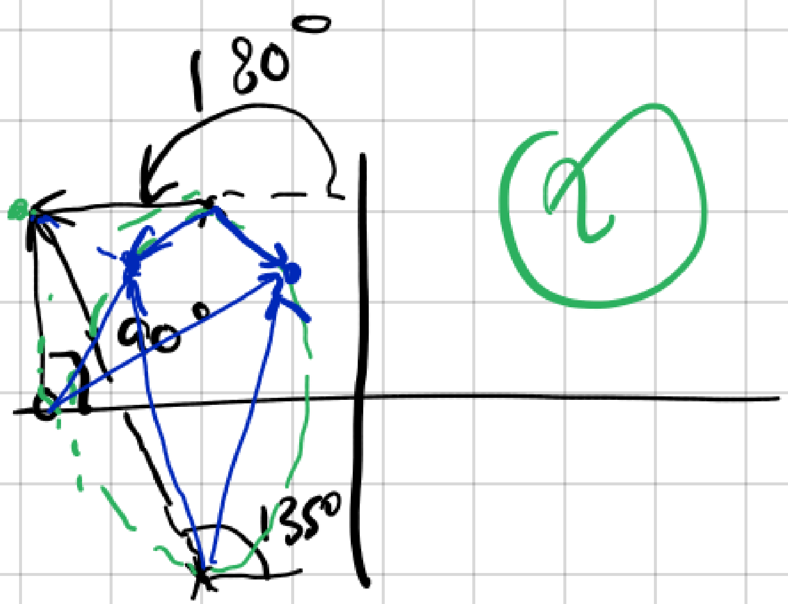
$\frac{-1+2j+4}{-1+2j+5} = \frac{3+2j}{4+2j}$

$\frac{4}{5} + \frac{1}{10}j$

if mapped from s-plane through $(1+G) \rightarrow$ ω-plane.

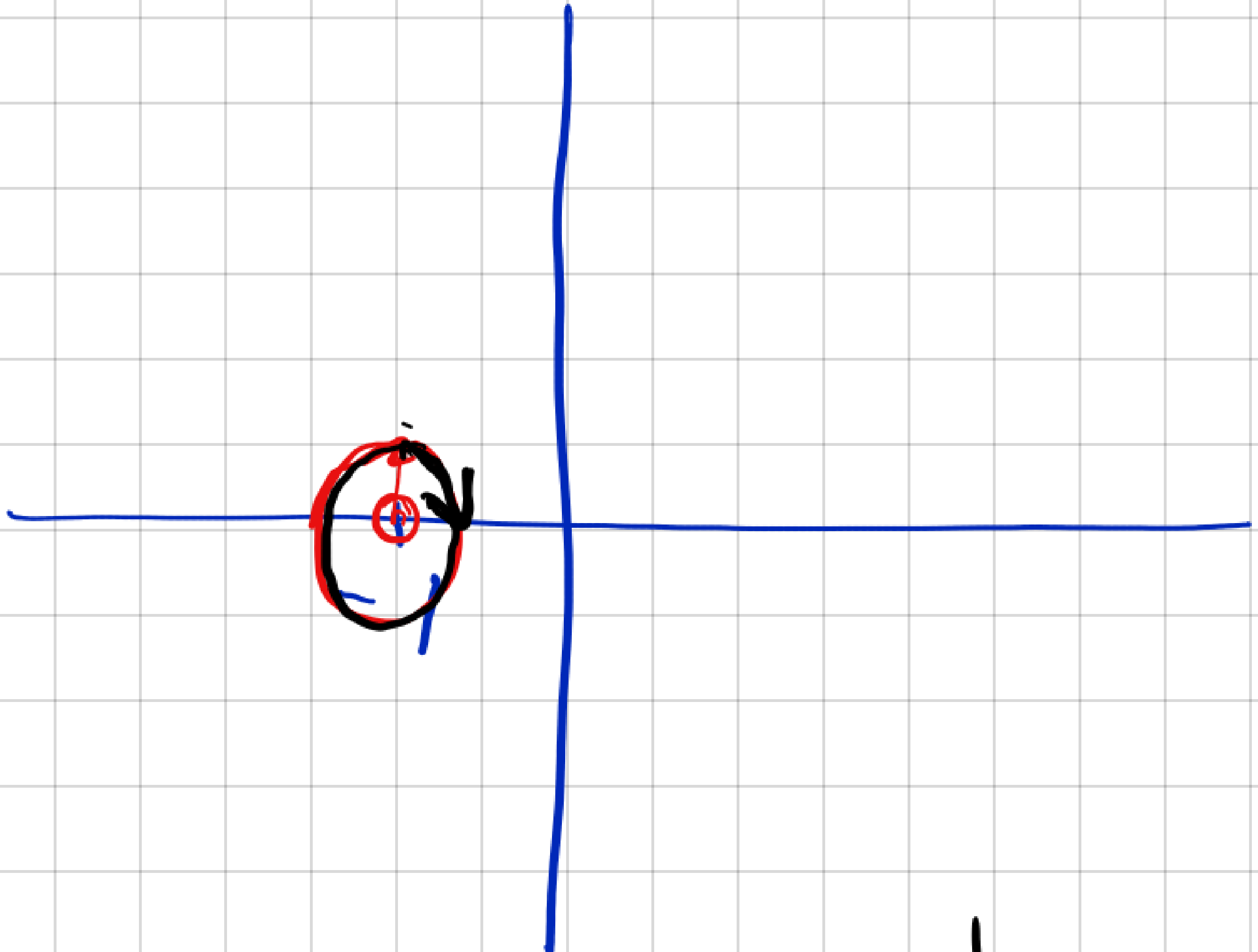


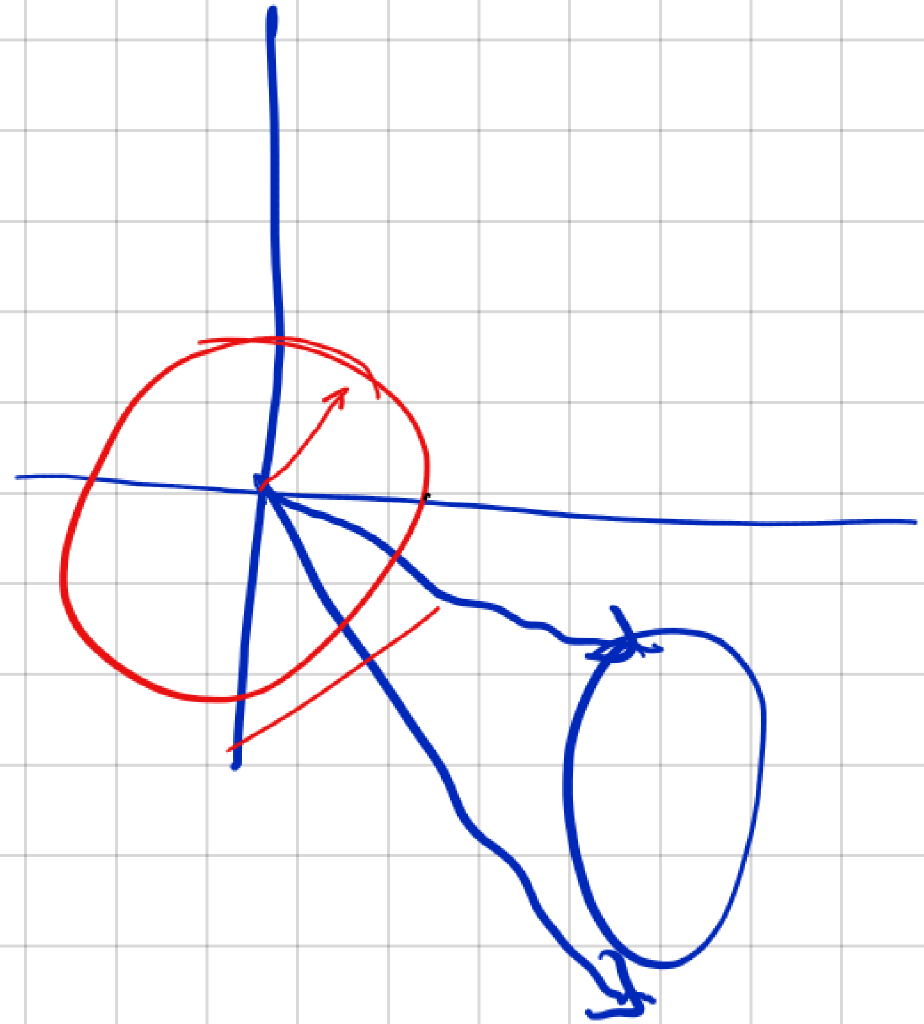




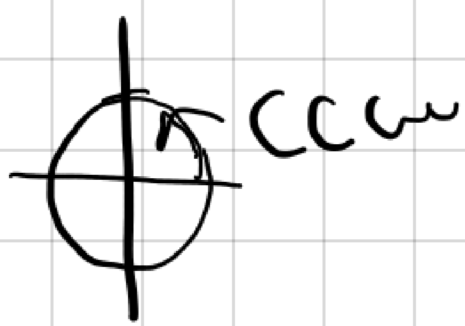
$G = \frac{1}{s+1}$

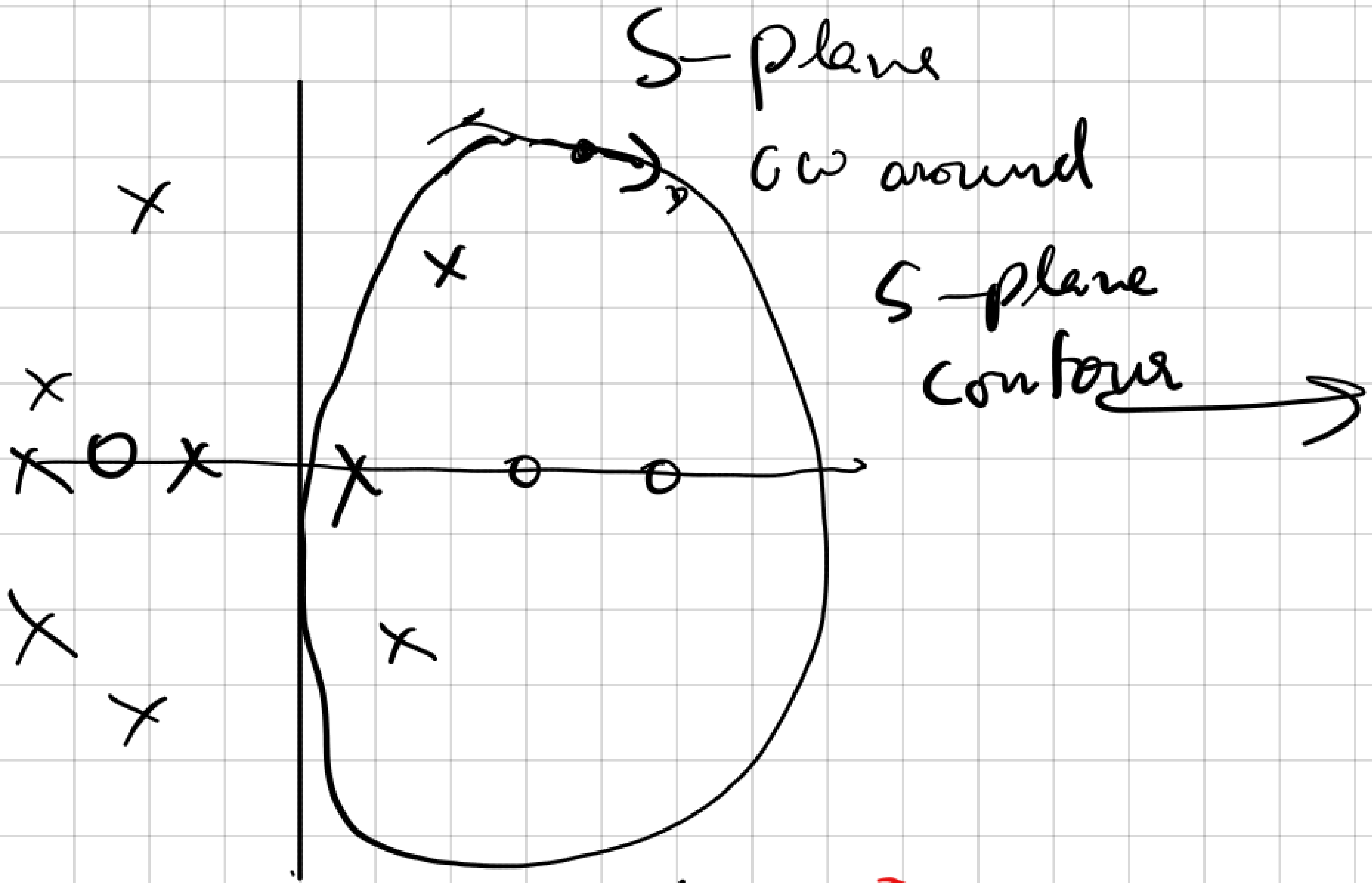
$\underline{s+1}$ → mapping clockwise.

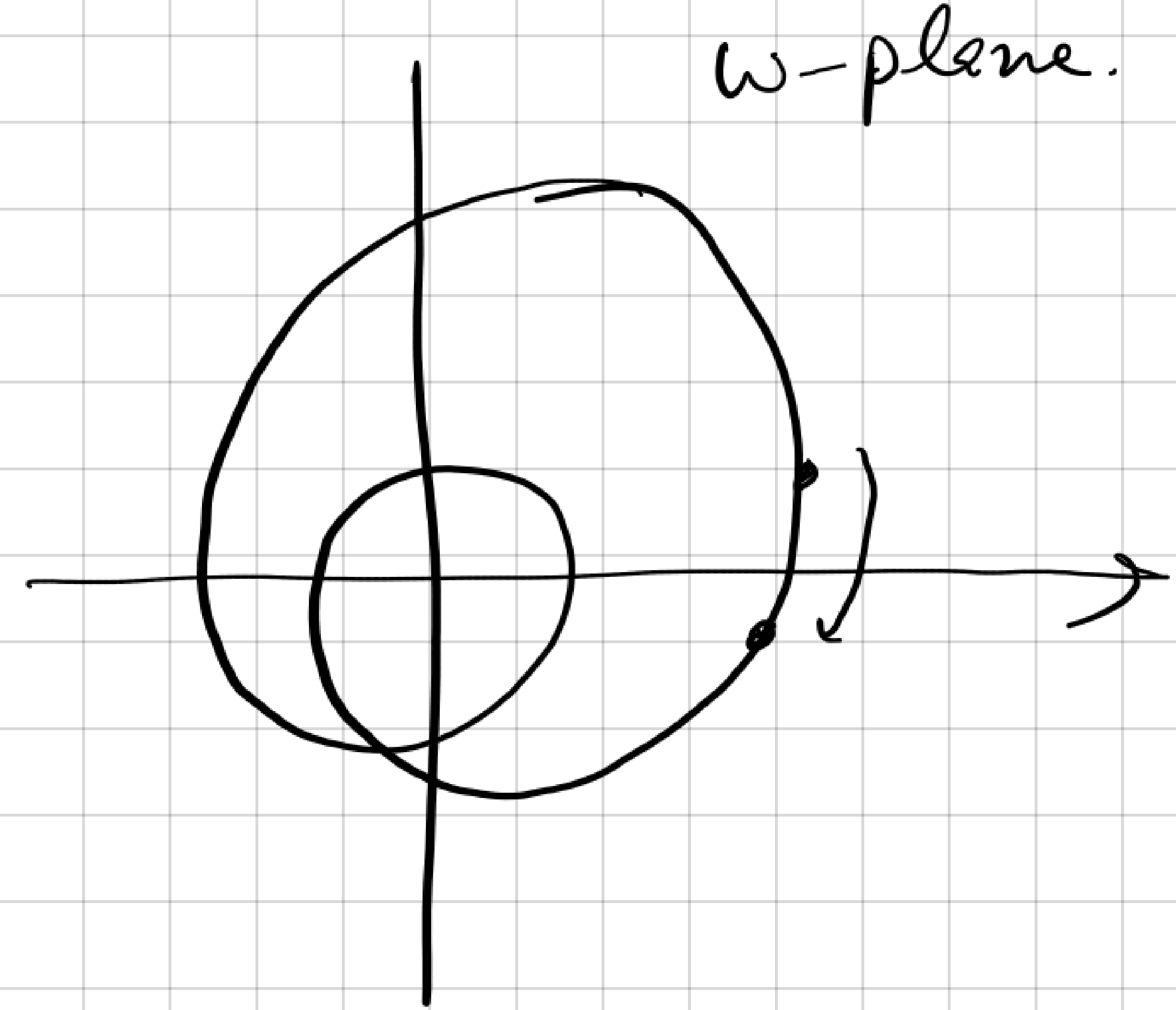


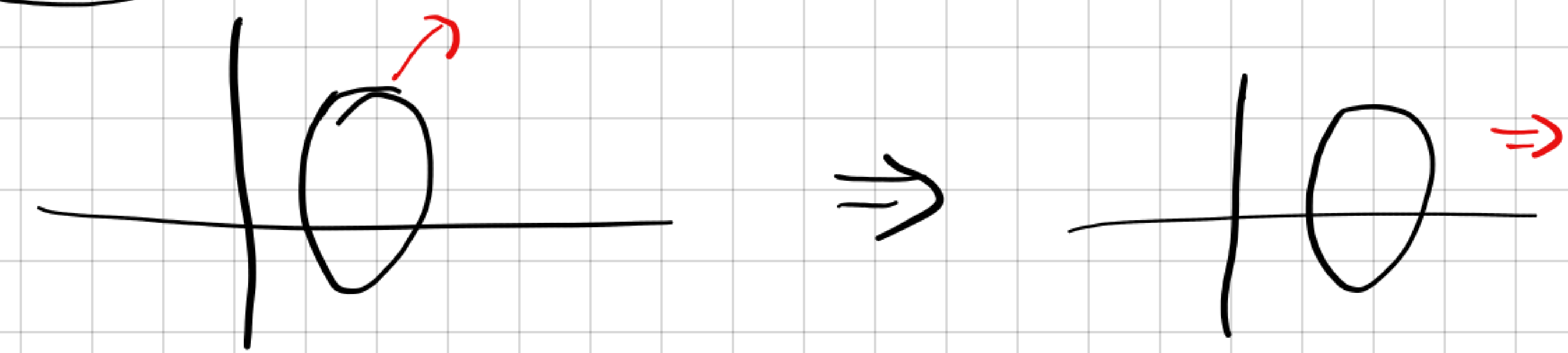


$\frac{1}{s+1}$ →

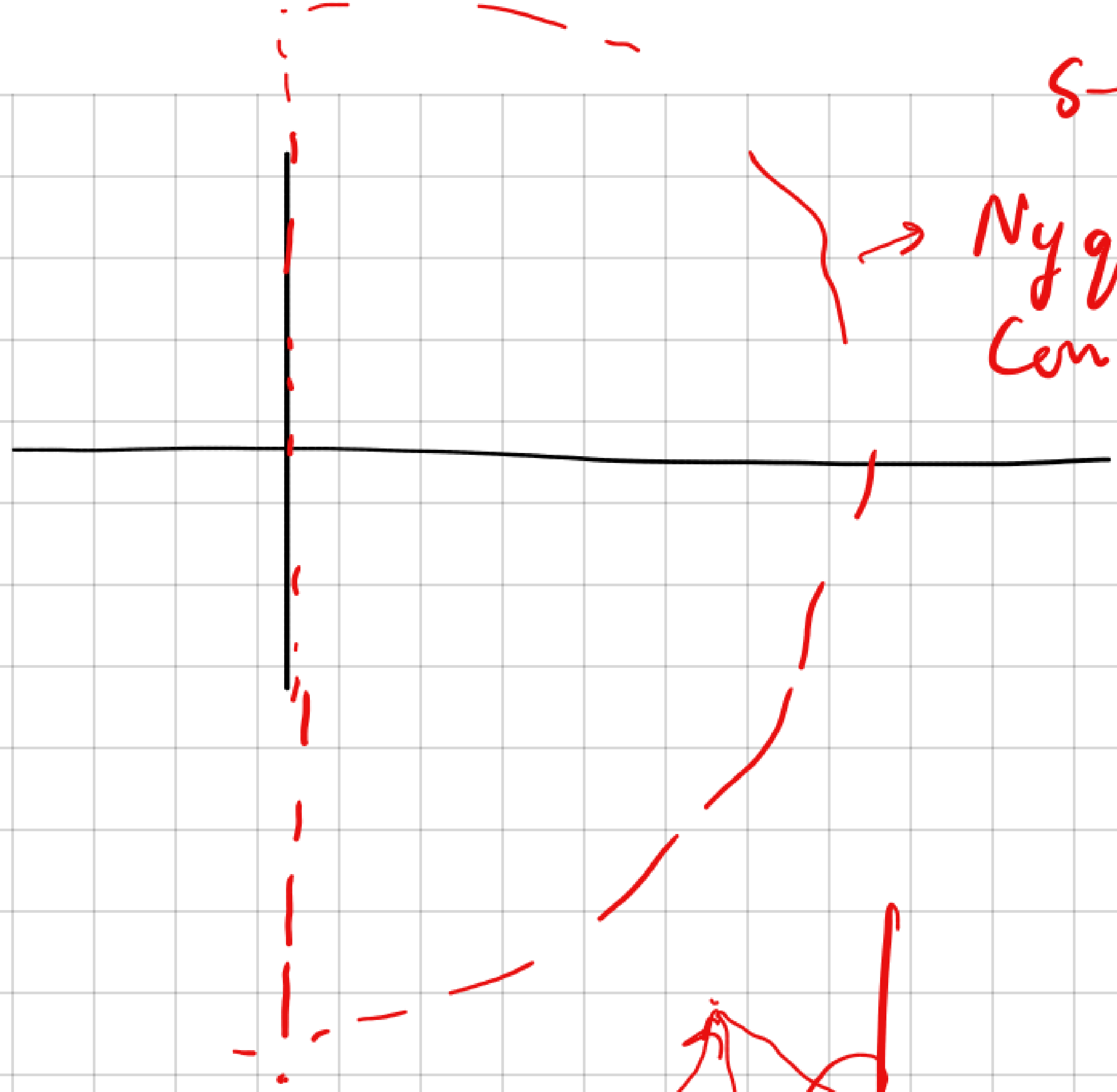








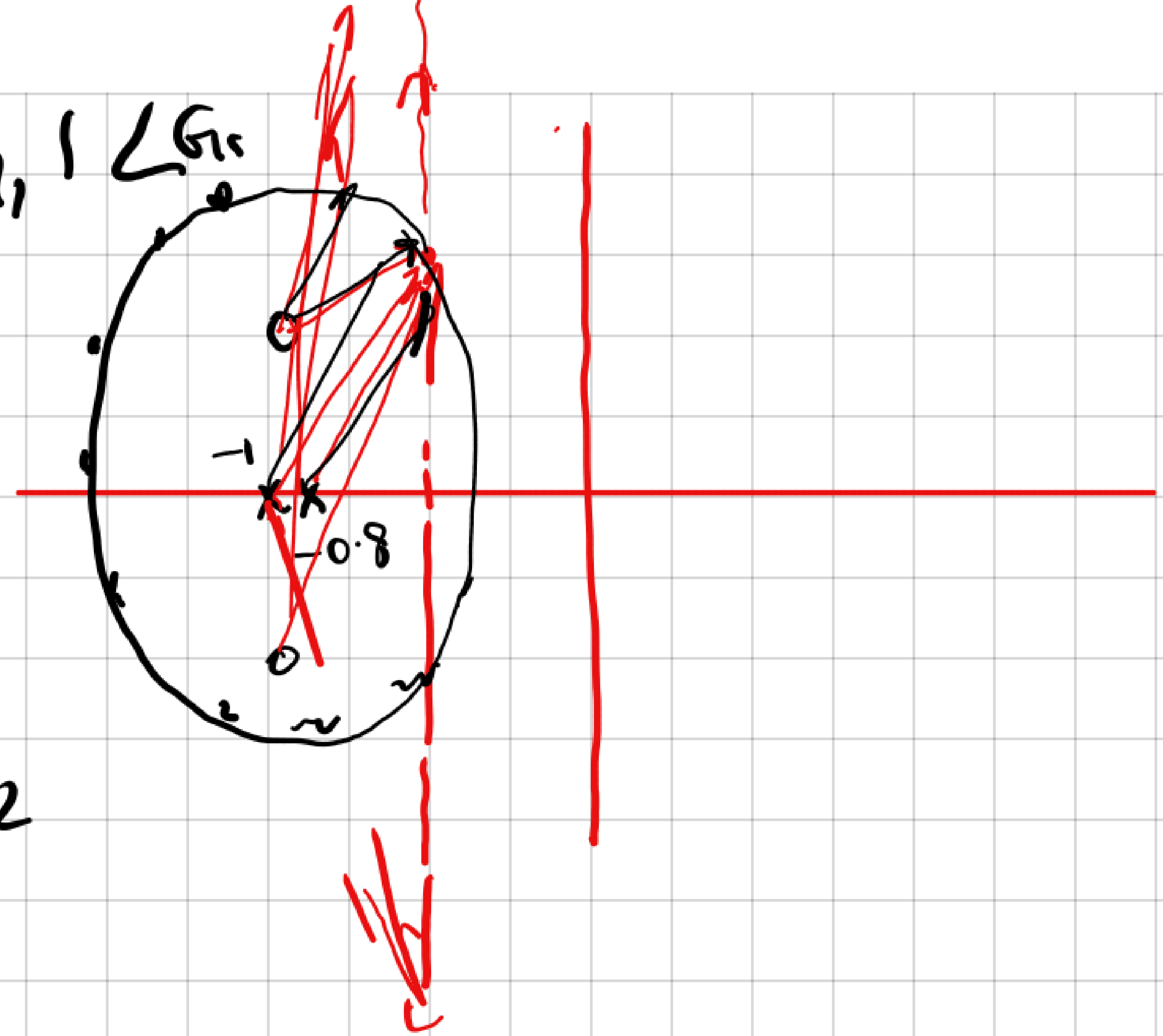
⇒ # poles = # zeros in the s-plane contour s-plane



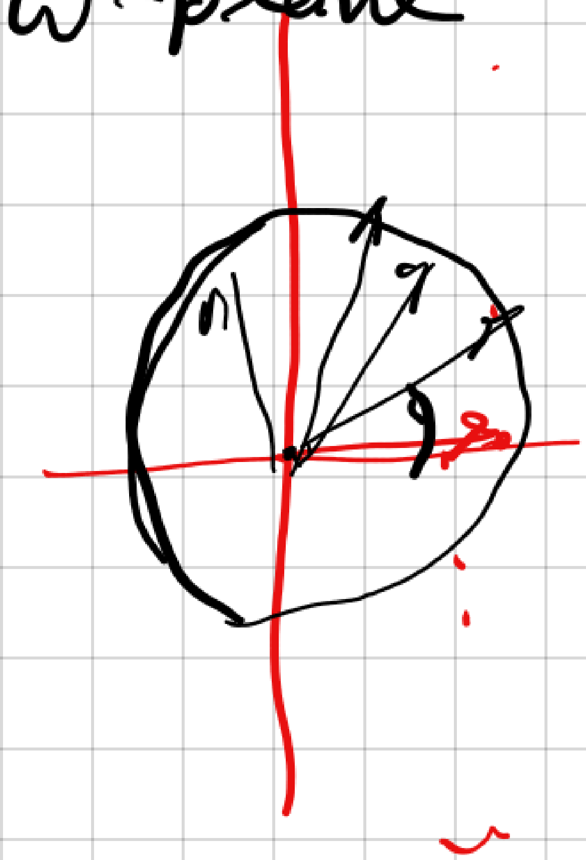
Nyquist Contour.

$$G_1 = \frac{s^2 + 1.5s + 0.8}{(s+1)(s+0.8)}$$

$|G_1| \angle G_1$



w-plane



$$G_2 = \frac{1}{(s+1)(s+0.8)}$$

$\rho = 2$

$$G_3 = \frac{1}{s+1}$$

$$G_4 = s+1$$

encirclements of CW origins in w plane = # zeros − # poles

$N \qquad\qquad Z \qquad P$

$$N + P = Z$$

$$\begin{array}{ccc} N & +P & = Z \\ \downarrow & \downarrow & \downarrow \\ -2 & 2 & 0 \end{array}$$

$$\frac{1}{S+1} = Z = 0$$

$$P = 1$$

$$N = -1$$

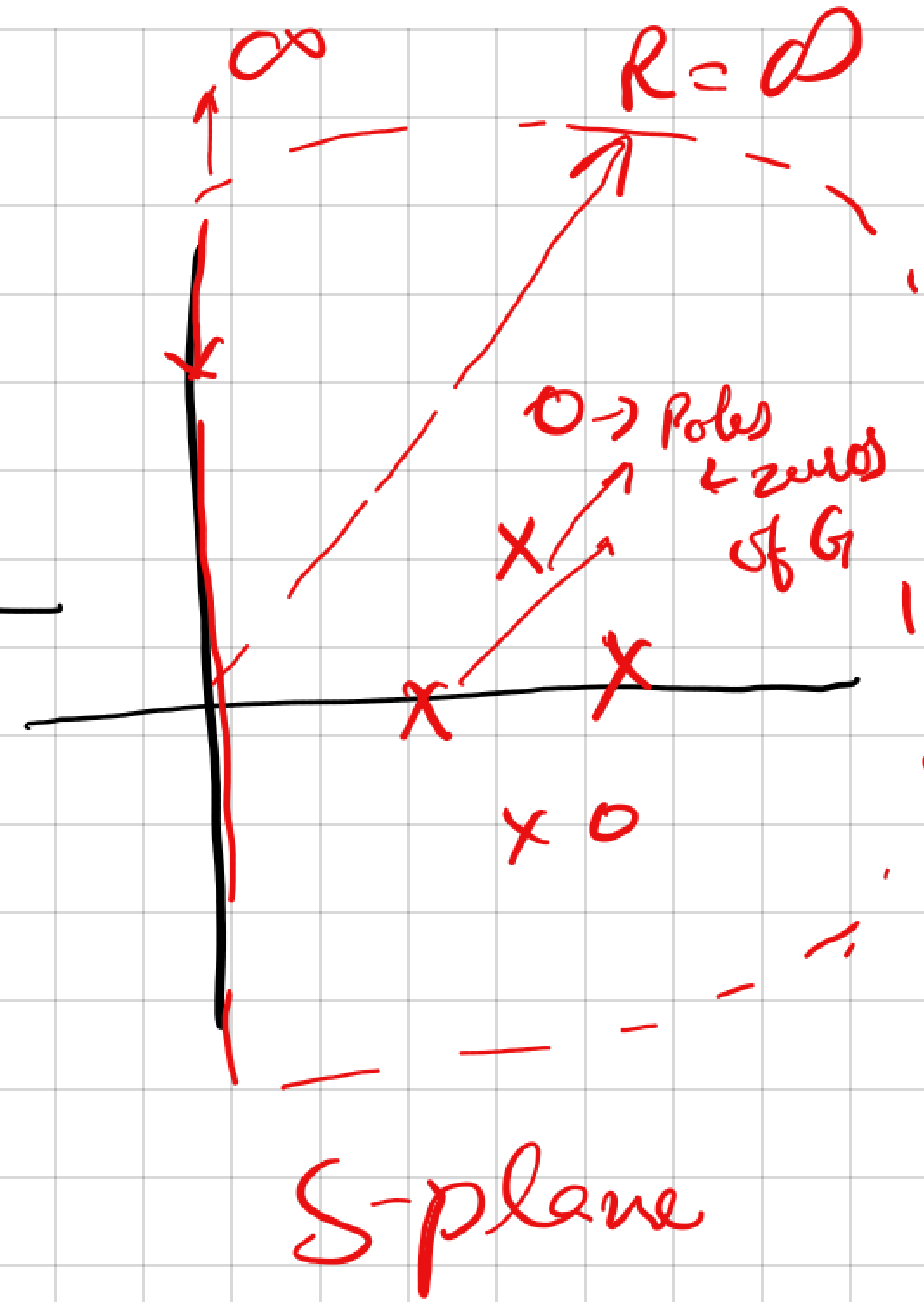
$$T = \frac{G}{1+G}$$

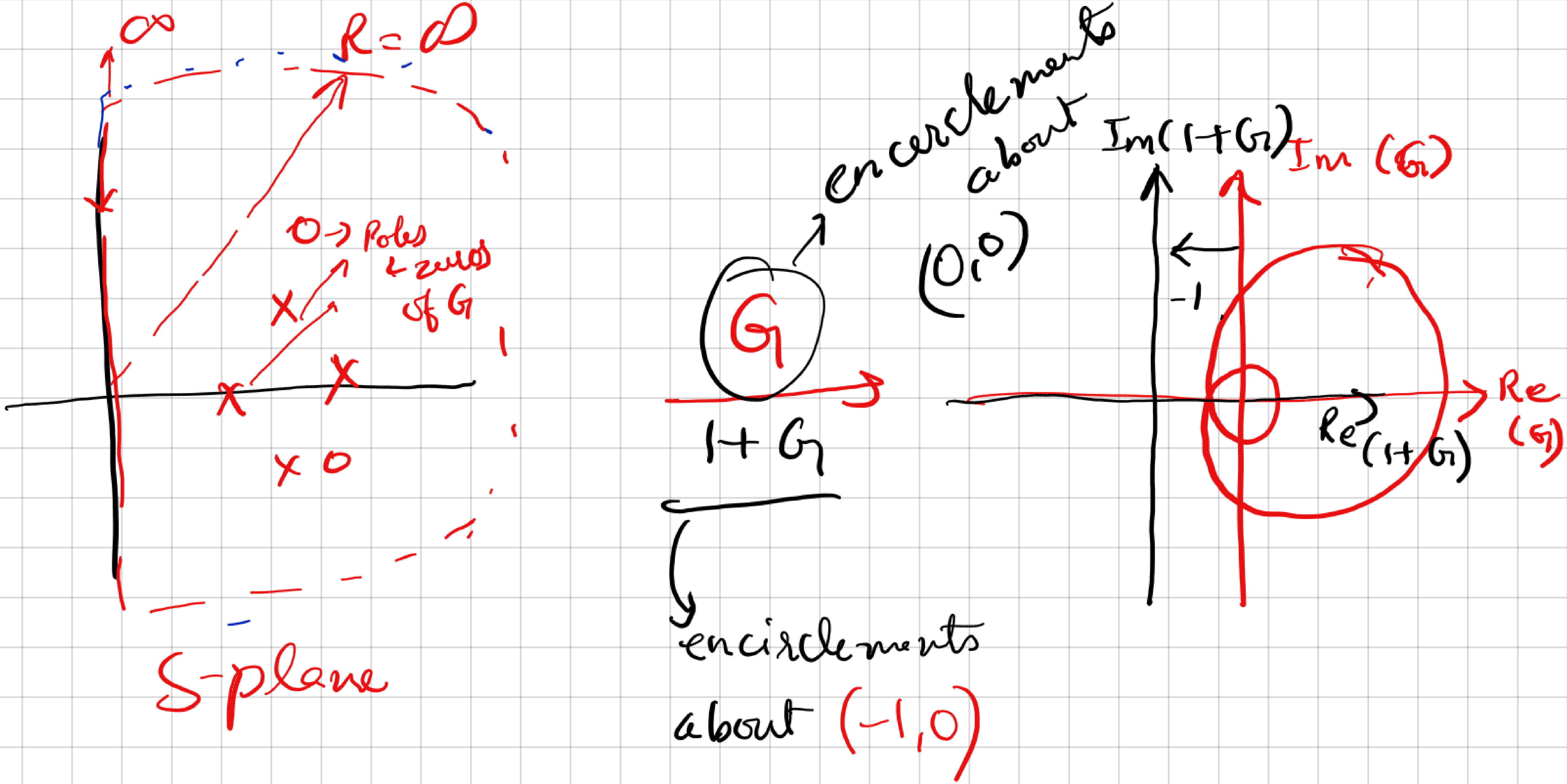
Poles of T = Zeros of $\underline{(1+G)}$

Poles of G = poles of $\underline{(1+G)}$

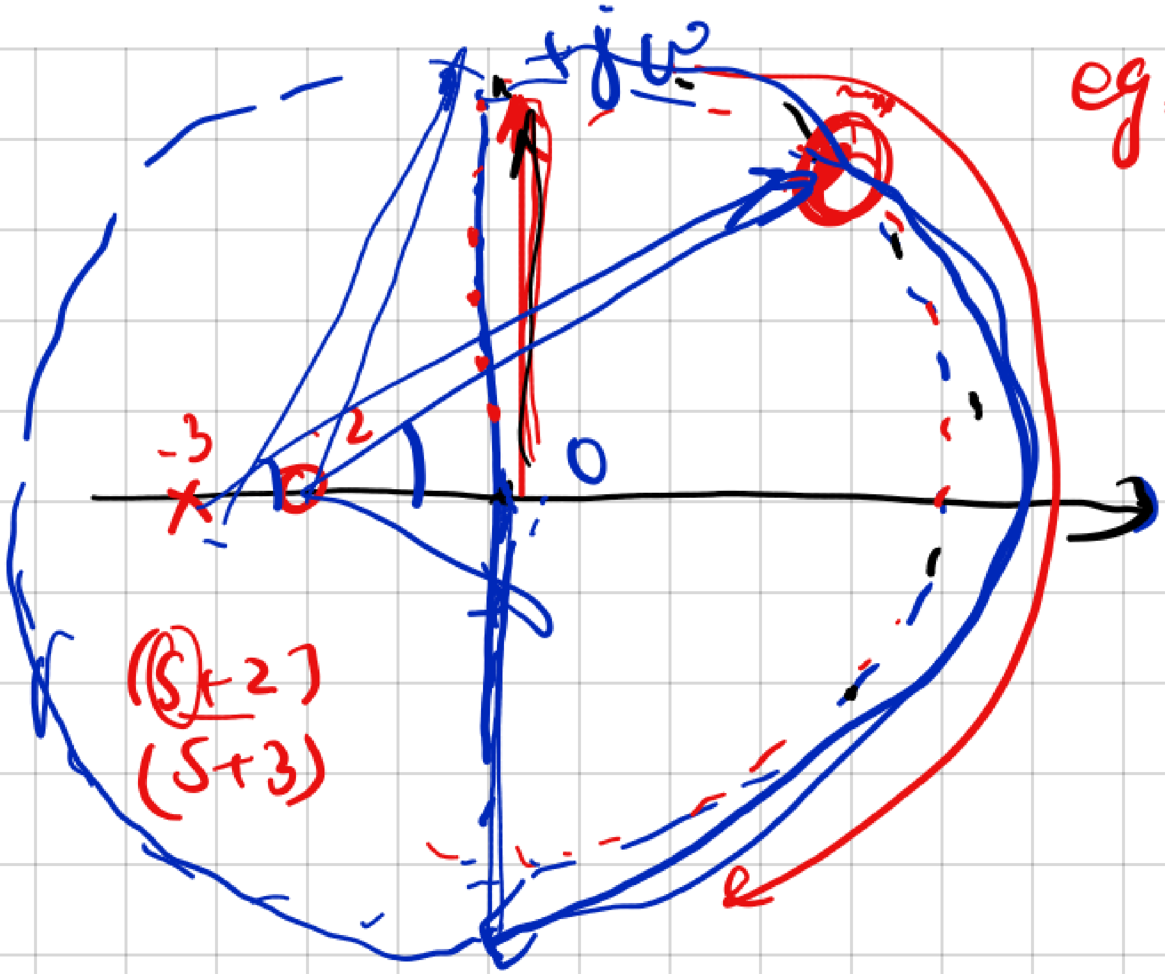
If I knew
\# poles of G in a RHP Contour

\# CW encirclements of 0

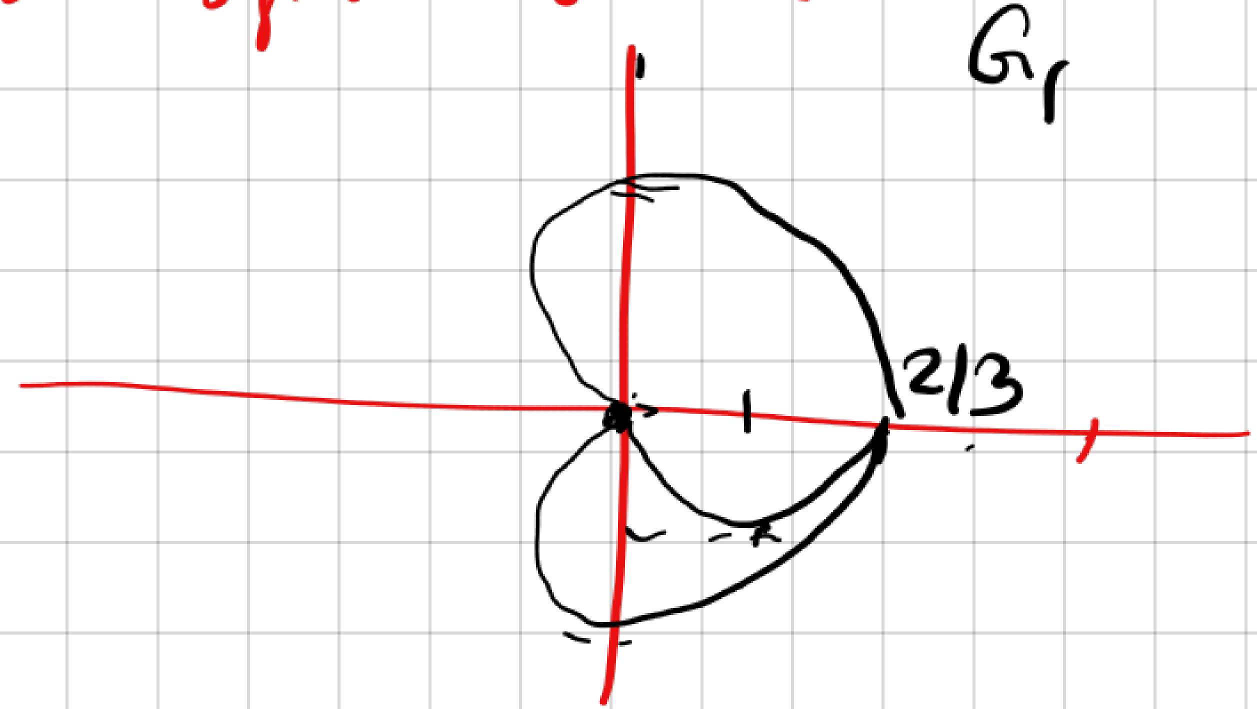


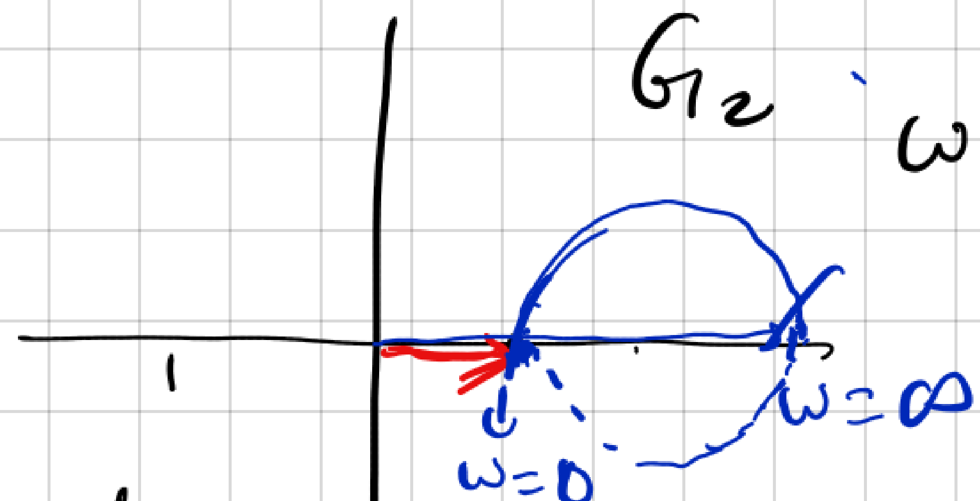


$\#$ zeros of $(1+G)$ = $\#$ CW enc. of -1 + $\#$ OL poles in RHP



eg. O.L. sys is stable.





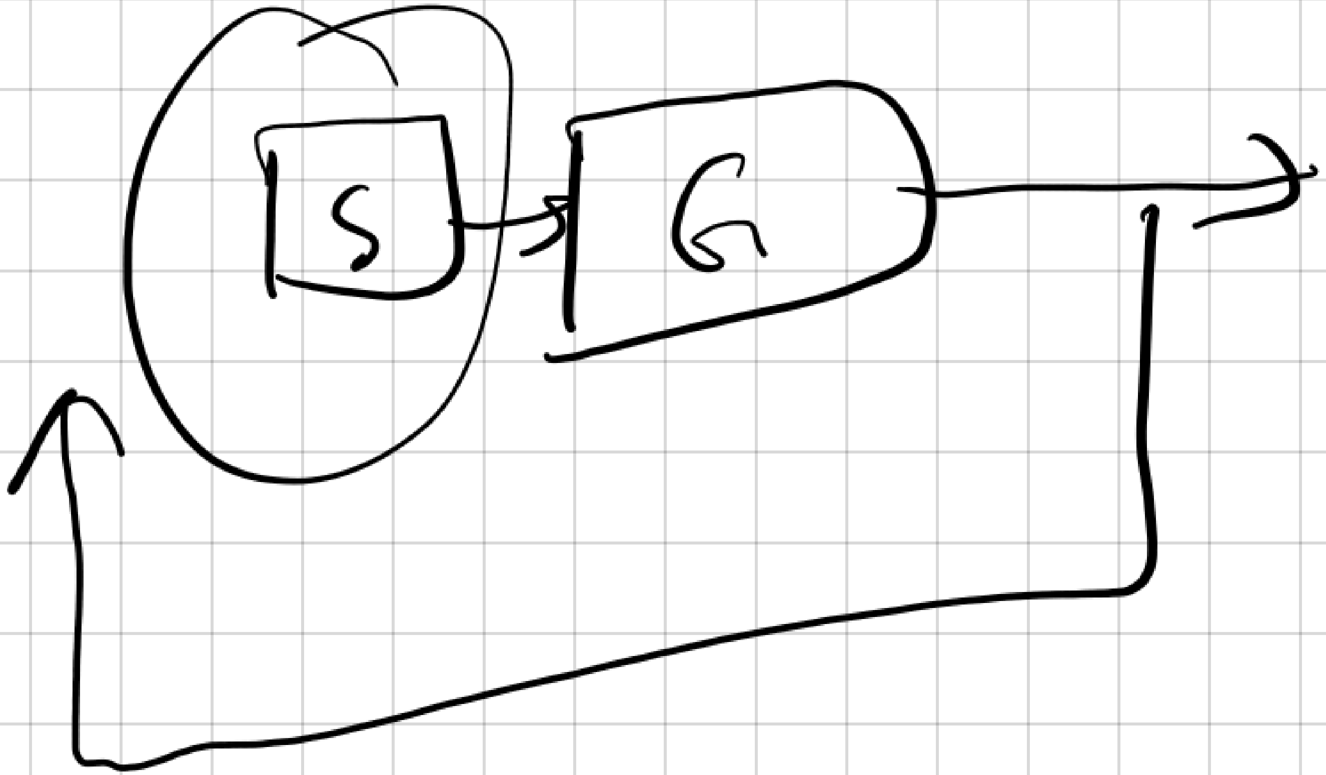
Strictly Proper T.f. ← $G_1 = \frac{s+2}{(s+1)(s+3)}$ → Subs. $s = j\omega$

$G_2 = \boxed{\frac{(s+2)}{(s+3)}}$

} Proper TF

order of den. $\geq$ order of num.

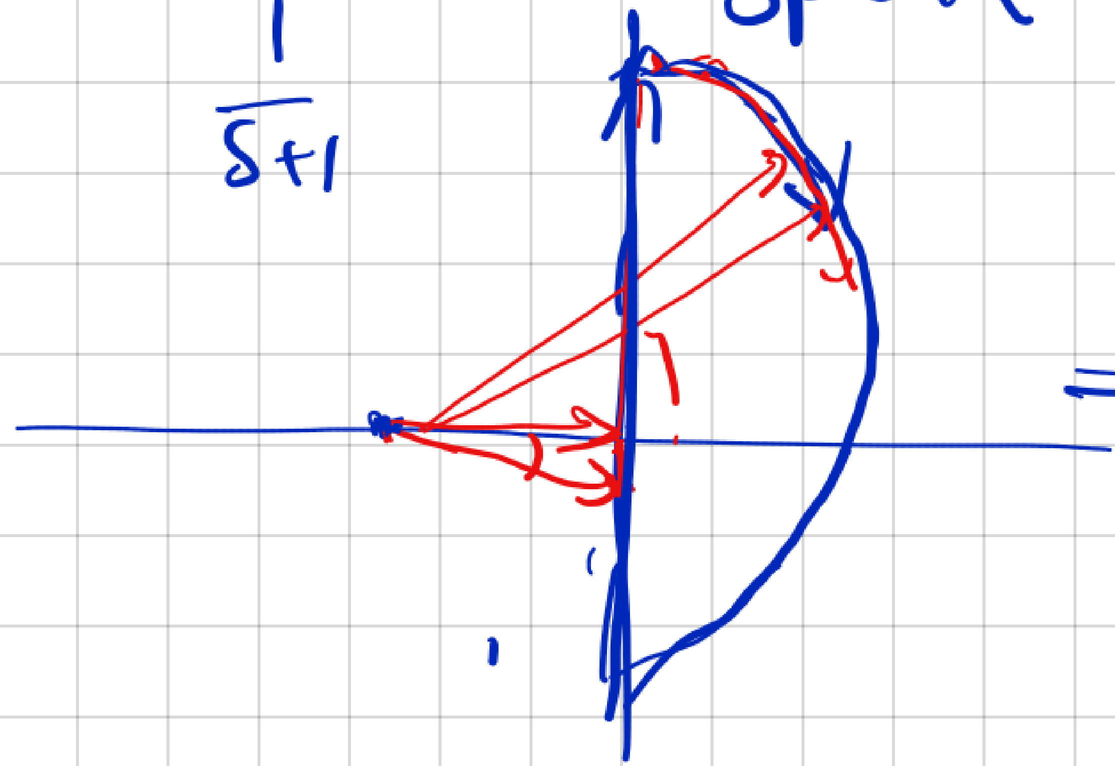
$G_3 = (s+2)$ $\quad 0 \to \infty$ Improper T.F.



$$\frac{1}{ms^2 + cs + k}$$

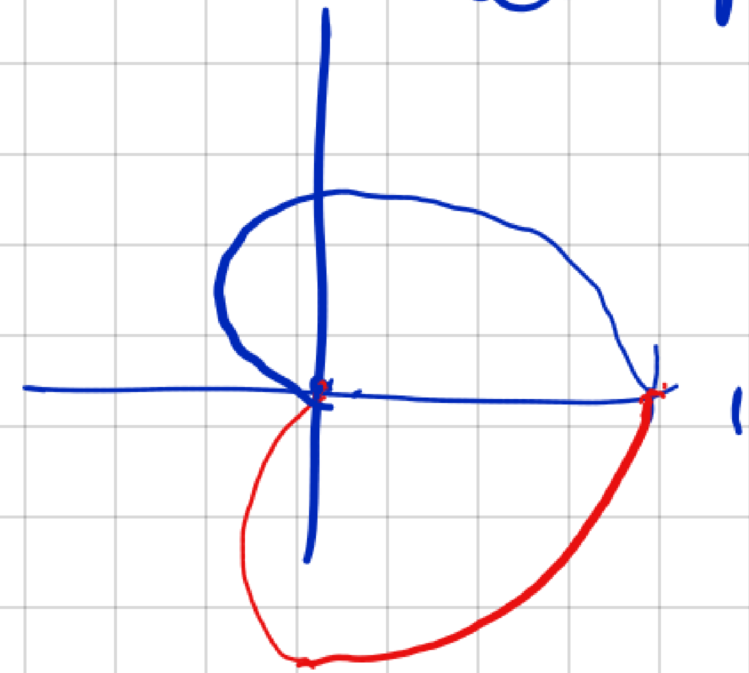
$\frac{1}{s+1}$

S plane



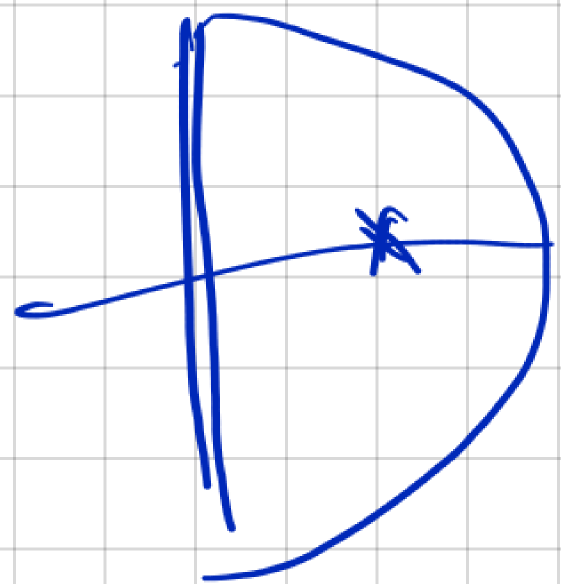
$\Rightarrow$

GH - plane.



$$\frac{G}{1+G} = \frac{1}{s+1}$$

$$G = \frac{1}{s}$$



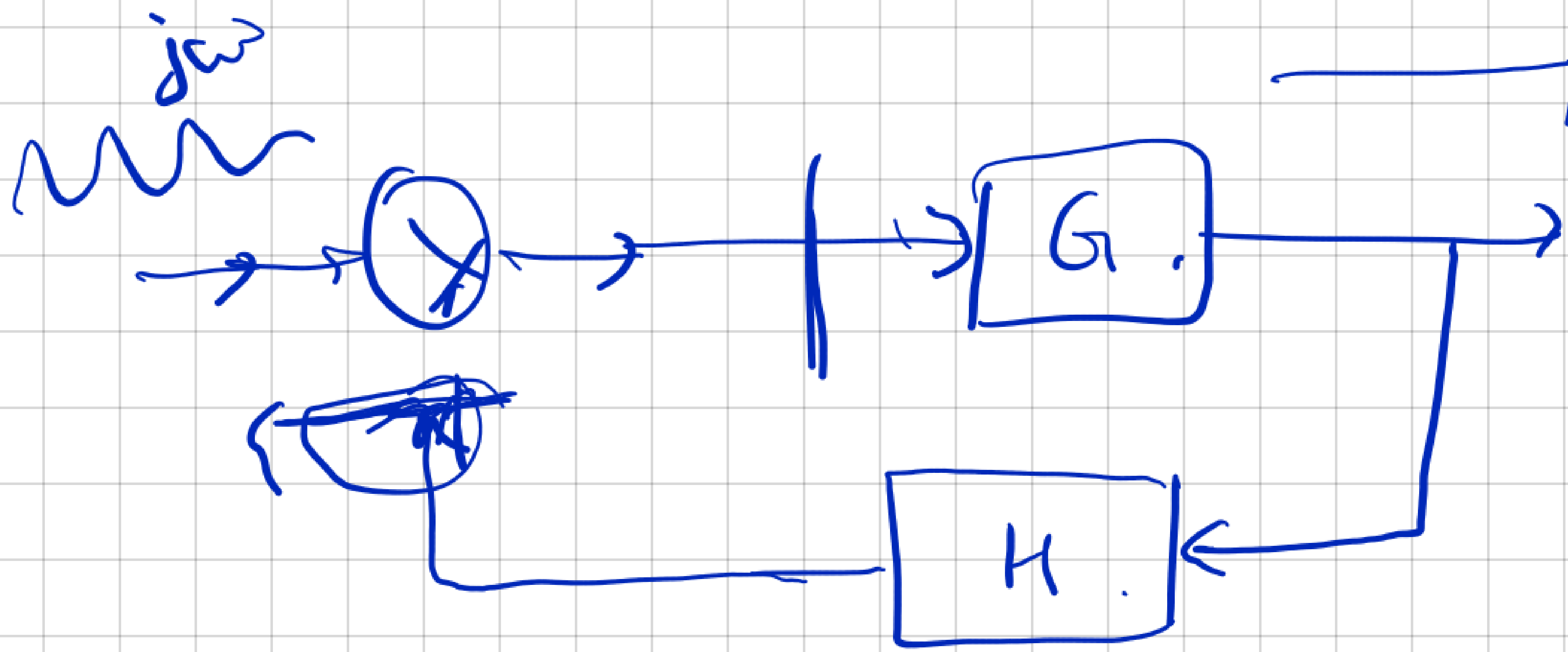
$$G = [-1, -2, -3, +1]$$

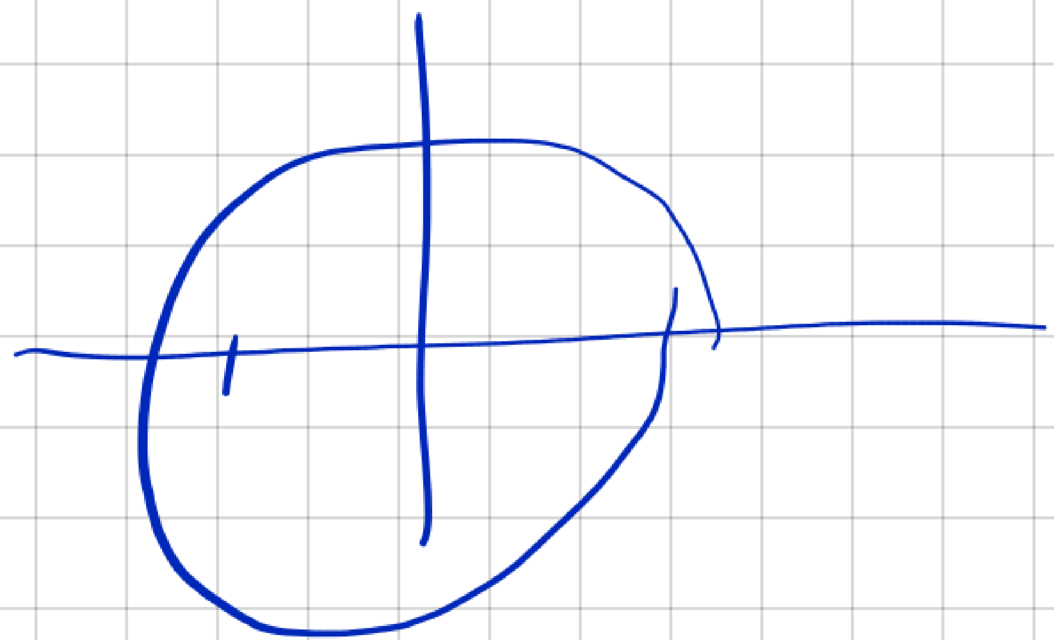
$P = 1$

$N = 0$

Cl. Poles = # zeros of $1 + G = P + N = 1$

$Z = N + P$

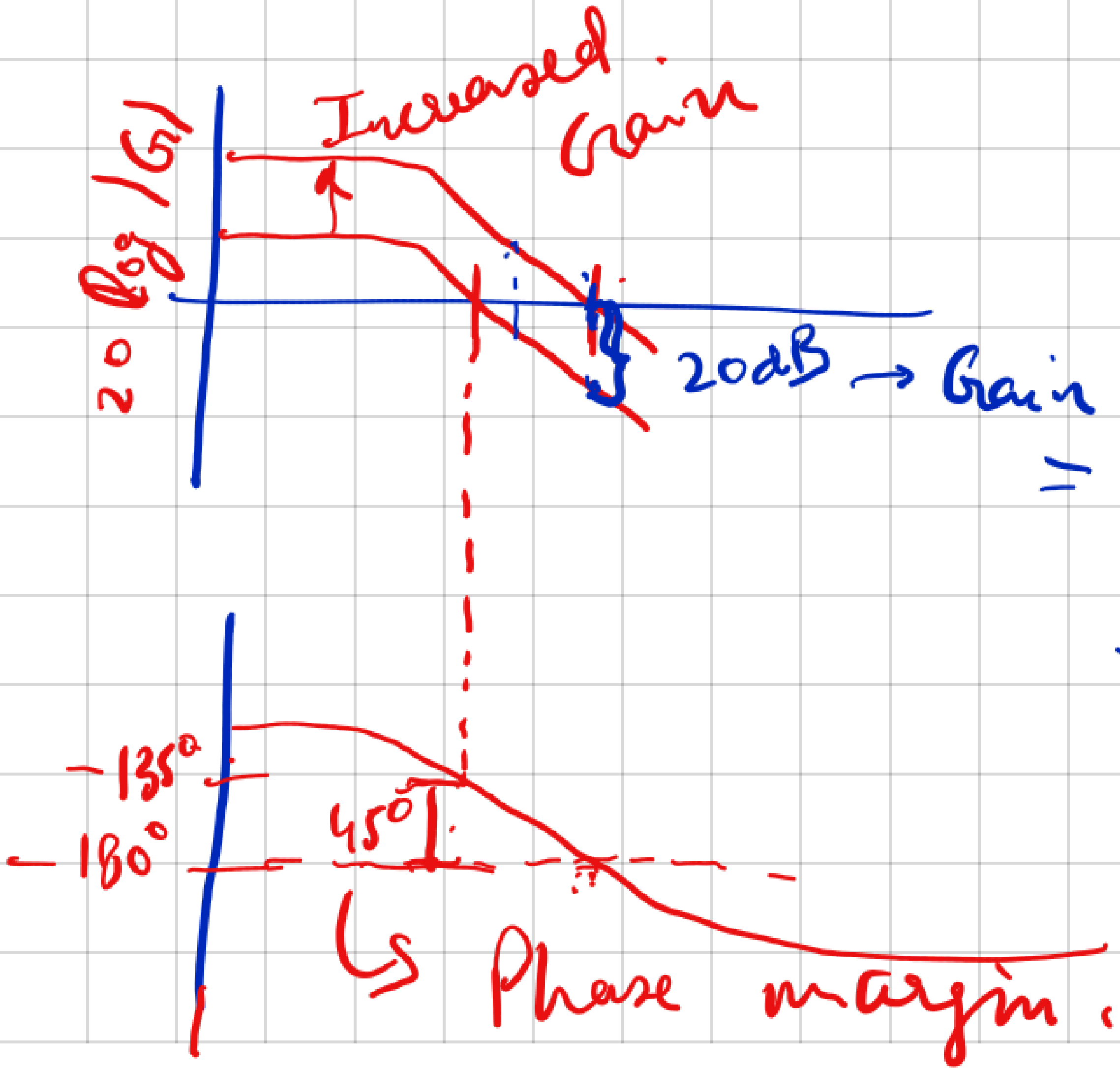

jω
G
H



$$R = \frac{KG}{1+KG}$$

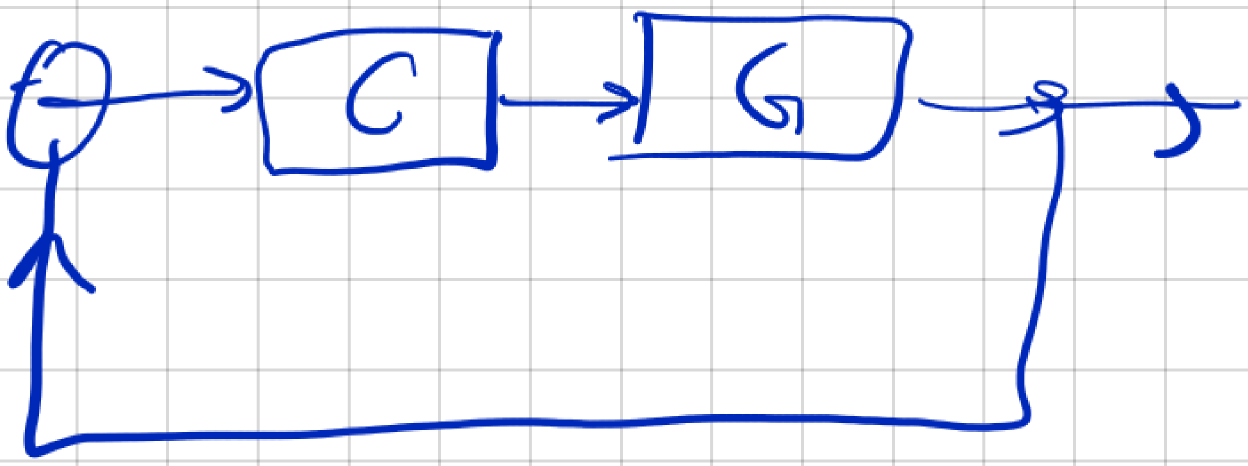
$$-1 = |-1| \angle -1 = 1 \angle 180^\circ$$

if $G = -1$, $R = \infty$



20dB → Gain margin $= 20 \log_{10} K$

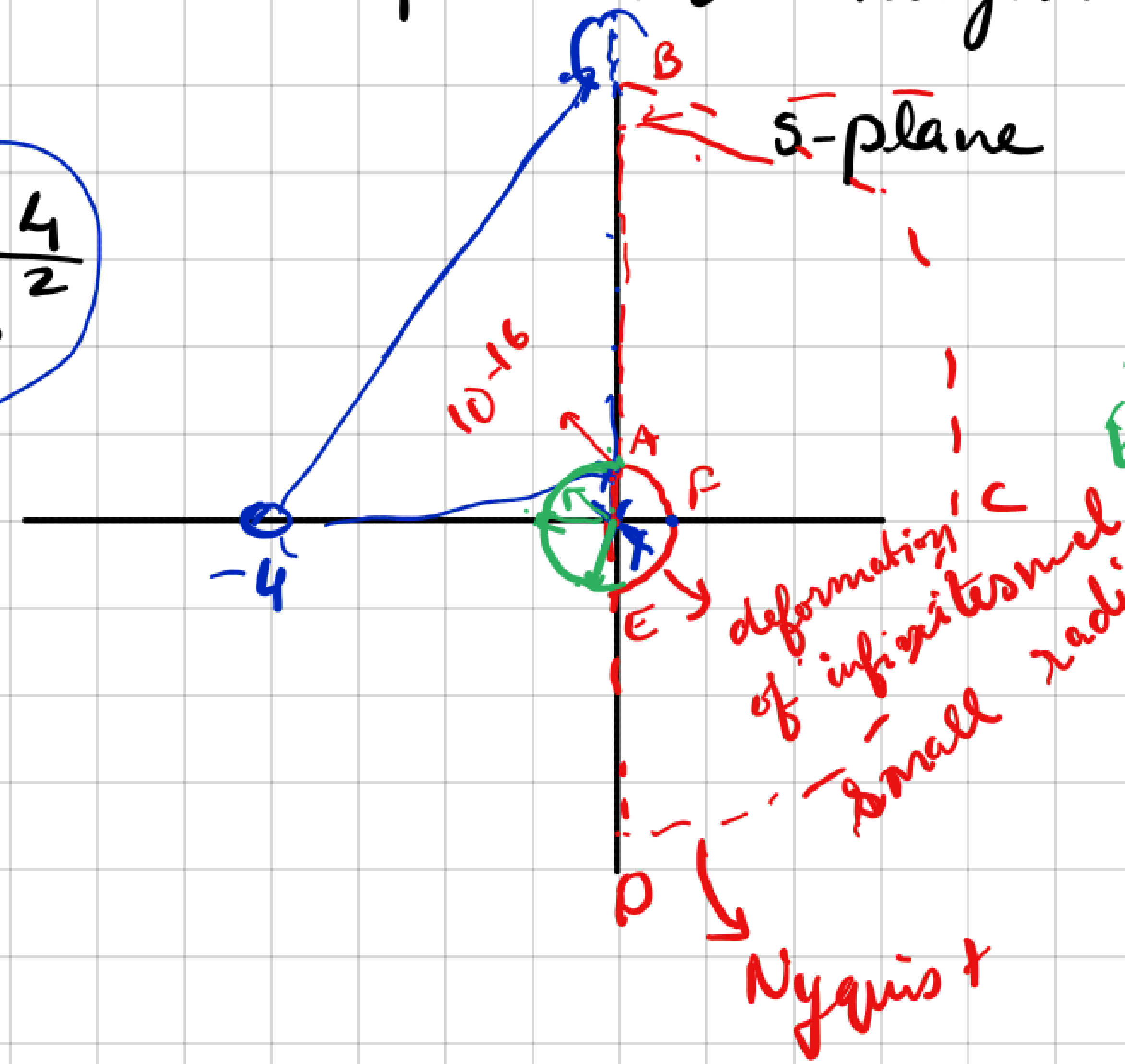
$=$ say 5dB Gain $< 10^{1/4}$

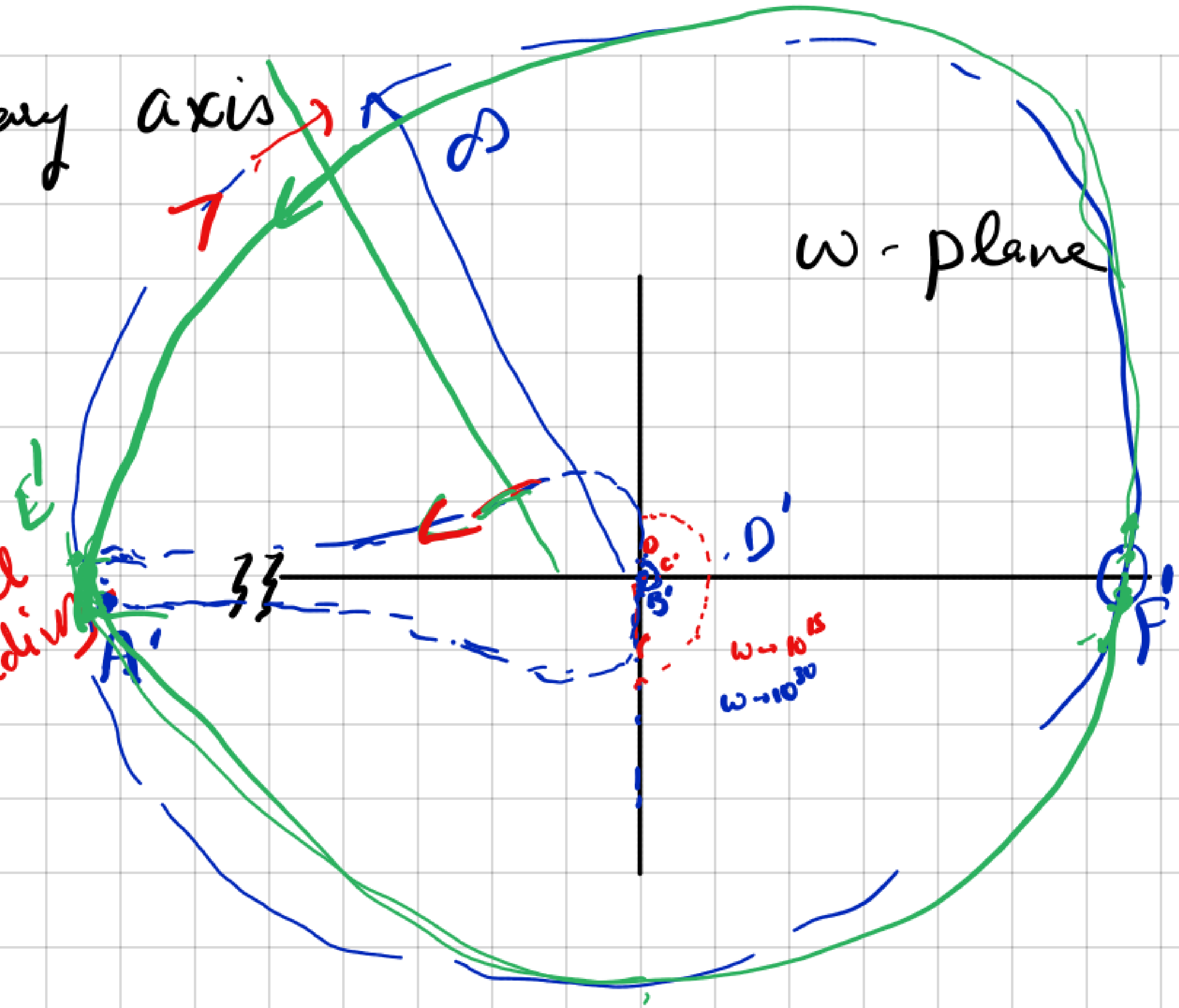


$G = \dfrac{N_G}{D_G}$ – Model has uncertainities in it

Nyquist with poles on imaginary axis

$$G = \frac{s+4}{s^2}$$





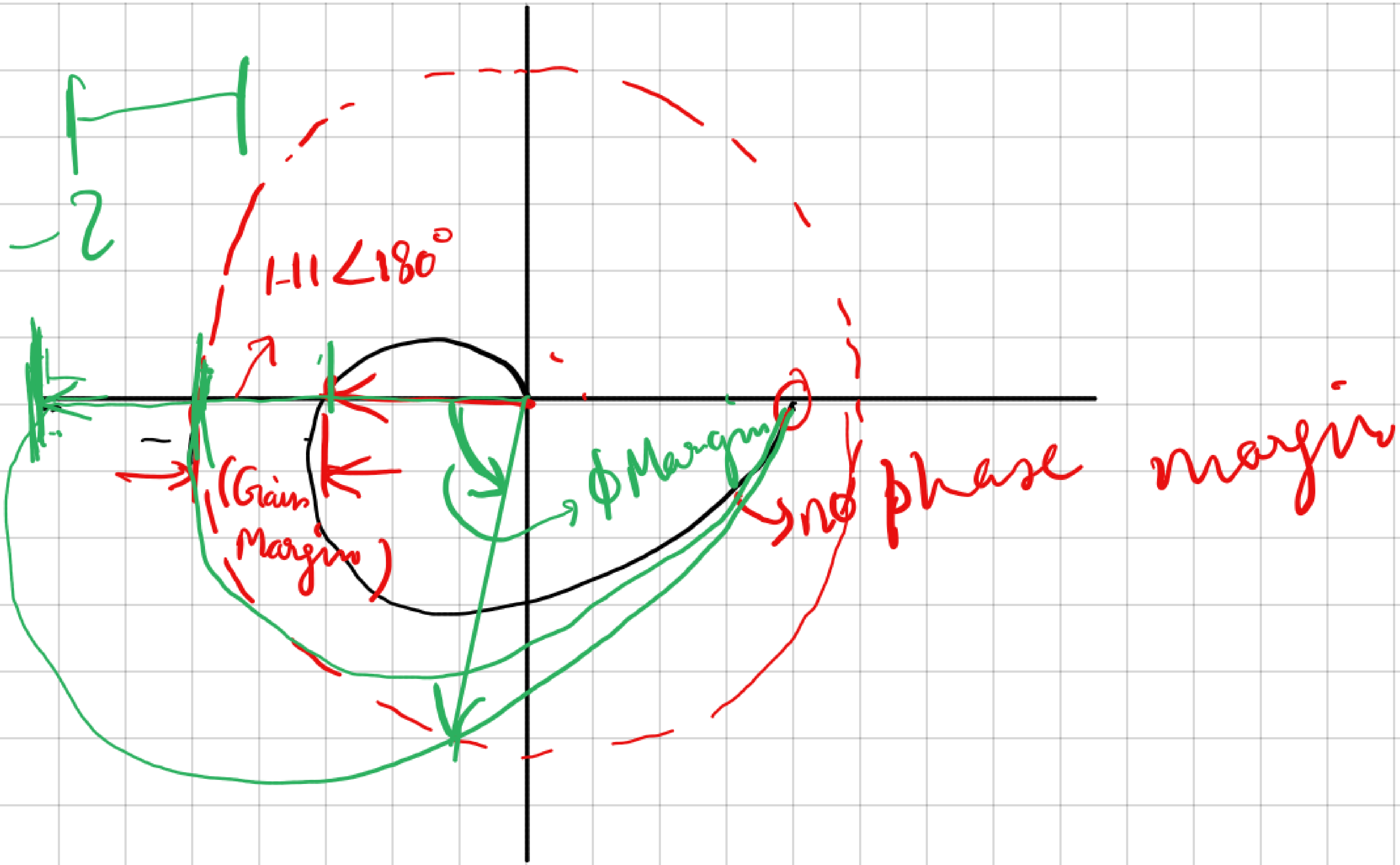
$+j\omega$ axis

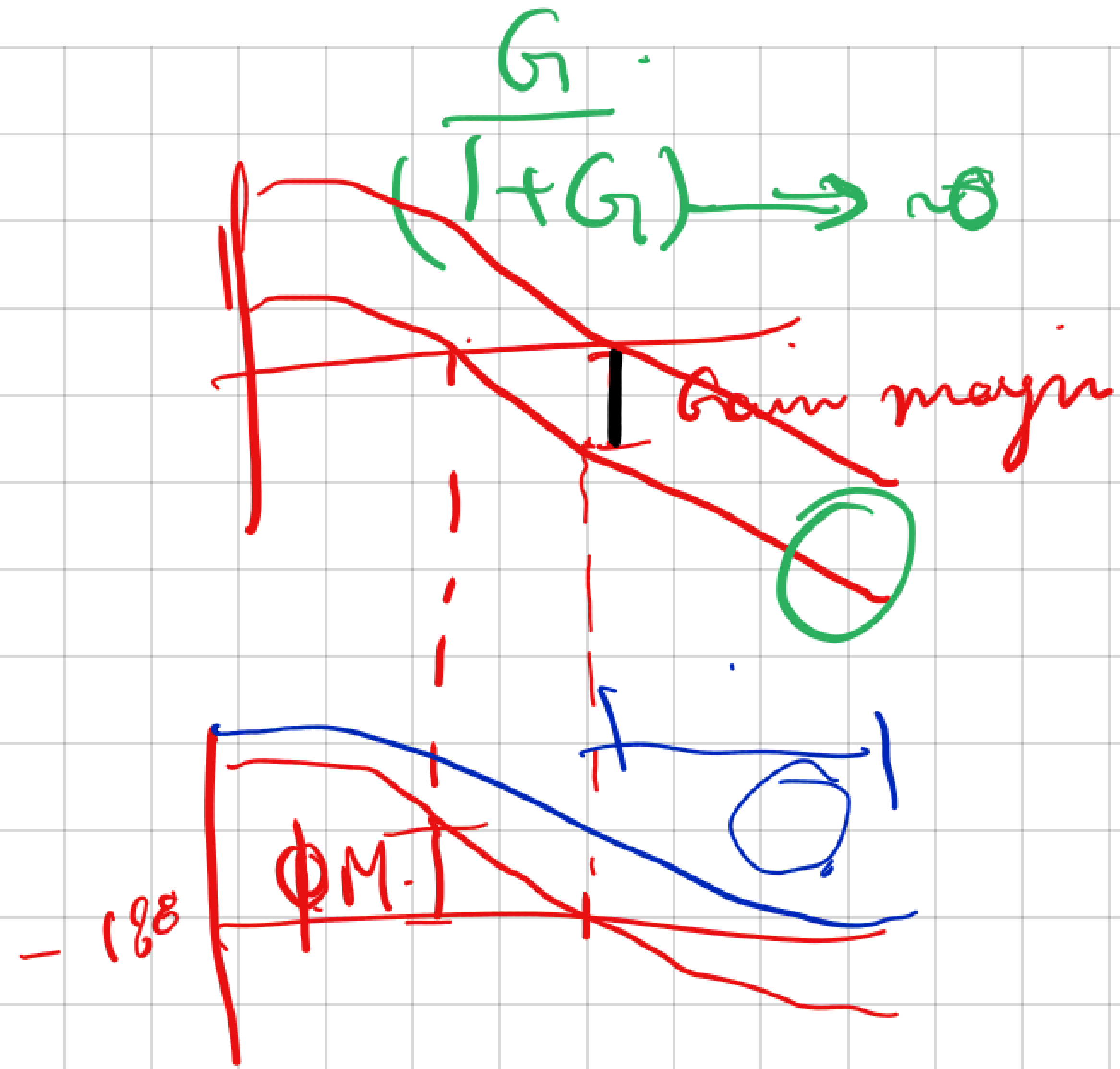
$\omega \to 0$

$|G| \sim \infty$

$\omega \to \infty \qquad \angle G = -90^\circ$

0 encirclements of -1


-2
|-1| ∠180°
(Gain Margin)
ϕ Margin
no phase margin


G/(1+G) → ∞
Gain margin
ϕM.
-180

Velocity error const $\boxed{K_v = \lim_{s \to 0} s\, G(s)}$ → [for ramp input]

$$e_{ss} = \frac{1}{K_v}$$ →

Acceleration error const. $\boxed{K_a = \lim_{s \to 0} s^2 G(s)}$ [for parabolic i/p]

$$e_{ss} = \frac{1}{K_a}$$

not to be confused with proportional gain const. in PID

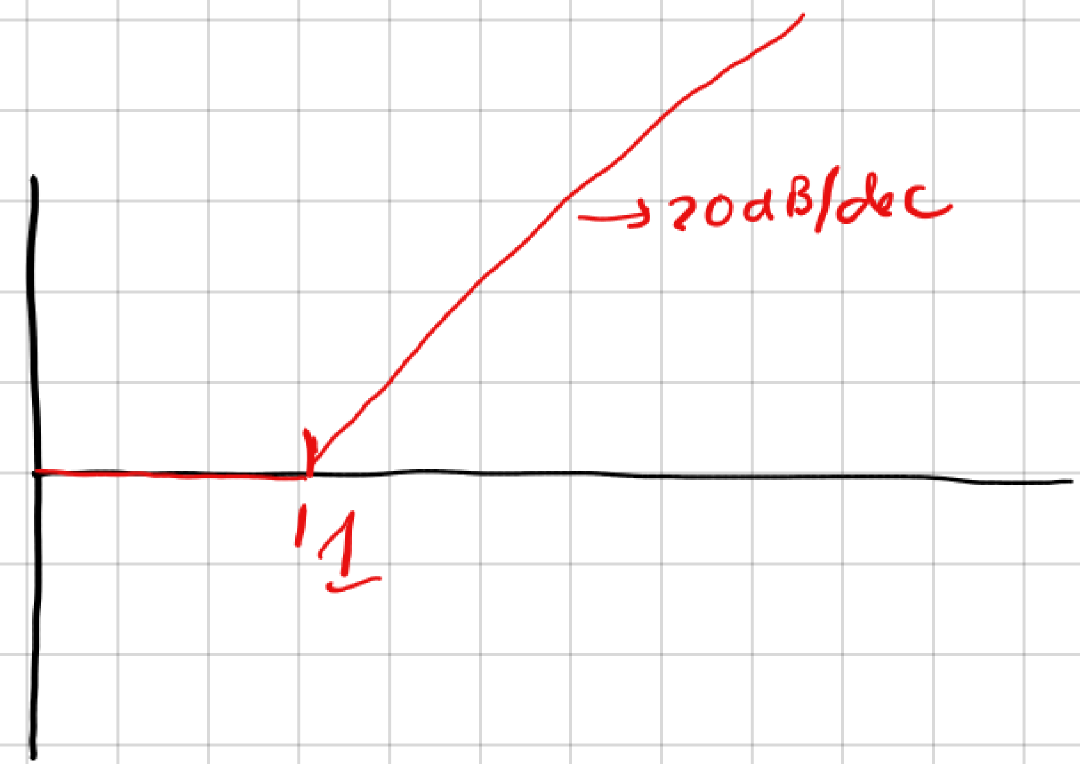
Position error const $\boxed{K_p = \lim_{s \to 0} G(s)}$

$$e_{ss} = \frac{1}{1 + K_p}$$

Specs. for the closed loop.

PID Controller Design using Frequency Response

$$C(s) = \boxed{\underbrace{\frac{(s+A)}{s}}_{PI}\ \underbrace{(s+B)}_{PD}} = \frac{K(as+1)(bs+1)}{s}$$



$$G(s) = \frac{1}{s^2+1}$$

Design a $C(s)$ such that CL sys: has following Charac.

Phase margin of 50°

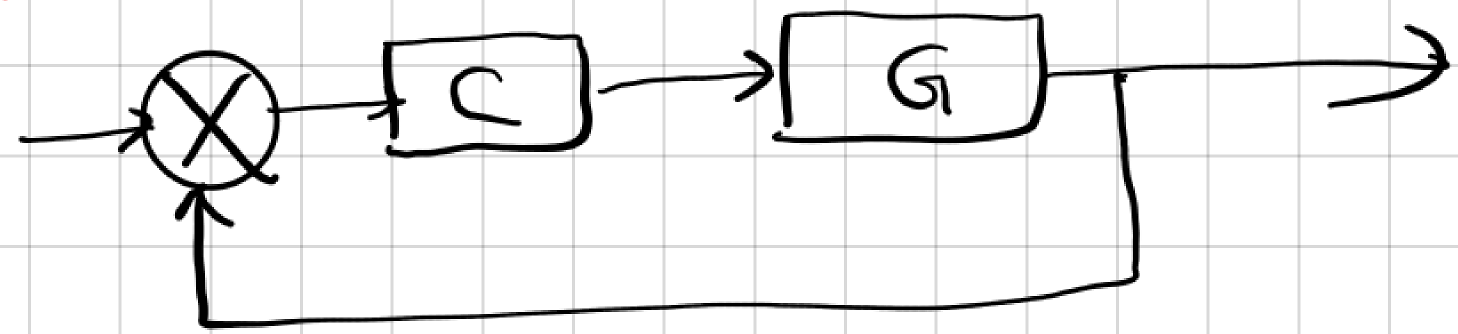
Gain margin of 10dB or more

✓ Velocity error constant 4/sec

$$\lim_{s \to 0} sCG = 4$$

$$= \lim_{s \to 0} s \underbrace{\frac{K(as+1)(bs+1)}{s}}_{C} \cdot \underbrace{\frac{1}{(s^2+1)}}_{G} = 4$$

$$\Rightarrow K = 4$$



$$CG = \frac{(as+1)(bs+1)4}{s(s^2+1)}$$

$$C = \frac{4(5s+1)(0.56s+1)}{s} \qquad G = \frac{1}{(s^2+1)} \rightarrow \text{Plot the unit ramp i/p response}$$

Steps to design

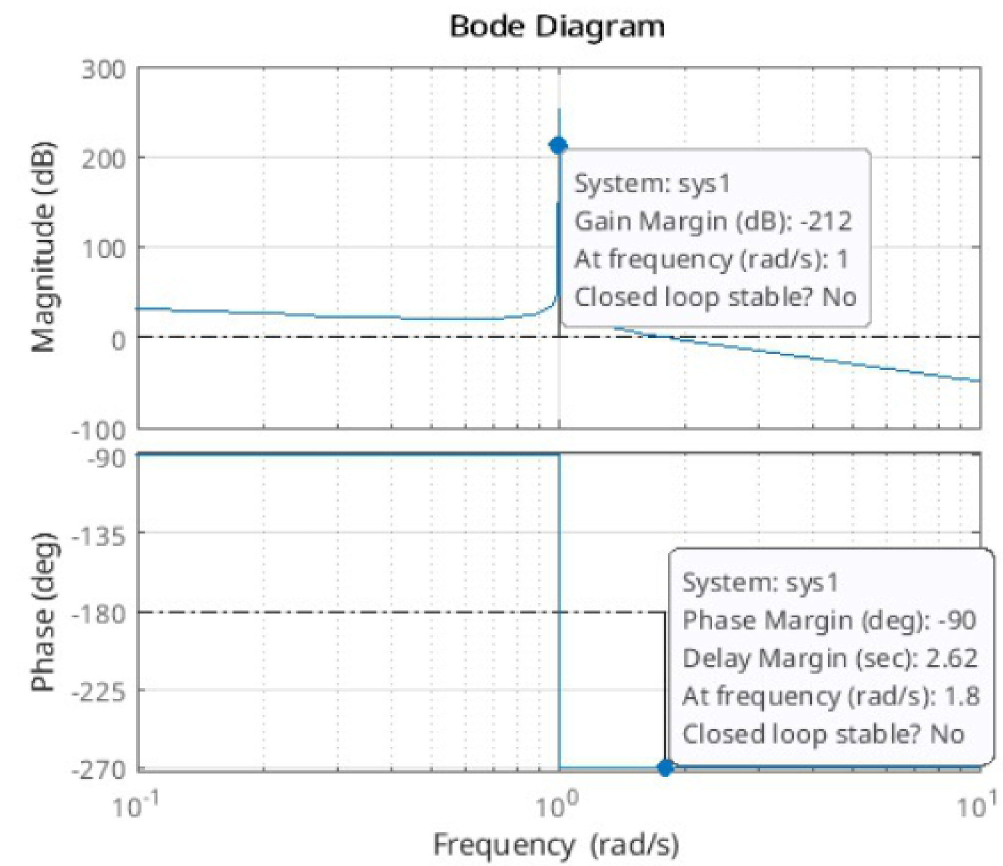
→ Determine the form of PID controller

i.e. $C = \dfrac{K\,(as+1)\,(bs+1)}{s}$ [note that the corner frequencies of the controller are $(1/a)$ & $(1/b)$]

→ Using the e_{ss} (or K_v in this case) requirement, determine K

$$K_v = 4/\text{sec} = \lim_{s \to 0} s \left[\frac{K(as+1)(bs+1)}{s} \right] \left[\frac{1}{s^2+1} \right] \Rightarrow K = 4$$

→ Generate the Bode plot of $\frac{K}{s(s^2+1)}$ to identify current phase & gain margins.



→ Current phase margin = $-90°$ @ 1.8 rad/sec

→ Current gain margin = -212 dB @ 1 rad/sec

→ We can add a max of ~$90° + 90°$ phase with the $(as+1)$ & $(bs+1)$ parts of the controller.

Say, I want to add a large phase gain ~ 90° with $(as+1)$.

$\Rightarrow$ corner freq. needs to be at least a decade below 1.8 rad/sec.

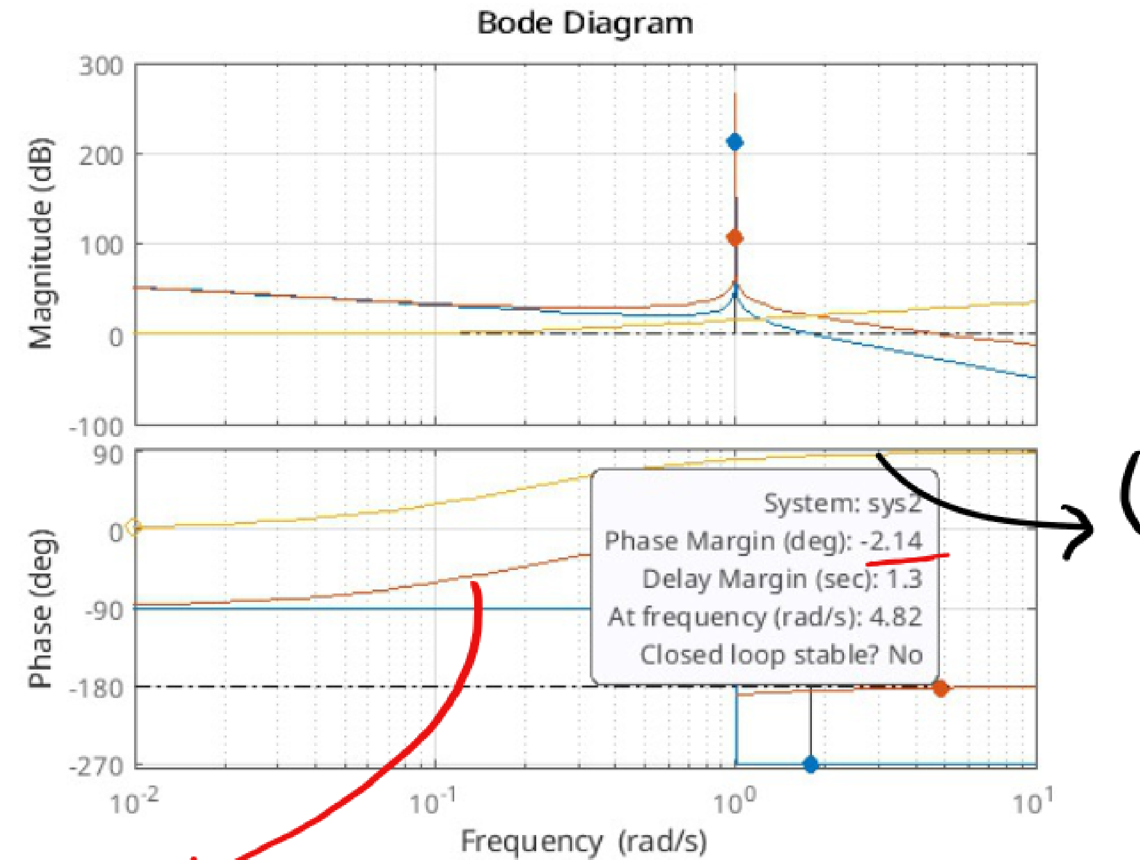
Let's choose 0.18 rad/sec

$\Rightarrow \dfrac{1}{a} = 0.18$ or $a = 5.56$

$C_1 = (as+1)$

New phase margin is -2.14 rad/sec. [an improvement but still does not meet the design req.]

Gain margin is still negative.

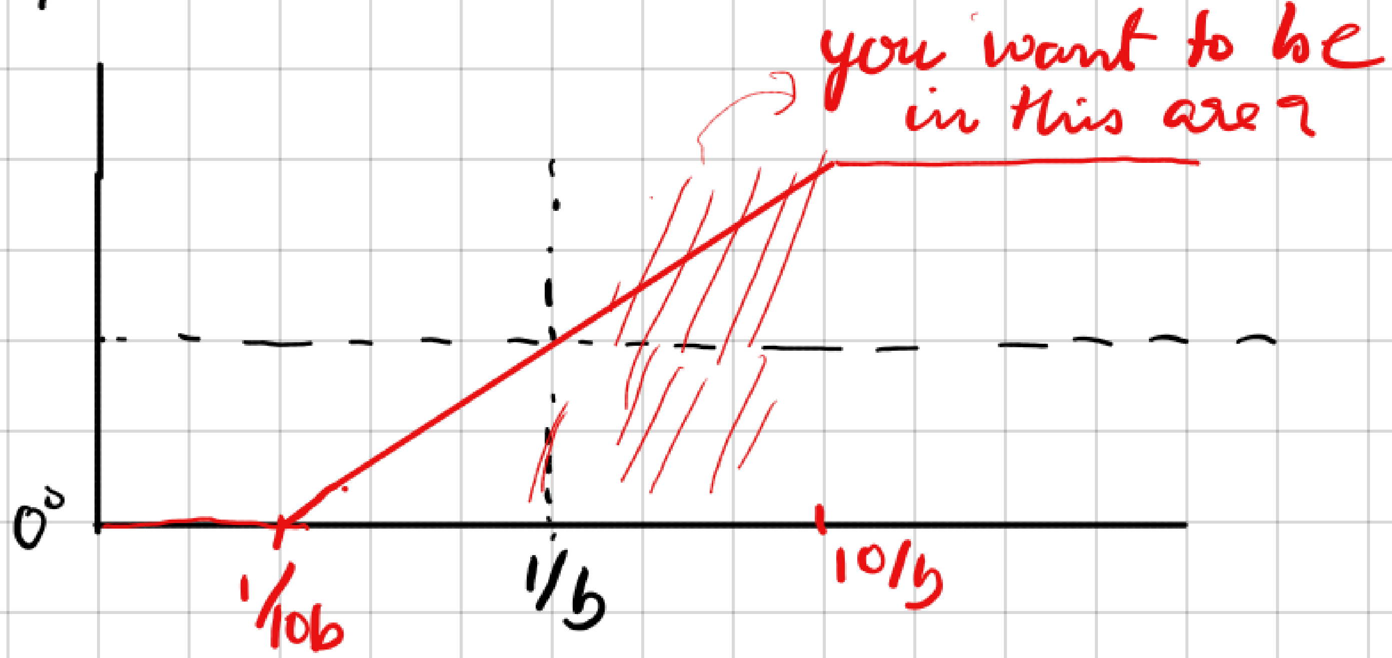


5.56

$(as+1)$

$C_1 \cdot \dfrac{4}{s(s^2+1)}$

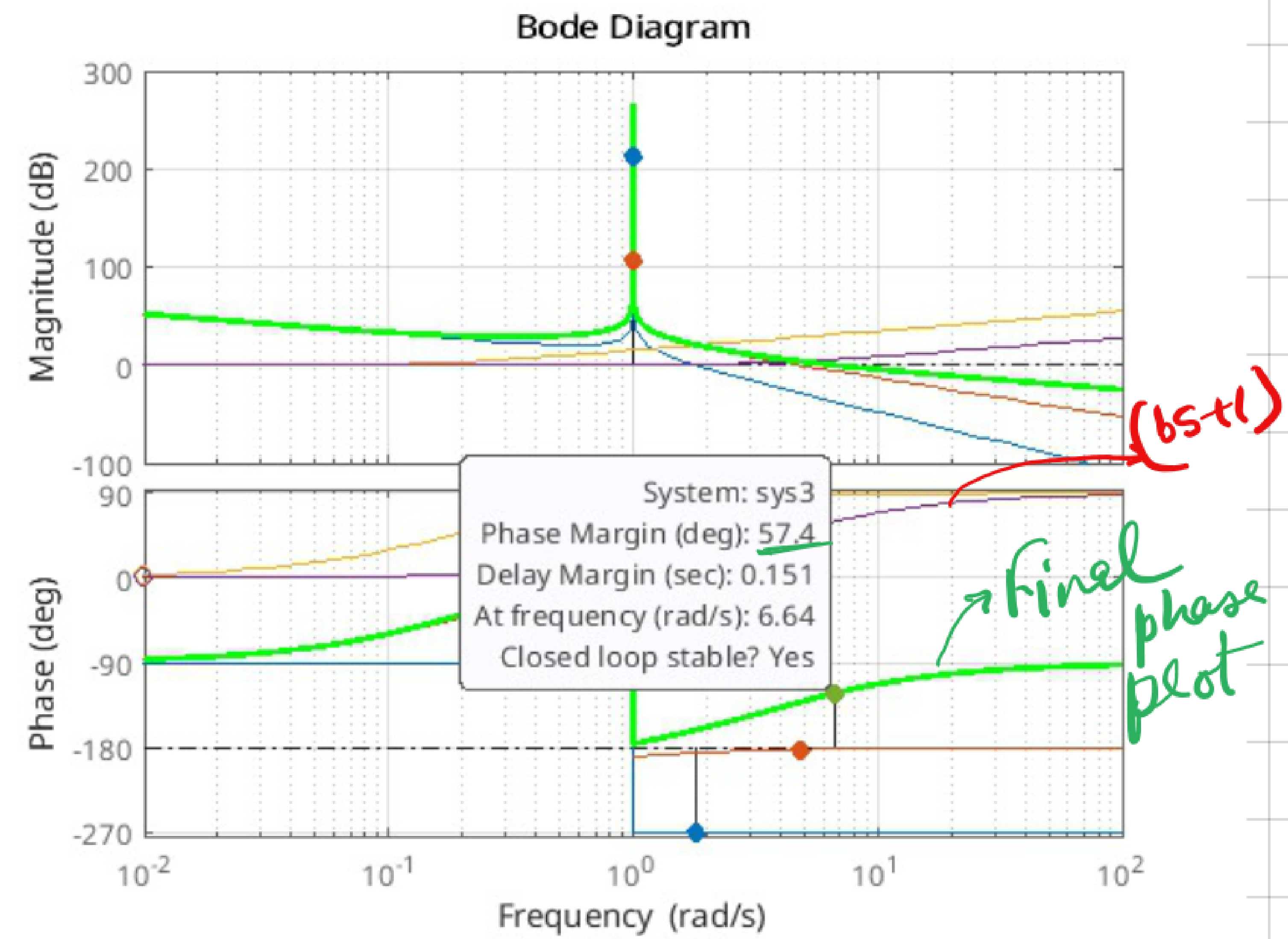
→ Lets add more phase using $(bs+1)$
We need at least ~53 degrees of phase to be added. i.e.



Let us choose a corner freq. of 4 rad/sec.

i.e. $1/b = 4$

$\boxed{b = 0.25}$



The net compensated system has a phase margin of ~57° 👍

& ∞ gain margin 👍👍

You can choose different combinations of corner freq. & design different controllers.

* Technically, you can calculate for the exact values by substituting the values in the gain & phase equations to solve for a & b.

Inherent ambiguity in Bode plots & general freq. domain approach attracts a bounded approach rather than exact values.