Modern Control Theory –

→ Uses State space Representation vs transfer function rep

→ Can represent & compute MIMO & nonlinear systems. vs only LTI SISO systems

→ Computationally efficient vs. Closed form solution efficient.

→ Can be easily extended to discrete time domain. i.e. digital systems vs Requires Z-Transform to account for sampling.

★ Time domain rep. (can also do freq.) vs. complex freq. domain rep.

State Space

What is state?

Smallest set of $\boxed{\text{State variables}}$ to convey full knowledge of sys. at $t = t_1$ if values of s.v. at $t = 0$ is known.

→ Any set of variables that can represent the systems characteristics.

The min # of state variables stays the same.

Completely define the characteristics of the sys.

State Vector :

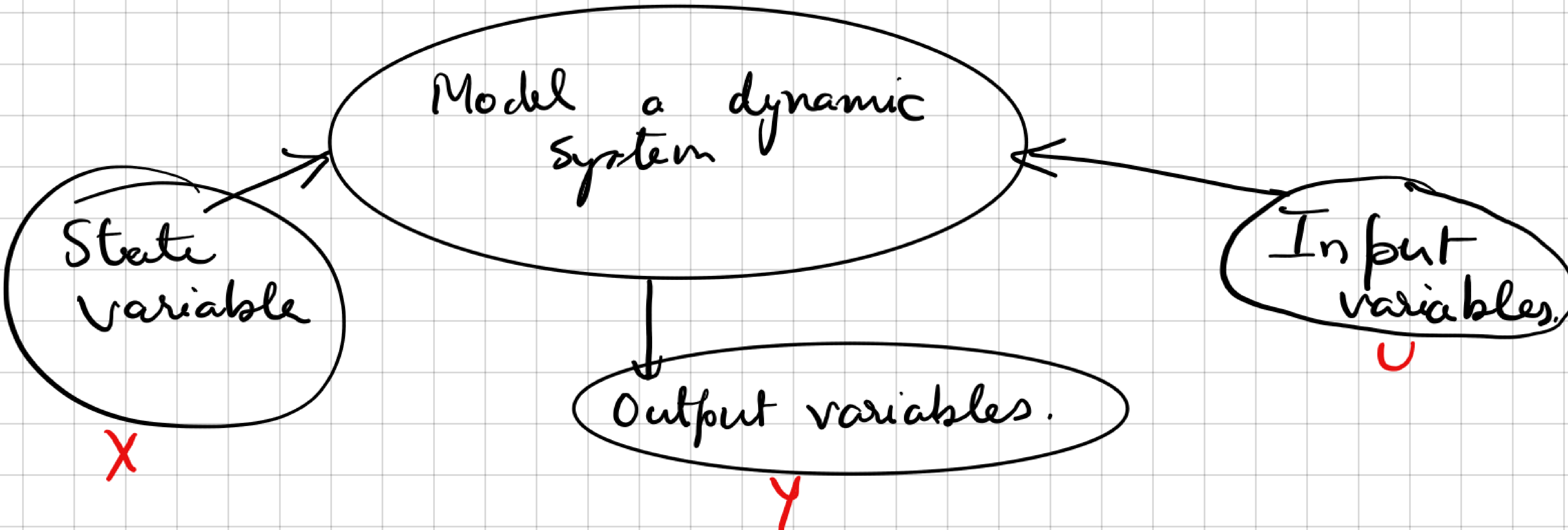
$$\begin{bmatrix} Svar_1 \\ Svar_2 \\ \vdots \\ \vdots \\ Svar_n \end{bmatrix} \rightarrow$$

Minimum.
n state variables are required to define some system

State Space :

n dimensional space whose co-ordinate axes consist of $Svar_1$, $Svar_2$ . . . $Svar_n$ axis defines the state space.

State Space Equations:



X → Matrices of functions

x → Individual variables or parameters.

$\dot{x}_1 = f_1(x_1, x_2, x_3 \ldots x_n, u_1 \ldots u_m)$ If $\dot{x}$ is known & I.C. of x is known at $t=0$

$\dot{x}_2 = f_2($ " $)$

$\vdots$

$\dot{x}_n = f_n($ " $)$

$\dot{x}(t) = f(x, u, t)$

State equation

then at $t = t_1$

$$\boxed{x_{t_1} = x_0 + \underbrace{(t_1 - t_0)}_{\to 0}\dot{x}}$$

$\dot{x}$ $\ddot{x}$ $\dot{x}$ $\dot{x}$

$y_1 = g_1(x_1, x_2 \ldots x_n, u_1 \ldots u_m)$

$y_2 = g_2($ " $)$

$\vdots$

$y_k = g_n($ " $)$

$y(t) = g(x, u, t)$

output equation

Here : We will deal with linear systems using theories of linear algebra.

$$\dot{X}(t) = A(t)\,x(t) + B(t)\,u(t) \rightarrow \text{State eqn.}$$

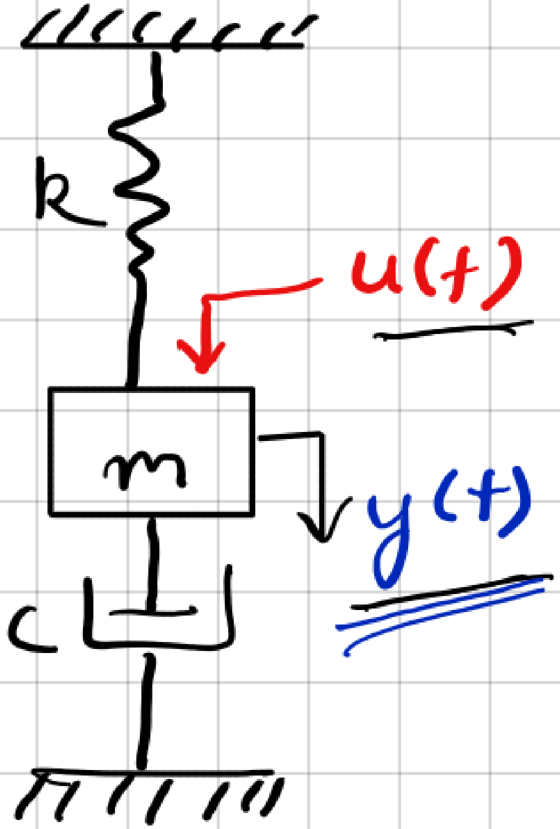
$$Y(t) = C(t)\,x(t) + D(t)\,u(t) \rightarrow \text{Output eqn.}$$

$A(t) \rightarrow$ State Matrix

$B(t) \rightarrow$ Input Matrix

$C(t) \rightarrow$ Output Matrix

$D(t) \rightarrow$ Direct transmission Matrix



$$m\ddot{y} + c\dot{y} + ky = u$$

$$\ddot{y} = \frac{1}{m}\left[-c\underbrace{\dot{y}}_{x_2} - k\underbrace{y}_{x_1} + u\right]$$

Order of differential eqn = # state variables

i.e. 2 state variables

$x_1(t)$

$x_2(t)$

$$\dot{x}_1 = \dot{y} = x_2$$

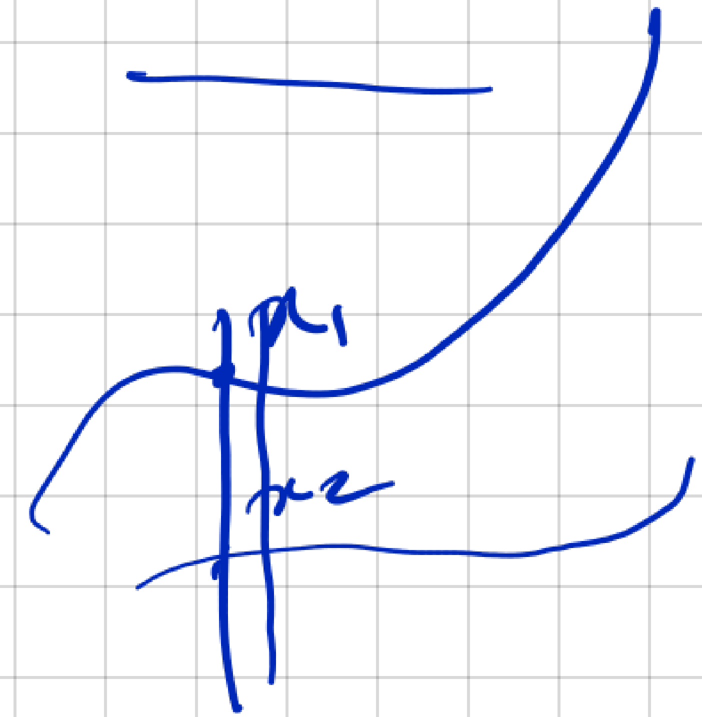
$$\dot{X} = AX + BU$$

$$Y = CX + DU$$

$$\dot{X} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}}_{B} u$$

$$Y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \begin{array}{l} \rightarrow y \\ \rightarrow \dot{y} \end{array} + \begin{bmatrix} 0 \end{bmatrix} u$$

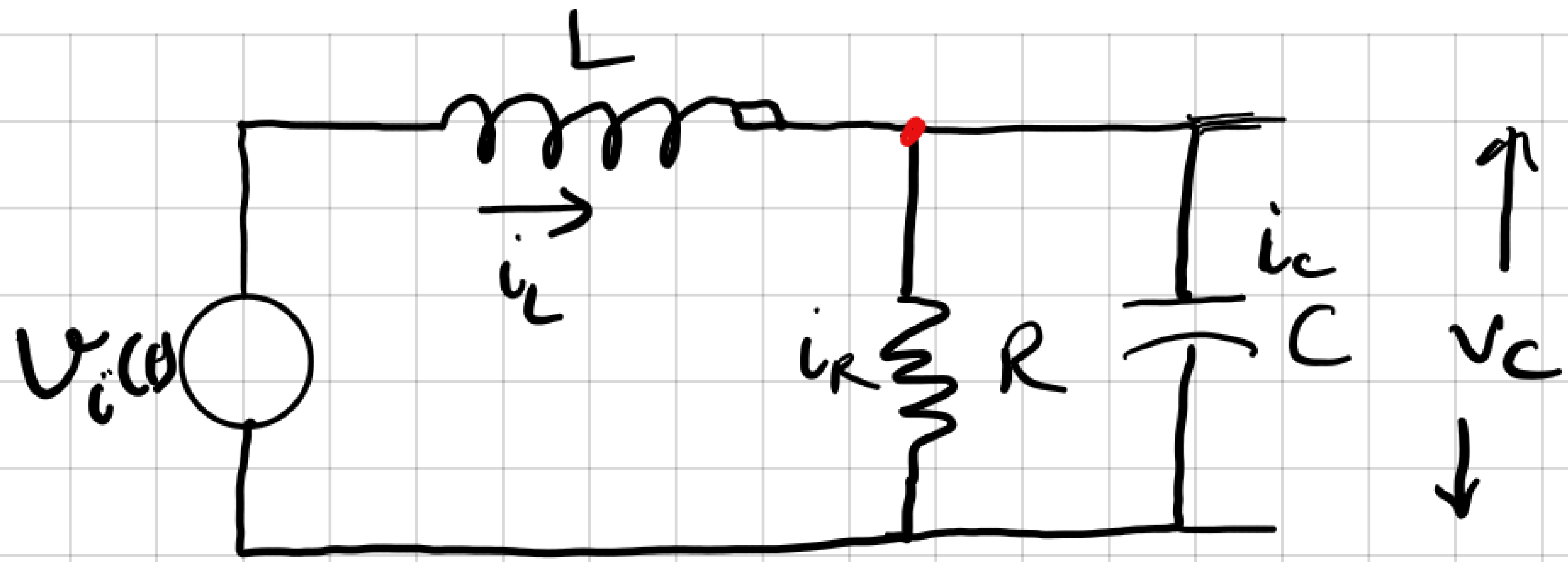
$$\boxed{\begin{gathered} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u \\ y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{gathered}}$$



State space representation of the spring mass damper system.

RLC Circuit

v_i , v_c , R, C , L



$$C\frac{dv_c}{dt} = i_c$$

$$L\frac{di_L}{dt} = v_L$$

$$i_L = i_R + i_c$$

$$i_c = i_L - i_R$$

$$= i_L - \frac{v_c}{R}$$

Chosen state variables

$$\begin{bmatrix} i_L \\ v_c \end{bmatrix}$$

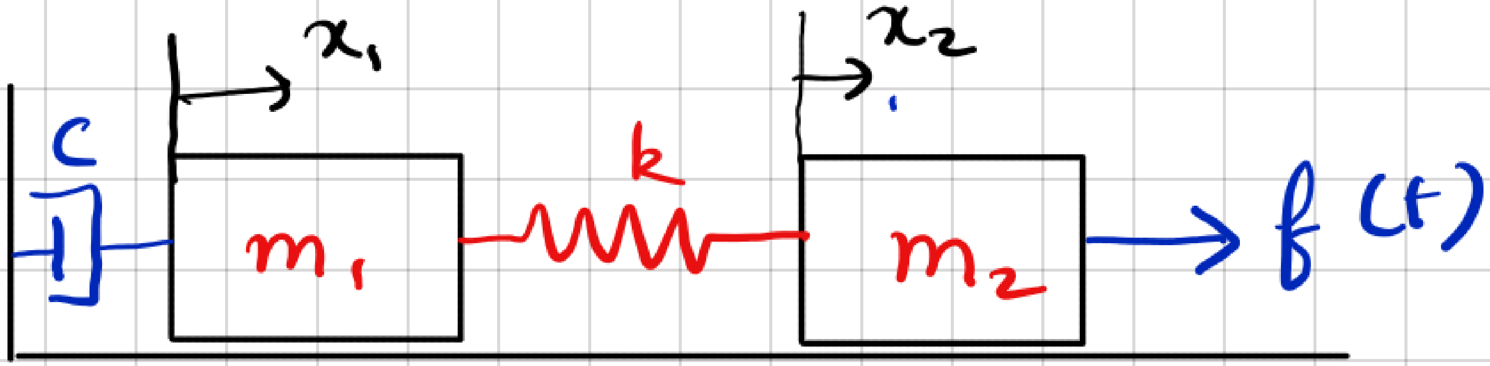
$$\dot{v}_c = \frac{i_c}{C} = \frac{1}{C}\left[i_L - \frac{v_c}{R}\right]$$

$v_L = ?$ $\quad v_i - v_c$

$$\dot{i}_L = \frac{1}{L} v_L = \frac{1}{L}\left[-v_c\right] + \frac{1}{L} v_i$$

$$\begin{bmatrix} \dot{v}_c \\ \dot{i}_L \end{bmatrix} = \begin{bmatrix} -\frac{1}{RC} & \frac{1}{C} \\ -\frac{1}{L} & 0 \end{bmatrix} \begin{bmatrix} v_c \\ i_L \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \quad u = v_i$$

$$y = v_c = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} v_c \\ i_L \end{bmatrix} + 0\, u$$



$(m_1, m_2, c, k) \to A$

$x_1, x_2 \to y$

$f(t) = u$

$\bar{X}$

$$\begin{bmatrix} m_1\ddot{x}_1 + c\dot{x}_1 + k(x_1 - x_2) = 0 \\ m_2\ddot{x}_2 + k(x_2 - x_1) = f(t) \end{bmatrix}$$

$A_2 - \left(-\frac{c}{m_1}A_1\right)$

$\lambda(A) = 0$

$\bar{X}$

$x_1 - (x_1 - x_2)$

$t=0 \quad = x_2$

$$\left.\begin{matrix} x_1 \\ x_2 \\ \dot{x}_1 \\ \dot{x}_2 \end{matrix}\right\} \text{State}$$

$$\begin{bmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k}{m_1} & -\frac{c}{m_1} & \frac{k}{m_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k}{m_2} & 0 & -\frac{k}{m_2} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/m_2 \end{bmatrix} f$$

$x_1, \dot{x}_2, \dot{x}_1$ $\qquad (x_2 - x_1)$

$\dot{x}_1 = \dot{x}_1$

$\dot{x}_2 =$

$\ddot{x}_1 = -\frac{c}{m_1}\dot{x}_1 - \frac{k}{m_2}x_1 + \frac{k}{m_2}x_2$

LINEAR SYSTEMS

$\dot{x} = Ax \rightarrow$ state vector $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$ $x \in \mathbb{R}^n$

$\bar{A}\bar{Z} = \lambda \bar{Z}$ → number (λ), → vector ($\bar{Z}$)

$\lambda \rightarrow$ eigen value

$z \rightarrow$ eigen vector

$(A - \lambda I)Z = 0$ ✓ characteristic equation

$|A - \lambda I| = 0 \rightarrow$ Relevant set of eigen values.

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$$

Find eigen values.

$$|A - \lambda I| = 0$$

$$\left| \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right|$$

$$= \begin{vmatrix} 2-\lambda & 1 \\ 3 & 2-\lambda \end{vmatrix}$$ solve for λ

$$(2-\lambda)^2 - 3 = 0$$

$$\lambda = \frac{2 + \sqrt{3}}{2 - \sqrt{3}}$$

$$\lambda_1 \bar{Z}_1 = \bar{\bar{A}} \bar{Z}_1$$

$$\lambda_2 \bar{Z}_2 = \bar{\bar{A}} \bar{Z}_2$$

$$\bar{\bar{T}} = \underbrace{\begin{bmatrix} \vdots & \vdots & & \vdots \\ z_1 & z_2 & \cdots & z_N \\ \vdots & \vdots & & \vdots \end{bmatrix}}_{\text{columns are eigen vectors}}$$

$$D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & 0 \\ & 0 & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

$$AT = TD$$

Diagonal matrix with eigen values as diagonals.

$$\boxed{T^{-1} A T = D}$$

$$\begin{bmatrix} \vdots & & \\ z_1 & \cdots & \\ \vdots & & \end{bmatrix} \begin{bmatrix} z_1 \\ \\ \end{bmatrix}$$

$$x = Tz$$

$$\dot{x} = Ax$$

$$\dot{x} = T\dot{z} = ATz$$

$$\dot{z} = T^{-1}ATz$$

$$\boxed{\dot{z} = Dz}$$

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \vdots \\ \dot{z}_n \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ 0 & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$$

$$\dot{z}_1 = \lambda_1 z_1$$

$$\dot{z}_2 = \lambda_2 z_2$$

$$\vdots$$

$$\dot{z}_n = \lambda_n z_n$$

$\Rightarrow$ Dynamics along the eigen vectors are decoupled.

$$z_1 = c_1 e^{\lambda_1 t}$$

$$z_2$$

$$\vdots$$

$$z_n$$

Generalized sol^n of sys. of Linear Diff. Eqn: $\dot{x} = Ax$

$x(t) = e^{At} x(0)$ ↳ Initial Value.

$$e^{At} = I + At + \frac{A^2 t^2}{2!} + \frac{A^3 t^3}{3!} + \cdots$$

$AT = TD$

$A = TDT^{-1}$

$$e^{TDT^{-1}t} = TT^{-1} + TDT^{-1}t + \underbrace{(TD\overbrace{T^{-1}T}^{I}DT^{-1})}_{TD^2T^{-1}}\frac{t^2}{2!} + TD^3T^{-1}\frac{t^3}{3!} + \cdots$$

$A^n = TD^nT^{-1}$

$$X = T\left[I + Dt + D^2\frac{t^2}{2!} + D^3\frac{t^3}{3!} + \cdots\cdots\right]T^{-1}$$

$$X = T\, e^{Dt}\, T^{-1}$$

mapped to eigenvectors.

Desired State vector

$$\begin{bmatrix} e^{\lambda_1 t}c_1 & & & \\ & c_2 e^{\lambda_2 t} & & \\ & & \ddots & \\ & & & c_n e^{\lambda_n t} \end{bmatrix}$$

Transfer F_{c^n}

$m\ddot{x} + c\dot{x} + kx = u$

$$\frac{X(s)}{U(s)} = \frac{1}{\boxed{ms^2 + cs + k}}$$

Roots of this eqn.

$\rightarrow$ poles of the sys.

State space

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix}$$

$$\begin{vmatrix} -\lambda & 1 \\ -\frac{k}{m} & \left(-\frac{c}{m} - \lambda\right) \end{vmatrix}$$

$$= \lambda\left(\frac{c}{m} + \lambda\right) + \frac{k}{m}$$

$$= \lambda^2 + \frac{c}{m}\lambda + \frac{k}{m}$$

$\lambda =$ eigen values

Pole-zero cancellation → Might miss out some dynamics.

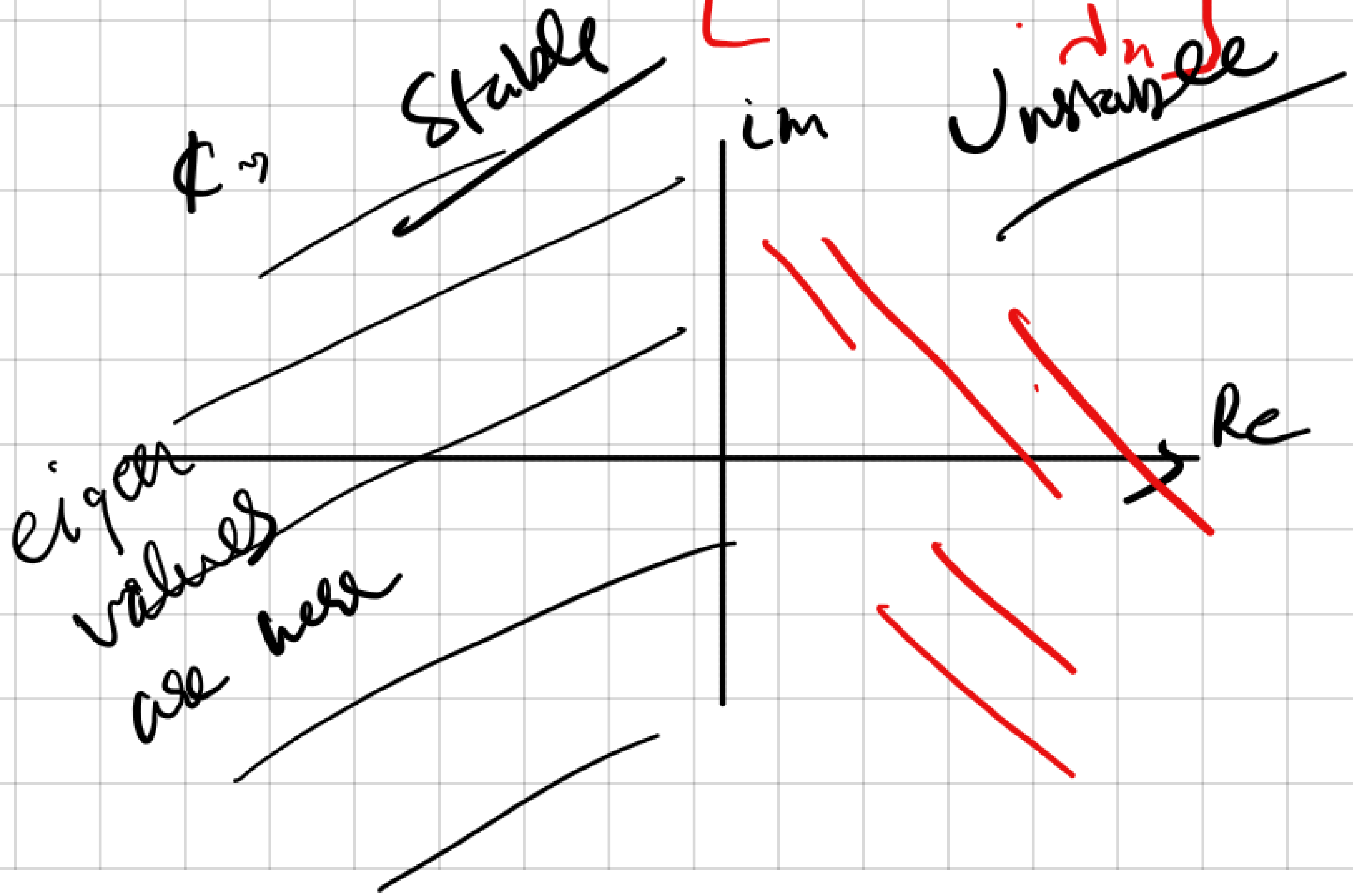
Some information is lost about INTERNAL STATES (values you could measure) in a T.F.

$\boxed{\dot{x} = Ax} + Bu$ → Actuator

STABILITY

$\dot{x} = Ax \qquad x \in \mathbb{R}^n$

$AT = TD \qquad D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$



$x = e^{At} x(0)$

$= T e^{Dt} T^{-1} x(0)$

$e^{Dt} = \begin{bmatrix} e^{\lambda_1 t} & & & 0 \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ 0 & & & e^{\lambda_n t} \end{bmatrix}$

if $Re(\lambda) < 0$, system is asymptotically stable.

Discrete time domain

$$x_{k+1} = \tilde{A} x_k \longrightarrow \neq \quad \dot{x} = A x$$

$x_k = x(k\Delta t)$ where Δt is sampling time & k is the sampled instance

Start at x_0

$$x_1 = \tilde{A} x_0$$

$$x_2 = \tilde{A} \cdot x_1 = \tilde{A}^2 x_0$$

$$x_3 = \tilde{A}^3 x_0$$

$$\vdots$$

$$x_m = \tilde{A}^m x_0$$

Let eigen values of $\tilde{A}$
$\lambda_1, \lambda_2 \ldots \lambda_n$

$$\tilde{A} = \tilde{T} \tilde{D} \tilde{T}^{-1} \quad \text{(Recall)}$$

$$x_m = \tilde{T} \tilde{D}^m \tilde{T}^{-1} x_0$$

$$\tilde{D}^m = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}^m$$

$$= \begin{bmatrix} \lambda_1^n & & & \\ & \lambda_2^n & & \\ & & \ddots & \\ & & & \lambda_n^m \end{bmatrix}$$

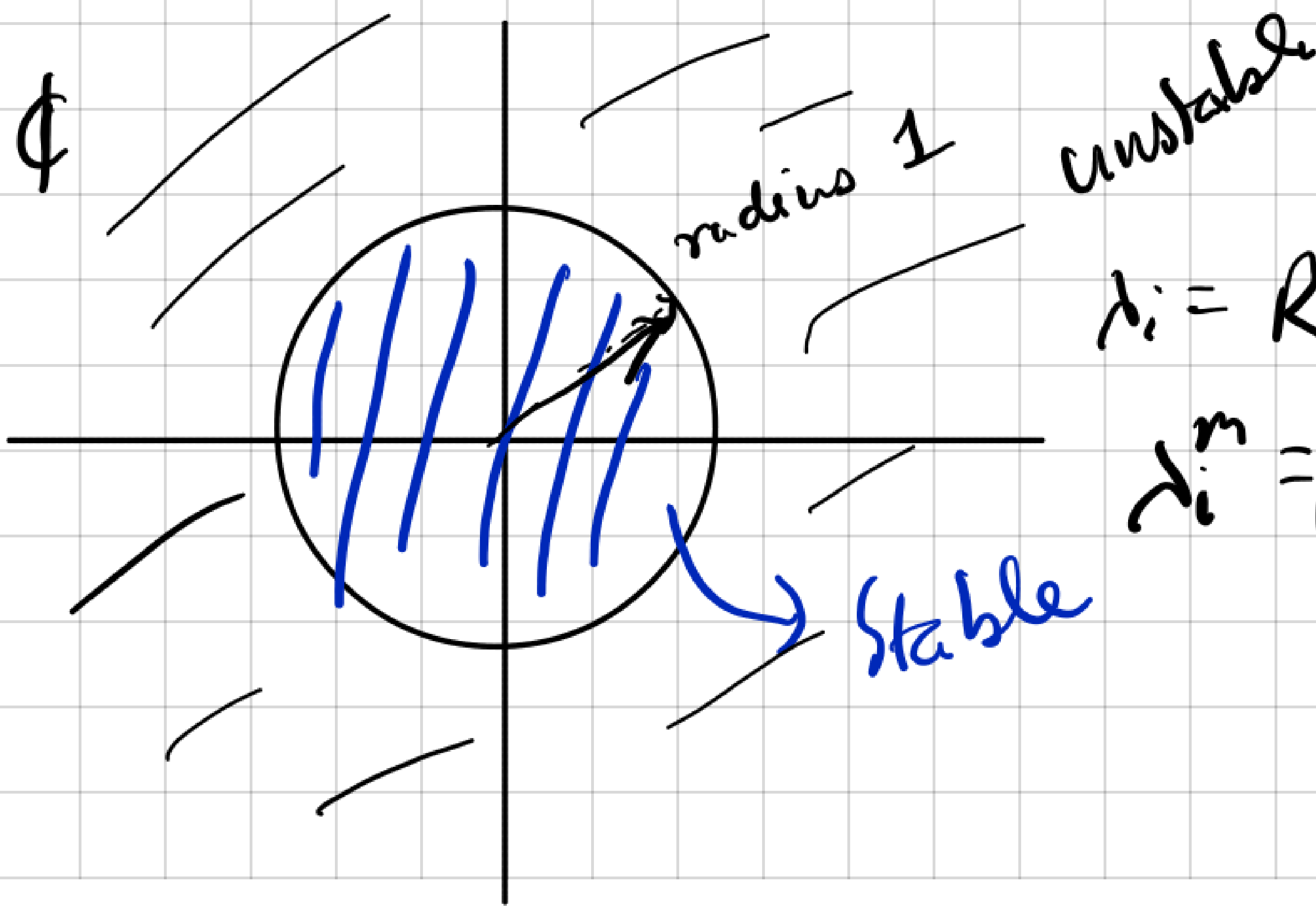
if you track any λ_i through time,

then at $t = t_m$

$$x_m = \tilde{A}^m x_0 = \tilde{T} D^m \tilde{T}^{-1} x_0$$

λ_i^m

If $\lambda_i > 1$, system response will blow up.

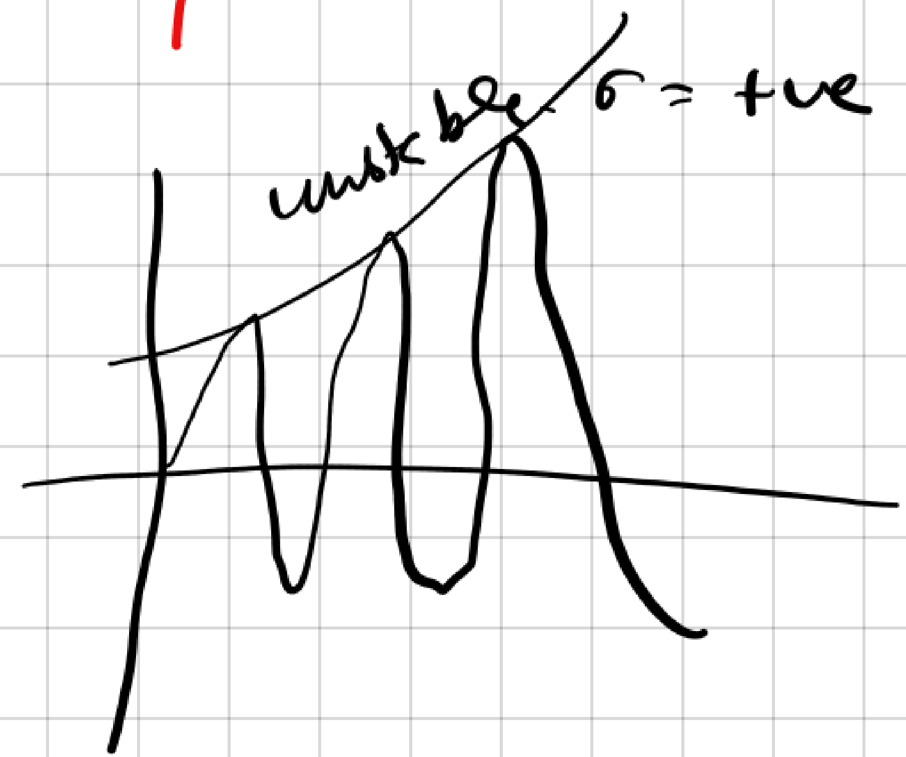


$\lambda_i = R e^{i\theta}$

$\lambda_i^m = R^m e^{im\theta}$

in cont. time domain

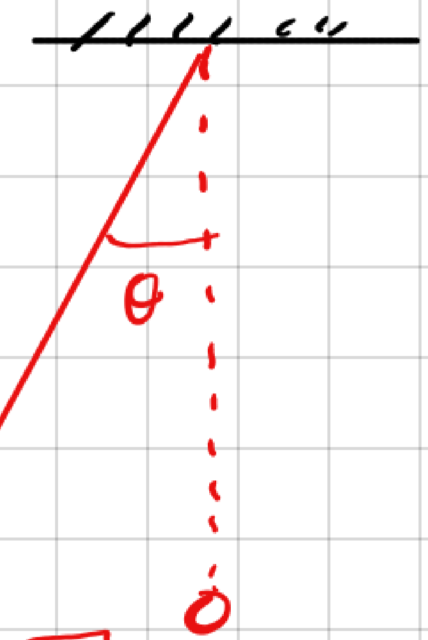
$e^{\sigma t}[\cos(\omega t + \phi)]$



Linearization

$$\dot{x} = f(x) \Rightarrow \dot{x} = Ax \qquad x \in \mathbb{R}^n$$

$$\dot{x}_1 = f_1(x_1)$$
$$\dot{x}_2 = f_2(x_2)$$
$$\vdots$$
$$\dot{x}_n = f_n(x_n)$$



fixed point $\bar{x}$ $\qquad f(\bar{x}) = 0$

Evaluate Jacobian of the sys. at $\bar{x}$ $\rightarrow$

$$J = \frac{Df}{Dx} = \left.\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \vdots \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}\right|_{\bar{x}}$$

Please study this from any engineering textbook

Example:

$$\dot{x}_1 = f_1(x_1, x_2) = x_1^2 + x_2^3$$

$$\dot{x}_2 = f_2(x_1, x_2) = x_1^2 x_2$$

find fixed points of f_1 & f_2

$$J = \begin{bmatrix} 2x_1 & 3x_2^2 \\ 2x_1 x_2 & x_1^2 \end{bmatrix}_{\bar{x}}$$

$$\bar{x} = \begin{bmatrix} a \\ b \end{bmatrix} \text{(say)}$$

$$J = \begin{bmatrix} 2a & 3b^2 \\ 2ab & a^2 \end{bmatrix}$$

$$\dot{x} = f(x) \rightarrow \text{about } \bar{X} \qquad \text{Taylor series.}$$

$$= f(\bar{x}) + \overbrace{\frac{Df}{Dx}\Big|_{\bar{X}}}^{J} (x-\bar{x})$$

$$+ \frac{D^2 f}{Dx^2}\Big|_{\bar{X}} (x-\bar{x})^2 \ldots\ldots$$

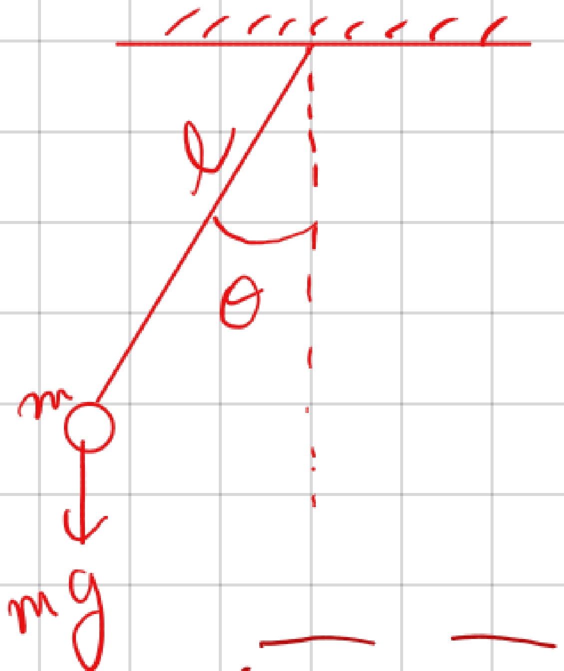
$$\boxed{(x-\bar{x}) \rightarrow 0}$$

higher order terms can be ignored.

$$\dot{x} = \overset{\nearrow 0}{f(\bar{x})} + J\,(x-\bar{x})$$

$$\boxed{\Delta\dot{x} = J\,\Delta x + \epsilon} \rightarrow \text{linear system.}$$

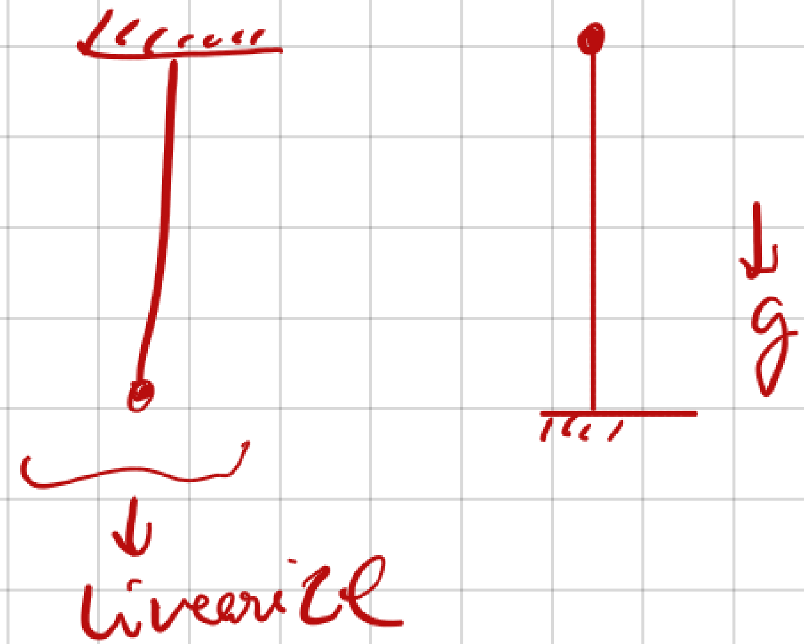
Pendulum :



$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$$

Fixed pt. $\bar{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \pi \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} 2\pi \\ 0 \end{bmatrix}$

$$\ddot{\theta} = -\frac{g}{l}\sin\theta - \underbrace{\delta\dot{\theta}}_{\text{small damping}}$$



Fixed Points = 0, π ......

$x_1 = \theta$

$x_2 = \dot{\theta}$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{g}{l}\sin(x_1) - \delta x_2 \end{bmatrix} \begin{matrix} \to f_1 \\ \to f_2 \end{matrix}$$

$$J = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l}\cos(x_1) & -\delta \end{bmatrix}$$

$$\bar{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$J = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & -\delta \end{bmatrix}$$

Say $l = 9.81\,m$

$\delta = 0.01$

$$J = \begin{bmatrix} 0 & 1 \\ -1 & -\delta \end{bmatrix}$$

eig $\to$ $-0.005 \pm i$

$$\bar{x} = \begin{bmatrix} \pi \\ 0 \end{bmatrix}$$

$$J = \begin{bmatrix} 0 & 1 \\ +1 & -\delta \end{bmatrix}$$

eig $\to$ -1.005
$+0.995$

State Space to Transfer Function

$\dot{x} = Ax + Bu$
$y = Cx + Du$

$\mathcal{L}$ on both eqns.

$sX(s) = AX(s) + BU(s)$ — 1
$Y(s) = CX(s) + DU(s)$ — 2

$\frac{Y(s)}{U(s)} = ?$

1 $\rightarrow$ $(sI - A)X(s) = BU(s)$

$X(s) = (sI-A)^{-1}BU(s)$

Subs. in 2

$Y(s) = [C(sI-A)^{-1}B + D]U(s)$

$$\boxed{G = \frac{Y}{U} = [C(sI-A)^{-1}B + D]}$$

$= \frac{Q}{|sI - Q|}$ → some polynomial
→ Characteristic eqn.

TF → S.S.

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}}{dt^{n-1}}y + \cdots\cdots + a_1\frac{dy}{dt} + a_0 y = b_0 u$$

$x_1 = y$

$x_2 = \dot{y}$

$x_3 = \ddot{y}$

$\vdots$

$x_{n-1} = \overset{n-1}{y}$

$$\begin{array}{c} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \\ \end{array} \begin{bmatrix} \dot{y} \\ \ddot{y} \\ \dddot{y} \\ \vdots \\ \overset{n}{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & \cdots & 0 \\ & & \vdots & & & \\ -a_0 & -a_1 & -a_2 & \cdots & & -a_{n-1} \end{bmatrix} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \end{bmatrix}}_{X} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ b_0 \end{bmatrix} u$$

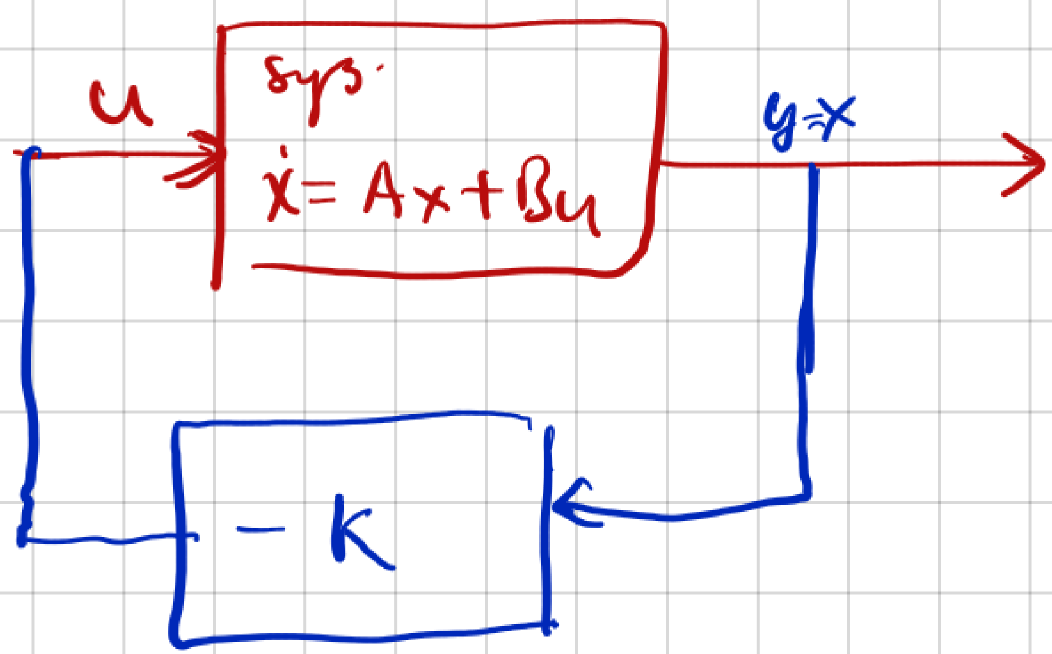
$$y = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} X$$

$$\underbrace{\dot{x}}_{\text{sys.}} = Ax + Bu$$

→ Matrix rep. actuator dynamics.

u → control i/p

$y = Cx$ → C → Identity matrix (say)



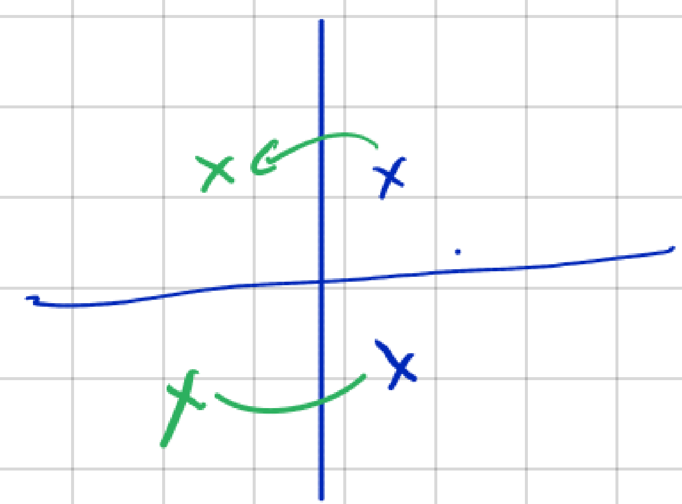
$u = -Kx$

$$\dot{x} = Ax + B(-Kx) = \underbrace{(A - BK)}_{\text{New system}}\, x$$

If the sys is

CONTROLLABLE

you can compute a K-matrix such that the eigen values are wherever you want them.



Which of the following systems are full-state controllable?

a) $$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 2 \end{bmatrix}}_{B} u$$

$$\dot{x}_1 = f_1(x_1) \leftarrow$$
$$\dot{x}_2 = f_2(x_2, u)$$

b) $$\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \rightarrow$$

c.) $$+ \begin{bmatrix} 1 \\ 2 \end{bmatrix} u = ✓$$

d) $$\begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u$$

CONTROLLABILITY MATRIX

$$\underline{\underline{R}} = \underbrace{\left[B \mid AB \mid A^2B \mid A^3B \mid \ldots\ldots \mid A^{n-1}B \right]}$$

If R has full column rank i.e. Rank $= n$

Sys. is $\underbrace{\text{fully state controllable}}$

if at t_0, $x(0)$ then it is possible to move the state

$x(0) \longrightarrow x_1(t_1)$ by using a suitable i/p.

@ $t = t_1$

$$x_{k+1} = Ax_k + Bu_k$$

Impulse i/p in discrete domain

$u_0 = 1 \qquad x_0 = 0$

$u_1 = 0 \qquad x_1 = B \rightarrow$

$u_2 = 0 \qquad x_2 = AB \rightarrow$

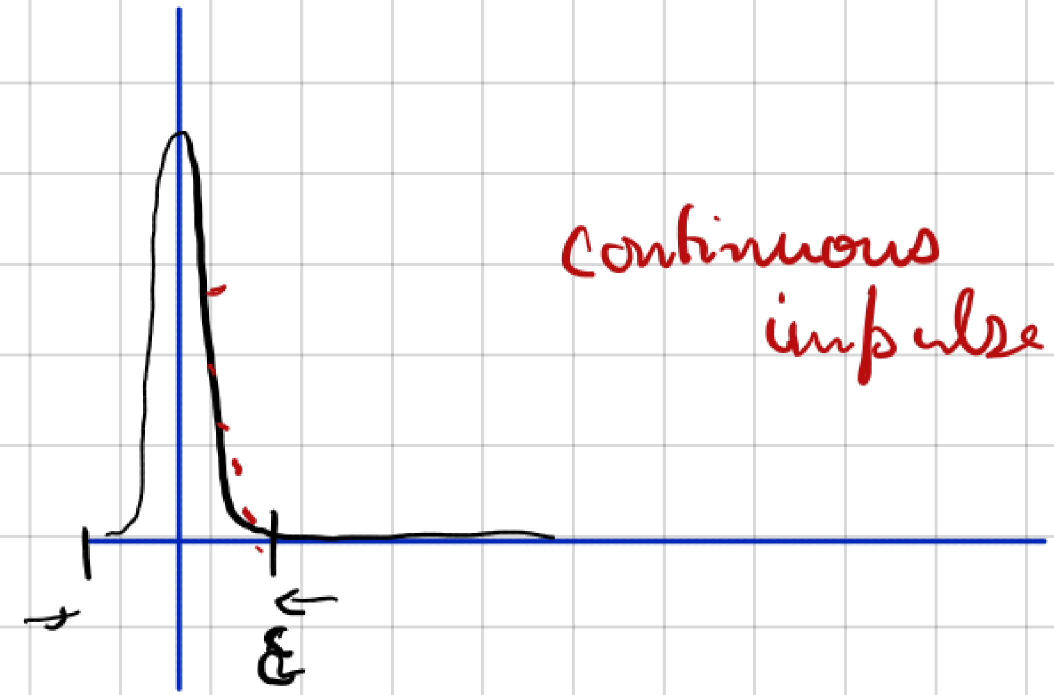
$u_3 = 0 \qquad x_3 = A^2B$

$\vdots \qquad \vdots$

$u_m = 0 \qquad x_m = A^{m-1}B$

Should be linearly independent for all states to be controllable.

say $\begin{bmatrix} 0 & 2 & 2 & 2 & \cdots & 2 \\ 1, & 3, & 3, & 3 & \cdots & 3 \end{bmatrix}$



continuous impulse

discrete impulse

Output Controllability

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

for the general case

matrix

$$R_0 = \left[\underset{m\times q}{CB} \;\middle|\; \underset{m\times q}{CAB} \;\middle|\; CA^2B \;\middle|\; CA^3B \;\middle|\; \ldots \ldots \;\middle|\; CA^{n-1}B \;\middle|\; D\right]$$

$A = n \times n$

$B = n \times q \rightarrow q$ i/ps

$C = \boxed{m} \times n$

$D = m \times q$

$x = n \times n$

$y = m \times 1$

$u = q \times 1$

System is completely o/p controllable if for a given y_0 @ t_0 o/p $y_n(t_n)$ @ $t = t_n$ can be achieved by selecting a suitable U i.e. control i/p

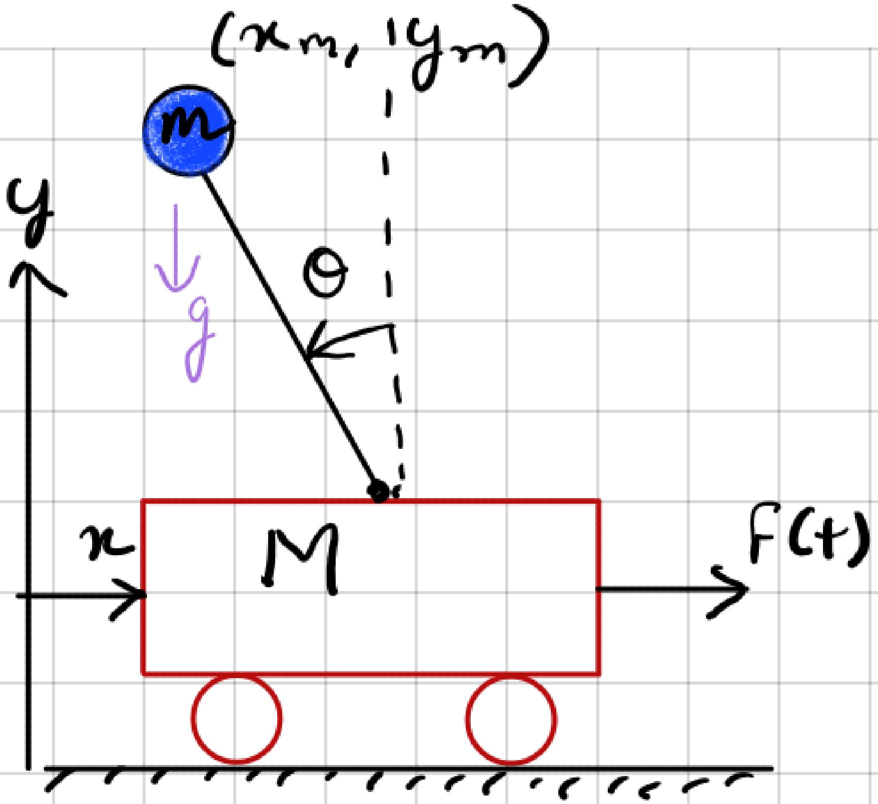
$\boxed{\text{Rank} = m}$

These rows are linearly independent

Stabilizability → if uncontrollable states of a sys are stable (i.e. (−)ve λ)

the sys is still stabilizable

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$



Kinematics

$$x_m = X - l\sin\theta \quad (1)$$

$$y_m = l\cos\theta \quad (2)$$

$$\dot{x}_m = \dot{x} - l\dot{\theta}\cos\theta \quad (3)$$

$$\dot{y}_m = -l\dot{\theta}\sin\theta \quad (4)$$

We'll use Lagrangian Method to write the Equations of Motion (EOM)

Need Kinetic & Potential energy for that.

P.E. $V = mg \cdot l\cos\theta \quad (5)$

KE. $T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left(\dot{x}_m^2 + \dot{y}_m^2\right)$

$$= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left(\dot{x}^2 - 2l\dot{\theta}\dot{x}\cos\theta + l^2\dot{\theta}^2\right)$$

$$\therefore T = \frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2}ml^2\dot{\theta}^2 - ml\dot{\theta}\dot{x}\cos\theta \quad (6)$$

GREAT!! We have everything for the $\boxed{\text{Lagrange's Eqn}}$?

Lagrange's Equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i \qquad (7)$$

where Lagrangian L is

$$L = T - V \qquad (8)$$

Subs. eqn. (5) & (6) in (8)

$$L = \frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2} m l^2 \dot{\theta}^2 - m l \dot{\theta} \dot{x} \cos\theta - mgl\cos\theta \qquad (9)$$

Equations of Motion

Along $\underline{\underline{x}}$ $\quad (M+m)\ddot{x} - [ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta] = F(t)$

EOM 1 $\quad \boxed{(M+m)\ddot{x} - ml\ddot{\theta}\cos\theta + ml\dot{\theta}^2\sin\theta = F(t)}$ $\quad$ (10)

Along θ $\quad ml^2\ddot{\theta} - ml\ddot{x}\cos\theta + \cancel{ml\dot{x}\dot{\theta}\sin\theta} - \cancel{ml\dot{\theta}\dot{x}\sin\theta}$

$- mgl\sin\theta = 0$

divide by ml

EOM-2 $\quad \boxed{l\ddot{\theta} - \ddot{x}\cos\theta - g\sin\theta = 0}$ $\quad$ (11)

States

$x \to x_1$

$\dot{x} \to x_2$

$\theta \to x_3$

$\dot{\theta} \to x_4$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = f_{u_1}(\underline{u(t)}, x_3, x_4)$$

$$\dot{x}_3 = x_4$$

$$\dot{x}_4 = f_{u_2}(u(t), x_3, x_4)$$

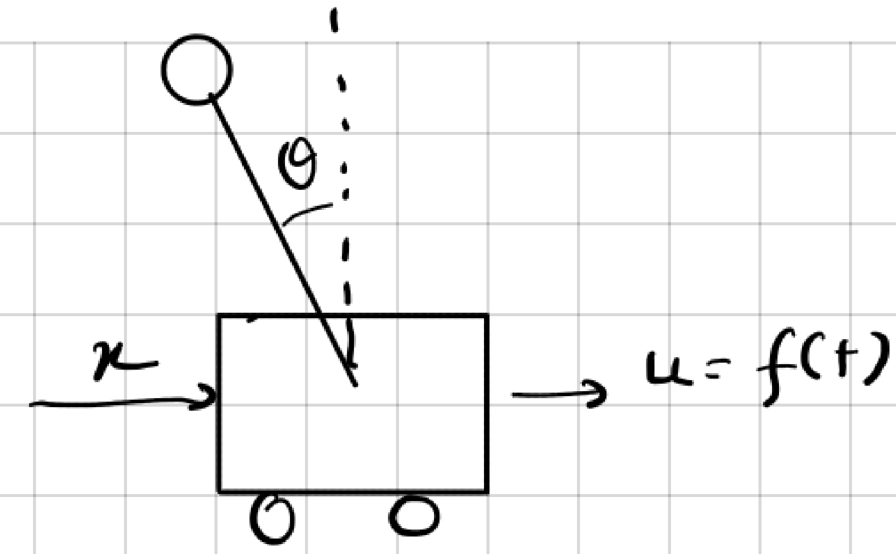
Control I/p $u = -Kx$

Linear Quadratic $\boxed{\text{Regulator}}$

LQR

$\dot{x} = (A - BK)x$

$K \rightarrow$ eigen values of this sys.



$$J = \int_0^{T\to\infty} x^T Q x + u^T R \underset{\cdot 01}{u}$$

$x \in \mathbb{R}^n$

$Q \rightarrow n \times n$

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A^*P + PA - PBR^{-1}B^*P + Q = 0$$ → Matrix Riccati eqn.

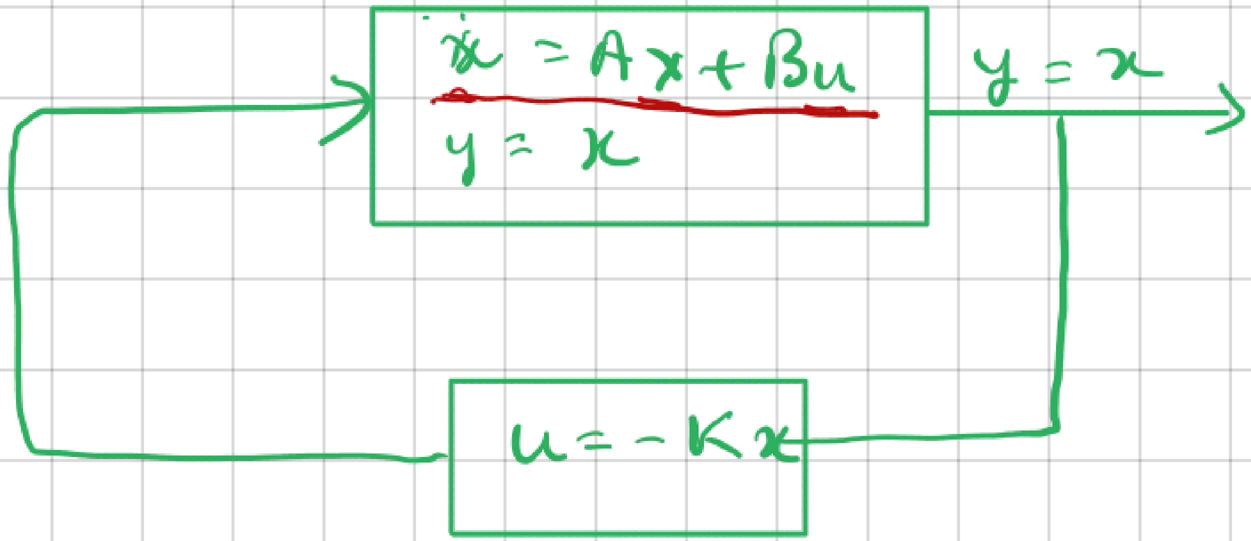
Solve for P & if P is +ve definite

$$(A - BK)^*P + P(A - BK) = -(Q + K^*RK)$$

Solve for K → optimal solution.

$\dot{x} = Ax + Bu \qquad A \in \mathbb{R}^{n \times n}$

$y = x \qquad B \in \mathbb{R}^{n \times q}$



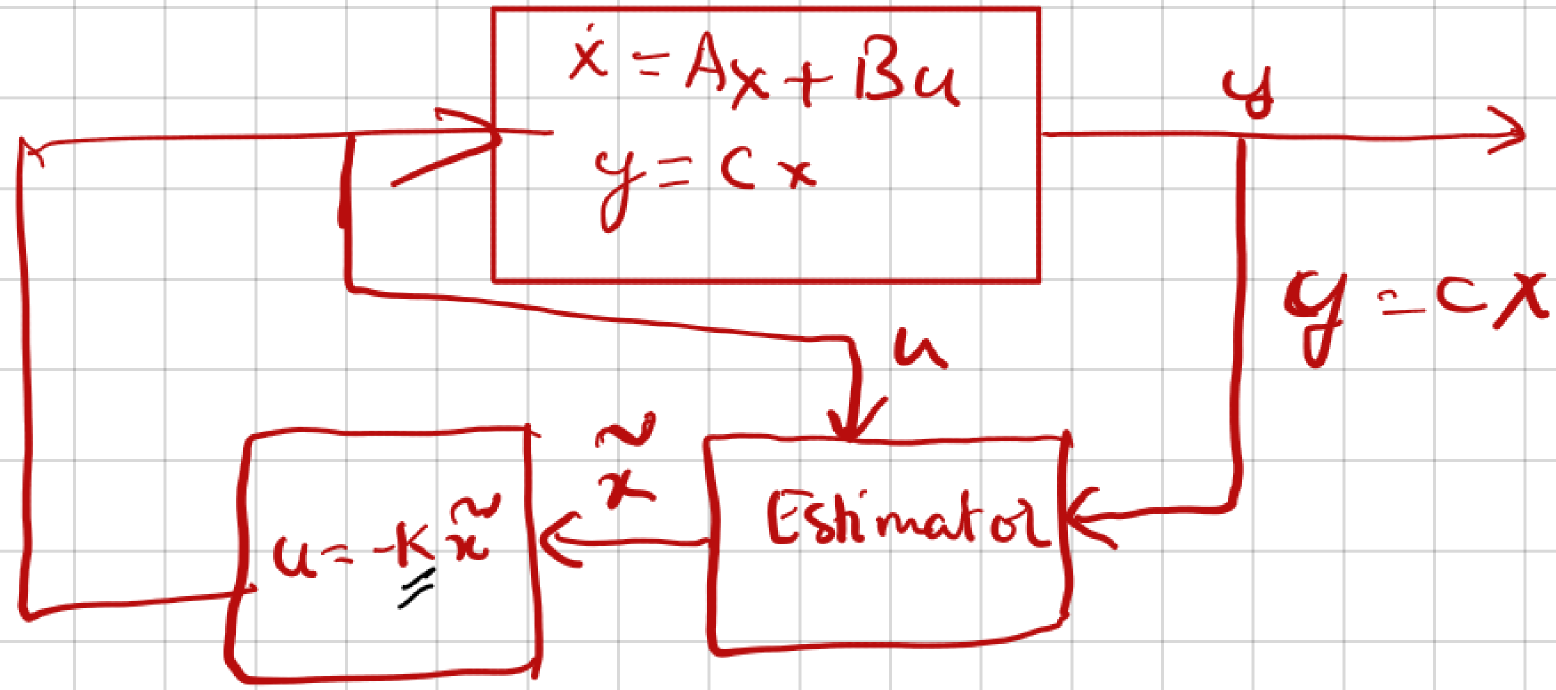
In reality, Sensors are expensive
Might not have full state measurement

$y = Cx \qquad C \in \mathbb{R}^{p \times n}$ (p measurements)

$p \leq n$

$\underbrace{\text{from a few measurements back out the values of states } x}_{}$

Estimation



$$\tilde{x} = f(y)$$

$$\dot{\tilde{x}} = \boxed{A\tilde{x} + Bu} + \underbrace{K_e(\overset{Cx}{y} - \overset{C\tilde{x}}{\tilde{y}})}_{\text{update}}$$

$$= (A - K_e C)\tilde{x} + Bu + K_e C x$$

$$e = x - \tilde{x}$$

$$\dot{e} = \dot{x} - \dot{\tilde{x}}$$

$$= (Ax + \cancel{Bu}) - (A\tilde{x} + \cancel{Bu} + K_e C(x - \tilde{x}))$$

$$= A\underbrace{(x - \tilde{x})}_{e} - K_e C\underbrace{(x - \tilde{x})}_{e}$$

$$\dot{e} = \underbrace{(A - K_e C)}_{\text{error dynamics.}} e$$

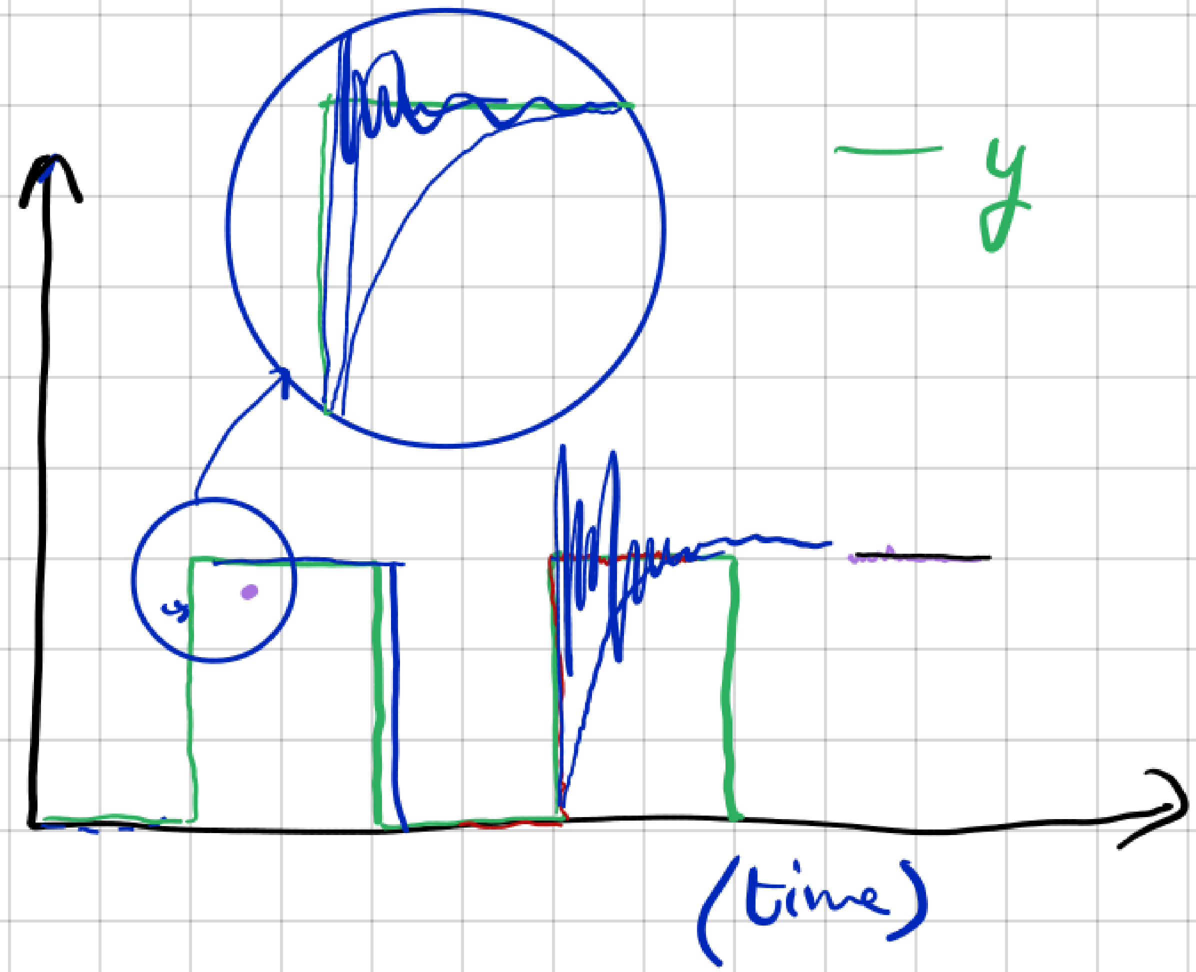
$eig(A-KC) \rightarrow (-)ve$ real parts.

$\Rightarrow e \rightarrow$ asymptotically converges to 0

If signal is clean

Place estimator Poles

5 times to the left of system's poles



If signal (y) is noisy

2 times to the left.

Observability

$$\text{obsv}(A, C) \rightarrow \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

Sys is observable if rank of obsv(A, C) of row space is $n \rightarrow A \in \mathbb{R}^{n \times n}$

$$\dot{z} = A^T z + C^T u$$

$$\underline{\begin{bmatrix} C^T & A^T C^T & A^{T^2} C^T & \dots & A^{T^{n-1}} C^T \end{bmatrix}}$$