Category theory summer course HW 1, meet and discuss 6.6.2019

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1. We'll define Rel to be the following category where objects of Rel are sets and morphisms of Rel are relations $R \subset X \times Y$. Define composition to category Rel such Set is its subcategory. Show that Rel is a category.

   Solution

2. Let's define a monoidoid. Let M be a class and $\cdot : M \times M \rightharpoonup M$. Pair $(M, \cdot)$ is called a monoidoid if the following holds:

   (a) If one of the expressions $(km)n$ and $k(mn)$ is defined, then so is the other and the expressions are equal for all $k, m, n \in M$. (associative)

   (b) For every $m \in M$ there exists neutral elements[1] $e, f \in M$ where me and fm are defined.

   (c) Let $m, n, e \in M$ where e is a neutral element. If the expressions me and en are defined then so is mn.

   Given a category $C = (O, M, \circ, dom, cod)$, show that $(M, \circ)$ is a monoidoid. Additionally show that every monoidoid defines canonically a category when one defines the objects to be the neutral elements.

3. Give definitions for the following categories:

   (a) Mon of monoids

   (b) Grp of groups

   (c) Ab of Abelian groups

   (d) Ring of rings

   (e) R-mod of R-modules where R is a fixed ring

   (f) Top of topological spaces

   page 3, example 1.1.3

4. Monoidoid is similar to a monoid. By analogy what are the natural morphisms between monoidoids?

5. Show that the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is epic and monic in Ring but not an isomorphism. Solution, ex 4

[1] $e \in M$ is a neutral element if $ef = f$ and $ge = g$ for all $f, g \in M$ where the expressions are defined

6. Let C be a category and $a \xrightarrow{f} b$ a morphism in C. Prove the following statements:
   (a) If f is a retraction, it's epic. Dual: If f is a section, it's monic.
   (b) If f is a retraction and monic, it's an isomorphims Dual: If f is a section and epic, it's an isomorphims.
   (c) The morphism f has at most one inverse.

   Similar to exercises 1 and 3

7. Let C be a category. Define C^* to consist of the following information:
   (a) objects of C
   (b) those morphisms f of C such that f is an isomorphism.

   Show that C^* is a subcategory of C. solution, ex 2

8. Let C be a category. Define a relation $\cong$ on the objects of C by setting $a \cong b$ if and only if a and b are isomorphic for all objects a and b in C. Show that the relation $\cong$ is an equivalence relation.

9. Let c, c' be initial objects in a category C. Show that the unique morphism $c \to c'$ is an isomorphism.

10. Category C is called skeletal, if $a \cong b$ implies that $a = b$ for all objects a, b in C. A subcategory $A \leq C$ is called dense, if A is full subcateogry and for any object c in C there exists an object a in A such that $c \cong a$. Show that for any category C there exists skeletal and dense subcategory. Such subcategory is called a skeleton of C. Hint: Use a very large axiom of choice and choose representatives of every isomorphism class.

11. Let $G \xrightarrow{f} H$ be an epimorphism in the cateogry Grp of groups. Show that f is surjective.

    outline of the proof, page 21 exercise 5