Category theory summer course HW 4, meet and discuss 27.6.2019

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24. kesäkuuta 2019

1. Let C, D and E be categories. Show that the horizontal composition $* : [D, E] \times [C, D] \to [C, E]$ is a bifunctor.

2. In this exercise we define a new meta categorical structure called the meta-category of natural transformations:

   Meta objects: categories $C, D, E, ...$

   Meta morphisms: Natural transformations $\eta : F \Rightarrow G : C \to D$ where the domain is C and codomain is D, and $F, G : C \to D$ are functors associated to the natural transformation η.

   Composition: Horizontal composition of natural transformations.

   Show that this definition satisfies the categorical axioms of associativity and identity.

3. Let C, D and E be categories and $B, B' : C \times D \to E$ be bifunctors. Let

   $$\eta = (\eta_{c,d} : B(c, d) \to B'(c, d))_{(c,d) \in obj(C \times D)}.$$

   Show that the follwing are equivalent:

   (a) $\eta : B \Rightarrow B' : C \times D \to E$

   (b) $\eta : \tilde{B} \Rightarrow \tilde{B}' : C \to [D, E]$., where $\tilde{B}$ and $\tilde{B}'$ are the associated functors by currying.

   (c) $\eta_{-,d} : B^d \Rightarrow B'^d : C \to E$ and $\eta_{c,-} : B_c \Rightarrow B'_c : D \to E$ for all $e \in C$ and $d \in D$.

4. Show that relation $\simeq$ is an equivalence relation, where $C \simeq D$ if and only if there exists functors $C \xrightarrow{F} D \xrightarrow{G} D$ where $F \circ G \cong I_D$ and $G \circ F \cong I_C$.

5. (Same exercises as in the previos week): Show that a functor $F : C \to D$ is an equivalence if and only if F is dense and fully faithful.

6. Give the represetations of the following functors:

   (a) The identity functor $I_{Set} : Set \to Set$.

   (b) The forgetful functor $U : Grp \to Set$.

   (c) The (contravariant) power set functor $P^* : Set^{op} \to Set$.

   (d) The functor $\tau : Top^{op} \to Set$, where $\tau(X)$ is the topology on X and for a continuous map $f : X \to Y$, $\tau(f) = f^* : \tau(Y) \to \tau(X)$ is the induced map.

   (e) The functor $F : Top^{op} \to Set$, where $F(X)$ is the set of closed subsets of X and where continuos map is taken to the induced map between the closed sets.

(f) The functor $Obj : Cat \to Set$, which takes a small category to it's underlying set of objects.

(g) The functor $Mor : Cat \to Set$, which associates a small category to its set of morphisms.

7. Show that the covariant power set functor $P_* : Set \to Set$ is not representable. (Hint: use Yoneda lemma).

8. Give all the natural transformations from the first functor to the second:

   (a) $I_{Set}, I_{Set} : Set \to Set$.

   (b) $P^*, P^* : Set^{op} \to Set$.

   (c) $I_{Set}, P_* : Set \to Set$

   (d) $U, U : Grp \to Set$, where U is the forgetful functor.

9. Let C be a category with sums, products and exponentials. We take it for granted that all of them define some kind of a functor. We may also assume that the following isomorphisms are natural in a, b and c:

   $C(a, b \times c) \cong C(a, b) \times C(a, c)$.

   $C(a + b, c) \cong C(a, c) \times C(b, c)$.

   $C(a \times b, c) \cong C(a, c^b)$.

   Show by using fully faithfulness of Yoneda embedding that the following isomorphisms hold[1].

   (a) $(a + b) \times c \cong (a \times c) + (b \times c)$.

   (b) $(a \times b)^c \cong a^c \times b^c$.

   (c) $(a^b)^c \cong a^{b \times c}$.

   (d) $a^{b+c} \cong a^b \times a^c$.[2]

10. Prove Yoneda lemma: Let C be a locally small category, c an object in C and $F : C \to Set$ a functor. Then

$$[C_c, F] \cong F(c)$$

naturally in F and c. In other words

$$Set^C \circ y^{op} \times I_{Set^C} \cong Ev : C \times Set^C \to Set,$$

where $y : C^{op} \to Set^C$ is the Yoneda functor (later proved to be an embedding), which is the functor associated to Hom-functor through currying.

   (a) Let $\eta \in [C_c, F]$ and so $\eta : C_c \Rightarrow F$. Show that given an object d in C and $f : c \to d$, we have

$$\eta_d(f) = Ff(\eta_c(id_c)).$$

   (b) Define a map $\psi_{c,F} : [C_c, F] \to Fc$.

   (c) Show that $\psi_{c,F}$ is a bijection.

   (d) Lastly demonstrate the naturality of of the collection of maps $(\psi_{c,F})_{c,F}$.

11. Show by using Yoneda lemma that the Yoneda functor $y : C^{op} \to [C, Set]$ is an embedding, meaning fully faithful and injective on objects.

[1] The isomorphims actually hold naturally as well

[2] Remark that if category C has coexponential objects, such as in $\mathcal{P}(X)$, then we get the dual results also and distributiveness hold: summation distributes over multiplication.