Category theory summer course HW 2, meet and discuss 13.6.2019

David Forsman, david.s.forsman@helsinki.fi

June 7, 2019

1. Show that the following concept pairs are dual:
   (a) monomorphism and epimorphim
   (b) retraction and section
   (c) isomorphism and isomorphism
   (d) initial and terminal
2. Show that
   (a) faithful functor need not be injective on morphisms.
   (b) the image of a functor need not be a subcategory

   Solution, ex 1 and 7
3. Functor F can preserve, reflect or create certain properties with respect to morphisms f or objects a, b:

   Preserve: if f has a property, then Ff has it also

   Reflect: Ff has a property, then f has it too

   Create: if Fa and Fb are in a relation, then a and b are in a corresponding relation. [1]

   Show that
   (a) Functors preserve sections and retractions.
   (b) Faithful functors reflects epis and monos.
   (c) Faithful functors need not preserve epis and monos.
   (d) Fully faithful functors reflect retractions and sections
   (e) Fully faithful functors create retracts and isomorphisms[2]

[1] The concepts of preservation, reflection and creation can be formalized, but it's better to leave it to be for now

[2] Let a, b be objects in C. Object a is a retract of object b if there exists a retraction $r : b \to a$.

4. In this exercise, one constructs a functor $Top \to Set$.

   (a) Let X be a topological space and $x, y \in X$. Give the definition for a path from x to y.

   (b) Given a topological space X, give the definition for a path connected subspace of X.

   (c) Show that if $f : X \to Y$ is in Top, then fA is path connected for all path connected $A \subset X$.

   (d) Let's define a relation $\sim$ on a topological space X. We say that $a \sim b$ whenever there exists a path from a to b. Show that the relation $\sim$ is an equivalence relation. The equivalence classes are called path components of X.

   (e) Let's define $F : Top \to Set$ followingly:

   $$F(X) = \{[x] : x \in X\}$$

   and

   $$F(f) : F(X) \to F(Y), F(f)([x]) = [f(x)]$$

   Show that this is well defined and actually yields a functor.

   Similar solution to ex 2

5. Is there a functor $F : Grp \to Ab$, where $F(G) = G/[G, G]$ where [G,G] is the subgroup of G that is generated by $\{ghg^{-1}h^{-1} : g, h \in G\}$. [3]

   Solution ex 3

6. Is there a functor $Z : Grp \to Ab$, where $Z(G) = \{g \in G : gh = hg \text{ for all } h \in G\}$.

   Hint: Given $n \in \mathbb{N}_{\geq 2}$, there exists a surjective homomorphism $sgn : S_n \to \mathbb{Z}_2$, where S_n is the symmetric group of n elements.

   Solution ex 4

7. Give the product and the coproduct of the following categories:

   (a) $B(\mathcal{P}(X), \subset)$, where X is a set.

   (b) $B(\mathbb{N}, |)$, where $a|b$ iff a divides b.

   (c) Set

   (d) Ab

   (e) Mon of monoids. (Hard)

8. Assume that C is a category where for any two given object there exists their product. Show that then there exists a functor $\times : C \times C \to C$, where $a \times b$ is the product object of objects a and b.

9. Let $F : C \to E$ and $G : D \to E$ be functors. Define the comma category $F \downarrow G$ and show that it actually is a category. Solution ex 5

[3] Let A be a subset of a group G. The subgroup H generated by A is the smallest subgroup of G that contains A. It can be shown that $H = \{\prod_{i=1}^{n} a_i : n \in \mathbb{N}, a_i \in A \cup A^{-1}\}$.