Category theory summer course HW 3, meet and discuss 20.6.2019

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1. Show that a natural transformation $\eta : F \Rightarrow G : C \to D$ is an isomorphims, if and only if $\eta_c : Fc \to Gc$ is an isomorphism for every object c in C.
2. Let $F, G : C \to D$ be functors. Assume $F \cong G$. Prove the following statements:
   (a) If G is faithful, so is F.
   (b) If G is full, so is F.
   (c) If G is dense, so is F.
3. Let $C \xrightarrow{F} D \xrightarrow{G} E$ functors. Show that the following statements hold:
   (a) If $G \circ F$ is faithful, then F is faithful as well.
   (b) If $G \circ F$ is dense, then G is dense.
   (c) If $G \circ F$ is full and F dense, then G is full.
   (d) Conclude that if F is an equivalence, it's dense and fully faithful.
4. (a) Given a bifunctor $B : C \times D \to E$, define functors $B_c : D \to E$ and $B^d : C \to E$ where $B(c, d) = B_c(d) = B^d(c)$ for all objects c in C and d in D.
   (b) Assume that $B_c : D \to E$ and $B^d : C \to E$ are functors for objects c in C and d in D. What property do you need that functors B^d and B_c define a bifunctor $B : C \times D \to E$, where $B(c, d) = B_c(d) = B^d(c)$ for all objects d in D and c in C?
   (c) Show that the construction given are other's inverses.
5. Show that
   (a) the category *Set* is Cartesian closed.
   (b) the meta category M-CAT is Cartesian closed.
6. In this exercise we give a new description for some natural transformations.
   (a) When homomorphisms between groups are thought of as functors, show that natural transformations between homomorphisms correspond to conjugation.
   (b) Show that natural transformations between G-actions are equivariant mappings, where G is a group.
   (c) Show that there exists a natural transformation $\eta : f \Rightarrow g : P \to Q$ between increasing maps, if and only if $f(p) \leq g(p)$ for all $p \in P$, where P and Q are preordered sets.

7. The following exercise is long and optional. Let C be a category with terminal object 1 and products. Previously we saw there exists a bifunctor $x : C \times C \to C$. We see that there are functors $F, G : C^3 \to C$, where $F(a, b, c) = (a \times b) \times c$ and $G(a, b, c) = a \times (b \times c)$, and $1 \times -$ and $- \times 1$.
   (a) Define isomorphisms $L_a : 1 \times a \to a$ and $R_a : a \times 1 \to a$ for all objects a in C.
   (b) Define an isomorphism $\alpha_{a,b,c} : (a \times b) \times c \to a \times (b \times c)$
   (c) Show that L, R and α become natural isomorphisms between the corresponding functors. L is then called the left unitor, R the right unitor and α associator.
   (d) Show that the following diagrams commute[1]:

      1. ***triangle identity*** (not to be confused with the triangle identities of an adjunction):

$$
\begin{tikzcd}
(x \otimes 1) \otimes y \arrow[rr, "\alpha_{x,1,y}"] \arrow[dr, "\rho_x \otimes 1_y"'] & & x \otimes (1 \otimes y) \arrow[dl, "1_x \otimes \lambda_y"] \\
& x \otimes y &
\end{tikzcd}
$$

      2. the ***pentagon identity*** (or ***pentagon equation***):

$$
\begin{tikzcd}
& (w \otimes x) \otimes (y \otimes z) \arrow[dr, "\alpha_{w,x,y \otimes z}"] & \\
((w \otimes x) \otimes y) \otimes z \arrow[ur, "\alpha_{w \otimes x,y,z}"] \arrow[d, "\alpha_{w,x,y} \otimes \mathrm{id}_z"'] & & (w \otimes (x \otimes (y \otimes z))) \\
(w \otimes (x \otimes y)) \otimes z \arrow[rr, "\alpha_{w,x \otimes y,z}"'] & & w \otimes ((x \otimes y) \otimes z) \arrow[u, "\mathrm{id}_w \otimes \alpha_{x,y,z}"']
\end{tikzcd}
$$

   This exerice shows that $(C, \times, 1, \alpha, L, R)$ defines a monoidal category. All basic diagrams constructed from only products, terminal object, associator and the unitors can be shown to commute.

8. Assume that category C is Cartesian closed and a, b, c objects in C.
   (a) Show that $Hom(c \times a, b) \cong Hom(c, b^a)$. This correspondence is called currying.
   (b) Show that $f : a \to b$ corresponds bijectively with $\tilde{f} : 1 \to b^a$.
   (c) (Difficult) Let $f : a \to b$ and $x : 1 \to a$. Now using the universal property of products, we have $(\tilde{f}, x) : 1 \to b^a \times a$. Show that $ev \circ (\tilde{f}, x) = f \circ x$.
   (d) Show that there exists a morphism $* : c^b \times b^a \to c^a$.The corresponding functor in M-CAT is called the horizontal composition.
   (e) (Very hard) Let $a \xrightarrow{f} b \xrightarrow{g} c$. Show that $* \circ (\tilde{g}, \tilde{f}) = \widetilde{g \circ f}$.

9. Let $F : C \to D$ be functor. Show that F is an equivalence between categories, if and only if F is dense and fully faithful.

[1] ρ and λ refer to the right and left unitors respectively