② Given the Taylor polynomial expansions of two functions around $x = 0$

$$\frac{1}{1-\Delta x} = 1+\Delta x+\Delta x^2+\Delta x^3+O(\Delta x^4)$$

and

$$\cosh\Delta x = 1+\frac{\Delta x^2}{2!}+\frac{\Delta x^4}{4!}+O(\Delta x^6)$$

Calculate their sum and product as well as the exponent that belongs in the O.

for small values of x:

$$f(\Delta x) = p(\Delta x) + O(\Delta x^n)$$
$$g(\Delta x) = q(\Delta x) + O(\Delta x^m)$$
$$r = \min(n, m)$$

$$f + g = p + q + O(\Delta x^r)$$

$$f \cdot g = p \cdot q + p \cdot O(\Delta x^m) + q \cdot O(\Delta x^n) + O(\Delta x^{n+m})$$
$$= p \cdot q + O(\Delta x^r)$$

exponent that belongs in the O: $r = \min(4,6) = \boxed{4}$

$r = \min(\underset{4}{n}, \underset{6}{m}) = 4$

sum: $f+g = p+q+O(\Delta x^r)$

$$\overset{f(\Delta x)}{\frac{1}{1-\Delta x}} + \overset{g(\Delta x)}{\cosh\Delta x} = \Big(\overbrace{1+\Delta x+\Delta x^2+\Delta x^3}^{p(\Delta x)} + \overbrace{O(\Delta x^4)}^{O(\Delta x^n)}\Big) + \Big(\overbrace{1+\frac{\Delta x^2}{2!}+\frac{\Delta x^4}{4!}}^{q(\Delta x)} + \overbrace{O(\Delta x^6)}^{O(\Delta x^m)}\Big)$$

$$\frac{1}{1-\Delta x} + \cosh\Delta x = \underbrace{2+\Delta x+\frac{3}{2}\Delta x^2+\Delta x^3}+\overset{O(\Delta r^n)}{\cancel{\frac{1}{24}\Delta x^4}} + \underbrace{O(\Delta x^4)}_{(\text{big } O)}$$

$$= \boxed{2+\Delta x+\frac{3}{2}\Delta x^2+\Delta x^3+O(\Delta x^4)}$$

product: $f\cdot g = p\cdot q + O(\Delta x^r)$

$$\frac{1}{1-\Delta x}\cdot\cosh\Delta x = (1+\Delta x+\Delta x^2+\Delta x^3)\cdot\left(1+\frac{\Delta x^2}{2!}+\frac{\Delta x^4}{4!}\right)+O(\Delta x^4)$$

$$= 1+\frac{\Delta x^2}{2!}+\frac{\Delta x^4}{4!}+\Delta x+\frac{\Delta x^3}{2!}+\frac{\Delta x^5}{4!}+\Delta x^2+\frac{\Delta x^4}{2!}+\frac{\Delta x^6}{4!}$$
$$+\Delta x^3+\frac{\Delta x^5}{2!}+\frac{\Delta x^7}{4!}+O(\Delta x^4)$$

$4\cdot3\cdot2 = 24$

$\frac{1+12}{24}$

$$= 1+\frac{3}{2}\Delta x^2+\frac{3}{2}\Delta x^3+\frac{13}{24}\cancel{\Delta x^4}+\frac{13}{24}\cancel{\Delta x^5}+\frac{1}{24}\cancel{\Delta x^6}+\frac{1}{24}\cancel{\Delta x^7}+\underbrace{O(\Delta x^4)}_{\text{big } O}$$

$$= \boxed{1+\frac{3}{2}\Delta x^2+\frac{3}{2}\Delta x^3+O(\Delta x^4)}$$

③ The great Exp challenge:

write a function to calculate e^x using its Taylor polynomial approximation expanded around $x_0 = 0$

$$e^x \approx T_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!}$$

s.t. the relative error of $f = e^x$ and $F = T_n(x)$ is of order $\varepsilon_{machine}$ for $x \in [-50, 50]$

Discretization Error

Taylor's Theorem: Let $f(x) \in C^{N+1}[a,b]$ and $x_0 \in [a,b]$, then for all $x \in (a,b)$ there exists a number $c = c(x)$ that lies between x_0 and x such that

$$f(x) = T_N(x) + R_N(x)$$

where $T_N(x)$ is the Taylor polynomial approximation

$$T_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(x_0) \cdot \overbrace{(x - x_0)}^{\Delta x}{}^n}{n!}$$

and $R_N(x)$ is the residual (the part of the series we left off)

$$R_N(x) = \frac{f^{(N+1)}(c) \cdot \overbrace{(x - x_0)}^{\Delta x}{}^{N+1}}{(N+1)!}$$

Start by replacing $x - x_0$ with Δx. The primary idea here is that the residual $R_N(x)$ becomes smaller as $\Delta x \to 0$ (at which point $T_N(x) = f(x_0)$).

$$T_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(x_0) \cdot \overbrace{\Delta x}^{(x - x_0)}{}^n}{n!}$$

and $R_N(x)$ is the residual (the part of the series we left off)

$$\boxed{R_N(x) = \frac{f^{(n+1)}(c) \cdot \Delta x^{n+1}}{(n+1)!}} \le M \Delta x^{n+1} = O(\Delta x^{n+1})$$

where M is an upper bound on

$$\frac{f^{(n+1)}(c)}{(n+1)!}$$

a) Assume $x > 0$ and show that the upper bound on the relative error at term n

$$r_n = \frac{\underbrace{|e^x - T_n(x)|}_{R_N(x)}}{|e^x|} \qquad (e^x = T_n(x) + R_n(x))$$

is given by $r_n \le \left| \dfrac{x^{n+1}}{(n+1)!} \right|$

(differentiated, not to be confused w/ exponent)

$$r_n = \frac{\underbrace{|e^x - T_n(x)|}_{|R_N(x)|}}{|e^x|} = \frac{|\overbrace{e^c}^{f^{(n+1)}(c)} \Delta x^{n+1}|}{|(n+1)!||e^x|} \le \left| \frac{\Delta x^{n+1}}{(n+1)!} \right|$$

upper bound as max val. occurs when $f^{(n+1)}(c) \ge |e^x|$

and we can't have $f^{(n+1)}(c) > |e^x|$

as c is defined as:

$\exists$ number $c = c(x)$ that lies b/w

$c \in [x, x_0]$ x_0 and x s.t. $f(x) = T_N(x) + R_N(x)$

$0 < c \leq x \Rightarrow 0 < e^c \leq e^x$

b) Show that for large $x >> 1$ and n, $r_n \leq \varepsilon_{machine}$ implies that we need approximately $n > e \cdot x$ terms in the series (where $e = \exp(1)$)

We need to show that:

$$\frac{x^{n+1}}{(n+1)!} \leq \varepsilon_{machine}$$

We then have $x^{n+1} < (n+1)!$

Taking log on both sides:

$$(n+1)\log x < (n+1)\log(n+1) - (n+1)$$

$$\log x < \log(n+1) - 1$$

$$\log x + \underset{e=\exp(1)}{1} < \log(n+1)$$

$$\log(e \cdot x) < \log(n+1)$$

for large n, $\log(n+1) \sim \log n$

$$\Rightarrow \log(e \cdot x) < \log n$$

$$e \cdot x < n$$

$$n > e \cdot x$$

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④ Storage of #'s is base 4

assuming we have $p=3$ for the mantissa & the exponent $E \in [-3,3]$,

& sign chirality:

a) how many #'s we can represent w/ this floating point system:

$N=4$, $p=3$, $E \in [-3,3]$

(normalized numbers $d_1 \neq 0$)

$d_1 = [1,2,3]$ $d_2 = d_3 = [0,1,2,3]$

$E = [-3,-2,-1,0,1,2,3]$ $|E| = 7$

$\boxed{f = \pm d_1 . d_2 d_3 \times 4^E \text{ w/ } E \in [-3,3]}$

$2 \times 3 \times 4 \times 4 \times 7 + \underset{\text{zero}}{1} = \boxed{673}$

underflow limit: $1.00 \times 4^{-3} = 0.015625$

overflow limit: $3.33 \times 4^3 = 213.12$

$e_{mach} = 4^{-1} = 0.25$

c) how many more #'s can we store in N base-pairs (base 4)
vs. N bits (base 2) where the mantissa and exponent are the same relative length
(e.g. $p=3$, $E \in [-3,3]$)

In base 4, our # of representations w/ $p=3$ and $E \in [-3,3]$ is

$$2 \times 3 \times 4 \times 4 \times 7 + 1 = 673$$

for base 2:

$d_1 = [1]$

$d_2 = [0,1]$

$d_3 = [0,1]$

$E = [-3,-2,-1,0,1,2,3]$

which gives us $2 \times 1 \times 2 \times 2 \times 7 + 1 = 57$ as the # of representations

Thus the difference is $673 - 57 = \boxed{616}$ more numbers