Interferometric Measurement of Angles

D. Malacara and O. Harris

A new interferometric device for measuring small angles or rotations with high accuracy is described. This instrument works by counting fringes formed by the rotation of a flat-parallel plate of glass illuminated with a collimated beam from a gas laser. Some possible applications are given.

Introduction

At the present time, there are two methods for measuring angles. The first method is mechanical and uses a highly precise protractor[1]; it achieves precisions up to 2.5 sec of arc. The second method is optical[2] but it is restricted to specific angles such as 30° or 45° and multiples of these angles. The accuracy that can be obtained with optical methods is within tenths of a sec of arc. Recently, new interferometric methods have been proposed to measure angles of the order of few degrees with an accuracy much better than tenths of a sec of arc.[3,4]

The basic principles and some considerations about the applications of a new interferometric instrument are described here. It is capable of measuring any angle from 0° to 360° with better precision than that given by mechanical devices.

Theory

The interferometer to measure angles is essentially the shearing interferometer devised by Murty[5] for testing optical systems, and the method is based on the variation of the optical path difference (OPD) between the reflections from the two faces of a glass plate with flat-parallel faces, when this plate is rotated.

The OPD between the front and the back surface reflections in a parallel plate of glass, as shown in Fig. 1, is given by[6]

$$\mathrm{OPD} = 2Nt \cos\theta', \tag{1}$$

where t is the thickness of the plate, N is the index of refraction and θ' is the angle of refraction corresponding to an angle of incidence θ.

To change the OPD, the plate is rotated about an axis parallel to the plate as shown by an arrow in Fig. 1.

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The difference in the OPD for two different angles of incidence θ_1 and θ_2 on the plate, (Eq. (1)) is given by

$$\mathrm{OPD}_1 - \mathrm{OPD}_2 = 2Nt(\cos\theta_1' - \cos\theta_2'). \tag{2}$$

Taking into account that by the Stokes relations one and only one of the reflections undergoes a phase change of π after the reflection, destructive interference will take place when:

$$\mathrm{OPD}_1 - \mathrm{OPD}_2 = m\lambda, \tag{3}$$

where λ is the wavelength of light and m is any integer.

Therefore, using the law of refraction, Eqs. (2) and (3) may be combined to give

$$m/t = (2/\lambda)[(N^2 - \sin^2\theta_1)^{\frac{1}{2}} - (N^2 - \sin^2\theta_2)^{\frac{1}{2}}]. \tag{4}$$

The total angle that the glass plate rotates is given by $\theta_2 - \theta_1$. If the measurement starts at normal incidence, that is, $\theta_1 = 0$, Eq. (4) reduces to

$$m/t = (2/\lambda)[N - (N^2 - \sin^2\theta_2)^{\frac{1}{2}}]. \tag{5}$$

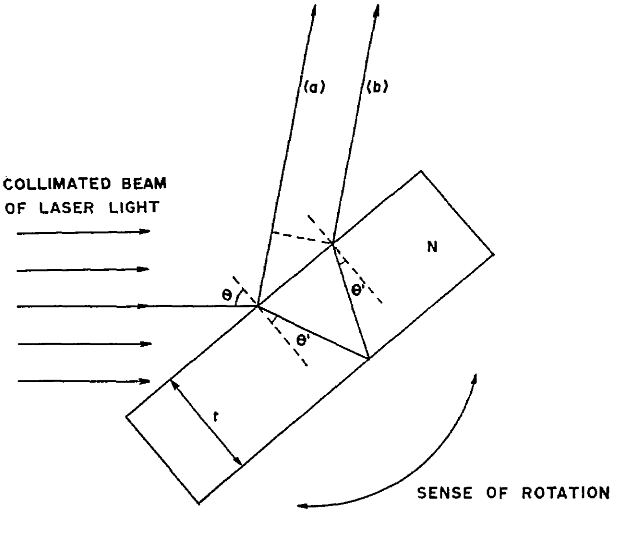

Fig. 1. Flat-parallel plate of glass.

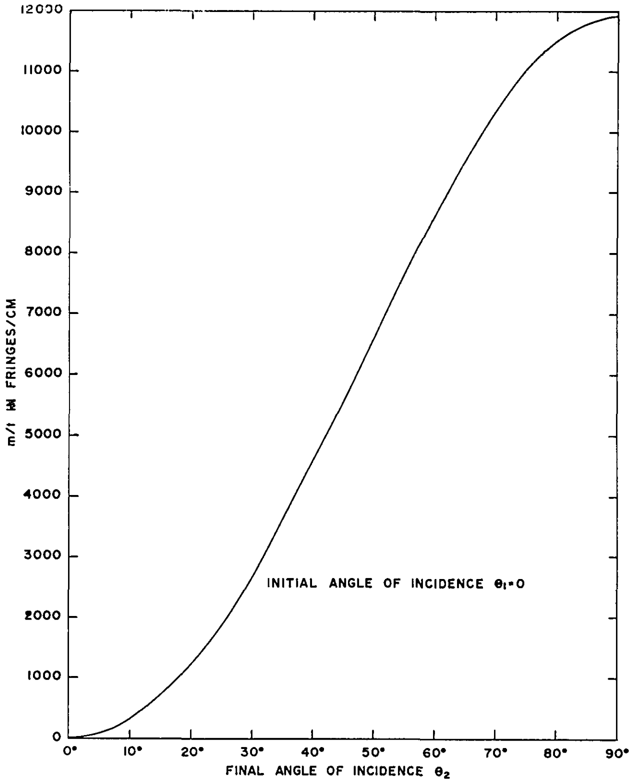

Fig. 2. Angle of rotation vs m/t for $\theta_1 = 0$.

A very common optical glass is BK7 whose index of refraction for red laser light (6328 Å) is 1.515. All the results and graphs of this work are referred to this glass. Plotting m/t vs θ_2 from Eq. (5) for this index of refraction, Fig. 2 is obtained. It can be noticed in this curve that m/t does not increase linearly with θ_2 except in the vicinity of $\theta = 45°$. Thus, the angle measured, $\theta_2 - \theta_1$, does not only depend on the number m/t but also on the initial angle θ_1 making, therefore, the accuracy of the final result depend on the accuracy with which the angle θ_1 is measured. This dependence of the final result $\theta_2 - \theta_1$ on the value θ_1 can be cancelled if

$$\partial(\theta_2 - \theta_1)/\partial\theta_1 = 0 \qquad (6)$$

or

$$\partial\theta_2/\partial\theta_1 = 1. \qquad (7)$$

From Eq. (4) this condition is satisfied if an initial value of θ_1 is taken such that

$$\sin\theta_1 \cos\theta_1/(N^2 - \sin^2\theta_1)^{\frac{1}{2}} = \sin\theta_2 \cos\theta_2/(N^2 - \sin^2\theta_2)^{\frac{1}{2}}; \qquad (8)$$

to analyze this condition, let us consider the function

$$G(\theta) = \sin\theta \cos\theta/(N^2 - \sin^2\theta)^{\frac{1}{2}}, \qquad (9)$$

and solving for $\sin^2\theta$,

$$\sin^2\theta = \tfrac{1}{2}(1 + G^2) \pm [\tfrac{1}{4}(1 + G^2)^2 - G^2N^2]^{\frac{1}{2}}. \qquad (10)$$

The value of $G(\theta)$ may be any quantity such that $\sin^2\theta$ is real and smaller than one. These two conditions are satisfied if

$$|G(\theta)| \leq N - [N^2 - 1]^{\frac{1}{2}}. \qquad (11)$$

Given a value of $G(\theta)$ there are two values of $\sin^2\theta$ that can be obtained from Eq. (10). These two values correspond to the angles θ_1 and θ_2 that satisfy the condition expressed by Eq. (8). The value of θ_1 such that $\theta_2 - \theta_1 = 0$ may be found equating expression (11). Substituting in Eq. (10), we obtain

$$\sin\theta m = [N^2 - N(N^2 - 1)^{\frac{1}{2}}]^{\frac{1}{2}}. \qquad (12)$$

The optimum value of θ_1 for any value of the angle $\theta_2 - \theta_1$ can be thus obtained from Eq. (10). Figure 3 shows a plot of θ_1 vs $\theta_2 - \theta_1$. It can be seen that this graph can be very well approximated by a straight line defined by

$$\theta_1 = (1 - \theta_2 - \theta_1/90°) \sin^{-1}[N^2 - N(N^2 - 1)^{\frac{1}{2}}]^{\frac{1}{2}}. \qquad (13)$$

Figure 4 shows a computer plot of m/t vs the angle to be measured.

At the maximum of this function $\theta_2 = 90°$ and $\theta_1 = 0°$; therefore, from Eq. (4), the maximum value of m/t is

$$m/t = (2/\lambda)[N - (N^2 - 1)^{\frac{1}{2}}]. \qquad (14)$$

Although not exactly, the value of m/t can in general be very well approximated by the function

$$m/t = (2/\lambda)[N - (N^2 - 1)^{\frac{1}{2}}] \sin(\theta_2 - \theta_1), \qquad (15)$$

which fits the computer curve in four significant figures.

Sensitivity

It is interesting to see how the order of interference varies with respect to the angle of incidence; in other words, what is the sensitivity S for a given plate of thickness t. This is equivalent to the following mathematical expression (S in fringes per deg):

$$S/t = (\pi/180°)(1/t)(\partial m/\partial\theta_2). \qquad (16)$$

Performing the differentiation, we get

$$S/t = (\pi/90°\lambda) \sin\theta_2 \cos\theta_2/(N^2 - \sin^2\theta_2)^{\frac{1}{2}}. \qquad (17)$$

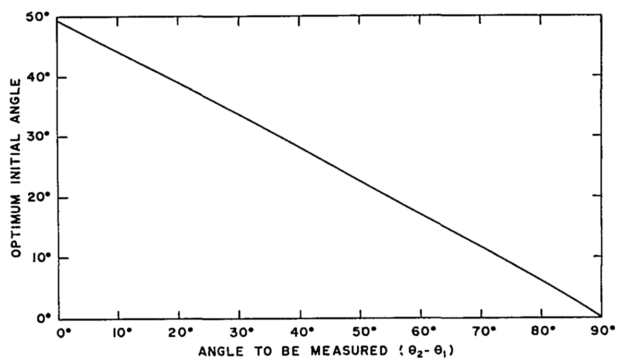

Fig. 3. Optimum angle θ_1 vs measured angle $(\theta_2 - \theta_1)$.

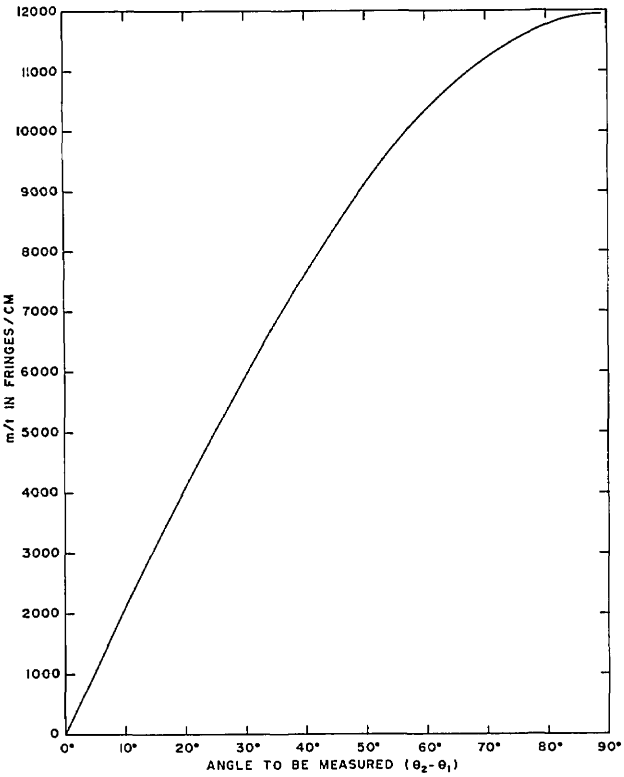

Fig. 4. m/t vs angle to be measured ($\theta_2 - \theta_1$).

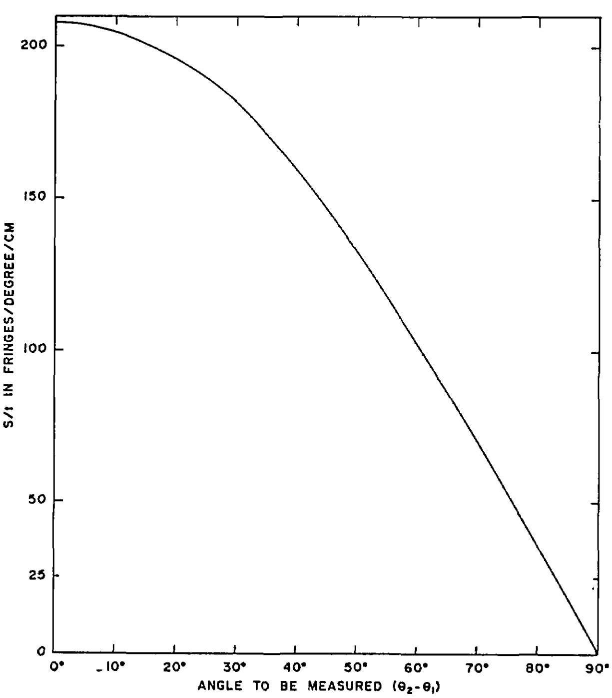

Fig. 5. Sensitivities vs angle to be measured ($\theta_2 - \theta_1$).

Table I. Maximum Error at Several Angles for a Plate 30 mm Thick

$\theta_2 - \theta_1$	S/t	S	Fringe Counting θ Seg.Arc. m=1	Fringe Position θSeg.Arc. m=±0.1
1°	209.0	627.0	5.741	0.574
5°	208.2	624.6	5.764	0.576
10°	205.5	616.5	5.839	0.584
30°	183.0	549.0	6.551	0.656
45°	148.0	444.0	8.108	0.811

Figure 5 is a plot of Eq. (17); the maximum sensitivity is obtained for small angles about the optimum value for

$$(\theta_1 - \theta_2)/2 \quad \text{(see Fig. 3).}$$

There are two ways of measuring the angle. The first one is a simple fringe counter; in this case, accuracy is a function of an integer, that is to say, the number of fringes; the error is ± one fringe. The second method implies the determination of the fringe position; for this case, the accuracy is a function of a fraction of the fringe separation. Table I shows the maximum error in the angle for the two cases $\Delta m = \pm 1$ and $\Delta m = \pm 0.1$. For angles greater than 45° the error increases rapidly, but with a suitable combination of plates any angle may be measured with a small error.

Tolerance to Variations in the Index of Refraction

The most important factor affecting the precision of the measurements is the index of refraction N of the plate. The flatness and the parallelism of the plate may be controlled to the necessary accuracy in the optics shop. The tolerance in the variation of the index of refraction can be found by computing the expression $\partial\theta/\partial N$, which will be given in degrees.

The differentiation of Eq. (4) gives

$$0 = \frac{N}{(N^2 - \sin^2\theta_1)^{\frac{1}{2}}} - \frac{N}{(N^2 - \sin^2\theta_2)^{\frac{1}{2}}} + \frac{\sin\theta_2 \cos\theta_2}{(N^2 - \sin^2\nu_2)^{\frac{1}{2}}} \frac{\partial\theta_2}{\partial N} \tag{18}$$

or

$$\partial(\theta_2 - \theta_1)/\partial N = \partial\theta_2/\partial N = N[(\sin\theta_2 \cos\theta_2)^{-1} - (\sin\theta_1 \cos\theta_1)^{-1}]. \tag{19}$$

Thus, obtaining $\Delta(\theta_2 - \theta_1)$ in degrees expressed by

$$\Delta(\theta_2 - \theta_1) = (180°N/\pi)[(\sin\theta_2 \cos\theta_2)^{-1} - (\sin\theta_1 \cos\rho_1)^{-1}]\Delta N. \tag{20}$$

Figure 6 represents expression 20 in a more useful form; there, the variation of the error in the angle as a

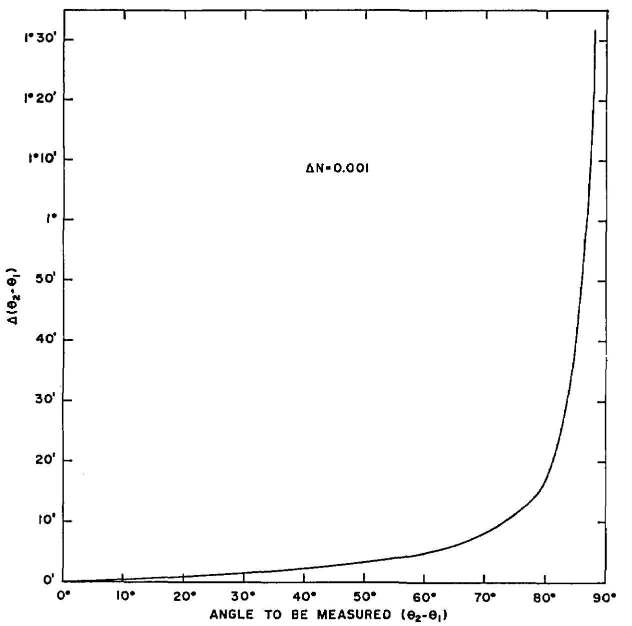

Fig. 6. Error in the angle to be measured for a given change in the index of refraction ($\Delta N = 0.001$).

function of the angle $(\theta_2 - \theta_1)$ for a given change ΔN in the index of refraction is given.

This method of measuring angles has, among others, two possible applications.

(1) The accurate measurement of angles of prisms and wedges. In this case, the measuring plate and the prism are rotated about a common axis. An autocollimator in front of the prism faces may be used to determine the starting and ending points in the fringe counting.

(2) The measurement of angular velocities of slowly rotating axes is another application of this method. The angular velocity ω in rad per min can be determined by measuring the frequency f of the signal in fringes per sec, using the formula

$$f = m/t = (s/60)\Delta(\theta_2 - \theta_1)/\Delta t = s\omega, \tag{21}$$

where s is the sensitivity.

A fringe counter gives a rough measurement of the angle (see Table I for $\Delta m = \pm 1$); the addition of a plotter at the beginning and ending of the counting increases the accuracy. We may conclude that this system achieves better precisions than any other mechanical instrument.

We wish to express our thanks to Arquímedes Morales, who made the drawings and graphs.

References

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3. R. Hoffman and L. Gross, J. Opt. Soc. Amer. **59**, 507 (1969).
4. J. Rholin, Appl. Opt. **2**, 762 (1963).
5. M. V. R. K. Murty, Appl. Opt. **3**, 531 (1964).
6. D. Malacara, "Testing of Optical Surfaces," (Ph.D. Thesis, Univ. of Rochester, 1965) p. 20.
7. F. Jenkins and H. White, *Fundamentals of Optics* (McGraw Hill Book Co., Inc., New York, 1957), p. 206.

Meetings Calendar *continued from page 1629*

May

7–11 Internat. Quantum Electronics Conf., Queen Elizabeth Hotel, Montreal *IEEE, 345 E. 47th St., New York, N.Y. 10017*

? Soc. for Exp. Stress Analysis, Olympic Hotel, Seattle, Wash. *SESA, 21 Bridge Sq., Westport, Conn. 06880*

September

24–29 SMPTE 112th Semiann. Conf., Los Angeles *D. A. Courtney, 9 E. 41st St., New York, N.Y. 10017*

October

8–13 SAS, Dallas, Tex.

9–13 ICO, Santa Monica, Calif. *R. Scott, Perkin-Elmer Corp., Norwalk, Conn.*

17–20 Optical Society of America, 57th Ann. Mtg., Jack Tar Hotel, San Francisco ***J. W. Quinn, OSA, 2100 Pa. Ave., N.W., Wash., D.C. 20037***

23–26 ISA 27th Ann. Conf. & Exhibit, New York *ISA HQ, 400 Stanwix St., Pittsburgh, Pa. 15222*

1973

October

1–5 SAS, Niagara Falls, N.Y.

1974

October

7–11 SAS, Indianapolis, Ind.

1976

October

8–12 SAS, Philadelphia, Pa.