Design of algorithms for phase measurements by the use of phase stepping

Yves Surrel

If the best phase measurements are to be achieved, phase-stepping methods need algorithms that are (1) insensitive to the harmonic content of the sampled waveform and (2) insensitive to phase-shift miscalibration. A method is proposed that permits the derivation of algorithms that satisfy both requirements, up to any arbitrary order. It is based on a one-to-one correspondence between an algorithm and a polynomial. Simple rules are given to permit the generation of the polynomial that corresponds to the algorithm having the prescribed properties. These rules deal with the location and multiplicity of the roots of the polynomial. As a consequence, it can be calculated from the expansion of the products of monomials involving the roots. Novel algorithms are proposed, e.g., a six-sample one to eliminate the effects of the second harmonic and a 10-sample one to eliminate the effects of harmonics up to the fourth order. Finally, the general form of a self-calibrating algorithm that is insensitive to harmonics up to an arbitrary order is given.
Key words: Interferometry, moiré, digital signal processing, waveform sampling.

1. Introduction

Many optical techniques provide their measurement results as fringe patterns. Phase stepping is now recognized as a very efficient method that allows automated fringe processing. The basic idea is to obtain different images of a fringe pattern, with a constant phase increment between two consecutive frames, so that a given pixel of the digitizing system is sampling a periodic intensity waveform over one period.[1–4] Another possibility is to introduce a so-called carrier into the fringe pattern so that a waveform period is sampled over adjacent pixels; this is called spatial phase stepping.[5]

It has been recognized that errors can be introduced to phase evaluations if the waveform is not a perfect sine and if the phase shift is different from its nominal value.[6,7] A number of algorithms have been proposed to deal with either or both causes, in specific cases.[8–11] Recently, Hibino *et al.*[12] proposed a general and systematic approach to algorithm design. The present paper reports another method and corrects an error that is present in Ref. 12. The present method allows one to generate arbitrary, customized algorithms with very simple algebraic calculations; in particular, no resolution of complex linear systems is needed. Customization may be in regard to the number of samples, algorithmic insensitivity to an arbitrary harmonic content, or algorithmic insensitivity to a phase-shift miscalibration up to an arbitrary order. We also provide in Appendix A the correspondence of the present method with the Fourier-transform approach.[9,13]

2. Theory

A. Characteristic Polynomials

In this subsection we show that a polynomial can be associated with any phase-shifting algorithm. The intensity $I(\varphi)$ recorded in the fringe pattern is a periodic function and can be expanded in a Fourier series as

$$I(\varphi) = \sum_{m=-\infty}^{\infty} \alpha_m \exp(im\varphi), \tag{1}$$

where the coefficient α_m is the complex Fourier coefficient of the mth harmonic and $i = \sqrt{-1}$.

If the intensity of the fringe pattern has a sine

The author is with the Department of Mechanical and Materials Engineering, École des Mines de Saint-Étienne, 158 Cours Fauriel, 42023 St-Étienne Cedex, France.

Received 10 April 1995; revised manuscript received 7 September 1995.
0003-6935/96/010051-10$06.00/0

profile, the usual expression is

$$
\begin{aligned}
I(\varphi) &= I_0(1 + \gamma \cos \varphi) \\
&= I_0 + \frac{I_0\gamma}{2} \exp(i\varphi) + \frac{I_0\gamma}{2} \exp[-(i\varphi)],
\end{aligned} \tag{2}
$$

where I_0 is the local average value and γ is the visibility of the fringes. From Eq. (2) we have

$$
\alpha_1 = \alpha_{-1} = \frac{I_0\gamma}{2}\cdot \tag{3}
$$

The fact that both the coefficients α_1 and α_{-1} are real is related to the implicit choice of the phase *origin,* which is made in writing Eq. (2): the phase origin is chosen at a point where the intensity reaches a maximum. Other coefficients are complex in general. Their arguments will represent the relative phase lags of the harmonics with respect to the fundamental.

We now consider that φ is a given phase at a point and is to be measured using a phase-shifting technique. That is, we will consider values $I(\varphi + \delta)$, where δ is the phase shift added to the initial phase φ. The expansion in a Fourier series of the intensity as a function of δ is

$$
\begin{aligned}
I(\varphi + \delta) &= \sum_{m=-\infty}^{\infty} [\alpha_m \exp(im\varphi)]\exp(im\delta) \\
&= \sum_{m=-\infty}^{\infty} \beta_m(\varphi)\exp(im\delta).
\end{aligned} \tag{4}
$$

From this equation, we can see that the main purpose of any phase-shifting algorithm is really to evaluate the argument of the Fourier coefficient β_1 corresponding to the fundamental harmonic of the intensity signal, considered to be a function of the phase shift at a given point.

M-step phase-shifting algorithms are usually written as the arctangent of the ratio between two linear combinations of values $I(\delta)$. That is, the measured phase that we denote φ^* to distinguish it from the actual phase φ is given by

$$
\varphi^* = \tan^{-1}\left[\frac{\sum_{k=0}^{M-1} b_k I(\varphi + k\delta)}{\sum_{k=0}^{M-1} a_k I(\varphi + k\delta)}\right]\cdot \tag{5}
$$

In the following discussions, it may appear preferable to interpret a phase-shifting algorithm as the computation of the argument of a complex linear combination:

$$
\varphi^* = \arg[S(\varphi)], \tag{6}
$$

with

$$
S(\varphi) = \sum_{k=0}^{M-1} c_k I(\varphi + k\delta), \tag{7}
$$

where $c_k = a_k + ib_k$. Using Eq. (4) one gets

$$
\begin{aligned}
S(\varphi) &= \sum_{m=-\infty}^{\infty} \left\{\alpha_m \exp(im\varphi) \sum_{k=0}^{M-1} c_k[\exp(im\delta)]^k\right\} \\
&= \sum_{m=-\infty}^{\infty} \{\alpha_m \exp(im\varphi)P[\exp(im\delta)]\},
\end{aligned} \tag{8}
$$

where $P(x)$ is a polynomial of degree $M - 1$:

$$
P(x) = \sum_{k=0}^{M-1} c_k x^k. \tag{9}
$$

As shown below, all the properties of any given phase-shifting algorithm can be deduced from the examination of the roots of $P(x)$. So we call $P(x)$ the *characteristic* polynomial of the algorithm. The number of coefficients of this polynomial is exactly equal to the number of intensity values involved in the algorithm. This introduction of a polynomial is similar to what is underlying in the z-transform theory.

B. Insensitivity to Harmonics

In this subsection it is shown that insensitivity to the harmonics depends on the number and on the locations of the roots of the characteristic polynomial. Equation (8) is nothing but the expansion of $S(\varphi)$ in a Fourier series and can be rewritten as

$$
S(\varphi) = \sum_{m=-\infty}^{\infty} \gamma_m \exp(im\varphi), \tag{10}
$$

where the coefficient γ_m of the mth harmonic is

$$
\gamma_m = \alpha_m P\,[\exp(im\delta)]. \tag{11}
$$

Thus, the necessary and sufficient condition for the measured phase φ^* [defined in Eq. (5)] to be equal to $\varphi + \psi$, where ψ is a constant, is that all coefficients γ_m must vanish for $m \neq 1$. So, the necessary and sufficient condition for a given algorithm to be insensitive to the presence of harmonics, up to the jth order, in the fringe signal is

$$
\begin{aligned}
&P[\exp(im\delta)] = 0, \\
&\qquad m = -j, -j + 1, \ldots, -1, 0, 2, 3, \ldots, j.
\end{aligned} \tag{12}
$$

So the characteristic polynomial associated with the algorithm must have its roots in the complex-unit circle, at positions of the polar angles $-j\delta$, $(-j + 1)\delta$, $\ldots$, $-\delta$, 0, 2δ, 3δ, $\ldots$, $j\delta$. More precisely, it is possible to state the following rule:

- Rule 1: Insensitivity to the mth harmonic present in the intensity signal can be achieved when the complex numbers $\exp(im\delta)$ (if $m \neq 1$) and $\exp(-im\delta)$ (where δ is the phase shift) are roots of the characteristic polynomial.

In other words, the monomials $[x - \exp(im\delta)]$ and $[x - \exp(-im\delta)]$ must appear in the factorization of the characteristic polynomial. These monomials may be identical, depending on the values of m and δ.

C. Discrete-Fourier-tranform Polynomial and Algorithm

We shall now show that the minimal number of intensity values necessary to evaluate the phase at any given point, if insensitivity to the harmonics up to the jth order that are present in the intensity profile is required, is $M = j + 2$. It is also shown that the corresponding algorithm is the classical N-bucket algorithm, with a phase shift of $2\pi/N$ and with $M = N$. This algorithm is sensitive only to harmonics $N \pm 1 + pN$, where p is an integer.

So the question now is which algorithm involves the lowest number of intensity values for a given value of j? First, we represent graphically in Fig. 1 the condition stated in Eq. (12) when $j = 4$ for the case of an arbitrary phase shift δ. In this figure, the dots indicate the roots of the polynomial, which can be written as

$$P(x) \propto \prod_{\substack{r=-j \\ r\neq 1}}^{j} (x - \xi^r), \tag{13}$$

where $\xi = \exp(i\delta)$. In Fig. 1 δ is equal to 20°. Equation (13) defines a polynomial of degree $2j$, with $2j + 1$ coefficients. So the corresponding algorithm involves $M = 2j + 1$ intensity values. For the example shown in Fig. 1, it would be a nine-step procedure. However, there is obviously a choice of δ that reduces this number, namely, $\delta = \pi/3$. This case is shown in Fig. 2, and it can be seen that the number of roots of the polynomial has been reduced from eight to five. This choice of δ has changed the nine-step procedure into a six-step one. The reason for the reduction is seen clearly from Fig. 2: with this choice of δ (i.e., $\delta = \pi/3$), some roots are associated with two different values of m. Diagrams such as those shown in Figs. 1 and 2 are hereafter called the characteristic diagrams associated with the algorithms.

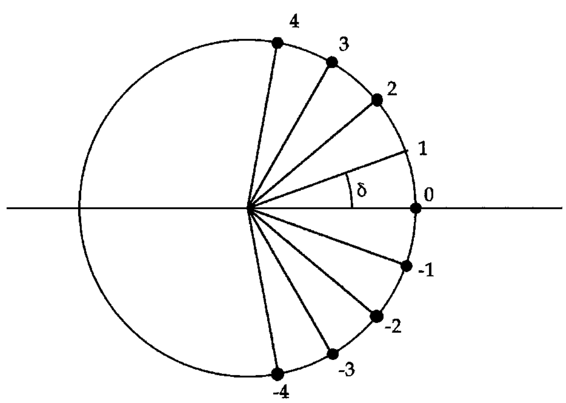

Fig. 1. Characteristic diagram for an arbitrary phase shift δ for the special case of $j = 4$: The numbers represent harmonics (m). The dots show the locations of the roots of the characteristic polynomial. The coefficients of the polynomial are those of an algorithm that is insensitive, up to the jth order, to the harmonics in the fringe signal. Next to each root is indicated the order of the harmonic whose effects are canceled by the presence of this root in the polynomial. The circle is the unit circle of the complex plane.

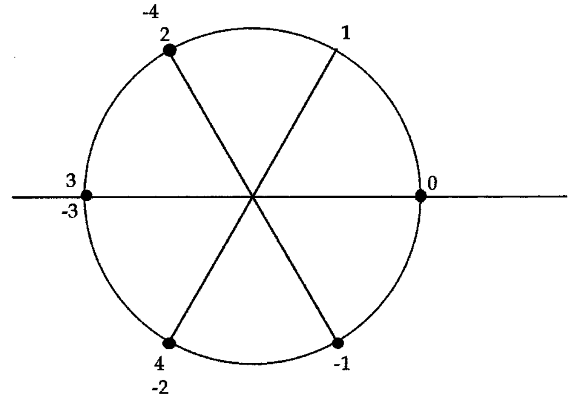

Fig. 2. Same diagram as for Fig. 1, but with a phase shift of $\delta = \pi/3$. Again, the numbers represent harmonics, and the dots their associated roots. Now, only five roots are required to cancel the same number of harmonics in the fringe signal. In this configuration, some roots cancel the effects of two harmonics simultaneously. This diagram is characteristic of the DFT algorithm.

In general, the best choice for δ corresponds to the configuration in which the root associated with $m = -j$ coincides with the one associated with $m = 2$. So the unit circle is divided into $j + 2$ parts and

$$\delta = \frac{2\pi}{j + 2} = \frac{2\pi}{N} \cdot \tag{14}$$

The corresponding polynomial is

$$P(x) \propto \prod_{\substack{k=-j \\ k\neq 1}}^{j} (x - \zeta^k), \tag{15}$$

where $\zeta = \exp(i2\pi/N)$. This polynomial is of the degree $N - 1$ and has N coefficients. So $M = N = j + 2$ intensity values are required. This polynomial can easily be shown to correspond to the classical N-bucket algorithm by the use of the fact that it has all the Nth roots of unity except for ζ:

$$\begin{aligned}\prod_{\substack{k=-j \\ k\neq 1}}^{j} (x - \zeta^k) &= \frac{x^N - 1}{x - \zeta} \\ &= \zeta^{-1}\frac{1 - (\zeta^{-1}x)^N}{1 - \zeta^{-1}x} \\ &= \zeta^{-1}[1 + \zeta^{-1}x + \zeta^{-2}x^2 + \cdots + \zeta^{-(N-1)}x^{N-1}].\end{aligned} \tag{16}$$

We call the discrete-Fourier-transform (DFT) polynomial the following:

$$\begin{aligned} P_N(x) &= 1 + \zeta^{-1}x + \zeta^{-2}x^2 + \cdots + \zeta^{-(N-1)}x^{N-1} \\ &= \zeta \frac{x^N - 1}{x - \zeta} \\ &= \sum_{k=0}^{N-1} \zeta^{-k}x^k. \end{aligned} \tag{17}$$

The discrete Fourier transform is referred to because the algorithm associated with $P_N(x)$ is simply the computation of the first coefficient of the DFT of the set of N values of intensity $I(0), I(\delta), \ldots, I[(N-1)\delta]$. Here, $c_k = \zeta^{-k}$. So the coefficients a_k and b_k of the arctangent form [Eq. (5)] of the corresponding algorithm are

$$\begin{aligned} a_k &= \cos\left(\frac{2\pi k}{N}\right), \\ b_k &= -\sin\left(\frac{2\pi k}{N}\right), \end{aligned} \tag{18}$$

which demonstrates that $P_N(x)$ is the characteristic polynomial of the N-bucket algorithm, which we hereafter propose to call the DFT algorithm to emphasize its intricate relation with the DFT itself. From Eq. (11), the fundamental Fourier coefficient of $S(\varphi)$ with this algorithm is

$$\gamma_1 = \alpha_1 P_N[\exp(i2\pi/N)] = \alpha_1 P_N(\zeta) = N\alpha_1, \tag{19}$$

and the argument of $S(\varphi)$ will be equal to φ. Remember that α_1 is real because of the phase origin chosen in Eq. (2), as was stated above.

We have shown that, when the phase shift δ is exactly equal to $2\pi/N$, the N-bucket algorithm is insensitive to harmonics up to the order $j = N - 2$. This algorithm is also insensitive to *other* harmonics. It is possible to rewrite Eq. (11) for the special case of the N-bucket algorithm:

$$\gamma_m = \alpha_m P_N(\zeta^m). \tag{20}$$

A harmonic of arbitrary order m has the Fourier coefficients α_m and α_{-m}. So the phase evaluation will be sensitive to the presence of the mth harmonic only when $P_N(\zeta^m)$ or $P_N(\zeta^{-m})$ is different from zero. This happens only when

$$m = N \pm 1 + pN, \tag{21}$$

where p is an integer. For example, in the case illustrated in Fig. 2, the associated six-step algorithm is insensitive to harmonics 2, 3, and 4, but also to harmonics 6, 8, 9, 10, 12, 14, ... It is sensitive only to harmonics $5 + 6p$ and $7 + 6p$. This subsection completes the results presented by Stetson and Brohinsky[14]; these results are reproduced in Table 1.

Table 1. Sensitivity of the DFT Algorithm to Harmonics as Deduced with Eq. (21)[a]

Number of Steps (N)	Harmonics (m) 2	3	4	5	6	7	8	9	10	11
3	2		4^b	5		7^b	8		10^b	11
4		3		5^b		7^b		9^b		11
5			4		6^b			9		11^b
6				5		7^b				11

[a]The table is presented as was the Table on p. 3633 of Ref. 14 to permit a direct comparison of the values.

[b]The superscript b denotes those values corresponding to $m = N + 1 + pN$; the other values correspond to $m = N - 1 + pN$.

D. Miscalibrated Phase Shift

The second property required for a phase-shifting algorithm is that it be insensitive to small errors in the value of the phase shift. We address this topic in this subsection and show that this property can also be characterized with the root locations of the characteristic polynomial.

We now assume that the real phase shift is

$$\delta' = \delta(1 + \varepsilon) = \delta + \delta\varepsilon, \tag{22}$$

instead of the nominal phase shift δ. On the basis of Eq. (11), the mth Fourier coefficient of $S(\varphi)$ is rewritten for the phase shift δ' as

$$\gamma_m = \alpha_m P[\exp(im\delta')]. \tag{23}$$

We now use the Taylor expansion

$$\begin{aligned} P\exp(im\delta') &= P[\exp(im\delta)] + (\delta' - \delta)[P'[\exp(im\delta)]im \\ &\quad \times \exp(im\delta) + O(\varepsilon^2) \\ &= P[\exp(im\delta)] + im\varepsilon\delta \exp(im\delta) \\ &\quad \times P'[\exp(im\delta)] + O(\varepsilon^2) \\ &= P[\exp(im\delta)] + im\varepsilon\delta\, \mathbf{D}P[\exp(im\delta)] \\ &\quad + O(\varepsilon^2), \end{aligned} \tag{24}$$

where

$$P'(x) = \frac{\mathrm{d}}{\mathrm{d}x}[P(x)] \tag{25}$$

and where the linear operator $\mathbf{D}$ is defined by

$$\mathbf{D} = x \cdot \frac{\mathrm{d}}{\mathrm{d}x} \cdot \tag{26}$$

Thus one obtains

$$\gamma_m = \alpha_m\{P[\exp(im\delta)] + im\varepsilon\delta\, \mathbf{D}P[\exp(im\delta)]\} + O(\varepsilon^2). \tag{27}$$

It should be noted that $\mathbf{D}P(x)$ is of the same degree as $P(x)$ and is given explicitly from Eq. (9) by

$$\mathbf{D}P(x) = \sum_{k=0}^{M-1} kc_k x^k. \tag{28}$$

If the insensitivity to the mth harmonic when the real phase shift differs from the nominal one is to be maintained, the coefficient γ_m in Eq. (23) and the corresponding γ_{-m} must vanish. It can be seen from Eq. (27) that this condition implies that the characteristic polynomial and its first derivative must have $\exp(im\delta)$ and $\exp(-im\delta)$ as roots. There are, however, two special cases:

- In the case of $m = 1$, only $\exp(-i\delta)$ has to be a double root, because γ_1 must not vanish, as it is the detection term.
- $m = 0$ requires 1 to be only a single root of the characteristic polynomial.

So another rule can be stated, as follows:

Rule 2: Insensitivity to the mth harmonic present in the intensity signal ($m \neq 0$) is achieved in the presence of a phase-shift miscalibration when the two complex numbers $\exp(im\delta)$ (if $m \neq 1$) and $\exp(-im\delta)$ (where δ is the phase shift) are double roots of the characteristic polynomial.

In other words, the squared monomials $[x - \exp(im\delta)]^2$ and $[x - \exp(-im\delta)]^2$ must appear in the factorization of the characteristic polynomial. Depending on the values of m and δ, those monomials may be identical.

E. Sensitivity of the *N*-Bucket Algorithm to a Phase-Shift Miscalibration

We now show how the preceding results can be used to evaluate the influence of a phase-shift miscalibration. For the N-bucket or DFT algorithm in the presence of the perfect sine signal given in Eqs. (2), Eq. (10) yields

$$S(\varphi) = \gamma_1 \exp(i\varphi) + \gamma_{-1} \exp(-i\varphi), \tag{29}$$

and from Eqs. (3) and (27) we obtain

$$\begin{aligned}
\gamma_1 &= \alpha_1[P_N(\zeta) + i\varepsilon\delta\, \mathbf{D}P_N(\zeta)] = \alpha_1\left[N + i\varepsilon\delta \frac{N(N-1)}{2}\right], \\
\gamma_{-1} &= \alpha_1[P_N(\zeta^{-1}) - i\varepsilon\delta\, \mathbf{D}P_N(\zeta^{-1})] \\
&= \alpha_1\left[\varepsilon\delta \frac{N \exp(i2\pi/N)}{2 \sin(2\pi/N)}\right].
\end{aligned} \tag{30}$$

The previous equalities result from

$$\begin{aligned}
P_N(\zeta) &= N, \\
\mathbf{D}P_N(\zeta) &= \frac{N(N-1)}{2}, \\
P_N(\zeta^{-1}) &= 0, \\
\mathbf{D}P_N(\zeta^{-1}) &= \frac{N\zeta}{\zeta^{-1} - \zeta} = \frac{Ni \exp(i2\pi/N)}{2 \sin(2\pi/N)}.
\end{aligned} \tag{31}$$

The last value [i.e., the fourth of Eqs. (31)] is found straightforwardly by the use of the second form of $P_N(x)$ given in Eqs. (17). So we have

$$\begin{aligned}
S(\varphi) = \alpha_1 N \exp(i\varphi)\bigg(1 &+ i\varepsilon\delta \frac{(N-1)}{2} \\
&+ \varepsilon\delta \frac{\exp\{-i[2\varphi - (2\pi/N)]\}}{2 \sin(2\pi/N)}\bigg).
\end{aligned} \tag{32}$$

From this expression and using $\delta = 2\pi/N$, we can deduce that, up to the first order in ε,

$$\Delta\varphi = \varphi^* - \varphi = \frac{\pi\varepsilon}{N}\left\{N - 1 - \frac{\sin[2\varphi - (2\pi/N)]}{\sin(2\pi/N)}\right\}, \tag{33}$$

which is the expression given by Surrel.[11]

F. (*N* + 1)-Bucket Algorithm

All algorithms that can be found in the literature can be analyzed in terms of the roots of their characteristic polynomials, and their respective properties as to the presence of harmonics in the intensity profile or of phase-shift miscalibration can be easily deduced from the location and multiplicity of these roots. For example, the $(N + 1)$-bucket algorithm proposed by Surrel[11] is claimed to be insensitive to a phase-shift miscalibration if the intensity signal is a perfect sine. It is written, in its arctangent form, as

$$\tan \varphi = \frac{\{[I(0) - I(N)]/2\}\cot(2\pi/N) - \sum_{n=1}^{N-1} I(n)\sin(2\pi n/N)}{\{[I(0) + I(N)]/2\} + \sum_{n=1}^{N-1} I(n)\cos(2\pi n/N)}, \tag{34}$$

and so the characteristic polynomial is

$$\begin{aligned}
P(x) &= \frac{1}{2}[1 + i \cot(2\pi/N)] \\
&\quad + \zeta^{-1}x + \zeta^{-2}x^2 + \cdots + \zeta^{-(N-1)}x^{N-1} \\
&\quad + \frac{1}{2}[1 - i\cot(2\pi/N)]x^N, \\
&= \frac{-\zeta^{-1}}{\zeta - \zeta^{-1}} + \zeta^{-1}x + \zeta^{-2}x^2 \\
&\quad + \cdots + \zeta^{-(N-1)}x^{(N-1)} + \frac{\zeta}{\zeta - \zeta^{-1}} x^N,
\end{aligned} \tag{35}$$

with $\zeta = \exp(i2\pi/N)$. An elementary computation shows that

$$P(x) = P_N(x) \frac{x - \zeta^{-1}}{\zeta - \zeta^{-1}}, \tag{36}$$

where $P_N(x)$ is still the DFT polynomial. Equation (36) shows that ζ^{-1} is a double root of the characteristic algorithm, which ensures the insensitivity of the characteristic algorithm to a phase-shift miscalibration, up to the first order in ε. The corresponding characteristic diagram is shown in Fig. 3 for the special case of $N = 6$ (a circled dot indicates a double root).

G. Improved Insensitivity to Phase-Shift Miscalibration

It is very easy to see how an improved insensitivity to phase-shift miscalibration can be achieved. First we have

$$\frac{d}{d\delta} P[\exp(im\delta)] = im \exp(im\delta) P'[\exp(im\delta)] = im\mathbf{D}P[\exp(im\delta)]. \tag{37}$$

This property is valid for *any* polynomial, and in particular for $\mathbf{D}P$. So,

$$\frac{d^2}{d\delta^2} P[\exp(im\delta)] = (im\mathbf{D})^2 P[\exp(im\delta)], \tag{38}$$

and in general

$$\frac{d^k}{d\delta^k} P[\exp(im\delta)] = (im\mathbf{D})^k P[\exp(im\delta)]. \tag{39}$$

Now the complete Taylor expansion initiated in Eq. (27) can be written as

$$\gamma_m = \alpha_m \left[\sum_{k=0}^{\infty} \frac{(im\varepsilon\delta\mathbf{D})^k P}{k!}\right][\exp(im\delta)], \tag{40}$$

with the convention that $\mathbf{D}^0$ is the identity operator.

It should now be clear that improvements in the insensitivity of the algorithm to the presence of the mth harmonic can be achieved by cancellation of the successive terms of increasing power in ε, both in γ_m and in γ_{-m}. For canceling the term of degree k in ε and in all preceding terms, all polynomials $\mathbf{D}^l P$ must have $\exp(im\delta)$ and $\exp(-im\delta)$ as roots for $0 \le l \le k$. If the following equalities are considered:

$$\begin{aligned} \mathbf{D}P(x) &= xP'(x), \\ \mathbf{D}^2P(x) &= x[P'(x) + xP''(x)] = xP'(x) + x^2P''(x), \\ \mathbf{D}^3P(x) &= x[P'(x) + xP''(x) + 2xP''(x) + x^2P'''(x)], \\ \mathbf{D}^4P(x) &= xP'(x) + 3x^2P''(x) + x^3P'''(x), \ldots, \end{aligned} \tag{41}$$

it can be seen that this cancellation condition is equivalent to the fact that $\exp(im\delta)$ and $\exp(-im\delta)$ must be roots of the order of $k + 1$ of the polynomial P because $\mathbf{D}^l P$ is expressed as a linear combination of the derivatives of P that are of an order less than or equal to l.

Hence the following rule can be stated (the demonstration of the last statement is left to the reader):

Rule 3: The algorithmic insensitivity to the mth harmonic ($m \neq 0$) is achieved in the presence of a phase-shift miscalibration when the two complex numbers, $\exp(im\delta)$ (if $m \neq 1$) and $\exp(-im\delta)$ (where δ is the phase shift), are roots of the order of $k + 1$ of the characteristic polynomial. The phase measured will contain no term in ε^l, $l \le k$, as a result of the presence of this harmonic.

In other words, the factors $[x - \exp(im\delta)]^k$ and $[x - \exp(-im\delta)]^k$ must appear in the factorization of the characteristic polynomial. These factors may be identical, depending on the values of m and δ.

H. Characteristic Polynomials of Some Algorithms

In this subsection are presented some algorithms that can be found in the literature and their corresponding characteristic polynomials. From the factorization, the reader should be able to evaluate the respective properties and interests of these algorithms and the relation between φ and φ^*.

1. Five-Sample Algorithms

The five-sample algorithm described here was developed by Hariharan *et al.*[15] It reads

$$\tan \varphi^* = \frac{2(I_1 - I_3)}{-I_0 + 2I_2 - I_4} \cdot \tag{42}$$

The corresponding characteristic polynomial is

$$\begin{aligned} P(x) &= -1 + 2ix + 2x^2 - 2ix^3 - x^4 \\ &= -(x - 1)(x + 1)(x + i)^2. \end{aligned} \tag{43}$$

Another five-sample algorithm was developed by Schmit and Creath.[16] It reads as

$$\tan \varphi^* = \frac{-I_0 + 4(I_1 - I_3) + I_4}{I_0 + 2I_1 - 6I_2 + 2I_3 + I_4}, \tag{44}$$

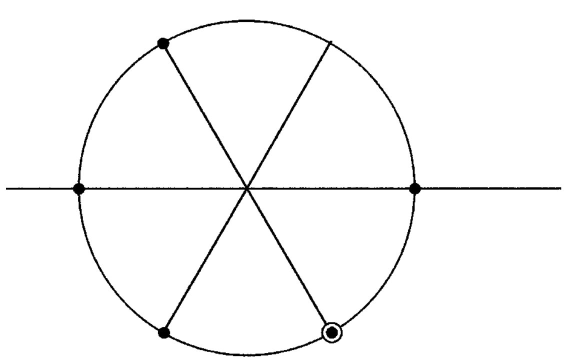

Fig. 3. Characteristic diagram for the special case of $N = 6$ for the $(N + 1)$-bucket algorithm proposed by Surrel.[11] The circled dot indicates a double root, and the presence of a double root provides insensitivity to a phase-shift miscalibration when the signal is a perfect sine.

and its corresponding characteristic polynomial is

$$P(x) = 1 - i + (2 + 4i)x - 6x^2 + (2 - 4i)x^3 + (1 + i)x^4 = (1 + i)(x - 1)(x - i)^3. \quad (45)$$

2. 6 + 1-Sample Algorithm

The (6 + 1)-sample algorithm below [Eq. (46)] was developed by Larkin and Oreb[9] and reads

$$\tan \varphi^* = \sqrt{3}\,\frac{I_1 + I_2 - I_4 - I_5}{-I_0 - I_1 + I_2 + 2I_3 + I_4 - I_5 - I_6}\cdot \quad (46)$$

Its corresponding characteristic polynomial is

$$\begin{aligned} P(x) &= -1 - (1 - i\sqrt{3})x + (1 + i\sqrt{3})x^2 + 2x^3 \\ &\quad + (1 - i\sqrt{3})x^4 - (1 + i\sqrt{3})x^5 - x^6 \\ &= -(x - 1)(x + 1)(x - \zeta^{-1})(x - \zeta^2)(x - \zeta^{-2})^2, \end{aligned} \quad (47)$$

where $\zeta = \exp(i\pi/3)$.

3. Seven-Sample Algorithm

The seven-sample algorithm presented here was developed by de Groot.[17] It reads

$$\tan \varphi^* = \frac{I_0 - 7(I_2 - I_4) - I_6}{4I_1 - 8I_3 + 4I_5}, \quad (48)$$

and its corresponding characteristic polynomial is

$$P(x) = i + 4x - 7ix^2 - 8x^3 + 7ix^4 + 4x^5 - ix^6 = -i(x - 1)(x + 1)(x + i)^4. \quad (49)$$

I. Minimal Algorithms: Examples for $j = 2$ and $j = 3$

It is now time to answer our final question: what is the algorithm

- that is insensitive to harmonics up to the order j,
- that is insensitive to a phase-shift miscalibration,
- that involves the minimum number of intensity values?

It should be clear from the preceding discussions that, for the characteristic diagram of this algorithm, which is given in Fig. 4 and corresponds to the special case of $j = 4$, the phase shift must be equal to $2\pi/N$, with $N = j + 2$, that the characteristic polynomial must have all the Nth roots of unity (except 1 and ζ) be double roots, and that the characteristic polynomial must also have 1 as a single root. The polynomial is of degree $2N - 3$, and so it has $2N - 2$ coefficients, that is, $2N - 2 = 2j + 2$ intensity values are needed. This is the minimal number of intensity values required. These conditions correct a conclusion from Hibino *et al.*[12] (i.e., that $2j + 3$ values are needed).

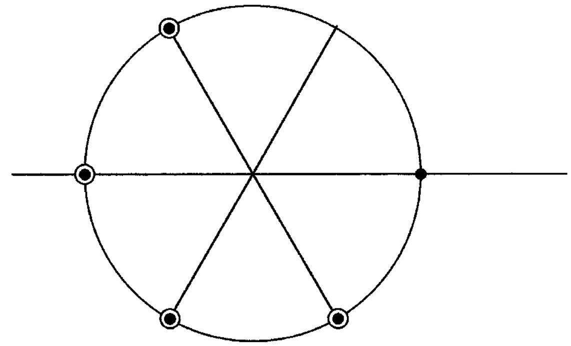

Fig. 4. Characteristic diagram for the special case of $j = 4$ of the minimal algorithm that involves the lowest number of intensity values and is insensitive to the harmonic content of the intensity profile up to the jth order, even in the presence of a phase-shift miscalibration. The circled dots indicate double roots.

We call the algorithm corresponding to these conditions the minimal algorithm. For example, to achieve insensitivity to the second harmonic ($j = 2$), one has $N = 4$, and the characteristic polynomial reads

$$\begin{aligned} P(x) &= -i(x - 1)(x + 1)^2(x + i)^2 \\ &= -i + (-2 - i)x + (-2 + 2i)x^2 \\ &\quad + (2 + 2i)x^3 + (2 - i)x^4 - ix^5. \end{aligned} \quad (50)$$

The proportionality factor $-i$ is there just to retain some amount of symmetry in the coefficients list.

Now, if the intensity signal contains only two harmonics, that is, if

$$\alpha_m = 0, |m| \geq 3, \quad (51)$$

one has, with the polynomial from Eqs. (50),

$$S(\varphi) = \alpha_1 P(i)\exp(i\varphi), \quad (52)$$

and, as $P(i) = 8(1 - i)$, the measured phase is

$$\varphi^* = \varphi - \frac{\pi}{4}\cdot \quad (53)$$

The polynomial in Eqs. (50) corresponds to the following arctangent form of a six-sample algorithm:

$$\varphi^* = \tan^{-1}\left[\frac{-I_0 - I_1 + 2I_2 + 2I_3 - I_4 - I_5}{2(-I_1 - I_2 + I_3 + I_4)}\right]\cdot \quad (54)$$

The seven-sample algorithm proposed by Hibino *et al.*[12] has exactly the same properties as the one in Eq. (54). It reads in the arctangent form (a minus sign has been added so that the detected phase has the proper sign) as

$$\varphi^* = \tan^{-1}\left[-\frac{I_0 - 3I_2 + 3I_4 - I_6}{2(-I_1 + 2I_3 - I_5)}\right], \tag{55}$$

and it corresponds to the polynomial

$$\begin{aligned} P(x) &= -i - 2x + 3ix^2 + 4x^3 - 3ix^4 - 2x^5 + ix^6 \\ &= i[(x-1)(x+1)(x+i)]^2, \end{aligned} \tag{56}$$

where 1 is a double root, which is not necessary. Because $P(i) = -16i$,

$$S(\varphi) = \alpha_1 P(i)\exp(i\varphi) = -16i\alpha_1 \exp(i\varphi), \tag{57}$$

and then

$$\varphi^* = \varphi - \frac{\pi}{2}\cdot \tag{58}$$

Similarly, if the third harmonic is to be eliminated, one finds the following 10-bucket polynomial, whose characteristic diagram is presented in Fig. 4:

$$P(x) = -2\zeta(x-1)(x-\zeta^{-1})^2(x-\zeta^{-2})^2(x+1)^2(x-\zeta^2)^2, \tag{59}$$

where $\zeta = \exp(i\pi/3)$. Equation (59) yields the following arctangent form:

$$\varphi^* = \tan^{-1}\left[\frac{\sqrt{3}(-I_0 - 3I_1 - 3I_2 + I_3 + 6I_4 + 6I_5 + I_6 - 3I_7 - 3I_8 - I_9)}{I_0 - I_1 - 7I_2 - 11I_3 - 6I_4 + 6I_5 + 11I_6 + 7I_7 + I_8 - I_9}\right]. \tag{60}$$

The 11-sample algorithm presented by Hibino *et al.*[12] (and provided with a minus sign as before) has a characteristic polynomial of

$$P(x) = -2(x-1)(x-\zeta^{-1})^2(x-\zeta^{-2})^2(x+1)^2(x-\zeta^2)^3, \tag{61}$$

with a surprising triple root at ζ^2. This 11-sample algorithm has the same properties as the 10-bucket algorithm shown in Eq. (60).

For an arbitrary value of N, the coefficients may become quite complicated. Fortunately, it is shown in Subsection 2.J. that simple results can still be obtained if an extra intensity value is permitted.

J. Windowed-DFT Algorithm

In our quest for algorithms insensitive to harmonics up to the order $j = N - 2$, even in presence of a phase-shift miscalibration, we now consider an algorithm involving one extra intensity value, which corresponds to a characteristic polynomial that also has 1 as a double root. Let us now consider the characteristic polynomial

$$\begin{aligned} P(x) &= [P_N(x)]^2 \\ &= 1 + 2\zeta^{-1}x + 3\zeta^{-2}x^2 \\ &\quad + \cdots + (N-1)\zeta^2 x^{N-2} + N\zeta x^{N-1} \\ &\quad + (N-1)x^N + (N-2)\zeta^{-1}x^{N+1} \\ &\quad + \cdots + 2\zeta^3 x^{2N-3} + \zeta^2 x^{2N-2} \\ &= \sum_{k=0}^{2N-1} \zeta^{-k} t_k x^k, \end{aligned} \tag{62}$$

where $\{t_k, k = 0, \ldots, 2N - 1\} = \{1, 2, \ldots, N-1, N, N-1, \ldots, 2, 1, 0\}$. The characteristic diagram for $N = 6$ is shown in Fig. 5.

From Eqs. (7), (9), and (62) and by denoting $I_k = I(\varphi + k\delta)$, one gets

$$S(\varphi) = \sum_{k=0}^{2N-1} t_k \zeta^{-k} I_k = \sum_{k=0}^{2N-1} (t_k I_k)\exp\left\{-i\left[\frac{2\pi(2k)}{2N}\right]\right\}\cdot \tag{63}$$

The argument of this last exponential function has been written in a form that shows that $S(\varphi)$ can be interpreted simply as the second DFT coefficient of a set of $2N$ intensity values extending over two periods and windowed by the triangle function represented by the set of coefficients t_k. As is well known from the DFT theory, the *first* coefficient corresponds to a signal frequency of one period over the sampling window and is of no interest. Thus, we call the corre-

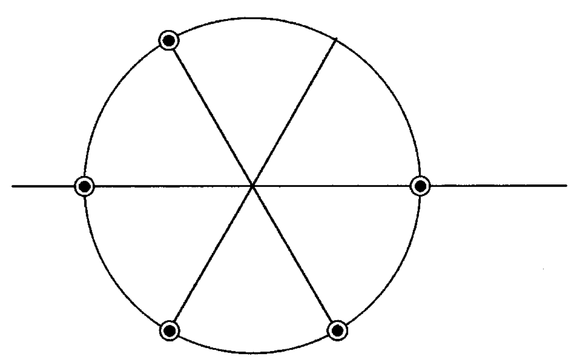

Fig. 5. Characteristic diagram of the WDFT algorithm. The algorithm has the same properties as that associated with the diagram in Fig. 4 (minimal algorithm), but its coefficients have a much simpler analytic form in the general case. The WDFT algorithm corresponds to the discrete Fourier transform of a set of intensities that extend over two signal periods and are windowed by a symmetrical triangle function.

Table 2. Comparison of the Present and Fourier-Transform Approaches to the Cancellation of Algorithmic Sensitivity to Harmonics m

Insensitivity	Present Approach	Fourier-Transform Approach
To the harmonic m	$P[\exp(im\delta)] = 0$	$F_1\left(\frac{m}{2\pi}\right) = -iF_2\left(\frac{m}{2\pi}\right)$
To miscalibration	$P'[\exp(im\delta)] = 0$	$\frac{dF_1}{d\nu}\left(\frac{m}{2\pi}\right) = -i\frac{dF_2}{d\nu}\left(\frac{m}{2\pi}\right)$

sponding algorithm the windowed-DFT (WDFT) algorithm.

It is to be noted that, for computational efficiency, it is a good choice to introduce a multiplying factor ζ^{-1} into $P(x)$ to obtain some amount of symmetry in the coefficients list. This factor enables us to write the following arctangent form of the W-DFT algorithm, which is insensitive to harmonics up to the order of $j = N - 2$, even in the presence of a phase-shift miscalibration:

$$\varphi^* = \tan^{-1}\left[-\frac{\sum_{k=1}^{N-1} k(I_{k-1} - I_{2N-k-1})\sin\left(\frac{2\pi k}{N}\right)}{NI_{N-1} + \sum_{k=1}^{N-1} k(I_{k-1} + I_{2N-k-1})\cos\left(\frac{2\pi k}{N}\right)}\right]. \tag{64}$$

The consequence of the introduction of the multiplying factor ζ^{-1} is that the measured phase is given by

$$\varphi^* = \varphi - \frac{2\pi}{N}\cdot \tag{65}$$

For $N = 4$ Eq. (64) reads

$$\varphi^* = \tan^{-1}\left[-\frac{(I_0 - I_6) - 3(I_2 - I_4)}{4I_3 - 2(I_1 + I_5)}\right], \tag{66}$$

which is nothing but the seven-sample algorithm proposed by Hibino *et al.*[12] and reported in Eq. (55).

3. Conclusion

A simple and straightforward method for deriving algorithms usable in phase-stepping data processing is presented in this paper. It is based on the one-to-one correspondence that can be made between an algorithm and a polynomial, which we have termed the characteristic polynomial, with the coefficients of the former being those of the latter. Quite simple rules apply to the location and multiplicity of the characteristic-polynomial roots to obtain algorithmic insensitivity with respect to an arbitrary harmonic content in the wave signal, with or without a constant phase-shift miscalibration. Hence, with a simple monomial product, a fully customized algorithm can be generated.

A graphical representation of the polynomial root locations and multiplicity, which provides a simple manner for summarizing the properties of the algorithm, has been introduced. Some classical algorithms have been analyzed in terms of their characteristic polynomials, and two new algorithms involving six and 10 samples have been derived. Generally, it is shown that the minimal number of intensity values required to obtain insensitivity, up to the jth order, to the harmonic content of the signal in the presence of a constant phase-shift miscalibration is $2j + 2$. The phase shift has to be $2\pi/(j + 2)$ between samples.

Finally, the general form of a self-calibrating algorithm involving samples extending over two periods has been proposed. Its relation with the windowing process in the discrete Fourier transform theory has also been demonstrated.

Appendix A.

The correspondence between our approach and the so-called Fourier-transform approach[9,13] is straightforward. In the Fourier-transform approach, one introduces the sampling functions related to the numerator and denominator of the fraction defining the tangent of the measured phase:

$$f_1(t) = \sum_{k=0}^{M-1} a_k\delta(t - t_k),$$
$$f_2(t) = \sum_{k=0}^{M-1} b_k\delta(t - t_k), \tag{A1}$$

and their inverse Fourier transforms (the direct transform is not used so that consistent signs would be preserved in the following) are

$$F_1(\nu) = \mathfrak{F}^{-1}f_1(\nu),$$
$$F_2(\nu) = \mathfrak{F}^{-1}f_2(\nu), \tag{A2}$$

We now assume that the shift parameter t is actually a phase, φ, so that Eqs. (A1) can be rewritten as

$$f_1(\varphi) = \sum_{k=0}^{M-1} a_k\delta(\varphi - k\delta),$$
$$f_2(\varphi) = \sum_{k=0}^{M-1} b_k\delta(\varphi - k\delta), \tag{A3}$$

where δ is still the phase shift. Note that the Fourier variable ν is no longer a temporal frequency. In this Fourier analysis, monochromatic signals are written as $\exp(i2\pi\nu\varphi)$.

Now we introduce the angular frequency:

$$\omega = \frac{\nu}{2\pi}\cdot \tag{A4}$$

Thus, the signal $\exp(i\varphi)$ corresponds to the angular frequency of $\omega = 1$. On the basis of the ideas presented in this paper, we can make the following definition:

$$g(\varphi) = f_1(\varphi) + if_2(\varphi). \tag{A5}$$

The inverse Fourier transform of $g(\varphi)$ is

$$\begin{aligned} G(\nu) &= \mathfrak{F}^{-1}g(\nu) = F_1(\nu) + iF_2(\nu), \\ &= \sum_{k=0}^{M-1} (a_k + ib_k)\exp(i2\pi\nu k\delta) = P[\exp(i2\pi\nu\delta)] \\ &= P[\exp(i\omega\delta)], \end{aligned} \tag{A6}$$

where $P(x)$ is the characteristic polynomial. These equations give the relationship between the filtering functions $F_1(\nu)$ and $F_2(\nu)$ used in the Fourier transform approach and the characteristic polynomial. In particular, if the sensitivity to a harmonic m is to be cancelled, with or without a miscalibration, the conditions listed in Table 2 must hold. From Eq. (A6), it can be seen that $2\pi\nu\delta = \omega\delta$ is the polar angle on the unit circle that corresponds to the angular frequency ω. So the graduation in m that appears around the circle in Figs. 1 and 2 is nothing but a graduation in ω. This is precisely why we have chosen an inverse, rather than a direct, Fourier transform in Eqs. (A2).

The author's e-mail address is surrel@emse.fr.

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