Vanishing tilt-to-length coupling for a singular case in two-beam laser interferometers with Gaussian beams

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The omnipresent tilt-to-length coupling in two-beam laser interferometers, frequently a nuisance in
precision measurements, vanishes for the singular case of two beams with identical parameters and
complete detection of both beams without clipping. This effect has been observed numerically and is
explained in this manuscript by the cancellation of two very different effects of equal magnitude and
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1. Introduction

One recurring noise source in precision interferometric length measurements is the parasitic coupling of misalignments (tilt) into the length readout, which arises due to straightforward geometrical path length changes of the beam axis. In this manuscript we show that in the special case of two identical fundamental Gaussian beams, a large detector without any clipping and a pivot placed directly on the beam axis, another effect of the same magnitude and the opposite sign occurs which, to first order, cancels the geometrical path length change in the interferometric measurement.

This manuscript does not investigate the effect of a lateral offset between the pivot and beam axis, but only the effect of a lever arm between the pivot and photodetector.

2. Geometrical Coupling

We consider a simplified interferometer reduced to its essential components. Only the reference beam,

the (tilted) measurement beam, and the photodiode are considered (Fig. 1). The photodiode detects the interference pattern between the measurement and reference beams, and from its photocurrent it is possible to determine the phase difference between the two beams by a variety of different readout schemes, both homodyne and heterodyne [1]. That phase difference can be translated to the longitudinal path length sensing signal (s_{LPS}) that describes the difference in the traveled path length between the two beams [1]. The result is independent of the interferometer type (homodyne or heterodyne). Any tilt of the measurement beam causes a tilt-to-length coupling.

Intuitively, one expects a coupling between the measurement beam angle and the path length change. The beam tilt results in a longer distance that the measurement beam has to travel from the pivot to the photodiode. Using straightforward geometry, this geometric path length change Δs_{geo} can be computed analytically,

$$\Delta s_{geo} = \left(\frac{1}{\cos(\alpha)} - 1\right)d_{pivot} \approx \frac{\alpha^2}{2}d_{pivot} + \mathcal{O}(\alpha^4). \quad (1)$$
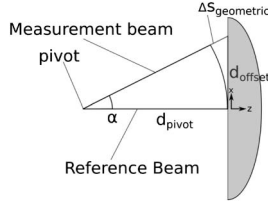
Fig. 1. Reference beam is centered on the detector and the measurement beam is tilted by the angle α around the pivot. The geometric path length change Δs is the additional distance the measurement beam has to propagate until it reaches the detector.

Here, α is the beam angle and d_{pivot} is the distance between the pivot and photodiode (as shown in Fig. 1). One would expect that this geometric path length change always appears in the measured s_{LPS}. This manuscript shows that this is indeed true for plane waves, but not for Gaussian beams.

3. Plane Waves

In this section, the relation between the geometrical path length change and the s_{LPS} is discussed for the case of two plane waves. The electric field for an infinite plane wave is given by

$$E_{\mathrm{plane}}(x,y,z) = A \exp(-i\omega t - ikz + i\Phi), \qquad (2)$$

if the plane wave propagates in the z direction, where $k = 2\pi/\lambda$ is the wave number, ω the frequency, A is the amplitude, and Φ is the initial phase. This expression is used for the reference beam $E_{\mathrm{ref}}(x,y,z) = E_{\mathrm{plane}}(x,y,z)$, and a rotated version is used for the tilted measurement beam. We denote the coordinate system of E_{ref} by $\vec{\mathbf{r}}_{\mathrm{ref}}$, the location of the pivot by $\vec{\mathbf{p}}_{\mathrm{pivot}}$, the rotation matrix for a rotation around the y axis is $\hat{\mathbf{m}}_{\mathrm{rot}}$, and the resulting coordinate system of the second electric field E_{meas} is called $\vec{\mathbf{r}}_{\mathrm{meas}}$:

$$\vec{\mathbf{r}}_{\mathrm{ref}} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \qquad \vec{\mathbf{p}}_{\mathrm{pivot}} = \begin{pmatrix} 0 \\ 0 \\ -d_{\mathrm{pivot}} \end{pmatrix},$$

$$\hat{\mathbf{m}}_{\mathrm{rot}} = \begin{pmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{pmatrix}, \qquad (3)$$

$$\vec{\mathbf{r}}_{\mathrm{meas}} = \hat{\mathbf{m}}_{\mathrm{rot}}^{-1} \cdot (\vec{\mathbf{r}}_{\mathrm{ref}} - \vec{\mathbf{p}}_{\mathrm{pivot}}) + \vec{\mathbf{p}}_{\mathrm{pivot}}. \qquad (4)$$

A more detailed explanation of this transformation can be found in [2]. The tilted electric field is now defined similar to E_{ref}, but with new coordinates:

$$E_{\mathrm{meas}}(x,y,z) = E_{\mathrm{plane}}(\vec{\mathbf{r}}_{\mathrm{meas}}(x,y,z)). \qquad (5)$$

Since the z position of the photodiode plane is arbitrary, it can be set to zero. The path length difference between the two beams is encoded in the intensity of the superposition between the two beams and thus also in the power as measured by a photodiode. By computing the integral of the intensity over the entire sensitive area, it is possible to extract the path length difference by analyzing the power fluctuation on the photodiode [1]. The same information is also covered in the complex phase of the integral over the overlap term,

$$\arg\left(\int_{\mathrm{pd}} E_{\mathrm{meas}} E_{\mathrm{ref}}^* \mathrm{d}r^2\right) = k s_{\mathrm{LPS}}. \qquad (6)$$

We prefer to extract the phase from the complex overlap term [Eq. (6)] instead of from the power variation as this reduces the computational effort. Since the s_{LPS} does not change in time and we are only interested in the variation of the phase difference between the two beams, we can set $t = 0$ and the initial total phase $\Phi = 0$. An integration of the overlap term over a square detector at position $z = 0$ (side length $2r_{\mathrm{pd}}$) gives the overlap integral for plane waves O_{ovi}^P, which corresponds to the complex amplitude in [1],

$$O_{\mathrm{ovi}}^P = A_{\mathrm{ref}} A_{\mathrm{meas}} \frac{4r_{\mathrm{pd}}\{\cos[kd_{\mathrm{pivot}}(-1 + \cos\alpha)] - i\sin(-kd_{\mathrm{pivot}} + kd_{\mathrm{pivot}}\cos\alpha)\}\sin(kd_{\mathrm{pivot}}\sin\alpha)}{k\sin\alpha}. \qquad (7)$$

The complex phase of this integral describes the phase difference between the two plane waves. This phase difference can be translated to the s_{LPS} using the wave number k:

$$s_{\mathrm{LPS}} = \frac{\arg(O_{\mathrm{ovi}}^P)}{k} \approx \frac{\alpha^2}{2} d_{\mathrm{pivot}} + \mathcal{O}(\alpha^4) \approx \Delta s_{\mathrm{geo}}. \qquad (8)$$

Thus, two plane waves on a detector show approximately the geometric coupling Δs_{geo}, confirming the intuitive results from Eq. (1).

4. Gaussian Beams

In laser interferometers, fundamental Gaussian beams are a more appropriate description than plane waves. We start with the special case of two identical fundamental Gaussian beams and an infinite detector (i.e., both beams are completely detected without

any clipping). The amplitude of the electric field is irrelevant for the path length signal and is therefore set to unity. The Gouy phase is also ignored, since its offset is negligible in the case of equal beams. The electric field can then be written as [3,4],

$$E_{\mathrm{Gauss}}(x,y,z) = \exp\left(-i\omega t - ik\frac{x^2+y^2}{2q} - ik(z-z_0)\right),$$
(9)

with the complex q parameter $q = (z-z_0) + iz_r$, where z_r is the Rayleigh range and z_0 denotes the position of the waist. The term $-ik(z-z_0)$ can be set to $-ikz$ since the z_0-dependent phase shift will not change. The expression in Eq. (9) is used for the reference beam $E_{\mathrm{ref}}(x,y,z)$, and a rotated version is used for the tilted measurement beam. The coordinate system of the tilted beam is computed analogously to Eq. (4). The real part of the q parameter changes only by propagation in the beam direction. For the reference beam, the propagation corresponds to an increase in z. For the measurement beam, the change in the direction of propagation corresponds to an increase in z but also a change of x [Eq. (4)]. This x dependence makes the 2D integration in the detector plane much harder. Therefore, the real part of the q parameter $z - z_0$ is set to the constant value $-z_0$. The changes in z due to the coordinate transformation over the detector surface are very small and cause a negligible change (therefore the error produced by a z-independent q parameter also becomes very small and is neglected). Furthermore, an infinite detector is assumed, which practically means any single element photodiode (SEPD) that is larger than three times the beam size. The integral of the overlap term over an infinite detector at position $z = 0$ yields

$$O_{\mathrm{ovi}}^{G} = \frac{2\pi(z_0^2 + z_r^2)}{k\sqrt{z_r[-iz_0 + 3z_r + (z_r + iz_0)\cos 2\alpha]}}$$
$$\times \exp\left[\frac{2ik\xi\,\sin(\alpha/2)^2}{-z_0 - 3iz_r + (z_0 - iz_r)\cos(2\alpha)}\right] \quad (10)$$

with:

$$\xi = (z_0 + d_{\mathrm{pivot}})^2 + 2id_{\mathrm{pivot}}z_r + z_r^2$$
$$+ [(z_0 + d_{\mathrm{pivot}})^2 - 2id_{\mathrm{pivot}}z_r + z_r^2]\cos\alpha.$$

This leads to the resulting path length change:

$$s_{\mathrm{LPS}} = \frac{\arg(O_{\mathrm{ovi}}^{G})}{k} \approx \frac{\alpha^2 z_0}{4z_r k} + \mathcal{O}(\alpha^4) \approx 0. \quad (11)$$

This result matches the expressions in [2] for the special case of equal beams. For two plane waves the resulting coupling [Eq. (8)] has the same form (proportional to α^2) with a prop. factor given by $d_{\mathrm{pivot}}/2$, which usually is a macroscopic quantity of magnitude between centimeters and meters. For
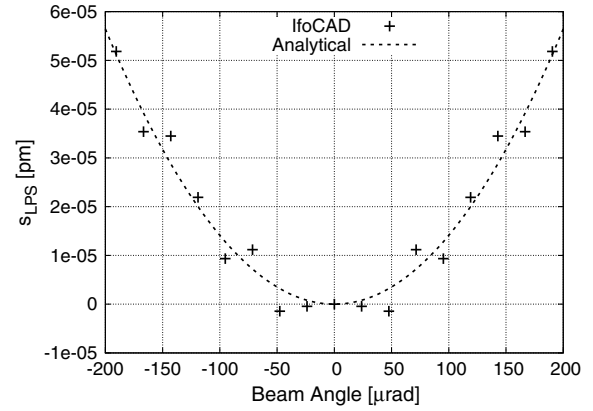
Fig. 2. Numerically/analytically computed s_{LPS} (second-order approximation) for Gaussian beams. The differences are due to numerical errors.

two Gaussian beams this factor becomes $z_0/(4z_r k)$, which is of the same order of magnitude as the wavelength, i.e., nanometers to micrometers for visible or infrared light. For typical parameters and beam angles, ≈ 1 mrad, the resulting length change is significantly below the picometer scale and thus below the sensitivity of most interferometers.

All results in this manuscript were confirmed by numerical simulations computed with IfoCAD [1,5]. Example results for the actual setup are shown in Fig. 2 for or a wavelength of 1064 nm, waist radii of 1 mm, 30 mm photodiode diameter, and the pivot and waists located 100 mm in front of the photodiode.

In the remaining part of the manuscript, we will show that the reason for the vanishing coupling for two Gaussian beams is an additional coupling effect that is caused by an angle-depending offset. A beam tilt generates, besides the geometric change of the path length, two other effects. The first one is a relative angle between the two beams on the photodiode and the other is an offset between them (d_{offset} in Fig. 1). Since infinite plane waves have no uniquely defined center, any shift orthogonal to
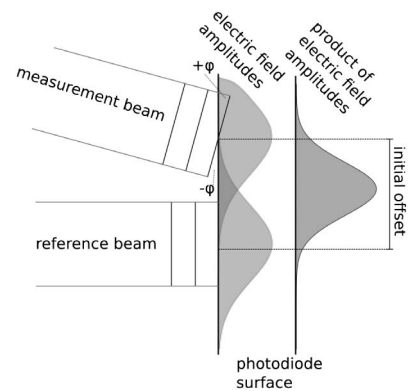

Fig. 3. Reference beam is centered on the SEPD, the measurement beam has an offset and is tilted around its center. This offset results in a different weighting of the phase differences; the negative phase part has a higher weighting and the resulting coupling shows a negative phase.

their direction of propagation maps the wave upon itself and therefore causes no effect. This is different for Gaussian beams. Due to the Gaussian intensity profile, there is a uniquely defined center. To investigate the effect of the generated offset in the case of Gaussian beams, the initial setup (Fig. 1) is changed to create a situation with an angle-invariant offset and no lever arm. We place the pivot directly on the detector. Furthermore, the measurement beam is placed with a transversal offset and tilted around its center on the SEPD (Fig. 3). According to Eq. (1) there is no coupling for plane waves ($d_{pivot} = 0$), and only the effect of the static offset remains.

For the analytical computation, we assume that the initial transversal offset changes the stationary reference beam $E_{ref}(x + d_{offset}, y, z)$ instead of the measurement beam, since it is unimportant which beam is moved and the transformation of the measurement beam would be more complicated with an additional offset. The distance between pivot and SEPD is set to zero ($d_{pivot} = 0$) and the measurement beam E_{meas} is rotated around zero [see Eq. (4)]. The expressions for the beams are the same as in Eq. (9). The resulting overlap integral O_{ovi}^{GO} for Gaussian beams with initial offset becomes:

is also caused by the beam tilt. Both effects generate the same amount of coupling, but with a different sign. In the special case of two identical Gaussian beams on an infinite single element diode, the resulting coupling between beam tilt and measured path length becomes negligible.

5. Discussion

For unequal beam parameters Eq. (34) in [2] shows that additional coupling terms appear that disturb the balance between both effects and lead to significant residual coupling. As an example, Fig. 4 shows numerical simulations and analytical expressions of the coupling for the same situation as in Section 4 but with slightly different waist positions for the measurement and reference beams. Similarly, incomplete detection, non-fundamental Gaussian beams, and misalignment cause non-negligible tilt-to-length coupling, as we have observed in numerical simulations and more complex analytic computations.

Therefore, the effect described in this manuscript only appears under very specific circumstances. However, it can be used in various situations, for example to stabilize an interferometer and investigate additional coupling effects. A manuscript on experiments that make use of this effect to measure

$$O_{ovi}^{GO} = \frac{\exp\left(-\frac{k\{-id_{offset}^2 \cos \alpha^2 + (z_r - iz_0) \sin \alpha[2d_{offset} + (z_0 - iz_r) \sin \alpha]\}}{-z_0 - 3iz_r + (z_0 - iz_r) \cos(2\alpha)}\right) 2\pi(z_0^2 + z_r^2)}{k\sqrt{z_r[-iz_0 + 3z_r + (iz_0 + z_r)\cos(2\alpha)]}}. \tag{12}$$

This leads to the path length change,

$$s_{LPS} \approx \frac{-\alpha d_{offset}}{2} + \mathcal{O}(\alpha^2). \tag{13}$$

This coupling is a result of a static offset. To compute the effect of the dynamic (angle depending) offset in the initial case (as shown in Fig. 1), the offset itself (d_{offset}) has to be replaced by its geometric expression,

$$d_{offset} = \tan(\alpha)d_{pivot} \approx \alpha d_{pivot}. \tag{14}$$

By combining Eqs. (13) and (14) the coupling caused by the offset in the initial setup becomes

$$s_{LPS} \approx \frac{-\alpha d_{offset}}{2} = \frac{-\alpha^2}{2}d_{pivot}. \tag{15}$$

Therefore, the negligible tilt-to-s_{LPS} coupling of Eq. (11) is the result of two effects. The first one is an obvious geometric effect [Eq. (8)], which is the geometrical distance change between the pivot (beam origin) and the photodiode. The second one is a result of the offset between the two beams, which

the tilt-to-length coupling caused by a quadrant photodiode is in preparation.

Another well-known way to explain this effect for very small tilt angles is to express the tilt as an excitation of the Hermite–Gaussian (HG) 01 mode
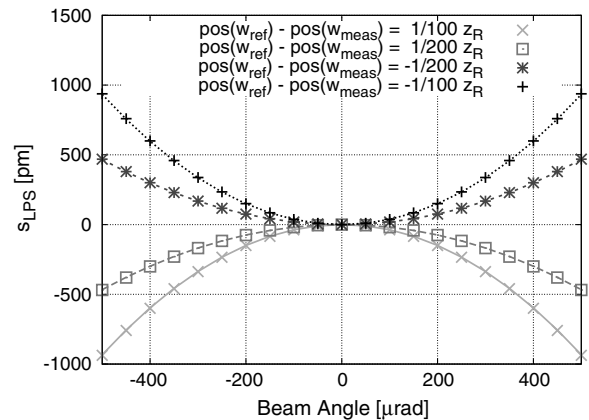

Fig. 4. Numerically and analytically [2] computed s_{LPS} for two Gaussian beams on an SEPD with differences in the waist position. The numeric values are marked with symbols, while the analytical expressions are lines. (The simulated setup is described in Section 5.)

as explained in [6]. Due to the orthogonality of the HG modes this excitation will not change the path length readout when the entire interference pattern is detected. However, this is only an approximation that is valid for very small angles (much smaller than the far field divergence [6]). The angles in the present examples exceed this limitation such that the field cannot be suitably expressed by an excitation of only the HG 01 mode.

When comparing the tilt-to-length coupling for plane waves, Eq. (8) to that of Gaussian beams Eq. (11) or Eq. (13), it should be pointed out that Eq. (8) is not a special case of Eq. (11) or Eq. (13). Due to the assumed complete detection (infinite integration limits) the Gaussian beams cannot be approximated by plane waves.

6. Conclusions

It was shown that the computed coupling between two plane waves matches exactly the expected geometric path length difference. In contrast, the coupling between beam tilt and longitudinal path length signal in an interferometer with two identical Gaussian beams and a large SEPD vanishes. It was shown that the reason for this disappearance is an additional coupling effect that is caused by the lever arm between the pivot and photodiode. Building an interferometer with two identical Gaussian beams and a large SEPD is a possible way to cancel out the tilt-to-path-length coupling, for example in a homodyne interferometer with one light source.

References

1. G. Wanner, G. Heinzel, E. Kochkina, C. Mahrdt, B. Sheard, S. Schuster, and K. Danzmann, "Methods for simulating the readout of lengths and angles in laser interferometers with Gaussian beams," Opt. Commun. **285**, 4831–4839 (2012).
2. G. Wanner and G. Heinzel, "Analytical description of interference between two misaligned and mismatched complete Gaussian beams," Appl. Opt. **53**, 3043–3048 (2014).
3. A. Yariv, *Quantum Electronics* (Wiley, 1989).
4. B. Saleh and M. Teich, *Fundamentals of Photonics* (Wiley, 1991).
5. E. Kochkina, G. Heinzel, G. Wanner, V. Mueller, C. Mahrdt, B. Sheard, S. Schuster, and K. Danzmann, "Simulating and optimizing laser interferometers," in *9th LISA Symposium*, Vol. **467** of ASP Conference Series (Astronomical Society of the Pacific, 2012), pp. 291–292.
6. D. Anderson, "Alignment of resonant optical cavities," Appl. Opt. **23**, 2944–2949 (1984).