Group Theory Behind Triadic Chordal Transformation of Music Theory

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1 Introduction to Music Theory

1.1 Motivation

The fields of mathematics and music have begun since the birth of humanity. While these two fields may seem distant, fundamental concepts of music are deeply intertwined with mathematics. Musical structures, chordal progressions, rhythmic repetition and much more can be represented via concepts existing in mathematics. In this essay, we will focus on the intersection between these two fields and especially how group theory operates structurally behind the triadic transformations of consonant chords in music theory. By demonstrating the isomorphism between the TI and PLR transformation groups and the dihedral group D_{24}, we will illustrate the underlying algebraic structure governing chord progressions which creates consonance. This analysis not only provides a rigorous framework for understanding musical relationships but also depicts the mathematical foundations upon which musical systems are built; how mathematical structures manifest in modern composition and analysis.

1.2 Pitches and Scales

Musical pitch is defined to be a perceptual quality of sound that allows listeners to order tones on a scale from low to high based on their frequency of sound waves created by the source.
In Western music, there are 12 pitches within an octave and the sequence of them separated by a semitone or a whole tone are called scales. Figure 1 below shows the chromatic scale where all pitches are a semitone away from each other. Pitches, the sounds we perceive, are linked to vibrational frequencies, and traditionally, their relationships were understood via frequency ratios.

Figure 1: Chromatic Scale [6]

Definition 1.1. We first define **Pitch Class** as the set of pitch equivalence classes in a logarithmically divided octave, where each class is represented by an integer from 0 to 11 under modulo 12 arithmetic. In this system, frequency of adjacent pitches in the chromatic scale are related by:

$$P_i = \sqrt[12]{2} * P_{i-1}$$

which is know as equal tempered tuning. [8, pp. 109–111]

Observation from this definition is that doubling the frequency of a sound wave corresponds to an upward octave shift. Mathematically, this can be expressed as:

$$f_{i+12} = 2 * f_i$$

Example 1.1. The note A4 is standardised at a frequency of 440Hz and acts as the tuning reference for a symphony orchestra. An octave below A4, the note A3 has a frequency of 220Hz and A5, an octave above A4 has a frequency of 880Hz.

1.3 Chords and Triads

Definition 1.2. **Chord** is composite wave synthesised by performing multiple pitches simultaneously that are combinations of various sound waves with distinct frequencies.

Definition 1.3. **Triad** is a specific type of chord which consists of three pitch classes:

- Root: The fundamental pitch class of the chord and becomes the name of the triad
- Third: A pitch class that is 3 or 4 semitones above the root (3 semitones above indicates a minor triad, while 4 semitones above is a major triad)
- Fifth: A pitch class that is 7 semitones above the root

Definition 1.4. **Pitch Class Set** is a set mapped from a collection of pitches $\mathbf{P} \to \mathbb{Z}_{12}$

Example 1.2. C Major chord consisted of [C, E, G] in pitch class set is [0,4,7] and C minor chord consisted of [C, $E\flat(D\sharp)$, G] in pitch class set is [0,3,7]

Theorem 1.1. *All 12 Major and Minor Triads are subsets of the Pitch Class Set and we denote this set to be* **M**.

Proof. Define the pitch class set P as the set of all integers modulo 12:
P = [0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11] and recall from definition of triads that each major and minor triads are in a form of $[r, r+4, r+7] \pmod{12}$ or $[r, r+3, r+7] \pmod{12}$ where r denotes the root of a triad. Since all root, third, fifth $\in \mathbf{P}$ as they are integer mod 12, $\mathbf{M} \subseteq \mathbf{P}$ □

This theorem depicts that pitch class set encompasses all possible triadic structures of major and minor chords in twelve-tone equal temperament. We will mainly focus on the structure of operations acting on the set **M** throughout this essay.

2 T_n and T_nI Transformations

[9, p. 9–11] This chapter delves into the relationship between the Transposition and Inversion operations established by Lewin's transformational theory, which are operations on the set of major and minor triads, **M**. It provides a powerful framework for analysing chord progressions. Here, we demonstrate that this TI group is, in fact, isomorphic to the dihedral group D_{24}. This isomorphism, where each transposition corresponds to a rotation and each inversion to a reflection, allows us to visualise these musical transformations geometrically.

2.1 T_n: Transposition of Triads

Definition 2.1. For any triad $t \in \mathbf{M}$, where $t = [x, y, z]$, T_n the transposition is the function $T_n : \mathbf{M} \to \mathbf{M}$ given by

$$T_n(t) = t + n = [x + n, y + n, z + n].$$

where $n \in \mathbb{Z}/12\mathbb{Z}$.

T_n translates pitches by n semitones preserving its parity. We observe the original triad remains stationary after applying T_0, as the transposition of 0 interval is the original triad. Also, T_{12} acts equivalently as T_0, because $T_{12}([x, y, z]) = [x + 12, y + 12, z + 12] = [x, y, z]$ which we can interpret as elevating up an octave.

2.2 T_nI: Inversion of Triads

Definition 2.2. For any triad $t \in \mathbf{M}$, where $t = [x, y, z]$, T_nI the inversion is the function $T_nI : \mathbf{M} \to \mathbf{M}$ given by

$$T_nI(t) = -t + n = [-x + n, -y + n, -z + n].$$

where $n \in \mathbb{Z}/12\mathbb{Z}$.

T_nI reflects pitches about C[0] then transposes them by n semitones. We note that the original triad does not remain stationary after applying T_0I; instead it is reflected. The property of modulo 12 is preserved as T_0I $= T_{12}I$.

Definition 2.3. TI Set is a set mapped from a collection of pitches $\mathbf{P} \to \mathbf{M}$. Where TI operations are the linear map on this set.

Theorem 2.1. *The set of composition between transposition and inversion operations denoted by the TI set forms a group where*

$$TI\ group = \{T_n \mid n \in \{0, 1, ..., 11\} \bigcup T_nI \mid n \in \{0, 1, ..., 11\}\}$$

Proof.

- *Closure*:
  First we have to check that compositions of transposition and inversion

lies within the TI set.

$$\begin{aligned}
T_m \circ T_n(t) &= T_m(T_n([x, y, z])) \\
&= T_m([x+n, y+n, z+n]) \\
&= [x+(n+m), y+(n+m), z+(n+m)] \\
&= T_{m+n}(t) \pmod{12} \\
T_m \circ T_nI(t) &= T_m([-x+n, -y+n, -z+n]) \\
&= [-x+(n+m), -y+(n+m), -z+(n+m)]) \\
&= T_{m+n}I(t) \pmod{12} \\
T_mI \circ T_n(t) &= T_mI([x+n, y+n, z+n]) \\
&= [-(x+n)+m, -(y+n)+m, -(z+n)+m]) \\
&= [-x+(m-n), -y+(m-n), -z+(m-n)]) \\
&= T_{m-n}I(t) \pmod{12} \\
T_mI \circ T_nI(t) &= T_mI([-x+n, -y+n, -z+n]) \\
&= [-(-x+n)+m, -(-y+n)+m, -(-z+n)+m]) \\
&= [x+(m-n), y+(m-n), z+(m-n)]) \\
&= T_{m-n}(t) \pmod{12}.
\end{aligned}$$

Therefore $\forall\, \alpha,\, \beta \in TI,\ \alpha \circ \beta = \gamma \in TI$ so the set TI is closed under composition.

- *Associativity*:
  If $\alpha,\, \beta,\, \gamma \in TI, (\alpha \circ \beta) \circ \gamma = \alpha \circ (\beta \circ \gamma)$ holds as it is composition of functions.

- *Identity*:
  T_0 acts as the identity element in TI set.

$$\begin{aligned}
T_0 \circ T_n(t) = T_{0+n}(t) = T_{n+0}(t) = T_n \circ T_0(t) = T_n(t) \\
T_0 \circ T_nI(t) = T_{0+n}I(t) = T_{n-0}(t) = T_nI \circ T_0(t) = T_nI(t).
\end{aligned}$$

- *Inverse*:
  Inverse of T_n functions:

$$T_n \circ T_{12-n}(t) = T_{n+(12-n)}(t) = T_{(12-n)+n}(t) = T_{12-n} \circ T_n(t) = T_{12}(t) = T_0(t).$$

  Inverse of T_nI functions:

$$T_nI \circ T_nI(t) = T_{(n-n)}(t) = T_0(t).$$

  Implies that T_nI functions are involutive.

Therefore TI functions form a group under composition. $\square$

Proposition 2.2. *TI group is **simply transitive** on set $\mathbf{M}$. [2, p. 62–67]*

Proof. A group is simply transitive if it is both free and acts transitively on a set. We show each properties:

- Transitivity on $\mathbf{M}$: Want to show for any $\alpha, \beta \in \mathbf{M}$, there exists a unique $T \in TI$ such that $T(\alpha) = \beta$.

  1. Consider transposition: Let $\gamma = |\beta - \alpha|$. Then $T_\gamma(\alpha) = \alpha + \gamma = \beta$.
  2. Consider inversion: Let $\gamma = |\beta + \alpha|$. Then $T_\gamma I(\alpha) = -(\alpha) + \gamma = \beta$.

  To show uniqueness of T, use Orbit-Stabilizer Theorem, for $\alpha \in \mathbf{M}$,

  $$[TI : \text{Stab}_{TI}(\alpha)] = |\text{Orb}_{TI}(\alpha)|$$

  Where we know that $\#TI = 24$ and $\#\text{Orb}_{TI}(\alpha) = \#\mathbf{M} = 24$ thus

  $$\#\text{Stab}_{TI}(\alpha) = \frac{\#TI}{\#\text{Orb}_{TI}(\alpha)} = \frac{24}{24} = 1.$$

  Thus if $T'(\alpha) = T(\alpha)$, $T^{-1}T'(\alpha) = \alpha \implies T^{-1}T'$ is identity. Therefore $T = T'$ so it is unique.

- Freeness of TI: Want to show for any $T \in TI$ and $\alpha \in \mathbf{M}$, if $T(\alpha) = \alpha$, then T must be the identity transformation T_0.
  Above we have shown that $|\text{Stab}_{TI}(\alpha)| = 1$ therefore it is trivial to show that the element is T_0.

□

We will later use the simply transitive nature of TI group to prove its duality to PLR group which will be introduced in Chapter 3.

2.3 Isomorphism Between TI and D_{24}

Now we will illustrate the isomorphism between the TI group defined above and Dihedral group when $n = 12$.

Theorem 2.3. *The TI group is isomorphic to D_{24}.*

Proof. First construct homomorphism ψ from D_{24} to TI by mapping the rotation r^k to T_n and mapping reflection s to T_nI.
Define the map ψ by,

$$\psi(r^k) = T_k \quad \text{for} \quad k = 0, 1, ..., 11$$
$$\psi(s) = T_0 I \quad \textit{and} \quad \psi(sr^k) = T_k I \quad \text{for} \quad k = 0, 1, ..., 11$$

- For rotations: If $g_1 = r^p$ and $g_2 = r^q$, then $g_1 * g_2 = r^p * r^q = r^{p+q}$

$$\psi(r^p * r^q) = \psi(r^{p+q}) = T_{p+q} = T_p \circ T_q = \psi(r^p) * \psi(r^q).$$

- For reflections and rotations: If $g_1 = s$ and $g_2 = r^p$, then $g_1 * g_2 = sr^p$

$$\psi(s * r^p) = \psi(sr^p) = T_p I = T_0 I \circ T_p = \psi(s) * \psi(r^p).$$

Now check if the map is bijective to prove isomorphism.

- Surjectivity: Every elements of transposition or inversion in the TI group either maps to rotation or reflection in D_{24}, all elements of TI group is covered by elements of D_{24}, thus surjective.

- Injectivity: Suppose $\psi(g_1) = \psi(g_2)$ $\forall g_1, g_2 \in D_{24}$. Want to show $g_1 = g_2$.

  1. If both g_1, g_2 are rotations, $\psi(g_1)$ and $\psi(g_2)$ are both transpositions say T_p and T_q. If $\psi(g_1) = \psi(g_2)$, $T_p = T_q \implies p = q$. Therefore $g_1 = g_2$.

  2. If both g_1, g_2 are reflections, with the same idea as rotations, $\psi(g_1) = \psi(g_2) \implies g_1 = g_2$.

  3. If g_1, g_2 are each rotation or reflection, $\psi(g_1) = \psi(g_2)$ does not hold as rotation can not be mapped to reflection. So this is not the case.

  Therefore the map is injective $\therefore TI \cong D_{24}$.

□

From the isomorphism illustrated between TI and D_{24}, we can see that each transposition maps to rotation and each inversion maps to reflection which hints on the underlying cyclic group of 12 transpositions and geometrical visualisation of compositional transformation.

3 Neo-Riemannian Transformations

Neo-Riemannian Theory was first introduced by the German musical theorist Hugo Riemann. Within this theory, Neo-Riemannian transformation consists of three operations including **Parallel**, **Leading-tone exchange** and **Relative**. Each describe relationships between triads and when applied to a triad in **M**, the transform changes the parity of the triad and is parsimony which means that it preserve 2 pitches.

3.1 Introduction to PLR Transformations

Definition 3.1. The **P** transformation maps given triad to its opposite parity. Specifically, for α, $\beta \in \mathbf{M}$ where $\alpha = [X, Y, Z]$ is a major triad and $\beta = [x, y, z]$ is a minor triad,

$$\mathbf{P}(\alpha) = \mathbf{P}[X, Y, Z] = [X, Y - 1, Z]$$
$$\mathbf{P}(\beta) = \mathbf{P}[x, y, z] = [x, y + 1, z].$$

Example 3.1. $\mathbf{P}(\mathbf{C}) = \mathbf{P}([0, 4, 7]) = [0, 3, 7]$ which is a c minor triad. $\mathbf{P}(\mathbf{c}) = \mathbf{P}([0, 3, 7]) = [0, 4, 7]$ which is a C major triad.

Definition 3.2. The **L** transformation exchanges the top note of a minor chord, raising it by a semi-tone and root of a major triad, lowering it by a semi-tone, which is known to be a leading tone of the scale.

$$\mathbf{L}(\alpha) = \mathbf{L}[X, Y, Z] = [Y, Z, X - 1]$$
$$\mathbf{L}(\beta) = \mathbf{L}[x, y, z] = [z + 1, x, y].$$

Example 3.2. $\mathbf{L}(\mathbf{C}) = \mathbf{L}([0, 4, 7]) = [4, 7, 11]$ which is an e minor triad. $\mathbf{L}(\mathbf{c}) = \mathbf{L}([0, 3, 7]) = [8, 0, 4]$ which is an A♭ major triad.

Definition 3.3. The **R** transformation builds the triad from the pitch that is three semi-tones up/down from the root of the triad which is known to be the original triad's relative major/minor.

$$\mathbf{R}(\alpha) = \mathbf{R}[X, Y, Z] = [Z + 2, X, Y]$$
$$\mathbf{R}(\beta) = \mathbf{R}[x, y, z] = [y, z, x - 2].$$

Example 3.3. $\mathbf{R}(\mathbf{C}) = \mathbf{R}([0, 4, 7]) = [9, 0, 4]$ which is an a minor triad. $\mathbf{R}(\mathbf{c}) = \mathbf{R}([0, 3, 7]) = [3, 7, 10]$ which is an A♭ major triad.

From the definition of PLR transformations, we identify that all three **P**, L and R transformations are involutive in other words, $\mathbf{P}^2 = \mathbf{L}^2 = \mathbf{R}^2 = i$.

Lemma 3.1. **P** *and* **L** *transformation can be represented accordingly,*

$$\mathbf{P} = \mathbf{R} \circ (\mathbf{L} \circ \mathbf{R})^3 = \mathbf{R} \circ (\mathbf{R} \circ \mathbf{L})^9$$
$$\mathbf{L} = \mathbf{R} \circ (\mathbf{L} \circ \mathbf{R})^{11} = \mathbf{R} \circ (\mathbf{R} \circ \mathbf{L})^1.$$

Proof. Proof is a simple calculation by applying **L** and **R** operations multiple times. □

We deduce the following relation which is known as the **Hexatonic Pole Relation** in Neo-Riemannian theory. This relationship is widely used in the context of romantic composition as the symmetrical voice-leading path creates smooth chromatic voice leading with minimal semitone movement.

Lemma 3.2. $(\mathbf{L}\circ\mathbf{R})^p = (\mathbf{R}\circ\mathbf{L})^{-p \pmod{12}}$

Proof. Proof by induction

- Base Case: when $p = 0$, $\quad (\mathbf{L}\circ\mathbf{R})^0 = (\mathbf{R}\circ\mathbf{L})^0 = i$
- Inductive Step: Assume the lemma holds for some $p = k \geq 0$. Will try to prove it holds for $k+1$.
  Note that as $\mathbf{L}^2 = \mathbf{R}^2 = i \implies (\mathbf{L}\circ\mathbf{R}) = (\mathbf{R}\circ\mathbf{L})^{-1}$

$$\begin{aligned}(\mathbf{L}\circ\mathbf{R})^{k+1} &= (\mathbf{L}\circ\mathbf{R})^k \circ (\mathbf{L}\circ\mathbf{R})\\ &= (\mathbf{R}\circ\mathbf{L})^{-k} \circ (\mathbf{L}\circ\mathbf{R}) = (\mathbf{R}\circ\mathbf{L})^{-k} \circ (\mathbf{R}\circ\mathbf{L})^{-1}\\ &= (\mathbf{R}\circ\mathbf{L})^{-(k+1) \pmod{12}}.\end{aligned}$$

Therefore if the lemma holds for k, it holds for $k+1$ as well completing the proof. □

Theorem 3.3. *PLR forms a group under composition and is generated by* $\mathbf{L}$ *and* $\mathbf{R}$ *where,*

$$PLR\ \ group = \{(\mathbf{L}\circ\mathbf{R})^n, \mathbf{R}\circ(\mathbf{L}\circ\mathbf{R})^n \ \mid\ n = 0, 1, ..., 11\}$$

Proof.

- *Closure*: There are four cases of combination of elements in PLR group,

  1. $(\mathbf{L}\circ\mathbf{R})^p \circ (\mathbf{L}\circ\mathbf{R})^q = (\mathbf{L}\circ\mathbf{R})^{p+q \pmod{12}} \in PLR$

  2. $(\mathbf{L}\circ\mathbf{R})^p \circ \mathbf{R} \circ (\mathbf{L}\circ\mathbf{R})^q = (\mathbf{R}\circ\mathbf{L})^{-p} \circ \mathbf{R} \circ (\mathbf{L}\circ\mathbf{R})^q$

  $= \mathbf{R}\circ(\mathbf{L}\circ\mathbf{R})^{-p} \circ (\mathbf{L}\circ\mathbf{R})^q = \mathbf{R}\circ(\mathbf{L}\circ\mathbf{R})^{-p+q \pmod{12}} \in PLR$

  3. $\mathbf{R}\circ(\mathbf{L}\circ\mathbf{R})^p \circ (\mathbf{L}\circ\mathbf{R})^q = \mathbf{R}\circ(\mathbf{L}\circ\mathbf{R})^{p+q \pmod{12}} \in PLR$

  4. $\mathbf{R}\circ(\mathbf{L}\circ\mathbf{R})^p \circ \mathbf{R} \circ (\mathbf{L}\circ\mathbf{R})^q$

  $= \mathbf{R}\circ(\mathbf{R}\circ\mathbf{L})^{-p} \circ \mathbf{R} \circ (\mathbf{L}\circ\mathbf{R})^q = (\mathbf{R}\circ\mathbf{R})\circ(\mathbf{L}\circ\mathbf{R})^{-p} \circ (\mathbf{L}\circ\mathbf{R})^q$
  $= (\mathbf{L}\circ\mathbf{R})^{-p+q \pmod{12}} \in PLR$

- *Associativity*: Associative as the operation is composition.
- *Identity*: There exists an identity where $i = (\mathbf{L}\circ\mathbf{R})^0$.

$$\begin{aligned}(\mathbf{L}\circ\mathbf{R})^p \circ (\mathbf{L}\circ\mathbf{R})^0 &= (\mathbf{L}\circ\mathbf{R})^{p+0}\\ (\mathbf{L}\circ\mathbf{R})^0 \circ (\mathbf{L}\circ\mathbf{R})^p &= (\mathbf{L}\circ\mathbf{R})^{p+0}\\ \mathbf{R}\circ(\mathbf{L}\circ\mathbf{R})^p \circ (\mathbf{L}\circ\mathbf{R})^0 &= \mathbf{R}\circ(\mathbf{L}\circ\mathbf{R})^{p+0}\\ (\mathbf{L}\circ\mathbf{R})^0 \circ \mathbf{R}\circ(\mathbf{L}\circ\mathbf{R})^p &= \mathbf{R}\circ(\mathbf{L}\circ\mathbf{R})^{0+p}\end{aligned}$$

- *Inverse*: Using the property found in closure, we deduce that the inverse to $(\mathbf{L} \circ \mathbf{R})^p$ and $\mathbf{R} \circ (\mathbf{L} \circ \mathbf{R})^p$ are themselves.

$\square$

As shown in Table 1, in appendix, we can observe that each operation of $(\mathbf{L} \circ \mathbf{R})^n$ and $\mathbf{R} \circ (\mathbf{L} \circ \mathbf{R})^n$ spans 5 intervals from the previous triad which corresponds to the structure of circle of fifths. This operation plays a significant role in music theory and harmonics, as these intervalic progression of chords are known as cadences. The V $\rightarrow$ I progression are referred as the perfect cadence which creates a strong sense of resolution and completeness when following the circle of fifths.

Furthermore, it is intriguing to note that the operation adheres to the sequence of sharps and flats, where if one moves around the circle of fifths, each step adds a sharp or flat, depending on whether the movement is clockwise or counterclockwise.

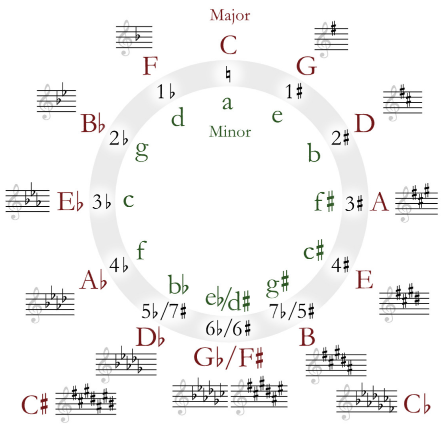

Figure 2: Circle of Fifth [7]

In fact, we are able to derive a geometric depiction of PLR group which is known as the **Tonnetz** which is german for tone-network. The Tonnetz is commonly used to visualise smooth voice leading in chromatic harmony, even in contexts that move beyond traditional tonal progressions. It is often depicted as a 2D lattice as in figure 4 the diagram is from [9, p. 9], where horizontal axis represents the circle of fifth and the diagonal axis represents circle of major and minors thirds. However, this structure can be extended to 3D; wrapping into a torus (doughnut shape) due to enharmonic equivalence (e.g., C$\sharp$ being the same as D$\flat$). This toroidal representation allows for the visualisation of more distant relationships and modulations.

Proposition 3.4. *PLR group is* ***simply transitive*** *on set* $\mathbf{M}$.

Proof. The proof is identical to *Prop 1.3*. $\square$

3.2 Isomorphism Between PLR and D_{24}

Theorem 3.5. *The PLR group is isomorphic to* D_{24}.

Proof. Construct homomorphism ϕ from PLR to D_{24} by defining the map ϕ by,

$$\phi((\mathbf{L}\circ\mathbf{R})^k) = r^k \quad \text{for} \quad k = 0, 1, ..., 11$$
$$\phi(\mathbf{R}\circ(\mathbf{L}\circ\mathbf{R})^k) = sr^k \quad \text{for} \quad k = 0, 1, ..., 11$$

Verify that the homomorphism preserves the group operation:
Note in D_{24}, $sr^p s = r^{-p} \implies sr^p = r^{-p} * s$.

1. $\phi((\mathbf{L}\circ\mathbf{R})^p \circ (\mathbf{L}\circ\mathbf{R})^q) = \phi((\mathbf{L}\circ\mathbf{R})^{p+q}) = r^{p+q} = r^p * r^q$
 $= \phi((\mathbf{L}\circ\mathbf{R})^p) \circ \phi((\mathbf{L}\circ\mathbf{R})^q)$.
2. $\phi((\mathbf{L}\circ\mathbf{R})^p \circ \mathbf{R} \circ (\mathbf{L}\circ\mathbf{R})^q) = \phi(\mathbf{R}\circ(\mathbf{L}\circ\mathbf{R})^{-p+q})$
 $= sr^{-p+q} = sr^{q-p} = r^{p-q} * s = r^p * (r^{-q} * s) = r^p * (sr^q)$
 $= \phi((\mathbf{L}\circ\mathbf{R})^p) \circ \phi(\mathbf{R}\circ(\mathbf{L}\circ\mathbf{R})^q)$.
3. $\phi(\mathbf{R}\circ(\mathbf{L}\circ\mathbf{R})^p \circ (\mathbf{L}\circ\mathbf{R})^q) = \phi(\mathbf{R}\circ(\mathbf{L}\circ\mathbf{R})^{p+q}) = sr^{p+q} = sr^p * r^q$
 $= \phi(\mathbf{R}\circ(\mathbf{L}\circ\mathbf{R})^p) \circ \phi((\mathbf{L}\circ\mathbf{R})^q)$.
4. $\phi(\mathbf{R}\circ(\mathbf{L}\circ\mathbf{R})^p \circ \mathbf{R} \circ (\mathbf{L}\circ\mathbf{R})^q) = \phi((\mathbf{L}\circ\mathbf{R})^{-p+q})$
 $= r^{-p+q} = r^{-p} * (s * s) * r^q = sr^p * sr^q$
 $= \phi(\mathbf{R}\circ(\mathbf{L}\circ\mathbf{R})^p) \circ \phi(\mathbf{R}\circ(\mathbf{L}\circ\mathbf{R})^q)$.

Determine the kernel and image of the map.

- $Ker(\phi) = \{x \in PLR \mid \phi(x) = i\}$
 $\phi((\mathbf{L}\circ\mathbf{R})^n) = r^n = i$ if and only if when $n = 0 \pmod{12}$
 $\phi(\mathbf{R}\circ(\mathbf{L}\circ\mathbf{R})^n) = sr^n \neq i \quad \forall n$
 Therefore $Ker(\phi) = \{i\}$.
- $Im(\phi) = \{\phi(x) \mid x \in PLR\}$
 $= \{r^k, sr^k \mid k = 0, 1, ..., 11\} = D_{24}$.

Now apply the First Isomorphism Theorem:

$$PLR/\{i\} \cong D_{24} \implies PLR \cong D_{24}.$$

$\square$

The isomorphism $PLR \cong D_{24}$ is significant because it shows that the PLR group inherits the symmetry and order of the dihedral group. In musical terms, it not only justifies the use of tools like the Tonnetz or circle of fifths as geometric representations of chordal space but also reflects the deep algebraic symmetry tonal harmony.

3.3 Transitivity of Isomorphisms

Theorem 3.6. *Isomorphism is Transitive therefore $PLR \cong TI$*

Proof. We have proven above that $TI \cong D_{24}$ and $PLR \cong D_{24}$
Define each homomorphism map in TI and PLR to be:

$$\phi : PLR \to D_{24}, \qquad \psi : D_{24} \to TI.$$

Now define a mapping from $\xi : PLR \to TI$ to be:

$$\xi(\alpha) \;=\; \psi\,(\,\phi\,(\alpha)\,).$$

Want to show that ξ is an isomorphism.

- Homomorphism of ξ:
  For any $x, y \in PLR$, $\xi(x * y) = \psi(\phi(x * y)) = \psi(\phi(x) * \phi(y))$
  $= \psi(\phi(x)) * \psi(\phi(y)) = \xi(x) * \xi(y)$.

- Injectivity of ξ:
  Assume $\xi(x) = \xi(y)$, then $\psi(\phi(x)) = \psi(\phi(y))$.
  Since both ϕ and ψ are bijective, their inverses ϕ^{-1} and ψ^{-1} exist. Applying them to both sides, we obtain $x = y$.

- Surjectivity of ξ:
  Since ψ is surjective, for all $\alpha \in PLR$, there exists $\beta \in D_{24}$ such that $\phi(\alpha) = \beta$. Similarly, since ψ is surjective, for all $\beta \in D_{24}$, there exists $\gamma \in TI$ such that $\psi(\beta) = \gamma$. Therefore, $\xi(\alpha) = \psi(\phi(\alpha)) = \psi(\beta) = \gamma$

□

Topological features are preserved over the sequences of isomorphisms which in music could be interpreted as a conservation of consonance when there is a key change or a modal interchange. Following from this result, we can construct the duality relation between the two groups acting on **M**. Table 2 illustrates mapping between PLR and TI transformations.

3.4 Lewin Duality Between PLR and TI Group

The PLR group and the TI group are both transformational systems that act on **M**. By studying their dual nature, it allows us to have a deeper understanding of their interrelations.

Definition 3.4. Two groups G and H acting on a set S are said to be **dual** in the sense of Lewin if:

1. Both G and H act simply transitively on S.
2. $G = C_{Sym(S)}(H)$ and $H = C_{Sym(S)}(G)$

3. $\#G = \#H = \#S$.

Lemma 3.7. *$\forall x \in PLR$ and $\forall y \in TI$, $xy = yx$*

Proof. Proof is series of calculations checking the commutativity of operations. $\square$

Theorem 3.8. *PLR and TI group acting on set $\mathbf{M}$ are dual.*

Proof.

1. Proven in Proposition 1.3 and 2.4

2. Want to show that $TI = C_{Sym(\mathbf{M})}(PLR)$ and $PLR = C_{Sym(\mathbf{M})}(TI)$. From the definition of centralizer,

$$C_{Sym(\mathbf{M})}(PLR) = \{t \in Sym(\mathbf{M}) \mid xt = tx \quad \forall x \in PLR\}$$

   - From Lemma 3.7 we know that every element of TI commutes with every element of PLR, meaning that TI is a subset of the centralizer of PLR in $Sym(\mathbf{M})$: $TI \subseteq C_{Sym(\mathbf{M})}(PLR)$

   - Let p be an element of $C_{Sym(\mathbf{M})}(PLR)$ and α be an element of $\mathbf{M}$ then $p(\alpha)$ results in another element in $\mathbf{M}$ say β. Due to the simple transitive nature of TI, there also exists a unique operation $q \in TI$ such that $q(\alpha) = p(\alpha) = \beta$.

     Want to show that $p = q$. Since p and q are both elements of $Sym(\mathbf{M})$, if they generate equivalent output for every element of $\mathbf{M}$, they must be the same function.

     Then for any $t \in PLR$, $p(\beta) = p(t(\alpha)) = t(p(\alpha))$ from the definition of $C_{Sym(\mathbf{M})}(PLR)$.
     Using $q(\alpha) = p(\alpha)$ and Lemma 3.7, $t(p(\alpha)) = t(q(\alpha)) = q(t(\alpha)) = q(\beta)$. Therefore $q(\beta) = p(\beta) \quad \forall \beta \in \mathbf{M} \implies p \in C_{Sym(\mathbf{M})}(PLR) \subseteq TI$ and $PLR = C_{Sym(\mathbf{M})}(TI)$ can be shown in a same way.

   Therefore as $TI \subseteq C_{Sym(\mathbf{M})}(PLR)$ and $C_{Sym(\mathbf{M})}(PLR) \subseteq TI$, we conclude that $C_{Sym(\mathbf{M})}(PLR) = TI$.

3. $\#PLR = \#TI = \#\mathbf{M} = 24$.

$\square$

As we have proven duality between PLR and TI groups, we can represent this duality as a **commutative diagram**, where the vertices represent triads and the edges represent the transformations. The commutativity of this diagram ensures that different sequences of PLR operations yield equivalent results to corresponding TI operations, and vice versa.

$$\begin{tikzcd} & \mathrm{Sym}(M) & \\ \mathrm{PLR} \arrow[ur] & & \mathrm{TI} \arrow[ul] \end{tikzcd}$$

$$C_{Sym(\mathbf{M})}(TI) = \mathrm{PLR}$$
$$C_{Sym(\mathbf{M})}(PLR) = \mathrm{TI}$$

Underlying dual structure of TI and PLR groups are not only frequently used in modern composition of music but also used in classical composition which can be identified in Johann Pachelbel's famous "Canon in D". [1, p. 20–21]

Example 3.4. Chord progression of Canon in D represented in commutative diagram

Figure 3: Canon in D [5]

$$\begin{tikzcd} D \arrow[r, "T_7"] \arrow[d, "R"'] & A \arrow[d, "R"] \\ b \arrow[r, "T_7"'] & f\sharp \end{tikzcd}$$

Commutative diagram depicts that all transformation paths yield equivalent results i.e $D \circ T_7 \circ R = D \circ R \circ T_7$. The dual structure in this piece establishes musical stability through consonance while providing a sense of dynamic progression that remains harmonically grounded.

4 Conclusion

In conclusion, this essay has explored the intersection of mathematics and music by examining how group theory can be used to model the transformations of consonant triadic chords. The next step from this essay will be to extend this approach by incorporating category theory. By defining the TI and PLR transformations as morphisms of a category could enlighten the new path for understanding structures behind dissonant chord transformations. This could lead to the expansion of this theory to include expansion of this theory onto 7th, 9th chords, and beyond.

Additionally, applying category theory enables us to stretch concept of Lewin duality that we have studied onto categorical duality which allows us to study their roles in musical contexts. By treating transformations as functors, we can preserve the underlying algebraic structure while gaining a more comprehensive view of how musical systems evolve. This layer of abstraction could bridge the gap between theoretical mathematics and practical music composition. As a result, providing an another perspective on musical forms and systems that move beyond traditional harmonic structures. [4, p. 18–24]

5 Appendix

$R(\langle 0,4,7\rangle) = \langle 4,0,9\rangle = a$	$R(LR)^6(\langle 0,4,7\rangle) = \langle 10,6,3\rangle = e\flat$
$LR(\langle 0,4,7\rangle) = \langle 5,9,0\rangle = F$	$(LR)^7(\langle 0,4,7\rangle) = \langle 11,3,6\rangle = B$
$R(LR)(\langle 0,4,7\rangle) = \langle 9,5,2\rangle = d$	$R(LR)^7(\langle 0,4,7\rangle) = \langle 3,11,8\rangle = g\sharp$
$(LR)^2(\langle 0,4,7\rangle) = \langle 10,2,5\rangle = B\flat$	$(LR)^8(\langle 0,4,7\rangle) = \langle 4,8,11\rangle = E$
$R(LR)^2(\langle 0,4,7\rangle) = \langle 2,10,7\rangle = g$	$R(LR)^8(\langle 0,4,7\rangle) = \langle 8,4,1\rangle = c\sharp$
$(LR)^3(\langle 0,4,7\rangle) = \langle 3,7,10\rangle = E\flat$	$(LR)^9(\langle 0,4,7\rangle) = \langle 9,1,4\rangle = A$
$R(LR)^3(\langle 0,4,7\rangle) = \langle 7,3,0\rangle = C$	$R(LR)^9(\langle 0,4,7\rangle) = \langle 1,9,6\rangle = f\sharp$
$(LR)^4(\langle 0,4,7\rangle) = \langle 8,0,3\rangle = A\flat$	$(LR)^{10}(\langle 0,4,7\rangle) = \langle 2,6,9\rangle = D$
$R(LR)^4(\langle 0,4,7\rangle) = \langle 0,8,5\rangle = f$	$R(LR)^{10}(\langle 0,4,7\rangle) = \langle 6,2,11\rangle = b$
$(LR)^5(\langle 0,4,7\rangle) = \langle 1,5,8\rangle = D\flat$	$(LR)^{11}(\langle 0,4,7\rangle) = \langle 7,11,2\rangle = G$
$R(LR)^5(\langle 0,4,7\rangle) = \langle 5,1,10\rangle = b\flat$	$R(LR)^{11}(\langle 0,4,7\rangle) = \langle 11,7,4\rangle = e$
$(LR)^6(\langle 0,4,7\rangle) = \langle 6,10,1\rangle = G\flat$	$(LR)^0(\langle 0,4,7\rangle) = \langle 0,4,7\rangle = C$

Table 1: Transformations of the CM triad under PLR transformations

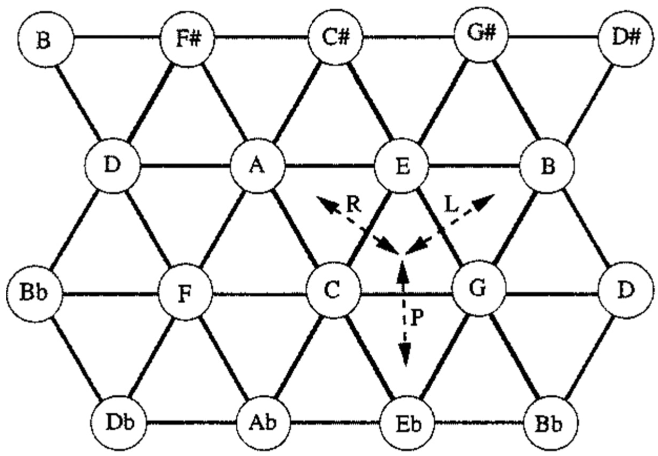

Figure 4: Tonnetz 2D Representation [3]

PLR $\to$ T/I		
R $\to$ I_0	$R(LR)^4 \to I_8$	$R(LR)^8 \to I_4$
LR $\to$ T_1	$(LR)^5 \to T_5$	$(LR)^9 \to T_9$
RLR $\to$ I_{11}	$R(LR)^5 \to I_7$	$R(LR)^9 \to I_3$
$(LR)^2 \to T_2$	$(LR)^6 \to T_6$	$(LR)^{10} \to T_{10}$
$R(LR)^2 \to I_{10}$	$R(LR)^6 \to I_6$	$R(LR)^{10} \to I_2$
$(LR)^3 \to T_3$	$(LR)^7 \to T_7$	$(LR)^{11} \to T_{11}$
$R(LR)^3 \to I_9$	$R(LR)^7 \to I_5$	$R(LR)^{11} \to I_1$
$(LR)^4 \to T_4$	$(LR)^8 \to T_8$	$(LR)^0 \to T_0$

Table 2: Mapping between PLR and TI transformations

References

[1] Alissa S. Crans, Thomas M. Fiore, and Ramon Satyendra. Musical actions of dihedral groups. *Journal of Mathematics and Music*, 3(1):1–16, 2009. doi: 10.1080/17459730802699757.

[2] Janine du Plessis. *Transformation groups and duality in the analysis of musical structure*. PhD thesis, 2005.

[3] E. Gollin. Some aspects of three-dimensional “tonnetze”. *Music Theory Online*, 1998. Published 23 January 1998.

[4] Tom Leinster. *Categories, Functors and Natural Transformations*. Cambridge University Press, 2014. doi: https://doi.org/10.1017/CBO9781107360068.003.

[5] Johann Pachelbel. Canone (per la sola mano sinistra), 2025. URL `https://ks15.imslp.org/files/imglnks/usimg/2/2f/IMSLP377245-PMLP04611-Pachelbel_-_Canone_(per_la_sola_mano_sinistra).pdf`. IMSLP Petrucci Music Library.

[6] Wikipedia. Chromatic scale, 2025. URL `https://en.wikipedia.org/wiki/Chromatic_scale`.

[7] Wikipedia. Circle of fifths, 2025. URL `https://en.wikipedia.org/wiki/Circle_of_fifths`.

[8] Julia Winterson. *Maths and Music*. University of Huddersfield Press, 1st edition, 2024.

[9] Ada Zhang. The framework of music theory as represented with groups. 2009.