MA3K7 Assignment 1 Rubric

Cathal Lee (2106946)

1 Entry

I KNOW

From the assumptions of the problem,

- We have a finite number of starting pairs of (a_0, a_1); 100 pairs as $a_0, a_1 \in \{0, \dots, 9\}$.
- Each new number in the bracelet is obtained by adding the previous two numbers and taking the result modulo 10
- Therefore, the next number is uniquely determined by the previous two numbers.

NOTATION

For brevity, I call

- Each bracelet as a sequence (a_n) generated by $a_0, a_1 \in \{0, \dots, 9\}$ and

$$a_{n+1} \equiv a_n + a_{n-1} \pmod{10}$$

where the bracelet ends if $a[0][0] == a[-1][0]$ and $a[0][1] == a[-1][1]$.

- Current position to be

$$pos_n := (a_{n-1}, a_n) \in \mathbb{Z}_{10}^2.$$

- Function T maps two elements to the next position

$$T(x, y) := (y, x + y \bmod 10).$$

so $pos_{n+1} = T(pos_n)$

I WANT

I want to understand what happens if the sequence (a_n) is continued indefinitely. In particular, I want to determine whether the sequence eventually becomes periodic.

Moreover, I want to find how many distinct number bracelets can be formed when the initial pair (a_0, a_1) ranges over all combinations in $\{0, \dots, 9\}$.

2 Attack

NOTE

I realise that the sequence is deterministic by the initial pair (a_0, a_1) because a_n follows the rule:

$$a_{n+1} \equiv a_n + a_{n-1} \pmod{10}$$

I FIND

and this only uses the two most recent terms. Thus, in order to predict the future terms, I do not need the entire history of the sequence but it suffices to know the $pos_n = (a_{n-1}, a_n)$. Therefore, we can perceive the bracelet as an evolution of pairs of digits.

TRY

I am not yet sure how to formalise the equivalence between bracelets generated by different initial pairs, since the same bracelet may be obtained by starting at different points along a cycle. To gain intuition, I first implemented a Python function which iterates T until the initial position is reached again.

INTRODUCE

```
def Sequence(a,b):
    pos = [(a,b)]

    while True:
        new = (pos[-1][0] + pos[-1][1])%10
```

```
        pos.append((pos[-1][1],new))
        if pos[-1][0] == a and pos[-1][1] == b:
            break
    return pos, len(pos)-1
```

If the same cycle is traversed starting from a different point, the resulting bracelet is the same up to a cyclic rotation. So I need to define an equivalence rule to account for this and this motivates the following definition.

Definition 2.1. *Two number bracelets are considered equivalent if their sequences differ only by a cyclic rotation.*

I FIND

Proposition 2.1. *If a bracelet corresponds to a cycle*

$$(\text{pos}_0, \text{pos}_1, \ldots, \text{pos}_{m-1}),$$

then starting from pos_k *produces a cyclic rotation of the same bracelet.*

Hence, counting distinct bracelets is equivalent to counting distinct cycles of T.

AHA This suggests that, to count distinct bracelets, I should not count initial pairs (a_0, a_1), but rather count distinct cycles of the transition map T.

JUSTIFY Recall that earlier I showed that the transition rule is deterministic. In fact we can deduce from this that the map T is invertible.

Hence, each position has a unique predecessor and a unique successor.

Proposition 2.2. *The map* T *is a bijection on* $\mathbb{Z}_{10}^2$.

Proof. Suppose $T(x, y) = (y, z)$. Then $z \equiv x + y \pmod{10}$, which implies

$$x \equiv z - y \pmod{10}.$$

Thus the inverse map exists and is given by

$$T^{-1}(y, z) = (z - y \bmod 10,\ y).$$

□

Therefore, T is a bijection on the finite set $\mathbb{Z}_{10}^2$.

AHA This observation shows that the question of "how many distinct bracelets are there" is equivalent to the question of how many disjoint cycles the map T has on $\mathbb{Z}_{10}^2$. Moreover, since two bracelets that differ only by a cyclic rotation are regarded as the same, each cycle should be counted only once.

PAUSE Now the problem has been reduced to a finite setting with well-defined counting problem.

3 Solution

Claim 1. *Every sequence* (a_n) *generated by the rule*

$$a_{n+1} \equiv a_n + a_{n-1} \pmod{10}$$

is periodic.

Proof. From the above, we model the evolution of the bracelet by the transition map

$$T(x, y) = (y, x + y \bmod 10)$$

acting on positions $(x, y) \in \mathbb{Z}_{10}^2$. Since T is a bijection on the finite set $\mathbb{Z}_{10}^2$ of size 100, the directed graph associated with T decomposes into disjoint directed cycles. Starting from any initial pair (a_0, a_1), the evolution follows one of these cycles and eventually returns to the starting position. Hence the corresponding sequence (a_n) is periodic. □

Claim 2. *The map T is a bijection on $\mathbb{Z}_{10}^2$.*

Proof. Suppose $T(x,y) = (y,z)$. By definition, $z \equiv x + y \pmod{10}$, hence

$$x \equiv z - y \pmod{10}.$$

Therefore, the inverse map exists and is given by

$$T^{-1}(y,z) = (z - y \bmod 10,\, y).$$

Thus T is bijective. □

Claim 3. *The number of distinct bracelets is equal to the number of disjoint cycles of the map T on $\mathbb{Z}_{10}^2$.*

Proof. Again, from above, two bracelets are considered equivalent if they differ only by a cyclic rotation. Such rotations correspond exactly to choosing a different starting position along the same directed cycle of T. Therefore, each directed cycle of T corresponds to exactly one distinct bracelet, and distinct cycles correspond to distinct bracelets. □

INTRODUCE

By explicitly enumerating the cycles of T using a Python implementation, we find that T decomposes $\mathbb{Z}_{10}^2$ into exactly six disjoint cycles.

```
def enumerate_cycles():
    visited = set()
    cycles = []

    for x in range(10):
        for y in range(10):
            start = (x, y)
            # Prevent equivalent bracelet a
            if start in visited:
                continue

            # Evolve until we see a repeated position
            order = {}
            path = []
            cur = start

            while cur not in order:
                order[cur] = len(path)
                path.append(cur)
                cur = T(*cur)

            # Extract the cycle part
            cycle_start_idx = order[cur]
            cycle = path[cycle_start_idx:]

            # Mark all positions encountered as visited
            for s in path:
                visited.add(s)

            cycles.append(cycle)

    return cycles
```

Hence there are six distinct number bracelets.

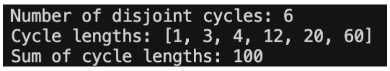

Figure 1: Output Result

4 Review

CHECK Throughout the solution, I tackled the argument in theoretical and computational perspective. In particular, I verified that the transition map T is bijective by constructing its inverse, and confirmed computationally that the resulting cycle decomposition covers all 100 possible positions.

REFLECT When first approaching the problem, it was not immediately clear what would happen if the sequence evolved indefinitely. I think the key insight of this problem was to look at ordered pairs of consecutive digits instead of individual numbers, which massively simplified the computation in python and justified the problem as a deterministic system on a finite state space. This greatly simplified both the conceptual understanding and the counting of distinct bracelets.

EXTEND One extension of this problem is to replace the modulus 10 by a general modulus m. In that case, the state space becomes $\mathbb{Z}_m^2$, and similar arguments suggest that the evolution again decomposes into disjoint cycles whenever the corresponding transition map is bijective. Another possible extension is to consider more general linear recurrences of the form

$$a_{n+1} \equiv pa_n + qa_{n-1} \pmod{m},$$

and investigate how the number and lengths of the resulting cycles depend on the parameters p, q, and m.

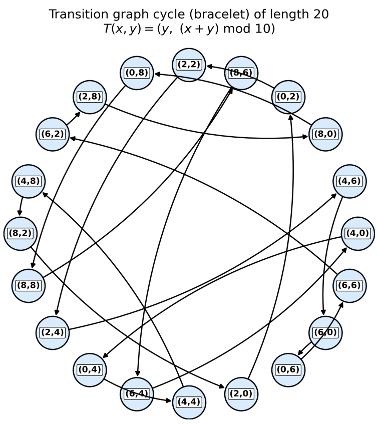

Figure 2: Transition graph cycle example

Code

The code for this assignment can be found on my GitHub page:
`https://github.com/cathal0317/Problem_Solving_RUBRIC`