Project 3

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Problem 1

We are given the mixed model

$$y_{ij} = \beta_0 + \gamma_i + \epsilon_{ij},$$

where γ_i are iid $\mathcal{N}(0, \tau^2)$ and ϵ_{ij} are iid $\mathcal{N}(0, \sigma^2)$ for $i = 1, \ldots, m, j = 1, \ldots n$. This means we have the same number of observations for each group. We can also write the model as

$$y_{ij} = \boldsymbol{x}_{ij}^T \boldsymbol{\beta} + \boldsymbol{u}_{ij} \boldsymbol{\gamma}_i + \epsilon_{ij},$$

where in this case $\boldsymbol{x}_{ij}^T = 1$ and $\boldsymbol{u}_{ij}^T = 1$. For each group $i = 1, \ldots, m$ we have

$$\boldsymbol{y_i} = \boldsymbol{X_i}\boldsymbol{\beta} + \boldsymbol{U_i}\boldsymbol{\gamma_i} + \boldsymbol{\epsilon_i}.$$

Here $\boldsymbol{y_i}$ is a n-dimensional vector of response values for group i, $\boldsymbol{X_i}$ is a $(n \times p)$-dimensional design matrix, and $p = 1$ since we only have intercept in the model. Hence, $\boldsymbol{X_i}$ is a $(n \times 1)$-dimensional vector with only ones. $\boldsymbol{U_i}$ in this case is also a $(n \times 1)$-dimensional design matrix with only ones. The p-dimensional vector of fixed effects $\boldsymbol{\beta} = \beta_0$ in this case, and since we only have a random intercept model the vector of group-specific effects $\boldsymbol{\gamma_i}$ has dimension (1×1). Moreover $\boldsymbol{\epsilon_i}$ is a n-dimensional vector of errors.

The model can be expressed in matrix notation as,

$$\boldsymbol{Y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{U}\boldsymbol{\gamma} + \boldsymbol{\epsilon}.$$

Here $\boldsymbol{Y} = [y_1 \ldots y_m]^T$, $\boldsymbol{X} = [x_1^T \ldots x_m^T]^T$, $\boldsymbol{\gamma} = [\gamma_1 \ldots \gamma_m]^T$, $\boldsymbol{\epsilon} = [\epsilon_1^T \ldots \epsilon_m^T]^T$ and $\boldsymbol{U} = blockdiag(U_1, \ldots, U_m)$. Here $\boldsymbol{X}$ and $\boldsymbol{U}$ are design matrices, $\boldsymbol{\beta}$ is the vector for fixed effects and $\boldsymbol{\gamma}$ the vector of random effects. In this case since γ_i and ϵ_{ij} are independent and they also are mutually independent we have

$$\boldsymbol{\gamma} \sim N(0, G) \quad \text{and} \quad \boldsymbol{\epsilon} \sim N(0, R),$$

where $G = \tau^2 I$ and $R = \sigma^2 I$.

We now implement a function that computes the maximum likelihood and restricted maximum likelihood estimates of the parameters of the given mixed model.

```
library(lme4)
```

```
## Loading required package: Matrix
```

```
data <- read.csv("https://www.math.ntnu.no/emner/TMA4315/2019h/random-intercept.csv",
  colClasses=c("numeric","factor"))
attach(data)
```

```
#Defining functions for beta(V), V(theta), l_p(theta), l_r(theta)

beta <- function(V){
  beta=solve(t(X)%*%solve(V)%*%X)%*%t(X)%*%solve(V)%*%y
  return(beta)
}

#Covariance matrix
```

```r
V <- function(theta){
  R <- theta[1]*diag(n*m)
  G <- theta[2]*diag(m)
  V = R + U%*%G%*%t(U)
  return(V)
}

#profile log-likelihood
l_p <- function(theta){
  V = V(theta)
  beta_est=beta(V)
  l_p = -1/2*(log(det(V))+(t(y-X%*%beta_est)%*%solve(V)%*%(y-X%*%beta_est)))
  return(l_p)
}
#restricted log-likelihood
l_r <- function(theta){
  l_r = l_p(theta)-1/2*log(det(t(X)%*%solve(V(theta))%*%X))
  return(l_r)
}
```

```r
#Define constands and design matrices X and U
  m<-nlevels(group)   # number of clusters/individuals
  n<-length(y)/m # number of measurements within each cluster
  U <- model.matrix(~0 + group)
  X <- matrix(1,m*n)

#Define mylmm
mylmm <- function(y, group,REML = FALSE){
  estimates=rep(0,3) #beta, sigma, tau
  theta <- rep(1,2) #sigma, tau

  #find theta trough numerical maximisation
  if (REML == FALSE) {
    obj = optim(par = theta,fn = l_p,control = list(fnscale=-1))
  }
  else {
    obj = optim(par = theta,fn = l_r,control = list(fnscale=-1))
  }

  #Extract estimated theta
  theta = obj$par
  #Estimated beta
  beta = beta(V(theta))

  estimates[1] = beta
  estimates[2] = sqrt(theta[1])
  estimates[3] = sqrt(theta[2])
  return(estimates)
}

#Check the estimates against computed values by lmer fitted with maximum likelihood
mylmm(y,group, REML = FALSE)
```

```
## [1] 10.426443  1.171092  1.306599
```

```r
lmer(y ~ (1|group), REML=FALSE)
```

```
## Linear mixed model fit by maximum likelihood  ['lmerMod']
## Formula: y ~ (1 | group)
##      AIC      BIC   logLik deviance df.resid
## 150.0288 155.0954 -72.0144 144.0288       37
## Random effects:
##  Groups   Name        Std.Dev.
##  group    (Intercept) 1.306
##  Residual             1.171
## Number of obs: 40, groups:  group, 10
## Fixed Effects:
## (Intercept)
##       10.43
```

```r
#Check the estimates against computed values by lmer fitted with restricted maximum likelihood
mylmm(y,group, REML=TRUE)
```

```
## [1] 10.426443  1.171128  1.390697
```

```r
lmer(y ~ (1|group), REML=TRUE)
```

```
## Linear mixed model fit by REML ['lmerMod']
## Formula: y ~ (1 | group)
## REML criterion at convergence: 143.724
## Random effects:
##  Groups   Name        Std.Dev.
##  group    (Intercept) 1.391
##  Residual             1.171
## Number of obs: 40, groups:  group, 10
## Fixed Effects:
## (Intercept)
##       10.43
```

The estimates from mylmm with maxmimum likelihood are $\hat{\beta} = 10,43$, $\hat{\sigma} = 1,171$ and $\hat{\tau} = 1,307$. When comparing to the computades values by lmer $\hat{\beta} = 10,43$, $\hat{\sigma} = 1,171$ and $\hat{\tau} = 1,306$ we see that the estimates are almost the same. The estimates from mylmm with restricted maximum likelihood are $\hat{\beta} = 10,43$, $\hat{\sigma} = 1,171$ and $\hat{\tau} = 1,391$. When comparing to the computades values by lmer $\hat{\beta} = 10,43$, $\hat{\sigma} = 1,171$ and $\hat{\tau} = 1,391$ we see that these are also almost the same.

Problem 2