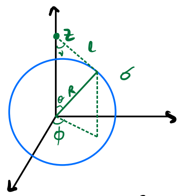


Small piece of electric field:

$$dE = \frac{1}{4\pi\varepsilon_0} \frac{\sigma\,(R^2 \sin\theta\, d\theta\, d\phi)}{\left(\sqrt{R^2+z^2-2Rz\cos\theta}\right)^2} \frac{(z-R\cos\theta)}{(R^2+z^2-2Rz\cos\theta)}$$

$$l^2 = R^2+z^2-2Rz\cos\theta$$

$$\cos\psi = \frac{z-R\cos\theta}{l}$$

$$E = \frac{1}{4\pi\varepsilon_0}\int_0^{2\pi}\int_0^{\pi} \frac{\sigma R^2 \sin\theta\,(z-R\cos(\theta))}{(R^2+z^2-2Rz\cos\theta)^{3/2}}\, d\theta\, d\phi$$

$$= \frac{\sigma R^2}{4\pi\varepsilon_0}(2\pi)\int_0^{\pi} \frac{\sin\theta\,(z-R\cos\theta)}{(R^2+z^2-2Rz\cos\theta)^{3/2}}\, d\theta$$

$u = \cos\theta \quad \frac{du}{d\theta} = -\sin\theta \Rightarrow -du = \sin\theta\, d\theta$ Substitution

function used for integration

$$\boxed{E = \int_{-1}^{1} \frac{(+du)(z-Ru)}{(R^2+z^2-2Rzu)^{3/2}}\, d\theta}$$

Another substitution: $R^2+z^2-2Rzu = y$

$$-2Rz = \frac{dy}{du} \Rightarrow du = \frac{dy}{-2Rz}$$

$$E = \int_{(R+z)^2}^{(R-z)^2} \frac{dy}{-2Rz}\,\frac{z^2-R^2+y}{(2z)\,y^{3/2}}$$

$$-2Rzu = y-R^2-z^2$$

$$z - Ru = \frac{y-R^2-z^2}{2z} + z = \frac{z^2-R^2+y}{2z}$$

$$E = +\int_{(R-z)^2}^{(R+z)^2} \frac{1}{(4Rz^2)}\left((z^2-R^2)\,y^{-3/2} + y^{-1/2}\right) dy$$

$$= \frac{1}{4Rz^2}\int_{(R-z)^2}^{(R+z)^2}\left[(z^2-R^2)\,y^{-3/2} + y^{-1/2}\right] dy$$

$$= \frac{1}{4Rz^2}\left[(z^2-R^2)(-2)\,y^{-1/2} + 2y^{1/2}\right]_{(R-z)^2}^{(R+z)^2}$$

$$= \frac{1}{2Rz^2}\left[(R^2-z^2)\,y^{-1/2} + y^{1/2}\right]_{(R-z)^2}^{(R+z)^2}$$

$$= \frac{1}{2Rz^2}\left[\underbrace{(R^2-z^2)}_{(R+z)(R-z)}\left(\frac{1}{R+z} - \frac{1}{|R-z|}\right) + (R+z-|R-z|)\right]$$

$$= \frac{1}{2Rz^2}\left(R-\cancel{z} - \frac{(R^2-z^2)}{|R-z|} + R + \cancel{z} - |R-z|\right)$$

$$|R-z|^2 = (R-z)^2$$

$$= \frac{1}{2Rz^2}\left(2R - \left(\frac{R^2-\cancel{z^2}+(R-z)^2}{|R-z|}\right)\right)$$

$$(R-z)^2 = R^2-2Rz+\cancel{z^2}$$

$$= \frac{1}{2Rz^2}\left(2R - \left(\frac{2R^2-2Rz}{|R-z|}\right)\right) = \frac{1}{z^2}\left(1-\frac{R-z}{|R-z|}\right)$$

Adding back the constants: $E = \frac{\sigma R^2}{2\varepsilon_0}\,\frac{1}{z^2}\left(1-\frac{R-z}{|R-z|}\right)$ *

① The $z > R$ case: $E = \frac{\sigma R^2}{2\varepsilon_0}\frac{1}{z^2}\left(1 - \cancelto{-1}{\frac{R-z}{z-R}}\right) = \frac{\sigma R^2}{2\varepsilon_0}\frac{1}{z^2}(2) = \frac{\sigma R^2}{\varepsilon_0 z^2}$

(outside of spherical shell)

$$= \frac{1}{4\pi\varepsilon_0}\frac{4\pi R^2\sigma}{z^2}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{Q}{z^2}$$

can be considered as a point particle.

② The $z < R$ case: $E = \frac{\sigma R^2}{2\varepsilon_0}\frac{1}{z^2}\left(1 - \cancelto{1}{\frac{R-z}{R-z}}\right) = 0$

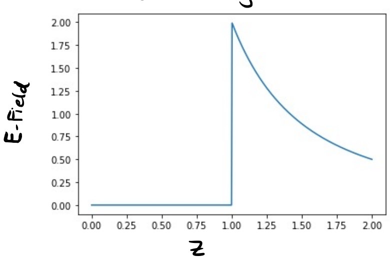
(inside of spherical shell)

∴ Electric field is 0 inside the sphere.

Singularity

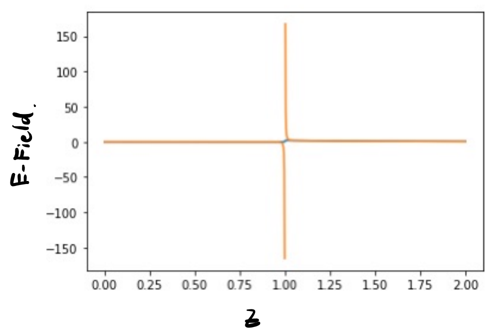
When $z = R$, the integral becomes undefined, since we would end up dividing by zero.

Quad Integrator



Quad doesn't seem to care about this singularity. at $R = z = 1$, the function simply jumps to the next defined integral value.

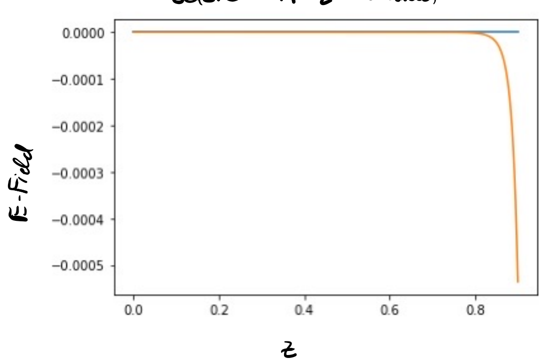
Simpson Integrator



However, my integrator does seem to care. As the function approaches zero, the electric field diverges.

More graphs when approaching 0:

Before $R = z$ (inside)



After $R = z$ (outside)