$\boxed{\text{QUESTION 1}}$

$$\frac{f(x+dx)-f(x-dx)}{2dx} \quad (*)$$

Taylor expansions:

① $f(x+dx) = \cancel{f(x)} + f'(x)\,dx + \cancel{\frac{1}{2}f''(x)\,dx^2} + \frac{1}{6}f^{(3)}(x)\,dx^3 + \cancel{\frac{1}{24}f^{(4)}(x)\,dx^4} + \frac{1}{120}f^{(5)}(x)\,dx^5 + \dots$

② $f(x-dx) = \cancel{f(x)} - f'(x)\,dx + \cancel{\frac{1}{2}f''(x)\,dx^2} - \frac{1}{6}f^{(3)}(x)\,dx^3 + \cancel{\frac{1}{24}f^{(4)}(x)\,dx^4} - \frac{1}{120}f^{(5)}(x)\,dx^5 + \dots$

Plugging into $(*)$: $\frac{f(x+dx)-f(x-dx)}{2dx} + \frac{1}{6}f^{(3)}(x)\,dx^2 + \frac{1}{120}f^{(5)}(x)\,dx^4$

we are left with this 3rd order term
How to cancel it? $\Rightarrow$ use other points.

$$\frac{f(x+2dx)-f(x-2dx)}{4dx} \quad (**)$$

③ $f(x+2dx) = \cancel{f(x)} + 2f'(x)\,dx + \cancel{\frac{4}{2}f''(x)\,dx^2} + \frac{8}{6}f^{(3)}(x)\,dx^3 + \cancel{\frac{16}{24}f^{(4)}(x)\,dx^4} + \frac{32}{120}f^{(5)}(x)\,dx^5 + \dots$

④ $f(x-2dx) = \cancel{f(x)} - 2f'(x)\,dx + \cancel{\frac{4}{2}f''(x)\,dx^2} - \frac{8}{6}f^{(3)}(x)\,dx^3 + \cancel{\frac{16}{24}f^{(4)}(x)\,dx^4} - \frac{32}{120}f^{(5)}(x)\,dx^5 + \dots$

Plugging into $(**)$: $\frac{f(x+2dx)-f(x-2dx)}{4dx} + \frac{(16/6)\,f^{(3)}(x)\,dx^3}{4dx} + \frac{(64/120)\,f^{(5)}(x)\,dx^5}{4dx} + \dots$

$$= \frac{f(x+2dx)-f(x-2dx)}{4dx} + \frac{2}{3}f^{(3)}(x)\,dx^2 + \frac{2}{15}f^{(5)}(x)\,dx^4 + \dots$$

$\frac{16}{6}\times\frac{1}{4} = \frac{4}{6} = \frac{2}{3}$

Now we have 2 equations:

① $f'(x) = \frac{f(x+dx)-f(x-dx)}{2dx} + \frac{1}{6}f^{(3)}(x)\,dx^2 + \frac{1}{120}f^{(5)}(x)\,dx^4$

$\frac{1}{120} - \frac{4}{120} = \frac{-3}{120} = \frac{-1}{40}$

② $\times\frac{1}{4}$ $f'(x) = \frac{f(x+2dx)-f(x-2dx)}{4dx} + \frac{2}{3}f^{(3)}(x)\,dx^2 + \frac{2}{15}f^{(5)}(x)\,dx^4$

$\frac{2}{60} = \frac{1}{30}$

$f'(x) - \frac{f'(x)}{4} = \frac{f(x+dx)-f(x-dx)}{2dx} - \left(\frac{1}{16dx}\right)\left(f(x+2dx)-f(x-2dx)\right) + \cancel{\frac{1}{6}f^{(3)}(x)\,dx^2} - \cancel{\frac{1}{6}f^{(3)}(x)\,dx^2} + \frac{1}{120}f^{(5)}(x)\,dx^4 - \frac{1}{30}f^{(5)}(x)\,dx^4$

$$= \frac{1}{16dx}\left[8f(x+dx) - 8f(x-dx) - f(x+2dx) + f(x-2dx)\right] \underbrace{- \frac{1}{40}f^{(5)}(x)\,dx^4}_{\text{leading error term.}} = \frac{3}{4}f'(x)$$

$$f'(x) = \frac{1}{12dx}\left[8\left(f(x+dx) - f(x-dx)\right) - f(x+2dx) + f(x-2dx)\right]$$

$\frac{1}{16}$

$\frac{1}{16}\cdot\frac{4}{3} = \frac{1}{4\cdot3} = \frac{1}{12}$

b) Total error: $\sim \frac{\varepsilon f(x)}{dx} + f^{(5)}(x)\,dx^4$

Differentiating wrt dx: $\frac{\varepsilon f(x)}{dx} + f^{(5)}(x)\,dx^4$

$$-\frac{\varepsilon f(x)}{dx^2} + 4f^{(5)}(x)\,dx^3 = 0$$

$\Rightarrow 4f^{(5)}(x)\,dx^3 = \frac{\varepsilon f(x)}{dx^2} \Rightarrow dx^5 = \frac{\varepsilon f(x)}{4f^{(5)}(x)} \Rightarrow dx = \left[\frac{\varepsilon f(x)}{4f^{(5)}(x)}\right]^{1/5}$

$\therefore dx \sim \left(\frac{\varepsilon f}{f^{(5)}(x)}\right)^{1/5}$

Optimal dx implies minimal error. Double precision: $\varepsilon = 2^{-52}$

$f(x) = e^x \Rightarrow f^{(5)}(x) = e^x \Rightarrow dx = \left(\frac{2^{-52}}{4}\right)^{1/5} \approx \frac{2^{-52/5}}{4^{1/5}} = \frac{2^{-10.4}}{4^{1/5}} = 0.00056088786 \quad \sim 10^{-3}$

$f(x) = e^{0.01x} \Rightarrow f^{(5)}(x) = (0.01)^5 e^{0.01x} \Rightarrow dx = \left(\frac{2^{-52}}{4(0.01)^5}\right)^{1/5} = \frac{2^{-52/5}}{4^{1/5}(0.01)} = 0.056088786 89 \quad \sim 10^{-2}$

Comparing errors from different dx values:

→ Using $x = 3$, $f = \exp(x)$ (error with dx described above is $\sim 3.659 \times 10^{-13}$)

- $2 \times dx$ yields an error of $\sim 1.1156 \times 10^{-12}$
- $\frac{1}{2} \times dx$ yields an error of $\sim 1.6679 \times 10^{-12}$

→ using $x = 18$, $f = \exp(0.01x)$ (error with dx described above is 0.011972173511 38365

- $2 \times dx$ yields an error of $\sim$ 0.011972173511 472126
- $\frac{1}{2} \times dx$ yields an error of $\sim$ 0.011972173511 380886

In the $f = \exp(x)$ case, it is obvious that we have found the optimal dx, since the dx yields the lowest error.
It is, however, slightly less obvious with $f = \exp(0.01x)$. Error seems to be a little bit better for 0.5 dx than for dx.

That being said, the derived dx seems to be an appropriate approximation.