Question 5

a) Show that $\sum_{x=0}^{N-1} e^{-2\pi i k x/N} = \frac{1-e^{-2\pi i k}}{1-e^{-2\pi i k/N}}$ (*)

Sum of finite geometric series: $S_n = \frac{a_1(1-r^n)}{1-r}$

$a_1 = 1$, $r = \alpha = e^{-2\pi i k/N}$

$$S_n = \frac{1-\alpha^N}{1-\alpha} = \frac{1-e^{-\cancel{N}2\pi i k/\cancel{N}}}{1-e^{-2\pi i k/N}} = \frac{1-e^{-2\pi i k}}{1-e^{-2\pi i k/N}} \quad \checkmark$$

b) (i) Show that this approaches N as k approaches 0

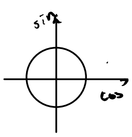
Taylor expansion: $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots$

So, as $k \to 0$: $\frac{1-(1-2\pi i k+\ldots)}{1-(1-2\pi i k/N+\ldots)} = \frac{\cancel{2\pi i k}}{\frac{\cancel{2\pi i k}}{N}} = N$

(ii) Show that this is 0 for any integer k that is not a multiple of N.

$$e^{-2\pi i k} = \cos(-2\pi i k) - i\sin(-2\pi i k)$$
$$e^{-2\pi i k/N} = \cos(-2\pi i k/N) - i\sin(-2\pi i k/N)$$

Plugging back into eqn.: $\frac{1-\cos(-2\pi i k)+i\sin(-2\pi i k)}{1-\cos(-2\pi i k/N)+i\sin(-2\pi i k/N)}$



If k was a multiple of N, then the denominator would go to 0, which is unphysical. $\hookrightarrow k = mN$

$$1-\cos(-2\pi i m)+i\sin(-2\pi i m)$$
$$= 1-1+0 = 0$$

However, if k is an integer BUT not a multiple of N, then the denominator would be non-zero and the numerator would go to 0:

$$1-\cos(-2\pi i k)+i\sin(-2\pi i k)$$
$$= 1-1+0 = 0$$

Meaning that (*) is 0 for any integer k that is not a multiple of N.

c) DFT of a non-integer sine-wave:

$$\sin(2\pi x k/N) = \frac{e^{i2\pi x k/N} - e^{-i2\pi x k/N}}{2i}$$

Fourier transform:

$$\mathcal{F}(k) = \sum_{x=0}^{N-1} \left(\frac{e^{i2\pi x k/N} - e^{-i2\pi x k/N}}{2i} \right) e^{-2\pi i k' x/N}$$

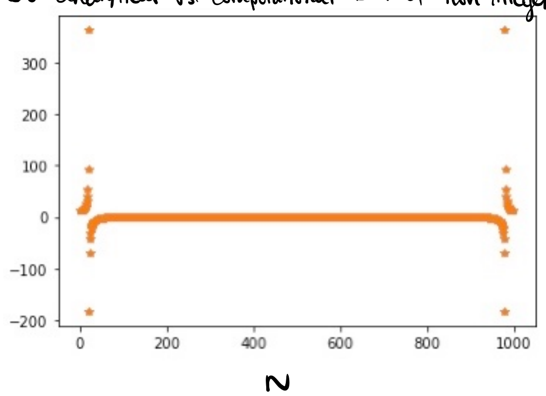
$$= \frac{1}{2i} \sum_{x=0}^{N-1} e^{-2\pi i x(k'-k)/N} - e^{-2\pi i x(k'+k)/N}$$

$$= \frac{1}{2i} \left[\frac{1 - e^{-2\pi i(k'-k)}}{1 - e^{-2\pi i(k'-k)/N}} - \frac{1 - e^{-2\pi i(k'+k)}}{1 - e^{-2\pi i(k'+k)/N}} \right]$$

→ Now I need to plot this vs. analytical solution.

Graph 1: Analytical vs. Computational DFT of non-integer sine wave.

$k = 20.34$



✱ = computational
● = analytical.

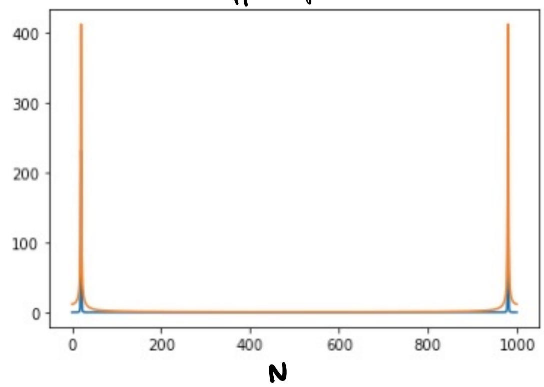
Notice that the analytical DFT is not seen. This is because the error between both methods is 3.67×10^{-13}. Thus, FFT agrees with analytical solution.

It is definitely close to the delta function, however this has finite width.

d) From Wikipedia, I am using the Hann window:

$$w[n] = 0.5\left[1 - \cos\left(\frac{2\pi x}{N}\right)\right]$$

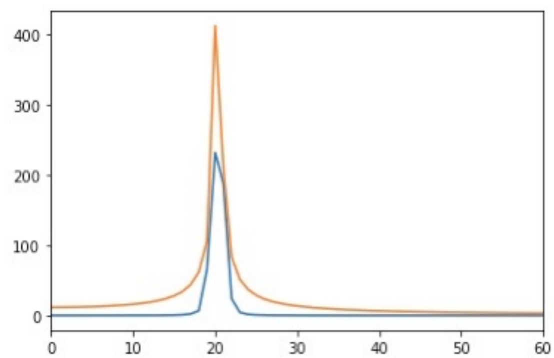
Graph 2: Effect of window fct.



orange = w/o window
blue = w/ window

We can see a difference, but it's better if we zoomed in onto one of the peaks:

Graph 3: Effect of window fct. (zoomed in)



orange = w/o window
blue = w/ window

Peak drops dramatically when we multiply input data by window function, thus spectral leakage for a non-integer period sine curve drops dramatically.

e) Show that Fourier transform of the window is $[N/2 \ -N/4 \ 0 \ldots 0 \ -N/4]$

$$w = 0.5\left[1 - \cos\left(\frac{2\pi x}{N}\right)\right]$$
$$= 0.5\left[1 - \left(\frac{e^{2\pi i x/N} + e^{-2\pi i x/N}}{2}\right)\right]$$

Taking DFT:

$$0.5 \sum_{x=0}^{N-1} \left[1 - \left(\frac{e^{2\pi i x/N} + e^{-2\pi i x/N}}{2}\right)\right] e^{-2\pi i k x/N}$$
$$= \frac{1}{2} \sum_{x=0}^{N-1} \left[e^{-2\pi i x/N} - \frac{e^{-2\pi i (k-1)x/N}}{2} - \frac{e^{-2\pi i (k+1)x/N}}{2}\right]$$

$$= \frac{1}{2}\left[\frac{\overset{(*)}{1 - e^{-2\pi i k}}}{\underbrace{1 - e^{-2\pi i k/N}}} - \frac{1}{2}\left(\frac{\overset{(*)}{1 - e^{-2\pi i(k-1)}}}{\underbrace{1 - e^{-2\pi i(k-1)/N}}}\right) - \frac{1}{2}\left(\frac{\overset{(*)}{1 - e^{-2\pi i(k+1)}}}{\underbrace{1 - e^{-2\pi i(k+1)/N}}}\right)\right]$$

So we know that k must be a multiple of N for a non-zero value.

$\rightarrow k = 0, N, 2N, \ldots$ $\rightarrow k = 1, N+1, \ldots$ $\rightarrow k = 1, N-1, \ldots$

when $k = 0$, $(*)$ becomes N, and other terms are 0.
so we have $\frac{1}{2}N$

when $k = 1$, $(*)$ becomes N, and other terms are 0.
so we have $-\frac{1}{2}N\left(\frac{1}{2}\right) = -\frac{N}{4}$

when $k = 2$, everything is 0,

when $k = N-1$, $(*)$ becomes N, and other terms are 0.
so we have $-\frac{N}{4}$

when $k = N$, $(*)$ becomes N, and other terms are 0.
so we get $\frac{N}{2}$

Thus, doing DFT on window function gives:

$$[\,N/2 \;\; -N/4 \;\; 0 \;\; 0 \;\ldots\; 0 \;\; -N/4 \;\; N/2\,]$$

$$\sum_{k=0}^{N} F(k)\, W(k'-k) \qquad k' = 0, 1, \ldots, N.$$

$k' = 0$: $\sum_{k=0}^{N} F(k)\, W(-k) = \frac{N}{2} F(0)$

$k' = 1$: $\sum_{k=0}^{N} F(k)\, W(1-k) = \frac{N}{2} F(1) - \frac{N}{4} F(0)$

$k' = 2$: $\sum_{k=0}^{N} F(k)\, W(2-k) = \frac{N}{2} F(2) - \frac{N}{4} F(1)$

... Pattern can be seen ....

$$\sum_{k=0}^{N} F(k)\, W(m-k) = \frac{N}{2} F(m) - \frac{N}{4} F(m-1)$$