Probabilistic graphical models

Lecture 3 of “Mathematics and AI”

Outline

1. Definition
2. Markov random fields
3. Bayesian networks
4. Inference via message passing

Factorizations of functions

$$p(x,y) = p(x)p(y)$$

$$p(x,y) \neq p(x)p(y)$$

$$e^{x+y} = e^x e^y$$

$$e^{(x+y)^2} = e^{x^2} e^{y^2} e^{2xy}$$

Definition

Definition

We want to describe complex relationships between an (ordered) set

$$S = \{X_1, X_2, \dots, X_n\}$$

of random variables $X_1, X_2, \dots X_n$. We can do so via the multivariate probability distribution

$$p(S = s) = p(X_1 = x_1, X_2 = x_2, \dots, X_n = x_n)$$

If S can be split into two independent subsets S_1 and S_2, $p(S = s)$ can be factorized:

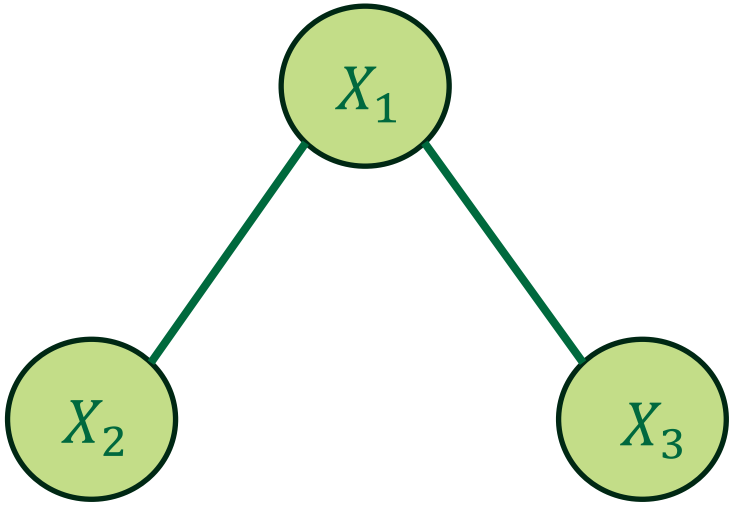
$$p(S = s) = p(S_1 = s_1)p(S_2 = s_2)$$

Definition

A probabilistic graphical model (PGM) includes:

- A multivariate probability distribution
- A graphical representation of its factorization properties

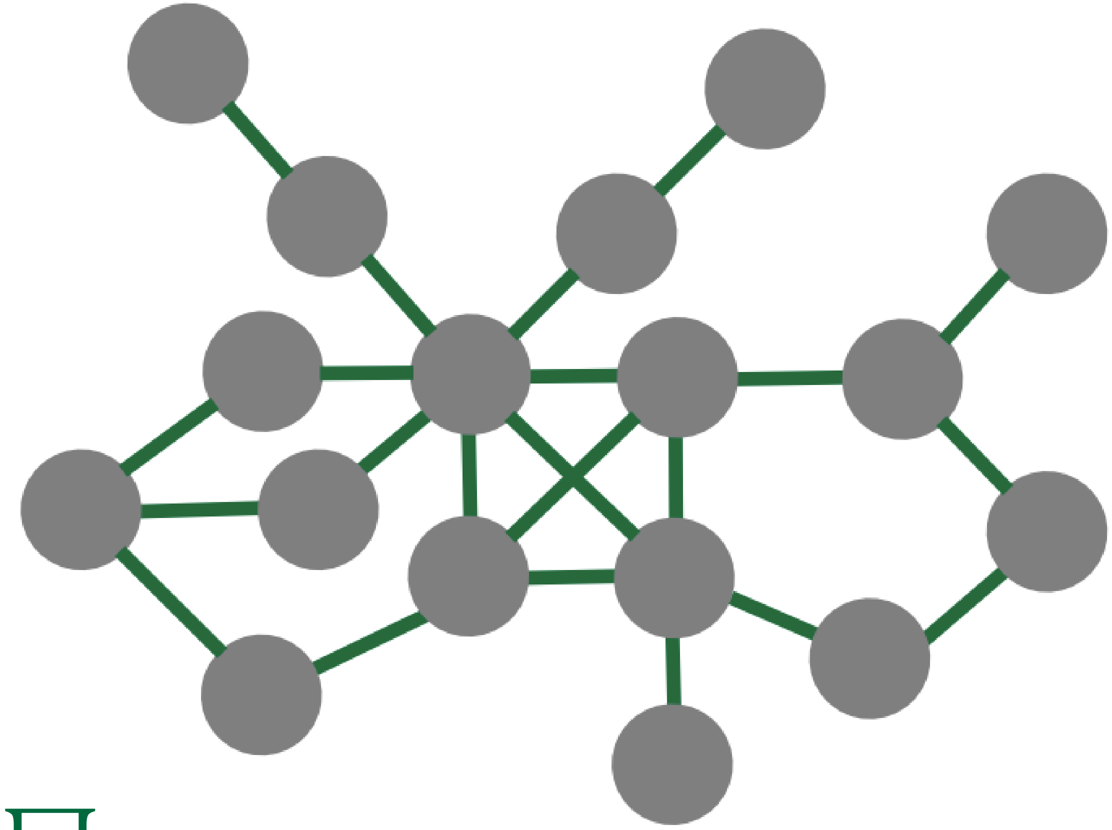
$$p(X_1, X_2, X_3)$$



Different variants of PGMs represent factorization properties differently

Markov random fields

Markov random field



- Undirected graph of statistical dependencies
- Edge = statistical dependence
- Missing edge = independence
- Probability distribution $p(S=s) = \frac{1}{Z} \prod_{c_i \in \text{Cliques}} \varphi(C_i)$
- Extension: Factor graphs

"potential" or "factor"

Normalization factor

Example

Rain Spring	False	True
False	$\frac{3}{4} \cdot \frac{4}{5}$	$\frac{3}{4} \cdot \frac{1}{5}$
True	$\frac{1}{4} \cdot \frac{1}{2}$	$\frac{1}{4} \cdot \frac{1}{2}$

Rain Outside	False	True
False	$\frac{5}{6} \cdot \frac{3}{4}$	$\frac{19}{20} \cdot \frac{1}{4}$
True	$\frac{1}{6} \cdot \frac{3}{4}$	$\frac{1}{20} \cdot \frac{1}{4}$