Orders of Growth

Computer Science Mentors 61A

March 7 to March 11, 2016

1. In big-O notation, what is the runtime for `foo`?

   (a)
   ```
   def foo(n):
       for i in range(n):
           print('hello')
   ```

   Solution: $O(n)$. This is simple loop that will run n times.

   (b) What's the runtime of `foo` if we change `range(n)`:

   i. To `range(n / 2)`?

   Solution: $O(n)$. The loop runs $n/2$ times, but we ignore constant factors.

   ii. To `range(10)`?

   Solution: $O(1)$. No matter the size of n, we will run the loop the same number of times.

   iii. To `range(10000000)`?

   Solution: $O(1)$. No matter the size of n, we will run the loop the same number of times.

2. What is the order of growth in time for the following functions? Use big-O notation.

   (a)
   ```
   def strange_add(n):
       if n == 0:
           return 1
       else:
   ```

```
        return strange_add(n - 1) + strange_add(n - 1)
```

Solution: $O(2^n)$. To see this, try drawing out the call tree. Each level will create two new calls to `strange_add`, and there are n levels. Therefore, 2^n calls.

(b)
```
def stranger_add(n):
    if n < 3:
        return n
    elif n % 3 ==  0:
        return stranger_add(n - 1) + stranger_add(n - 2) +
           stranger_add(n - 3)
    else:
        return n
```

Solution: $O(n)$ is n is a multiple of 3, otherwise $O(1)$.
The case where n is not a multiple of 3 is fairly obvious – we step into the else clause and immediately return.
If n is a multiple of 3, then neither $n-1$ nor $n-2$ are multiples of 3 so those calls will take constant time. Therefore, we just run `stranger_add`, decrementing the argument by 3 each time.

(c)
```
def waffle(n):
    i = 0
    sum = 0
    while i < n:
        for j in range(50 * n):
            sum += 1
        i += 1
    return sum
```

Solution: $O(n^2)$. Ignore the constant term in $50 * n$, and it because just two for loops.

(d)
```
def belgian_waffle(n):
    i = 0
    sum = 0
    while i < n:
        for j in range(n ** 2):
            sum += 1
        i += 1
    return sum
```

Solution: $O(n^3)$. Inner loop runs n^2 times, and the outer loop runs n times. To get the total, multiply those together.

(e)
```
def pancake(n):
    if n == 0 or n == 1:
        return n
    # Flip will always perform three operations and return
      -n.
    return flip(n) + pancake(n - 1) + pancake(n - 2)
```

Solution: $O(2^n)$. Flip will run in constant time. Therefore, this call tree looks very similar to fib! (which is 2^n)

(f)
```
def toast(n):
    i = 0
    j = 0
    stack = 0
    while i < n:
        stack += pancake(n)
        i += 1
```

```
    while j < n:
        stack += 1
        j += 1
    return stack
```

Solution: $O(n2^n)$. There are two loops: the first runs n times for 2^n calls each time (due to pancake), for a total of $n2^n$. The second loop runs n times. When calculating orders of growth however, we focus on the dominating term – in this case, $n2^n$.

3. Consider the following functions:

```
def hailstone(n):
   print(n)
   if n < 2:
    return
   if n % 2 == 0:
        hailstone(n // 2)
   else:
        hailstone((n * 3) + 1)

def fib(n):
   if n < 2:
      return n
   return fib(n - 1) + fib(n - 2)

def slow(n):
    i, j, k = 0, 0, 0
    while i < n:
        while j < n:
            while k < n:
                fib(k)
                k += 1
            fib(j)
            j += 1
        fib(i)
        i += 1

def foo(n, f):
    return n + f(500)
```

In big-O notation, describe the runtime for the following:

(a) `foo(10, hailstone)`

Solution: $O(1)$. $f(500)$ is independent of the size of the input n.

(b) `foo(3000, fib)`

Solution: $O(1)$. See above.

(c) `foo(99999999999, slow)`

Solution: $O(1)$. See above.

4. **Fast Exponentiation:** in this problem, we will examine a real-world algorithm used to improve the speed of calculating exponents.

   (a) First, express the runtime of the naive exponentiation algorithm in big-O notation.

```
def exp(b, n):
    if n == 0:
        return 1
    else:
        return b * exp(b, n - 1)
```

Solution: $O(n)$. n decreases by 1 each call, so there are naturally n calls.

   (b) Now, express the runtime of the fast exponentiation algorithm in big-O notation.

```
def fast_exp(b, n):
    if n == 0:
        return 1
    elif n % 2 == 0:
        return square(fast_exp(b, n // 2))
    else:
        return b * fast_exp(b, n - 1)
```

Solution: $O(\log n)$. n is halved each call, so the number of calls is the number of times n must be halved to get to 1. This is $\log n$.

   (c) What about this slightly modified version of `fast_exp`?

```
def fast_exp(b, n):
    for _ in range(50 * n):
        print("Killing time")
    if n == 0:
        return 1
    elif n % 2 == 0:
        return square(fast_exp(b, n // 2))
    else:
        return b * fast_exp(b, n - 1)
```

Solution: $O(n)$. Ignore the constant term. The first call will perform n operations, the second call will perform $n/2$ operations, the third will perform $n/4$ operations, etc. Using geometric series, we see this adds up to $2n$, which is n if we ignore constant terms.

5. **Mysterious loops:** What is the order of growth in time for the following functions? Use big-O notation.

   (a)
```
def mystery(n):
    for i in range(n):
        while i % 2 != 0:
            print(i)
            i = i - 1
        print("Done")
```

   Solution: $O(n)$. The work for when i is divisible by two is constant. Subtracting one will immediately allow us to exit the `while` loop. Therefore, we can concentrate on just the outer loop.

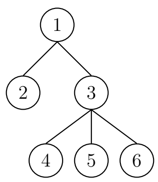
   (b)
```
def fun(n):
    for i in range(n):
        for j in range(n * n):
            if j == 4:
                return -1
            print("Fun!")
```

   Solution: $O(1)$. Inner loop always immediately exits after running for 4 iterations, independent of n.

6. **Orders of Growth and Trees:** Assume we are using the non-mutable Tree implementation introduced in discussion. Consider the following function:

```
def word_finder(t, n, word):
    if label(t) == word:
        n -= 1
        if n == 0:
            return True
    for child in children(t):
        if word_finder(child, n, word) == True:
            return True
    return False
```

(a) What does this function do?
Hint: A path is a sequence of connected nodes. For example, here are four paths in the tree below: $1 \rightarrow 2$, $1 \rightarrow 3 \rightarrow 4$, $1 \rightarrow 3 \rightarrow 5$, $1 \rightarrow 3 \rightarrow 6$



Solution: This function take a Tree `t`, an integer `n`, and a string `word` in as input. Then, `word_finder` returns True if the word appears as a label in any path in the Tree `n`-times and False otherwise.

(b) If a tree has n total nodes, what is the total runtime for all searches in big-O notation?

Solution: $O(n)$. At worst, we must visit every node of the tree.

7. **Orders of Growth and Linked Lists:** Consider the following linked list function:

```
def insert_at_end(lst, x):
    if lst.rest is Link.empty:
        lst.rest = Link(x)
    else:
        insert_at_end(lst.rest, x)
```

(a) What does this function do?

> **Solution:** Inserts a value `x` at the end of linked list `lst`.

(b) Say we want to repeatedly insert some numbers into the end of a linked list:

```
def insert_many(lst, n):
    for i in range(n):
        insert_at_end(lst, i)
```

i. Assume `lst` is initially length 1. How long will it take to do the first insertion? The second? The nth?

> **Solution:** Notice that the list gets longer with each insertion, so each operation will make it harder to do the next. Therefore, the first insertion will take about 1 unit of time. The second will take about twice as long, at two units of time. The nth insertion will take n units of time.

ii. In big-O notation, What is the total runtime to do all the inserts? (total runtime of `insert_many`)

> **Solution:** The total runtime will be the sum of all the inserts: 1 + 2 + 3 + $\ldots + n = \frac{n(n+1)}{2} = O(n^2)$