Infinite sparse factor stochastic volatility model

Martina Danielova Zaharieva

CUNEF Universidad

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Multivariate Stochastic Volatility

- Stochastic volatility models: flexible alternative to the GARCH
- Large literature on univariate modeling
- No straightforward extension to multivariate models
  - $\rightarrow$ Appropriate factorization of the covariance matrix
  - $\rightarrow$ Wishart processes
- Exploring the latent factor structure as a scalable alternative

Factor Stochastic Volatility

- Generalization of the univariate SV model
- Dealing with the *curse of dimensionality*
- Imposing a lower dimensional latent factor structure
- Time-varying volatility for the idiosyncratic term, driving the univariate dynamics
- Time-varying volatility for the factors, driving the multivariate dynamics
- Straight forward Bayesian inference for latent factor models

Sparse Factor Stochastic Volatility

- Prevents overfitting
- Sparsity can improve the predictive performance
- More room for interpretation
- **Sparsity on the loadings** is a plausible assumption
- Not all factors are relevant for each dimension

Infinite Sparse Factor Stochastic Volatility

- Bayesian non-parametric prior for the loadings matrix
- Indian Buffet Process (IBP) for the factor loadings matrix (Griffiths and Ghahramani, 2006)
- A flexible way to impose sparsity and at the same time to avoid making assumptions on the number of factors
- The complexity of the model is determined by the data

Outline

Latent factor models

- Data generating process

$$\mathbf{y}_t \sim N(0, \Omega)$$

- Add factor structure

$$\mathbf{y}_t = \Lambda \mathbf{f}_t + \boldsymbol{\epsilon}_t$$

(i) $\mathbf{f}_t \sim N(0, I_K)$
(ii) $\boldsymbol{\epsilon}_t \sim N(0, \Sigma)$, where $\Sigma = \text{diag}(\sigma^2_{1:N})$
(iii) $\boldsymbol{\epsilon}_t$ and $\mathbf{f}_t$ are independent
(iv) Λ is the $N \times K$ factor loading matrix

- The covariance matrix is constrained

$$\begin{aligned}\Omega =& \text{Var}(\mathbf{y_t}|\Omega) = \text{Var}(\mathbf{y_t}|\Lambda, \Sigma)\\ \Omega =& \Lambda\Lambda' + \Sigma\end{aligned}$$

Factor stochastic volatility

- Straight forward multivariate extension of the univaraite SV
- Reduce the modeling to $N + K$ univariate stochastic volatility models (N series and K factors)

$$\mathbf{y}_t = \Lambda \mathbf{f}_t + \boldsymbol{\epsilon}_t,$$
$$\boldsymbol{\epsilon}_t \sim N_N(\mathbf{0}, \mathbf{H}_t),$$
$$\mathbf{f}_t \sim N_K(\mathbf{0}, \mathbf{V}_t).$$

$$\mathbf{H}_t(\mathbf{h}_t) = \text{diag}(\exp\{h_{1,t}\}, \ldots, \exp\{h_{N,t}\}),$$
$$\mathbf{V}_t(\mathbf{h}_t) = \text{diag}(\exp\{h_{N+1,t}\}, \ldots, \exp\{h_{N+K,t}\})$$

with

$$h_{i,t} - \mu_i = \phi_i(h_{i,t-1} - \mu_i) + \sigma_i^{\eta} \eta_{i,t}, \text{ for } i = 1, \ldots, N + K$$

Factor stochastic volatility

- The overall covariance matrix of $\mathbf{y}_t$ is then given by

$$\Omega_t = \Lambda \mathbf{V}_t(\mathbf{h}_t)\Lambda' + \mathbf{H}_t(\mathbf{h}_t)$$

- $\eta_{i,t} \sim N(0,1)$
- To account for the fat tails

$$\begin{aligned} \mathbf{y}_t &= \Lambda \mathbf{f}_t + \boldsymbol{u}_t, \\ u_{i,t} &= \lambda_{i,t}^{-1/2} \epsilon_{i,t}, \\ \lambda_{i,t} &\sim Gamma\left(\frac{\nu_i}{2}, \frac{\nu_i}{2}\right) \end{aligned}$$

FSV: Bayesian inference

1. Step Sample $\mathbf{H}_t, \mathbf{V}_t | \Lambda, \mathbf{f}, \mathbf{y}$ by defining

$$\tilde{y}_{i,t} \begin{cases} = y_{i,t} - \Lambda_{(i)} f_{i,t}, & \text{for } i = 1, \dots, N, \\ = f_{i,t}, & \text{for } i = N+1, \dots, N+K, \end{cases}$$

   where $\Lambda_{(i)}$ denotes the ith row of Λ
2. Step Sample $\mathbf{f} | \mathbf{H}_t, \mathbf{V}_t, \Lambda, \mathbf{y}$
3. Step Sample $\Lambda | \mathbf{H}_t, \mathbf{V}_t, \mathbf{f}, \mathbf{y}$

– Step 2,3 are easily derived from Bayesian linear regression theory (e.g Lopes and West, 2004)

Infinite sparse FSV

- An infinite dimensional process driving the factor loadings
  - (i) Achieve sparsity in the loadings
  - (ii) Avoid making apriori assumption on the number of the factors
- A "spike and slab"-type of prior for the elements of the loadings matrix Λ (Knowles and Ghahramani, 2011):
- A Bayesian nonparametric prior on the binary matrix $\mathbf{Z}$
- $\mathbf{Z}$ with infinitely many columns

$$p(\lambda_{ik}|z_{ik}) = z_{ik}N(0, c_0) + (1 - z_{ik})\delta_0(\lambda_{ik}),$$

- Assigning potentially infinite number of factors

Indian Buffet Process (1)

- Bayesian nonparametric prior, defining a distribution over infinite binary matrices
- The elements of the binary matrix $\mathbf{Z}$, z_{ik} represent which dimension i includes any contribution from factor k
- $\mathbf{Z}$ has infinitely many columns but finite number of non-zero entries

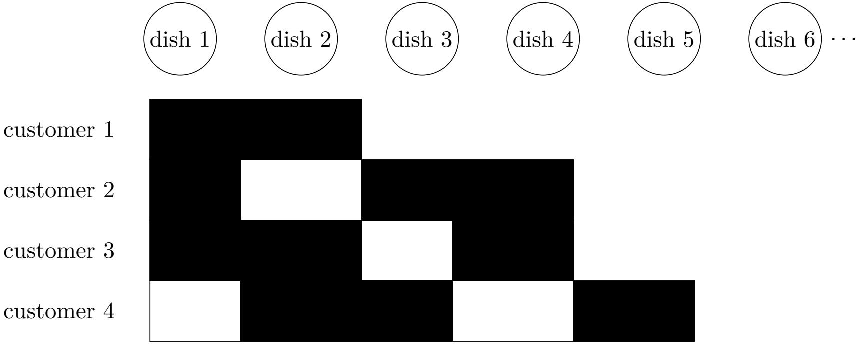
$\Rightarrow$ The number of factors is not determined a-priori

Indian Buffet Process (2)

- N customers entering the restaurant
- First customer samples $Poisson(\alpha)$ dishes
- The i-th customer samples already sampled dishes with probability m_k/i
- m_k is the number of customers previously sampled k
- The customer samples at the end $Poisson(\alpha/i)$ new dishes
- The sampled dishes are collected in the binary matrix $\mathbf{Z}$ with dimensions $N \times \infty$
- Element $z_{ik} = 1$ if customer i sampled dish k

Indian Buffet Process (3)

- Lunch buffet with apparently infinite number of dishes
- N customers entering and selecting by moving from the left to the right

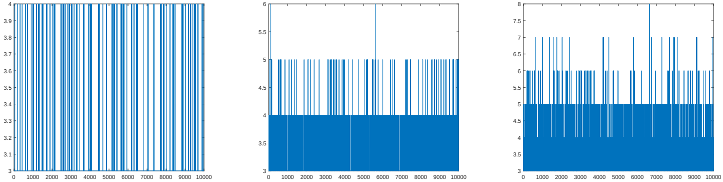


Simulation exercise

Data generation process:

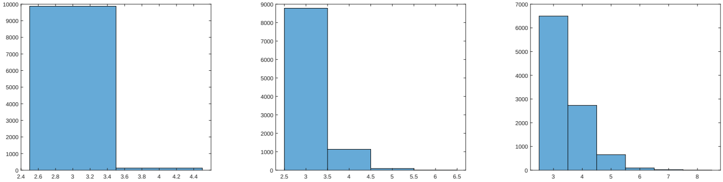
- 3-factor model (true $K = 3$), $N = 9$, $T = 1500$
- Sparsity 44% including identification restrictions
- α is fixed to $\alpha = 0.1, 1, 5$

Simulation exercise



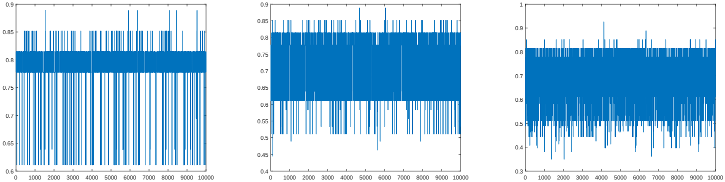
Trace of posterior number of factors when $\alpha = 0.1, 1$ and 5 from left to right

Simulation exercise



Posterior number of factors when $\alpha = 0.1, 1$ and 5 from left to right

Simulation exercise



Sparsity in % when $\alpha = 0.1, 1$ and 5 from left to right

Real data results

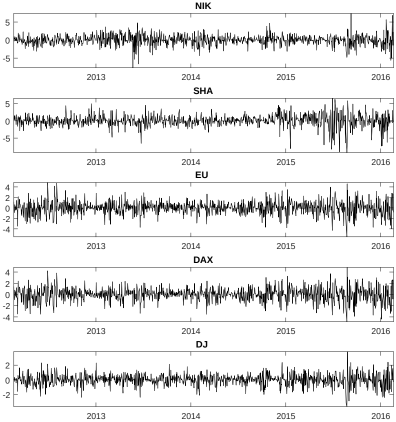


Figure: Daily returns in %

Real data results

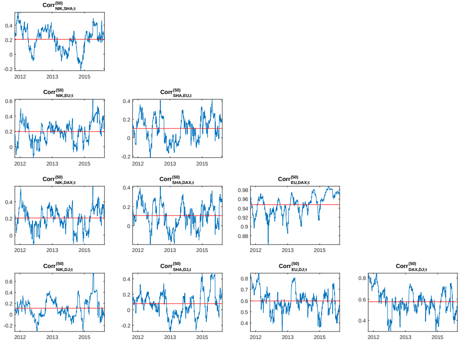
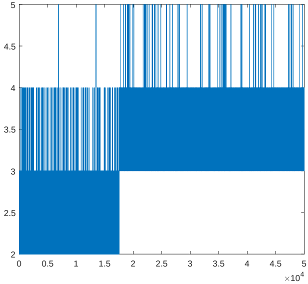


Figure: Sample correlations obtained from a rolling window of size 50 centered around the actual observation with the sample-correlation (horizontal line).

Real data results



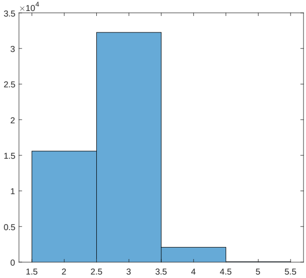


Figure: The number of active factors over iterations.

Real data results

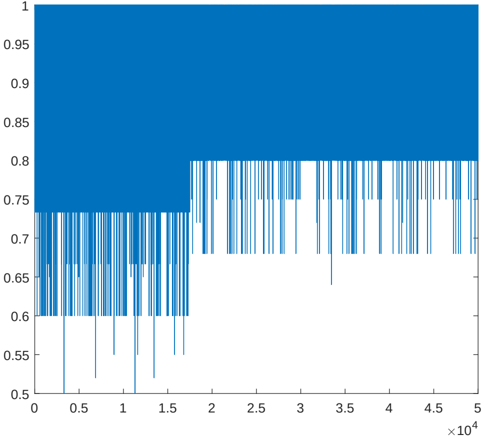


Figure: The sparsity of the loadings matrix over iterations in % (non-zero entries).

Model comparison

Cumulative log-predictive likelihoods (CPLs)

Model	CPL	pred. log BF
∞-FSV	-486.0189	
2-FMSV	-487.5190	1.5001
3-FMSV	-486.3124	0.2935
4-FMSV	-487.4388	1.4199

Out-of-sample period: 22 February 2016 – 8 July 2016 (100 observations).

Main issues:

- Identification:
  - Popular solution: $\lambda_{ii} = 1$ and $\lambda_{ij} = 0$ for $i < j$, $i \leq K$
  - More attractive solution: final re-sampling of $\lambda_{ij} = 0$ for $i < j$ and letting $\lambda_{ii} \neq 1$
  - Ensure full rank of the loadings (Poisson prior)
- Interpretation:
  - Identifying sparsity patterns
  - Prediction
  - High number of factors

Outlook

- Interpretability
- Application to Portfolio allocation
- Predictive performance investigation

Thank you!