Coverage of credible intervals under multivariate monotonicity

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Bayesian Networking Workskshop 2023, Monash University, Melbourne, Australia

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Preliminaries

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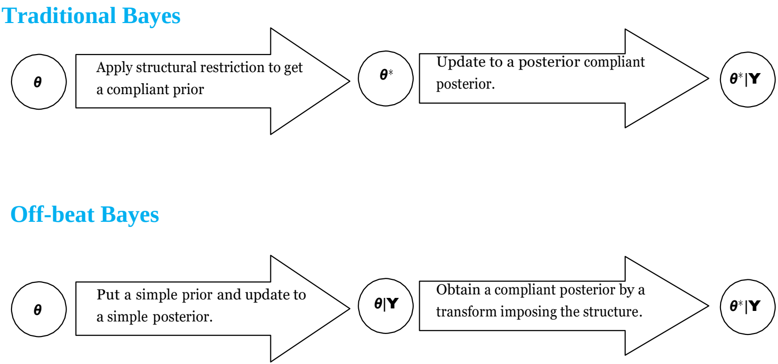
- Posterior analysis is simpler on Θ aided by conjugacy.
- Consider a map that takes an element of Θ to an element of Θ_0. This may be used to induce a prior distribution on Θ_0, but the corresponding posterior distribution is complicated.
- The key idea is to switch the order of a map $\iota : \theta \mapsto \theta^*$ enforcing the desirable restriction, and the posterior updating, and use the induced posterior distribution to make an inference.

Traditional Bayes vs the Off-beat Bayesian Idea



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- An immersion map only needs to be defined on the support of the prior/posterior. Oftentimes, priors are put using basis expansion. Then the immersion map will typically be a finite-dimensional optimization.

Univariate Monotone Regression

Chakraborty and G. (2021, EJS; 2021, AoS)

- A finite random series of step functions: $f(x) = \sum_{j=1}^{J} \theta_j \mathbb{1}((j-1)/J < x \leq j/J]$, $\theta_1, \ldots, \theta_J$ are the coefficients given a prior distribution, J is the number of terms.

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- Monotone projection reduces to isotonization of the θ-coefficients, for which efficient algorithms like the PAVA exist.
- Work with a deterministic choice of J, depending on the sample size n.

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- Consider an equal tailed $(1-\alpha)$-credible interval for $f(x_0)$ at an interior point x_0. Does its coverage go to $1-\alpha$?

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- **Cox phenomenon:** Agreement of credibility and coverage happens in parametric problems by the Bernstein-von Mises theorem, but for smoothing problems, the coverage may be arbitrarily low (Cox (1993, Ann Stat)).
- We shall obtain coverage of the monotone regression problem with the empirical $\mathbb{L}_2$-projection-posterior, which corresponds to the weighted isotonization problem of minimizing $\sum_{j=1}^{J} N_j(\theta_j - \theta_j^*)^2$ subject to $\theta_1^* \leq \cdots \leq \theta_J^*$, where $N_j = \sum_{i=1}^{n} \mathbb{1}\{(j-1)/J < X_i \leq j/J]\}$.

Limiting Coverage of a Credible interval

Let $\Delta_f^* = \arg\min\{f(t) + t^2 : t \in \mathbb{R}\}$, W_1, W_2 be independent two-sided Brownian motions, $C_0 = 2b\,(a/b)^{2/3}$ with $a = \sqrt{\sigma_0^2/g(x_0)}$, $b = f_0'(x_0)/2$, g the density of X. Let $\hat{f}_n$ be the sieve-MLE.
Let $F^*_{a,b}(z|w) = \mathrm{P}(C_0\Delta^*_{w+W_2} \le z)$.

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Theorem

(a) *for every* $z \in \mathbb{R}$, $\Pi(n^{1/3}(f^*(x_0) - f_0(x_0)) \leq z|\mathbb{D}_n) \rightsquigarrow F^*_{a,b}(z|W_1)$;

(b) *Limiting coverage of* $I_{n,\gamma}$*: with* $Z_B = \mathrm{P}(\Delta^*_{W_1+W_2} \geq 0 \big| W_1)$,
$\mathrm{P}_0\left(f_0(x_0) \in I_{n,\gamma}\right) \to \mathrm{P}(\gamma/2 \leq Z_B \leq 1 - \gamma/2)$.

Note that the nuisance parameters a, b magically vanish from the limit.

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- If the target coverage is $(1-\alpha)$, starting with a $(1-\gamma)$-credible interval, where $A(\gamma/2) = \alpha/2$, the **limiting coverage** $(1-\alpha)$ **is attained exactly.**

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- For instance, 93.2% equal-tailed credible interval has 95% limiting coverage.

Multivariate Monotone Regression

Prior and projection-posterior

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- $Y = f(\boldsymbol{X}) + \varepsilon$ with $\boldsymbol{X} \sim G$ on $[0,1]^d$, f multivariate monotone, that is, $f(\boldsymbol{x}_1) \leq f(\boldsymbol{x}_2)$ if $\boldsymbol{x}_1 \preceq \boldsymbol{x}_2$, where $\boldsymbol{x}_1 \preceq \boldsymbol{x}_2$ means that $x_{1,k} \leq x_{2,k}$, $k = 1, \ldots, d$.

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- Prior: On piecewise constant functions $f = \sum_{\boldsymbol{j}} \theta_{\boldsymbol{j}} \mathbb{1}_{I_{\boldsymbol{j}}}$ on the blocks $I_{\boldsymbol{j}} = \prod_{k=1}^{d} ((j_k - 1)/J, j_k/J]$ with a prior on the random heights $\theta_{\boldsymbol{j}}$.

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- This also leads to a consistent Bayesian test as before.

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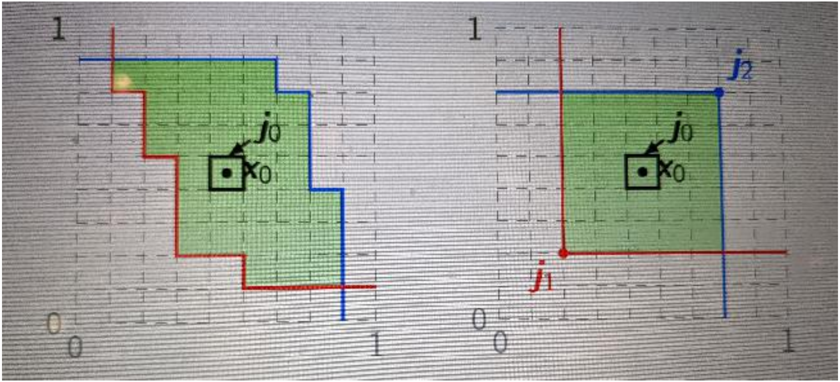
- The clue comes from the operation used in the construction of the maxi-min estimator of Han and Zhang (2020, Ann Stat).
- A major difference with theirs is that the operation is in a discrete domain, and hence is also simpler, because of the binning through the hypercubes used in constructing the basis of step functions.



1
j_0
x_0
0
0
1
1
j_2
j_0
x_0
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0
0
1

- A natural alternative is obtained by switching the maximization and the minimization operations

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- Both operations $\underline{\iota}$ and $\bar{\iota}$ are asymmetric in terms of the direction. A symmetric operation is obtained by averaging: $\iota(f)(\boldsymbol{x}_0) = (\underline{\iota}(f)(\boldsymbol{x}_0) + \bar{\iota}(f)(\boldsymbol{x}_0))/2$.

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- Denote the images under the corresponding immersion maps by $f_*(\boldsymbol{x}_0)$, $f^*(\boldsymbol{x}_0)$, and $\tilde{f}(x_0)$ respectively.
- In the univariate case, all three operations coincide and reduce to the empirical $\mathbb{L}_2$-projection on monotone functions for stepwise functions given by the standard isotonization procedure for the step-heights.

Key assumptions

- Local smoothness: let β_k be the order of the first non-zero derivative of f at $\boldsymbol{x}_0$ along the kth coordinate, that is, $\beta_k = \min_{l \geq 1}\{l : \partial_k^l f(\boldsymbol{x}_0) \neq 0\}$.

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- For a positive sequence $\omega_n \downarrow 0$, set $\boldsymbol{r}_n = (\omega_n^{1/\beta_1}, \dots, \omega_n^{1/\beta_s}, 1, \dots, 1)^{\mathrm{T}}$. For any $t > 0$,

$$\sup_{|x_k - x_{0,k}| \leq t r_{n,k}} \left| f_0(\boldsymbol{x}) - f_0(\boldsymbol{x}_0) - \sum_{\boldsymbol{l} \in L} \frac{\partial^{\boldsymbol{l}} f_0(\boldsymbol{x}_0)}{\boldsymbol{l}!} (\boldsymbol{x} - \boldsymbol{x}_0)^{\boldsymbol{l}} \right| = o(\omega_n),$$

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- β_k is odd, $k \leq s$; $\partial^{\boldsymbol{l}} f_0(\boldsymbol{x}_0) = 0$ if $\sum l_k/\beta_k < 1$.

Notations

Let H_1, H_2 be independent centered Gaussian processes on $\mathbb{R}^d_{\geq 0} \times \mathbb{R}^d_{\geq 0}$ with covariance kernel
$K(\boldsymbol{u}, \boldsymbol{v}) = \prod_{k=1}^{s}(u_k \wedge u'_k + v_k \wedge v'_k) D_s(\boldsymbol{u} \wedge \boldsymbol{u}', \boldsymbol{v} \wedge \boldsymbol{v}')$, where

$$D_s(\boldsymbol{u} \wedge \boldsymbol{u}', \boldsymbol{v} \wedge \boldsymbol{v}') = \int_{\substack{x_k \in [(\boldsymbol{x}_0 - \boldsymbol{u})_k, (\boldsymbol{x}_0 + \boldsymbol{v})_k] \cap [0,1] \\ s+1 \leq k \leq d}} g(x_{0,1}, \ldots, x_{0,s}, x_{s+1}, \ldots, x_d) \mathrm{d}x_s$$

with $D_d(\boldsymbol{u}, \boldsymbol{v}) = g(\boldsymbol{x}_0)$, where g is the probability density function of $\boldsymbol{X}$.

Let

$$U(\boldsymbol{u},\boldsymbol{v}) = \frac{\sigma_0 H_1(\boldsymbol{u},\boldsymbol{v})}{\prod_{k=1}^{s}(u_k+v_k)D_s(\boldsymbol{u},\boldsymbol{v})} + \frac{\sigma_0 H_2(\boldsymbol{u},\boldsymbol{v})}{\prod_{k=1}^{s}(u_k+v_k)D_s(\boldsymbol{u},\boldsymbol{v})} + \sum_{\boldsymbol{l}\in L^*} \frac{\partial^{\boldsymbol{l}} f_0(x_0)}{(\boldsymbol{l}+\boldsymbol{1})!} \prod_{k=1}^{s} \frac{v_k^{l_k+1} - (-u_k)^{l_k+1}}{u_k+v_k}.$$

$$Z_* = \sup_{\substack{\boldsymbol{u}\succeq\boldsymbol{0} \\ u_k\le x_{0,k} \\ s+1\le k\le d}} \inf_{\substack{\boldsymbol{v}\succeq\boldsymbol{0} \\ v_k\le 1-x_{0,k} \\ s+1\le k\le d}} U(\boldsymbol{u},\boldsymbol{v}), \quad Z^* = \inf_{\substack{\boldsymbol{v}\succeq\boldsymbol{0} \\ v_k\le 1-x_{0,k} \\ s+1\le k\le d}} \sup_{\substack{\boldsymbol{u}\succeq\boldsymbol{0} \\ u_k\le x_{0,k} \\ s+1\le k\le d}} U(\boldsymbol{u},\boldsymbol{v}).$$

Weak limit

Theorem

Let $\omega_n = n^{-1/(2+\sum_{k=1}^{s}\beta_k^{-1})}$. *Let* $J_k \gg r_{n,k}^{-1}$, *for each* $k = 1, \ldots, d$, *and* $\prod_{k=1}^{d} J_k \ll n\omega_n$. *For any* $z \in \mathbb{R}$, *we have*

$$\Pi(\omega_n^{-1}(f_*(\boldsymbol{x}_0) - f_0(\boldsymbol{x}_0)) \leq z|\mathbb{D}_n) \rightsquigarrow \mathrm{P}(Z_* \leq z|H_1);$$
$$\Pi(\omega_n^{-1}(f^*(\boldsymbol{x}_0) - f_0(\boldsymbol{x}_0)) \leq z|\mathbb{D}_n) \rightsquigarrow \mathrm{P}(Z^* \leq z|H_1);$$
$$\Pi(\omega_n^{-1}(\tilde{f}(\boldsymbol{x}_0) - f_0(\boldsymbol{x}_0)) \leq z|\mathbb{D}_n) \rightsquigarrow \mathrm{P}((Z_* + Z^*)/2 \leq z|H_1).$$

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- Tightness of the functional — max/min activity within a compact domain with high probability.

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  - the univariate case considered earlier.

Coverage of a credible interval

- Consider the $(1-\gamma)$-quantile of the $\underline{\iota}$-immersion posterior $Q^{(1)}_{n,\gamma} = \inf\{z : \Pi(f_*(\boldsymbol{x}_0) \leq z | \mathbb{D}_n) \geq 1-\gamma\}$, and the corresponding one-sided credible interval $(-\infty, Q^{(1)}_{n,\gamma}]$.

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- Its asymptotic coverage is $\mathrm{P}(\underbrace{\mathrm{P}(\sup_{\boldsymbol{u}\succeq\mathbf{0}} \inf_{\boldsymbol{v}\succeq\mathbf{0}} \tilde{U}(\boldsymbol{u},\boldsymbol{v}) \leq 0|\tilde{H}_1)}_{Z_B^{(1)}} \leq 1-\gamma)$.

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- Similar statements for the other two immersion maps with corresponding changes in the limiting process.

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- Compared with the Han-Zhang procedure, the corrected intervals are significantly shorter (even the uncorrected ones are shorter), and have better coverage too, in most cases.

Thank you