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Acknowledgment:

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Section: 1

Homework # 4

Problem 1, Section 6.2-40 (pg 406)

Prove that at a party where there are at least two people, there are two people who know the same number of other people there.

For any person there is at least zero and at most $n-2$ other people they can know.[1] Thus, each person can fall into $n-1$ containers based on the number of other people they know. Because there are n-people and $n-1$ containers, we know, by the pigeonhole theorem, that there must be at least two people that know the same amount of other people.

Problem 2, Section 6.3-42 (pg 415)

Find a formula for the number of ways to seat r of n people around a circular table, where seatings are considered the same if every person has the same two neighbors without regard to which side these neighbors are sitting on.

The total number of ways they can sit is defined as:

$$\frac{n!}{(n-r)!}$$

However, this does not take into account reflections and rotations. To fix repetitions caused by rotation, we can hold one of the seats in place by dividing by r. To fix reflections, we can divide by 2. Thus, our solution is

$$\frac{n!}{2r(n-r)!}$$

Problem 3,Section 6.4-22 (pg 422)

Prove the identity $\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}$, whenever n, r and k are nonnegative integers with $r \leq n$ and $k \leq r$.

a. Using a combinatorial argument.
   Let there be a set A such that $\mid A \mid= n$. Then $\binom{n}{r}$ is the number of ways to choose a subset B of A such that $\mid B \mid= r$ and $\binom{r}{k}$ is the number of ways to choose a subset C of B such that $\mid C \mid= k$. Thus, $\binom{n}{r}\binom{r}{k}$ is the number of ways to choose a pair of B and C where $C \subset B \subset A$, $\mid B \mid= r$, and $\mid C \mid= k$. Now we will show that the right-hand side is counting the same thing.

   On the RHS, we are choosing our C first and then the rest of the elements in B. There are $\binom{n}{k}$ ways to choose a subset C of A such that $\mid C \mid= k$. The set B must already contain C, so k elements of B are already chosen. Thus, we must choose the remaining $r-k$ elements of B out of the remaining $n-k$ elements of A. There are $\binom{n-k}{r-k}$ ways to do this. Therefore, there are $\binom{n}{k}\binom{n-k}{r-k}$ ways to choose a pair of B and C where $C \subset B \subset A$, $\mid B \mid= r$, and $\mid C \mid= k$.

b. Using an argument based on the formula for the number of r-combinations of a set with n elements.
   Using the formula we get,

$$\frac{n!}{r!(n-r)!} \times \frac{r!}{k!(r-k)!} = \frac{n!}{k!(n-k)!} \times \frac{n-k!}{(r-k)!(n-k-r+k)!}$$

   Simplifying on both sides,

$$\frac{n!}{k!(n-r)!(r-k)!} = \frac{n!}{k!(n-r)!(r-k)!}$$

[1] This is to exclude themselves and the one other person who's being compared to them

Problem 4, Section 6.4-36 (pg 422)

Use Exercise 33 to prove Pascals identity. [Hint: Show that a path of the type described in Exercise 33 from $(0,0)$ to $(n+1-k,k)$ passes through either $(n+1-k,k-1)$ or $(n-k,k)$, but not through both.]

Assume that a path from $(0,0)$ to $(n+1-k,k)$ passes through both $(n+1-k,k-1)$ and $(n-k,k)$. It must pass through one of the points first. If it were to pass through point $(n+1-k,k-1)$ first, then it would no longer be able to pass through $(n-k,k)$ because left movements are not allowed. If it were to pass through point $(n-k,k)$ first, then it would no longer be able to pass through $(n+1-k,k-1)$ because down movements are not allowed. We have a contradiction because passing through either point makes the other point inaccessible.Thus, a path from $(0,0)$ to $(n+1-k,k)$ passes through either $(n+1-k,k-1)$ or $(n-k,k)$, but not through both.
Furthermore, seeing as those two points are adjacent to $(n+1-k,k)$, all paths to $(n+1-k,k)$ must pass through either one of those points. Thus, using the formula from Problem 33, Part b, we have

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k},$$

which is Pascal's Identity.