EXHIBITING AUTOMORPHISM GROUPS VIA TESSELLATIONS

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ABSTRACT. Given a curve with cone metrics, we introduce a new conjectural method for computing its automorphism group by constructing a hyperbolic tessellation that exhibits the automorphism group of the curve. The curves of our interest are cyclic covers over the Riemann sphere, as we can pin down explicit cone metrics on such curves. Given a cone metric, we locate all Weierstrass points on the curve, with which we construct a tessellation. We calculate the automorphism group of the tessellation combinatorially, and conjecture that it is isomorphic to the automorphism group of the curve. Examples we explore include Klein's quartic and Fermat's quartic, where all Weierstrass points have the same weight, and also Schoen's I-WP minimal surface, where the Weierstrass points have different weights. We verify this conjecture in several key examples with programs based on the algorithms of Bruin-Sijsling-Zotine.

1. INTRODUCTION

In [5], the first author studies the underlying curve of an embedded triply periodic polyhedral surface $\Pi \subset \mathbb{R}^3$ equipped with a polyhedral cone metric. By triply periodic, we mean that Π is invariant under a rank-three lattice $\Lambda \subset \mathbb{R}^3$ by translations. With identifications, the underlying curve $C = \Pi/\Lambda$ is a closed genus three curve and with the induced polyhedral cone metric, it is invariant under an order-eight rotation. This order-eight symmetry is not visible on the embedding of the polyhedral surface $\Pi \subset \mathbb{R}^3$. The quotient under the $\mathbb{Z}/8\mathbb{Z}$ action on C yields the Riemann sphere where the cover is branched over three points. We say that a curve C is a d-fold cyclic cover branched over a sphere if $C/(\mathbb{Z}/d\mathbb{Z}) = \mathbb{CP}^1$ for some positive integer d. A cyclic cover over a sphere gives rise to various cone metrics with which one can compute a basis of holomorphic 1-forms on C.With explicit cone metrics and symmetries induced from the polyhedral surface, the author locates all Weierstrass points on C. Then by finding a hyperbolic tessellation Δ on the curve where all vertices are Weierstrass points, the author computes $\mathrm{Aut}(\Delta)$ and shows that $\mathrm{Aut}(Delta) \simeq \mathrm{Aut}(C)$. [[the author computes the automorphism group of the tessellation that acts freely and transitively on the tiles]]. We call a tessellation with this property **all-seeing**. This all-seeing tessellation provides a concrete way of presenting the automorphism group of a curve, as opposed to presenting the automorphism group as an action on the coefficients of a polynomial describing the curve (appendix A, [8]). Moreover, the author also shows that the curve is in fact Fermat's quartic (Theorem 5.1, [5]).

Although we do not prove any theorem in this paper, we claim that given a cyclic cover branched over $\mathbb{CP}^1$, there exists a tessellation that exhibits the automorphism group of the curve. Further, we present an algorithm to construct this tessellation. We heavily rely on the first author's thesis [6] where they discuss cone metrics on cyclic covers over $\mathbb{CP}^1$ and classify all of them up to genus five. We summarize in section 2 how to find cone metrics, find a basis of holomorphic 1-forms, find a plane curve model, and locate Weierstrass points given only a construction of a cyclic cover over a sphere. Furthermore, from cone metrics we can compute period matrices, which we use in a sequel to this paper.

This project began from the second author wondering if the first author's method for computing the automorphism group could be applied in more generality to exhibit the automorphism group of any principally-polarized Jacobian variety. As Torelli's theorem gives the relation between the automorphism group of a genus g curve C_g and its Jacobian Jac(C_g, the aim was to find a combinatorial way to exhibit the automorphisms of the Jacobian via an all-seeing tessellation. The initial naive approach was finding an all-seeing tessellation of the Jacobian $\mathrm{Jac}(C_g)$ by taking the image of an all-seeing tessellation of C under one of two maps: either an Abel-Jacobi map $C_g \hookrightarrow \mathrm{Jac}(C_g)$, or embedding the g-th symmetric product of the tessellation on $C_g \hookrightarrow \mathrm{Sym}^g(C_g) \to \mathrm{Jac}(C_g)$, where the second map is a birational equivalence. We found that it is computationally easier to find the all-seeing tessellation on the underlying curve and use the precise Torelli theorem to get a concrete presentation of the automorphism group of its principally-polarized Jacobian. We thus focus on formulating the all-seeing tessellation in more generality.

In this paper, we conjecturally extend the first author's "all-seeing" method to curves that have Weierstrass points of different weights. In section 2, we discuss how examples that naturally yield cone metrics appear in the wild, that is, cyclic covers that are branched over $\mathbb{CP}^1$. Then given a cone metric, we discuss how to achieve plane curve models and find Weierstrass points. In section 3, we show an algorithm that yields a tessellation given only the construction of a cyclic cover over $\mathbb{CP}^1$. Then we compute the automorphism group of this tessellation. We conjecture that this automorphism group of the tessellation is the same as the automorphism group of the original curve. In section 4, we verify this conjecture for examples of cyclic covers over $\mathbb{CP}^1$. Independently, we will compute the automorphism group of the curves using a program based on the algorithm of Bruin-Sijsling-Zotine [1]. This program requires only the plane curve model which we obtain from cone metrics.

This paper proves no theorems, but rather demonstrates the strength and potential of a computational technique, and a new way to present the automorphisms of a curve and its Jacobian.

2. Background

In [5], the first author investigates a triply periodic polyhedral surface Π embedded in $\mathbb{R}^3$. By triply periodic, we mean that Π is invariant under a rank-three lattice $\Lambda \subset \mathbb{R}^3$. We take its fundamental piece $C = \Pi/\Lambda$ and by appropriate identification, we get a genus three curve where we pin down the conformal structure by descending the polyhedral cone metric to its quotient. Although it is not visible on the embedded surface, C is intrinsically invariant under an order-eight rotational symmetry, and its quotient under $\mathbb{Z}/8\mathbb{Z}$ yields $\mathbb{CP}^1$. We focus on curves that are cyclic covers over $\mathbb{CP}^1$, hence we will devote this section to providing the background. This section is a summary of chapter 3 from the first author's thesis [6].

2.1. **Cone metrics.** First, we discuss the topological construction of cyclic covers over $\mathbb{CP}^1$. This will naturally yield cone metrics on the curves.

Construction of cyclic covers over $\mathbb{CP}^1$*.*

Definition 1. A curve C is a d-fold cyclic cover over $\mathbb{CP}^1$ if $C/(\mathbb{Z}/d\mathbb{Z}) = \mathbb{CP}^1$.

We construct such a curve in the following way: for n distinct points $p_i, \dots, p_n \in \mathbb{CP}^1$, let $Y := \mathbb{CP}^1 \backslash \{p_1, \dots, p_n\}$ and let γ_i be a branch cut from p_i to some $q \in Y$ so that γ_i are mutually disjoint. To each i, we assign a positive integer d_i and call it the **branching index**. Let d be the degree of the covering map and use j to label $Y_1, \dots, Y_d$. For each i and j, we identify the "left side" of γ_i of Y_j to the "right side" of γ_i of $Y_{j+d_i \pmod d}$. We denote such a covering C by a d-tuple $d(d_1, \dots, d_n)$.

Remark. Given a d-fold cyclic cover over Y with branching indices $(d_1, \dots, d_n)$, a covering is uniquely defined up to homeomorphism. That is, the construction only depends on d_i and is independent of p_i, γ_i, and q. The covering is closed if and only if $\sum_{i=1}^{n} d_i \equiv 0 \pmod d$, and is connected if and only if $\gcd(d_1, \dots, d_n) = 1$. We assume these two conditions. Then the genus of the curve $g(C) = \frac{d(n-2)}{2} + 1 - \frac{1}{2}\sum_{i=1}^{n} \gcd(d, d_i)$ follows from Riemann-Hurwitz formula.

Branching indices on Octa-4. As a running example, we will look at the curve from [5]. It is there called Octa-4 due to the formation of Π. The underlying genus three curve $C = \Pi/\Lambda$ (Figure 1 (reprinted from [5])) is invariant under an order-eight rotational symmetry. The curve is an eightfold cyclic cover over $\mathbb{CP}^1$ denoted as $8(1, 2, 5)$.

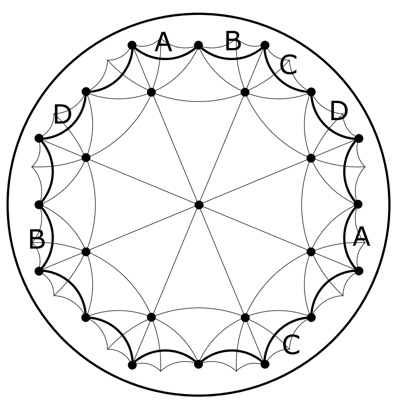


FIGURE 1. Hyperbolic tessellation on $8(1, 2, 5)$

The euclidean triangles on the fundamental piece of Π have a one-to-one correspondence to the hyperbolic triangles.

For each i, there are $\gcd(d, d_i)$ preimages $\widetilde{p_i}$ of p_i on C. Hence, each $\widetilde{p_i}$ is a cone point. To pin down a flat metric on C, that is, a holomorphic 1-form on C, we pull back a cone metric from $\mathbb{CP}^1$. We say that a cone metric on $\mathbb{CP}^1$ is **admissible** if its pullback yields a flat structure on C. The following proposition is a version of Gauss-Bonnet theorem.

Proposition. *Given a compact Riemann surface of genus g with a cone metric, let $p_1, \ldots, p_n$ be distinguished points with respective cone angles θ_i. Then* $\sum_{i=1}^{n} \theta_i = 2\pi(2g - 2 + n)$.

In other words, the sum of cone angles on a genus 0 curve is $2\pi(n-2)$. Given branching indices such that $\sum_{i=1}^{n} d_i = d(n-2)$, we get a cone metric on the Riemann sphere with cone angles $\frac{2\pi d_i}{d}$ at each p_i. Our goal is to find other cone metrics that realize the same underlying curve. If $\sum_{i=1}^{n} d_i \neq d(n-2)$, then we modify the cone angles by $\frac{2\pi a_i}{d}$ so that where $a_i \equiv d_i \pmod{d}$ for each i. As each cone metric corresponds to a holomorphic 1-form, we use the following notion of multipliers that give rise to other cone metrics in order to get a basis of 1-forms.

Definition 2. Given branching indices $(d_1, \ldots, d_n)$, we say $a \in \{1, \ldots, d-1\}$ is a **multiplier** if the cone metric given by cone angles $\frac{2\pi}{d}(a \cdot d_1 \pmod{d}, \ldots, a \cdot d_n \pmod{d})$ is admissible.

It is proved in [6] (Theorem 3.15) that for $n = 3$, there are exactly g multipliers.

Admissible cone metrics on Octa-4. Given branching indices $(1, 2, 5)$, multipliers 1, 2, and 5 give rise to cone metrics with cone angles $\frac{2\pi}{8}(1, 2, 5)$, $\frac{2\pi}{8}(2, 4, 2)$, and $\frac{2\pi}{8}(5, 2, 1)$, respectively. These cone metrics yield a basis of holomorphic 1-forms with the following divisors:

$$(\omega_1) = 4\widetilde{p_3}, \qquad (\omega_2) = \widetilde{p_1} + \widetilde{p_{2_1}} + \widetilde{p_{2_2}} + \widetilde{p_3}, \qquad (\omega_3) = 4\widetilde{p_1}.$$

Remark. Note that 3, 6, and 7 are not multipliers as their corresponding forms do not yield canonical divisors. On the other hand, 4 is not a multiplier as the cone metric derived from cone angles $\frac{2\pi}{8}(4, 0, 4)$ is not admissible. This induces a meromorphic form whose divisor is $3\widetilde{p_1} - \widetilde{p}_{21} - \widetilde{p}_{22} + 3\widetilde{p_3}$.

2.2. **Plane curve model.** Given brancing indices (d_i), we write C as $y^d - (x-p_1)^{d_1}(x-p_2)^{d_1} \cdots (x-p_n)^{d_n}$. In section 4, we will show examples where the degree of the polynomial can be reduced and hence the polynomial is not uniquely defined. However, different polynomials representing the same curve yield the same output when put in the program in [1].

Plane curve model on Octa-4. With branching indices $(1, 2, 5)$, the underlying curve of Octa-4 is written as $y^8 - x(x-1)^2$. However, Theorem 5.1 from [5] shows that by change of coordinates, it can also be written as $y^4 - x(x-1)(x+1)$.

2.3. **Weierstrass points and how to find them.** A Weierstrass point $p \in C$ is a point where there are functions on C with unusual pole orders at p and nowhere else. In other words, they are points where the dimension-count of the Riemann-Roch theorem is not generic. Hurwitz showed that for genus $\geqslant 2$, there are finitely many Weierstrass points on C and the number of Weierstrass points is bounded by $2g+1 \leqslant |W(C_g)| \leqslant (g-1)g(g+1)$ where $W(C_g)$ is the set of Weierstrass points on a curve of genus g. In characteristic zero, this can be proved by showing that every automorphism that fixes more than $2g+2$ points is the identity (by the Riemann-Roch theorem), from which we obtain a faithful representation of $\mathrm{Aut}(C_g)$ as permutations on $W(C_g)$. Moreover, the number of Weierstrass points on C_g is $2g+2$ if and only if the curve is hyperelliptic. That is, the hyperelliptic involution fixes $2g+2$ points. We remark that in general, the action of the automorphism group on the set of Weierstrass points is not transitive [4].

In this section, we will discuss how to find all Weierstrass points on cyclic covers over $\mathbb{CP}^1$. In the previous sections, we learned that these covers are defined by a d-tuple of branching orders which naturally gives rise to a cone metric. Then given a cone metric on the covering, we obtain g admissible cone metrics, in other words, a basis of holomorphic 1-forms. Another way of defining a generic point is as follows: given a basis of holomorphic 1-forms, a generic point p is where the order of zeros at p form the sequence $0, 1, \dots, g-1$. If the sequence differs from this at p, we call p a Weierstrass point. The weight of a point measures how far it is from a generic point and is defined by the difference between the two sequences. For example, if a point p yields a sequence $n_0 = 0 < n_1 < \cdots < n_{g-1}$, then the weight at p is defined as $\mathrm{wt}_p = \sum_{i=0}^{g-1}(n_i - i)$. On a compact Riemann surface of genus $\geqslant 2$, there are finitely many such points and $\sum_{p \in C_g} \mathrm{wt}_p = (g-1)g(g+1)$ [2].

Weierstrass points on Octa-4. Previously, we found the basis of holomorphic 1-forms that arise from the cone metrics, so we have $\mathrm{wt}_{\widetilde{p}_i} = 2$ for all i. This gives us only four Weierstrass points of weight 2 each. The following definition of the Wronski metric yields Weierstrass points that are not preimages of any p_i.

Given a basis, we define the Wronski metric given by the Wronskian.

Definition 3. Given a basis of holomorphic forms $\omega_i = f_i dz$ on C, the Wronskian defined by

$$\mathcal{W}(z) := \det\left(\frac{d^j f_k(z)}{dz^j}\right)_{j=0,\dots,g-1,\,k=1,\dots,g}$$

is a non-trivial holomorphic function on C that induces a cone metric which we call the **Wronski metric.**

The zeros of the Wronksi metric correspond to the Weierstrass points and the order at each zero corresponds to the weight at each point [2].

Weierstrass points on Octa-4 (continued). We use the Mathematica codes from [6]. This code is also available on our github page

`https://github.com/catherineray/aut-jac/wronskian.m`

Given branching indices, we compute all admissible cone metrics and obtain a basis of holomorphic 1-forms. We write $(p_i) = (0, 1, -1)$, then the wronski metric $(1+3z)^2$

yields a double order zero that is not a preimage of any p_i. That is, the (eight) preimages of $-\frac{1}{3}$ are Weierstrass points of weight two.

3. Exhibiting the Automorphism Group via Tessellation

In [5], the first author computed the automorphism group of the underlying curve of Octa-4 by finding a hyperbolic tessellation where all vertices corresponded to the Weierstrass points. By **tessellation** Δ of a curve C, we mean a polygonal decomposition of C where the polygons are either disjoint or share an edge or a vertex, and their union is the entire curve. First, the author finds maps from the tessellation to itself that permute the Weierstrass points, then pins down the relation between the maps. This leads to finding the automorphism group in a generator-relation format. As a result, the automorphism group of the underlying curve of Octa-4 as $\mathrm{Aut}(C) = \langle a, b \mid a^8 = b^3 = (ab)^2 = (a^2b^2)^3 = (a^4b^2)^3 = 1\rangle$. By the GAP small group identification function, this is equivalent to ${C_4}^2 \rtimes S_3$. In the previous section, we showed that all Weierstrass points on this particular curve have uniform weight. Here, we conjecturally generalize this algorithm to curves with Weierstrass points of different weights.

Remark. A tessellation Δ of a curve Ccan be considered on its own, without C, as a combinatorial object.

Definition 4. Given a tessellation Δ, we define the **automorphism group of the tessellation** $\mathrm{Aut}(\Delta)$ to be the group of automorphisms of the tessellation. These are invertible maps which are are orientation-preserving, send vertices to vertices, edges to edges, and faces to faces.

Definition 5. We call a tessellation Δ of a curve C an **all-seeing tessellation** if we have a group isomorphism:

$$\mathrm{Aut}(\Delta) \simeq \mathrm{Aut}(C)$$

Remark. In this case, all of the combinatorial automorphisms $\mathrm{Aut}(\Delta)$ lift to automorphisms of C, and every automorphism of C is uniquely determined by its action described combinatorially.

Definition 6. Let C be a d-fold cyclic cover over $\mathbb{CP}^1$ branched at n points. We define the **base tessellation** of C as a tessellation tiled by n-gons with valency d at every vertex.

Remark. For $n \geqslant 3$, $d > \frac{n}{2}$, and $g > 1$, there exists a unique base tessellation on C_g, for $g > 1$, tiled by $N(= \frac{4d(g-1)}{(n-2)(d-2)-4})$ regular hyperbolic n-gons. Every angle of each n-gon is $\frac{2\pi}{d}$. This follows from Euler's characteristic formula.

Definition 7. We say that a tessellation Δ' is a **refinement** of Δ if $\Delta \subset \Delta'$.

With the following algorithm, we refine the base tessellation to find what we conjecture to be the all-seeing tessellation.

Algorithm: A conjectural all-seeing tessellation Δ_T on S.
Input: A curve C which is a cyclic cover of $\mathbb{CP}^1$ branched at n points.
Output: A conjectural all-seeing tessellation Δ_T.

(1) Construct a base tessellation Δ of C.
(2) Separately, find the Weierstrass points of C using Definition 3.
(3) Refine Δ until all Weierstrass points of C occur as vertices and all tiles are congruent. Call this new tessellation $\widetilde{\Delta}$.
(4) Restrict our attention to a tile T of $\widetilde{\Delta}$. Let G_T be the orientation-preserving automorphism group of a tile $T \in \widetilde{\Delta}$. (Recall that all tiles are congruent[1]). Find a refinement of T by adding edges until each tile T' (or $T' \cup R(T')$ where R is a reflection about an edge) is the fundamental domain of G_T acting on T. Apply the same refinement to all $T \in \widetilde{\Delta}$, and call this tessellation Δ_T.

Conjecture 1. *The output tessellation Δ_T of this algorithm always exists and*

$$\mathrm{Aut}(\Delta_T) \simeq \mathrm{Aut}(C)$$

Tessellations on Octa-4. We perform this algorithm to the underlying curve of Octa-4 as done in [5]. We begin with the base tessellation (Figure 1). The base tessellation of this curve is tiled by 24 regular hyperbolic $\frac{2\pi}{8}$-triangles. Next, we locate all Weierstrass points. On C, all vertices of the base tessellation correspond to Weierstrass points, that is $\Delta = \widetilde{\Delta}$. Each tile T of $\widetilde{\Delta}$ is a regular triangle with all vertices the same weight, hence $G_T = \mathbb{Z}/3\mathbb{Z}$. We add edges to each T so that for the smallest tile T', $T' \cup R(T')$ (where R is a reflection about an edge) represents a fundamental domain for the action of G_T on T (Figure 2). We include an additional edge to mimic the $(2,3,7)$-tiling on Klein's quartic. This captures orientation-reversing automorphisms.

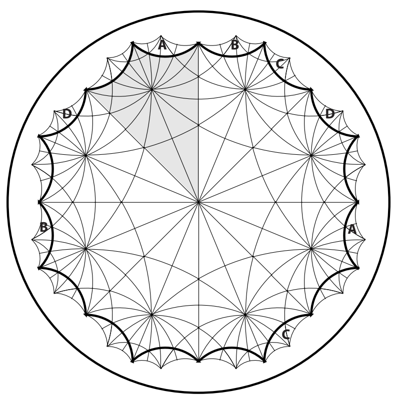


Figure 2. The refined tessellation on $8(1,2,5)$

4. Example Calculations

This section is devoted to the computation of automorphism groups of our Riemann surfaces by all-seeing tessellations using the algorithm shown in section 3. We checked the results programmatically with programs based on the algorithms of Bruin-Sijsling-Zotine [1], which we discuss in section **??**.

[1] Note that the automorphism group G_T encodes the weight of the vertices of T. As automorphisms preserve weights, vertices of different weights cannot be mapped to each other. Since all T are congruent, G_T does not depend on T.

As our test subjects, we consider a subset of examples from the classification of cyclic covers found in [6] that we expect to have large automorphism groups.

Table 1 displays our results. The genus and (non)-hyperelliptic nature are calculated by tools from section 2.

TABLE 1. Plane Curve Automorphism Groups

Curve C	Plane Curve Model	Genus	Aut(C)	|Aut(C)|
$7(1,2,4)$ (Klein's quartic)	$y^3x + x^3 + 1$	3	$GL_3(F_2)$	168
$*8(1,1,6)$	$y^2 - (x^8 - 1)$	3	$D_4 \rtimes C_4$	32
$8(1,2,5)$ (Fermat's quartic)	$y^4 - x(x+1)(x-1)$	3	$C_4^{\ 2} \rtimes S_3$	96
$*12(1,5,6)$	$y^2 - (x^7 - x)$	3	$C_4 \times S_3$	24
$5(1,2,4,3)$ (Bring's curve)	$y^5 - x(x-1)^2(x+1)^3$	4	S_5	120
$12(1,4,7)$ (I-WP)	$y^3 - (x^5 - x)$	4	$C_3 \times S_4$	72

An $*$ indicates that the curve is hyperelliptic

In each of the following individual sections, we derive tessellations Δ_T to show that our examples satisfy Conjecture 1.

7(1, 2, 4) Klein's quartic. Klein's quartic is a genus three curve that is a sevenfold cover over $\mathbb{CP}^1$ with branching indices $(1,2,4)$ [3]. Its base tessellation is tiled by 56 triangles (Figure 3(a)). With the cone metric that arises from the branching indices and its multipliers, we get a basis of holomorphic 1-forms with the following divisors:

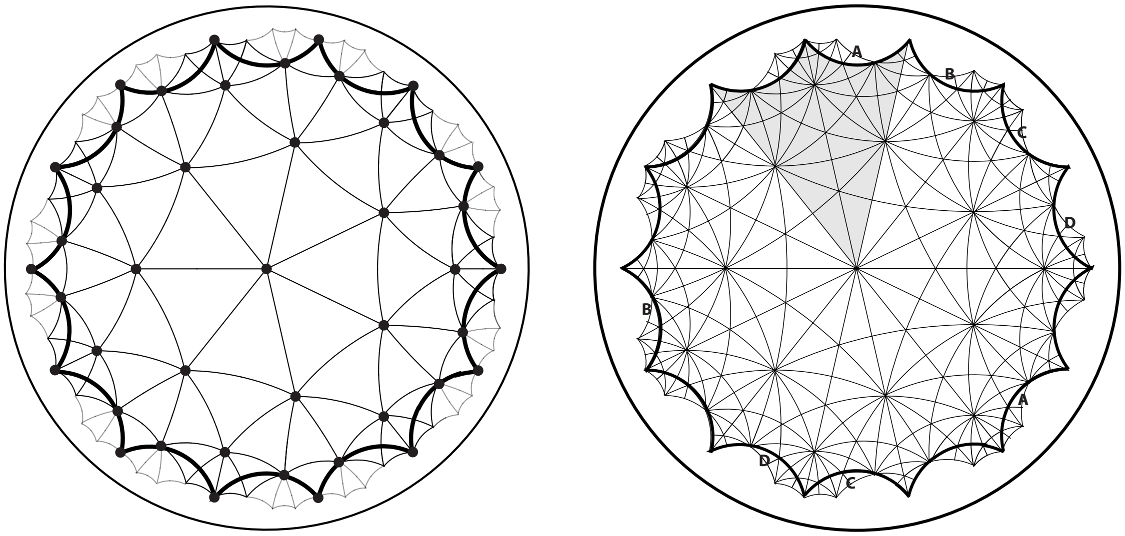
$$(\omega_1) = \widetilde{p_2} + 3\widetilde{p_3}, \qquad (\omega_2) = \widetilde{p_1} + 3\widetilde{p_2}, \qquad (\omega_3) = 3\widetilde{p_1} + \widetilde{p_3}.$$

All $\widetilde{p_i}$ are Weierstrass points with weight 1. Moreover, the `Resultant` of the wronski metric in `wronski.nb` tells us that the remaining Weierstrass points have weight one that arise from the midpoint of $\overline{p_i p_j}$. We refine the tessellation with additional Weierstrass points as vertices and achieve the well-known tiling by $(\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{7})$-triangles (Figure 3(b)). Note that this tesellation exhibits orientation reversing automorphisms (reflections).

8(1, 1, 6). In this and the following subsection, we will look at two different hyperelliptic curves of genus three. First, we look at the eightfold cover defined by branching indices $(1,1,6)$. The branching indices along with the multipliers yield a basis of holomorphic 1-forms with divisors

$$(\omega_1) = 4\widetilde{p_3}, \qquad (\omega_2) = \widetilde{p_1} + \widetilde{p_2} + \widetilde{p_{3_1}} + \widetilde{p_{3_2}}, \qquad (\omega_3) = 2\widetilde{p_1} + 2\widetilde{p_2}.$$

Note that no $\widetilde{p_i}$ is a Weierstrass point. The wronski metric tells us that the Weierstrass points come from the midpoint of $\overline{p_1 p_2}$. Hence we refine the tessellation to show all symmetries (Figure 4). These points are fixed under the hyperelliptic involution, whose quotient is a doubled octagon. In other words, we can rewrite the plane curve model as $y^2 - (x^8 - 1)$.



(A) The base tessellation Δ (B) The refined tessellation Δ_T

FIGURE 3. Klein's quartic

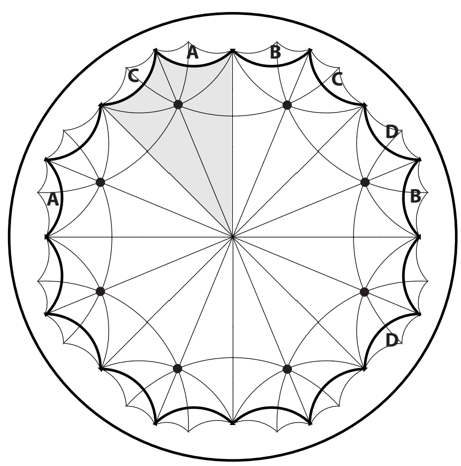


FIGURE 4. Hyperbolic tessellation on $8(1,1,6)$

12(1, 5, 6). This curve is yet another genus three hyperelliptic curve that is defined as a twelvefold cover over a sphere with branching indices $(1,5,6)$. We achieve a basis of 1-forms with divisors

$$(\omega_1) = 4\widetilde{p_2}, \qquad (\omega_2) = 2\widetilde{p_1} + 2\widetilde{p_2}, \qquad (\omega_3) = 4\widetilde{p_2},$$

hence $\widetilde{p_1}$ and $\widetilde{p_2}$ are Weierstrass points of weight 3 each. Recall that if q for $q \neq p_i$ is a zero of the Wronski metric, then there must be 12 preimages $\widetilde{q}$ on the genus three curve, which contradicts the total weight theorem $(g^3 - g)$. However, the wronski metric tells us that p_3 is in fact a triple order zero, hence all six $\widetilde{p_{3_i}}$ are Weierstrass points. All Weierstrass points are marked as either $\bullet$ or $\circ$ as they have different valency (Figure 5).

The surface is a double cover over a Riemann sphere branched at six points which are located at the North and South Pole, and six equidistributed points on the Equator. Hence, the plane curve model can also be written as $y^2 - (x^7 - x)$.

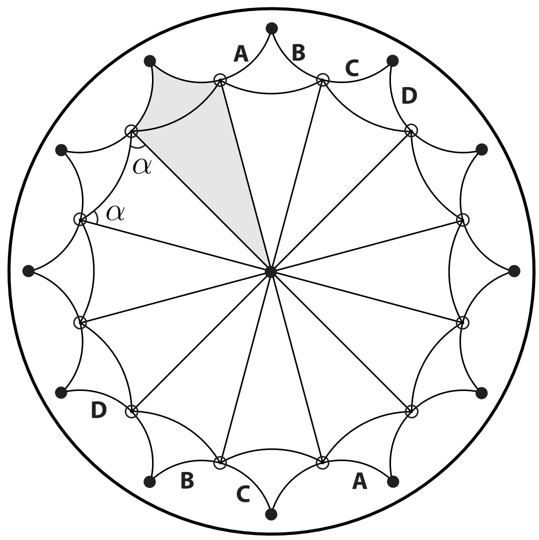
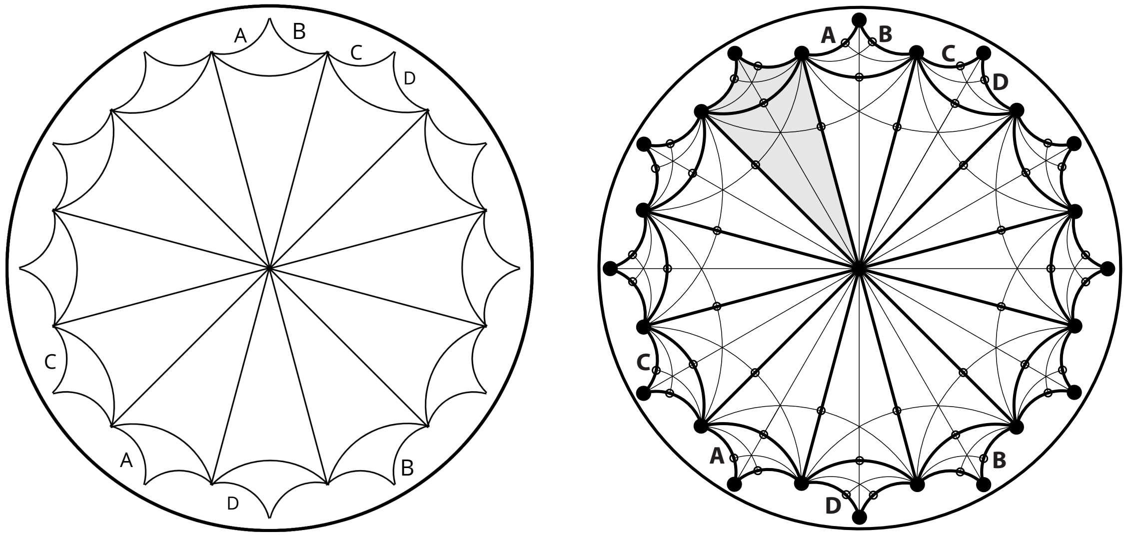


Figure 5. Hyperbolic tessellation on $12(1, 5, 6)$

12(1, 4, 7) Schoen's I-WP Surface. In this section, we look at another genus four non-hyperelliptic curve. This curve appears in [6] as the underlying curve of a triply periodic polyhedral surface called Octa-8. The fundamental piece of the polyhedral surface is tiled by 24 triangles which appear in Figure 6(a). The curve is a twelvefold cover over $\mathbb{CP}^1$ with branching indices $(1, 4, 7)$. By admissible cone metrics, we obtain a basis of 1-forms with divisors

$$(\omega_1) = 6\widetilde{p_3}, \qquad (\omega_2) = \widetilde{p_1} + \widetilde{p_{2_1}} + \widetilde{p_{2_2}} + \widetilde{p_{2_3}} + \widetilde{p_{2_4}} + \widetilde{p_3}, \qquad (\omega_3) = 2\widetilde{p_1} + 2\widetilde{p_3}, \qquad (\omega_4) = 6\widetilde{p_1}.$$

The wronski metric tells us that three simple zeros are located at the midpoint of $\overline{p_i p_j}$ for all i and j. We obtain a refined tessellation (Figure 6(b)) where vertices marked as $\bullet$ have weight four, and those marked as $\circ$ have weight one.



(A) The base tessellation Δ (B) The refined tessellation Δ_T

FIGURE 6. 12(1,4,7)

5. Programmatic Method of Checking Conjecture

We verify this conjecture with programs based on the algorithms of Bruin-Sijsling-Zotine.

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