COMPUTING AUTOMORPHISM GROUPS VIA CONSTRUCTING TESSELLATIONS

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Abstract. Given a symmetric surface, we introduce a new conjectural method for computing its automorphism group by constructing and counting a particular tessellation of the surface. More specifically, we consider any cyclically branched surface with either equal or unequal weight Weierstrass points, and construct a regular hyperbolic tessellation on this surface which is preserved under the automorphism group. Examples we explore include Klein's quartic, Fermat's quartic, and Bring's curve. We verify this conjecture up to genus 5 with programs based on the algorithms of Bruin-Sijsling-Zotine.

1. Introduction

In [5], the first author develops a method to compute the automorphism group of Fermat's quartic by constructing a particular tesselation. This method works particularly well on highly symmetric surfaces, e.g. those whose Weierstrass points have uniform weight.

Lee presents Fermat's quartic as an eightfold cyclically branched cover over a thrice punctured sphere and finds a regular hyperbolic tessellation on the surface that is preserved under the automorphism group. Using this "all-seeing" tessellation, Lee computes the automorphism group of Fermat's quartic in a generator-relation format. Lee's method (all-seeing method?) provides a different way to present automorphisms of a curve concretely, without presenting the automorphism group as an action on the coefficients of a polynomial describing the curve (as is done in Silverman, appendix A).

This project began from the second author wondering if the first authors method for computing the automorphism group of Fermat's quartic could be applied in more generality, to model the automorphism group of the principally-polarized Jacobian of a curve (where curve here means cyclically branched cover of the punctured sphere). The aim was to embed the all-seeing tessellation from the original curve C into the Jacobian, and extend it to a tesselation on the entire Jacobian. This embedding would be done by naively trying to extend the all-seeing tessellation's image under an Abel-Jacobi Map $C \hookrightarrow \mathrm{Jac}(C)$, or by embedding the gth symmetric product of the all-seeing tesselation of a genus g curve $C \hookrightarrow \mathrm{Sym}^g(C) \to \mathrm{Jac}(C)$, where the last map is a birational equivalence. We found that it is computationally easier, and conceptually less confusing, to instead calculate the all-seeing tesselation on the underlying curve, and use the precise Torelli theorem to get a concrete presentation of the automorphism group of its principally-polarized Jacobian. We thus focus here on extending the "all-seeing" method to more curves.

In this paper, we conjecturally extend the first authors "all-seeing" method to surfaces that have Weierstrass points of different weights. We verify this conjecture for a class of examples. Lee classified genus ¡ 5 surfaces which are cyclically branched covers of punctured spheres. We check these surfaces by finding a plane curve model, computing the automorphism group via finding a tessellation, and seperately computing the automorphism group as a plane curve using a program based on the algorithm of Bruin-Sijsling-Zotine [1].

This paper proves no theorems, but rather demonstrates the strength and potential of a computational technique.

1.1. Exhibiting the Automorphism Group via Tessellation.

2. Background

2.1. Cylically branched covers of the Punctured Sphere.

2.2. Weierstrass points and how to find them. We will locate Weierstrass points on the coverings. This information will be used in subsection 1.1. These are points where the dimension-count of the Riemann-Roch theorem is not generic. On a compact Riemann surface of genus > 1, there are finitely many such points and we will find all of them using the Wronski metric. For example, Klein's quartic has three holomorphic 1-forms with the following divisors

$$(\omega_1) = \widetilde{p_2} + 3\widetilde{p_3}, \qquad (\omega_2) = \widetilde{p_1} + 3\widetilde{p_2}, \qquad (\omega_3) = 3\widetilde{p_1} + \widetilde{p_3}.$$

We define the weight of a point which measures how far the point is from being generic. A generic point is where the order of zeros form the sequence $0, 1, \ldots, g-1$ given a basis. For Klein's quartic, at $\widetilde{p_i}$, this sequence is $0, 1, 3$. The weight at each $\widetilde{p_i}$ is computed as $(0-0) + (1-1) + (3-2) = 1$.

Proposition. [2] *Let S be a compact Riemann surface of genus $\geqslant 2$. Let wt_p be the weight of $p \in S$. Then* $\sum_{p \in S} wt_p = (g-1)g(g+1)$.

Proposition. [2] *Let $W(S)$ be the finite set of Weierstrass points on S. If $\phi \in \mathrm{Aut}(S)$, then $\phi(W(S)) = W(S)$. In fact, the sequences of the order of zeros at $p \in S$ and $\phi(p)$ are the same.*

In other words, there is a faithful representation of $\mathrm{Aut}(S)$ as permutations of the set of Weierstrass points $L : \mathrm{Aut}(S) \to \Sigma_{W(S)}$. However, in general, the action of $\mathrm{Aut}(S)$ on $W(S)$ is not transitive [**?**]. Next, we use the Wronski metric to find all Weierstrass points that are not necessarily $\widetilde{p_i}$.

2.2.1. *Wronski Metric.*

Definition 1. Given a basis of holomorphic differentials $\omega_i = f_i dz$ on a Riemann surface X of genus g, the Wronskian defined by

$$\mathcal{W}(z) := \det\left(\frac{d^j f_k(z)}{dz^j}\right)_{j=0,\ldots,g-1,\, k=1,\ldots,g}$$

is a non-trivial holomorphic function on X that induces a metric which we call the *Wronski metric.*

By induction, one can show that a zero of the Wronskian is a Weierstrass point on X and the order of a zero at a point equals its weight. In our case, the Wronskian has simple zeros at the preimages of the midpoint of two p_i.

3. Main Conjecture

In this section, we wish to find a tessellation Δ on a surface S which exhibits the automorphism group of S. Formally, we wish for $\mathrm{Aut}(S)$ to act freely and transitively on the tiles of Δ; we denote this with a slight abuse of notation:

$$\mathrm{Aut}(\Delta) \simeq \mathrm{Aut}(S)$$

This notation is partially justified by defining $\mathrm{Aut}(\Delta)$ to be the group of automorphisms which are orientation-preserving, send vertices to vertices, edges to edges, and faces to faces.

Definition 2. A **tessellation** Δ is a polygonal decomposition of the surface where the polygons are either disjoint or share an edge or vertex, and their union is the entire surface. We say Δ' is a refinement of Δ if $\Delta \subset \Delta'$.

Definition 3. Let S be a d-fold cyclic cover over an n-punctured sphere. We define the **base tessellation** of S as a tessellation tiled by n-gons with valency d at every vertex.

For $n \geqslant 3$ and $d > \frac{n}{2}$, there exists a unique base tessellation on a surface of genus $g > 1$. This follows from the Gauss-Bonnet formula. The base tessellation is tiled by N-polygons where $2\pi(2-2g) = -N(\pi - n \cdot \frac{2\pi}{d})$.

Algorithm: A particular tessellation Δ_T on S.

Input: A surface S which is a cyclically branched cover of a sphere.

Output: A particular tessellation Δ_T.

(1) Construct a base tessellation Δ of S.
(2) Separately, find the Weierstrass points of S using Definition 1.
(3) Refine Δ until all tiles are similar and all Weierstrass points of S occur as vertices. Call this new tessellation $\widetilde{\Delta}$.
(4) Let G_T be the orientation-preserving automorphism group of a tile T of $\widetilde{\Delta}$ (recall that all tiles are similar[1]). Restrict our attention to a tile T of $\widetilde{\Delta}$. On this tile T, add lines until any tile of the refinement T' is the fundamental domain of G_T acting on T. Doing this to every tile T gives a new tessellation, which we call Δ_T.

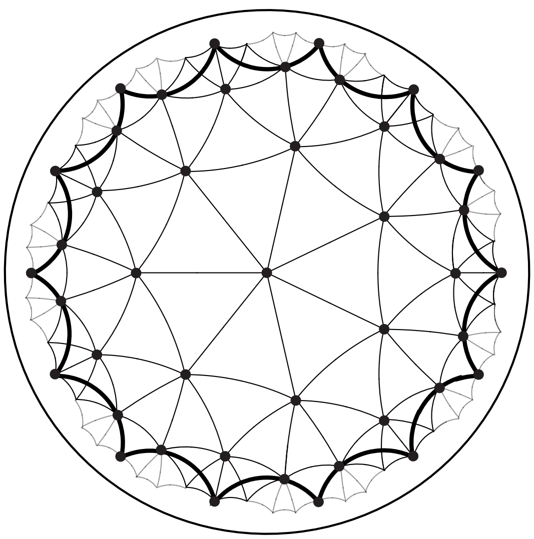
Conjecture 1. *The output tessellation Δ_T of this algorithm always exists and*

$$\mathrm{Aut}(\Delta_T) \simeq \mathrm{Aut}(S)$$

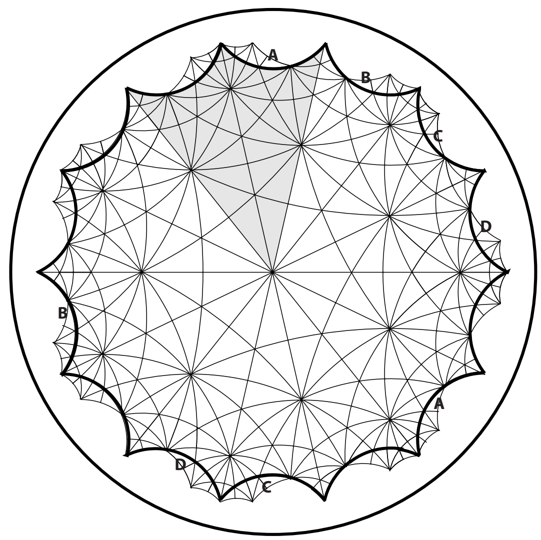
[1]Note that the automorphism group G_T encodes the weight of the vertices of T. As automorphisms preserve weights, vertices of different weights cannot be mapped to each other. Since all T are similar, all G_T are the same.

By checking against the automorphism groups obtained via the `autplane` program described in Section **??**, we have shown this conjecture to be true for the cyclically branched surfaces of genus < 5 mentioned in Section **??**.

Example 2. We perform an example of this algorithm for Klein's quartic. We begin with the base tessellation Figure 1(a). Both d and n play a role as we already know the existence of the d-fold map and the order-n map that permutes the branched values. Due to the Gauss-Bonnet formula, the base tessellation of Klein's quartic is tiled by hyperbolic $\frac{2\pi}{7}$-triangles (Figure 1(a)). Since the genus is three, we have $2\pi(2-2\cdot 3) = -56(\pi - 3\cdot\frac{2\pi}{7})$ hence we need 56 triangles. Next, we locate all Weierstrass points. On Klein's quartic, all vertices of the base tessellation correspond to Weierstrass points, that is $\Delta = \widetilde{\Delta}$. We move to the last step. Since each tile T of $\widetilde{\Delta}$ is a regular triangle with all vertices the same weight, we get $G_T = \mathbb{Z}/3$. Thus, we add lines to each tile T such that each tile T' is now a fundamental domain for the action of G_T on T. (Figure 1(b)).



(A) The base tessellation Δ



(B) The refined tessellation Δ_T

FIGURE 1. Klein's quartic

A Weierstrass point on a non-singular irreducible algebraic curve S is a point P for which there exist functions on the curve with unusual pole orders at P and no poles everywhere else. Hurwitz showed that if the genus g is greater than 2, there there do exist Weierstrass points, and the number of Weierstrass points is bounded by $2g+2 \leqslant \omega \leqslant g(g+1)(g-1)$. Further, when genus $g \geqslant 2$, then $\mathrm{Aut}(S)$ is finite. In characteristic zero, this can be proved by showing that every automorphism fixing more than $2g+2$ points is the identity (by the Riemann-Roch theorem), and then obtaining a faithful representation of $\mathrm{Aut}(S)$ as permutations of the set of Weierstrass points $L : \mathrm{Aut}(S) \to \Sigma_\omega$. Note that in general, the action of $\mathrm{Aut}(S)$ on the set of Weierstrass points is not transitive.

We calculate the Weierstrass points of a surface by finding the zeros of the Wronskian.

Definition 4. Given a basis of holomorphic functions $\{f_1, \dots, f_g\}$ on a Riemann surface X of genus g, the Wronskian defined by

$$\mathcal{W}(z) := \det\left(\frac{d^j f_k(z)}{dz^j}\right)_{j=0,\dots,g-1,\, k=1,\dots,g}$$

is a non-trivial holomorphic function on X that induces a metric which we call the *Wronski metric.*

4. Verifications

The results in Table **??** are proved as follows. The genus, their (non)-hyperelliptic nature, and their Weierstrass points are calculated by tools from section **??**(??). The all-seeing tessellation Δ_T isconstructed using Section **??**.

We verify Conjecture 1 by using a seperate method of calculating automorphism groups. We derive the plane curve models of the cyclic covers of the sphere. With this input, calculate the automorphism group of the plane curve with: (1) our `autplane` program based on the algorithm in Jeroen-Sijsling-Zotine [1], and (2) a group identification algorithm (written by the second author). The code used in this paper can all be found at `https://github.com/catherineray/aut-jac`.

Table 1. Plane Curve Automorphism Groups

Curve C	Plane Curve Model	Genus	Aut(C)	|Aut(C)|
$*8(1,3,4)$	$y^2-(x^5-x)$	2	$GL_2(F_3)$	48
$*6(1,1,4)$	$y^2-(x^6-1)$	2	$S_3 \rtimes D_4$	24
$7(1,2,4)$ (Klein's quartic)	y^3x+x^3+1	3	$GL_3(F_2)$	168
$8(1,2,5)$ (Fermat's quartic)	$y^4-x(x+1)(x-1)$	3	$C_4^2 \rtimes S_3$	96
$12(1,3,8)$	$y^3-(x^4-1)$	3	$C_4 \cdot A_4$	48
$*8(1,1,6)$	$y^2-(x^8-1)$	3	$D_4 \rtimes C_4$	32
$*12(1,5,6)$	$y^2-(x^7-x)$	3	$C_4 \times S_3$	24
$*4(1,3,3,1)$	$y^4-(x^2-1)(x^2-a^2)^3$	3	$C_2 \times D_8$	16
$5(1,2,4,3)$ (Bring's curve)	$y^5-x(x-1)^2(x+1)^3$	4	S_5	120
$12(1,4,7)$ (I-WP)	$y^3-(x^5-x)$	4	$C_3 \times S_4$	72

An $*$ indicates that the curve is hyperelliptic

(for each example:) – W points – plane curves – hyperbolic tessellation – aut group

References

[1] N. Bruin, J. Sijsling, A. Zotine, *Numerical Computation of Endomorphism Rings of Jacobians*, `https://arxiv.org/pdf/1807.02605.pdf`

[2] H. Farkas, I. Kra, *Riemann Surfaces*, Springer-Verlag, New York, 1992.

[3] H. Karcher, M. Weber, *On Klein's Riemann Surface*, The Eightfold Way, MSRI Publications, Vol. 35, 1998, pp.9–49.

[4] Z. Laing, D. Singerman, *Transitivity on Weierstrass Points* Annales Academiae Scientiarum Fennicae Mathematica, Vol. 37, 2012, pp.285–300

[5] D. Lee, *On a triply periodic polyhedral surface whose vertices are Weierstrass points*, Arnold Mathematical Journal, Vol. 3, Issue 3, 2017, pp.319–331.

[6] D. Lee, *Geometric realizations of cyclically branched coverings over punctured spheres*, `https://arxiv.org/abs/1809.06321`, preprint.

[7] R. Lercier, C. Ritzenthaler, J. Sijsling, *Fast computation of isomorphisms of hyperelliptic curves and explicit Galois descent*, Proceedings of the Tenth Algorithmic Number Theory Symposium, 2013, pp. 463-486.

[8] M. Weber, *Kepler's small stellated dodecahedron as a Riemann surface*, Pacific Journal of Mathematics, Vol. 220, 2005, pp.167–182.