THE STUDY OF PRINCIPAL POLARIZATIONS THROUGH AUTOMORPHISMS OF THE JACOBIAN

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1. Introduction/Summary of Prior Work

The main result of this paper is finding multiple non-autoequivalent principal polarizations on many different Jacobians by using a new method. Before we state our result we summarize what is known about abelian varieties with several principal polarizations.

Narasimhan nori. Abelian varieties with several principal polarizations

The previous work toward the Narasimhan-Nori conjecture can be divided into the simple and non-simple cases (in which we can further indicate char 0 or char p cases). An abelian variety is **simple** if it is not isogenous to a product of abelian varieties of lower dimension.

A technique developed by Lange first for *simple* varieties in characteristic 0. In [12] Theorem 1.5, Lange establishes for simple varieties that the order of $\mathrm{Aut}(A)$ with certain restriction conditions and equivalence relations is equal to the number of principal polarizations on A up to isomorphism, $\pi(A)$. One could in principal compute the size of this specially carved out version of $\mathrm{Aut}(A)$ by hand using Lange's theorem, however, it is computationally infeasible. This is incredible, it gives us a a set equivalence between a slighly carved automorphism group of a variety, and its set of principal polarizations.

Remark. In Theorem 3.1, Lange further establishes bounds on $\pi(A)$ in terms of the class group of $End_{\mathbb{Q}}(A)$, if $End_{\mathbb{Q}}(A)$ is a *totally real* number field K (and thus the variety A is simple).

ADD OTHER RECENT WORK OF LANGE ON products of elliptic curves without complex multiplication.

Remark. His bijection between sets is induced by considering an element in the Neron-Severi group of A, and representing such elements as endomorphisms of A preserved under the Rosati involution with respect to a chosen polarization. Since this bijection is induced by a principal polarization, one must know that one exists to implement this theorem.

All other previous works known to the authors on finding multiple principal polarizations on abelian varieties have been done by finding two non-isomorphic curves with the same (unpolarized) Jacobian. Therefore, their associated canonical polarizations must be different by the Torelli theorem. Otherwise, the curves would be isomorphic. All papers that the we know of using this technique do so only in characteristic p.

Remark. There is only a canonical principal polarization on A if a curve C is specified so that $A = \mathrm{Jac}(C)$. If C is not specified, there is no canonical choice – knowing that A is in the image of the functor Jac is not enough. Thus, there can be several "canonical" principal polarizations on one Jacobian, its canonicality only refers to the fact that it comes from a curve.

The papers using this non-isomorphic curve technique discuss the case of *nonsimple* Jacobians of curves of genus two [8] and three [Brock, *Superspecial curves of genera two and three*], though the second author was unable to find a copy of the latter.

This technique is again used by E. Howe [6] and [7] which gives examples of non-isomorphic genus two curves with the same *simple* Jacobian. He finds such examples in characteristic p by playing with isogeny classes of abelian varieties which correspond to special Weil numbers.

To finish up the discussion on what was known before our paper, we wish to make a remark on how to use techniques for simple varieties to bound below the number of principal polarizations of nonsimple varieties.

Remark. Since every abelian variety is isogenous to a product of simple abelian varieties

$$A \simeq A_1 \times ... \times A_k$$

it is reasonable to ask how the numbers of principal polarizations on A_i are related to that of A.

Let's quickly establish some vocabulary to discuss this intuitively. Recall that we may also define a principal polarization on A as an isogeny which is also an isomorphism between $A \to A^\vee$, where $A^\vee$ denotes the dual variety. Let A and B be arbitrary abelian varieties. Note that $\mathrm{Corr}(A, B) \simeq \mathrm{Hom}(B, A^\vee)$, where we take a correspondence from A to B to be a line bundle $\mathcal{L}$ over the product $A \times B$ which is trivial when restricted to A or B.

We are interested in $\mathrm{Aut}(A, A^\vee)$, which is isomorphic to $\mathrm{Corr}(A, A)^\times$ but the problem of comparision arises immediately and obviously without having to pass to isomorphisms.

We wish to compare

$$\mathrm{Corr}\left(\prod_{j=1}^{k} A_j, \prod_{i=1}^{k} A_i\right) \qquad \text{and} \qquad \prod_{i,j} \mathrm{Corr}(A_j, A_i)$$

Let C and D be abelian varieties. Given a line bundle on C and on D, we get a line bundle on $C \times D$, but not vice-versa. Intuitively, the product $C \times D$ may have many more interesting cycles than the product of the cycles of C and D, and may not necessarily restrict to a line bundle on C or D. Therefore, in general the number of principal polarizations of A is at least the product of the principal polarizations of the simple components A_i, that is,

$$\pi(A_1 \times ... \times A_k) \geqslant \prod_{i=1}^{k} \pi(A_i)$$

as observed. We saw that Jac(I-WP) is isogenous to a product of 4 elliptic curves. Since these curves are all isomorphic, we only get one principal polarization from this decomposition, but we found at least 9 principal polarizations on their product.

In this paper, we introduce an entirely different technique to find different principal polarizations on Jacobians, exposited in Section 5.4, which treats both simple and non-simple cases in characteristic 0. This technique, for example, gives us 9 nonisomorphic principal polarizations on the Jacobian of Schoen's I-WP Surface. This is a very interesting result, especially since the variety Jac(I-WP) itself factors into a product of 4 elliptic curves[1], so the remaining principal polarizations must come from interesting new cycles in the product of these elliptic curves. Other such surprising results from this technique are shown in Table **??**.

The key insight(?) of our technique is as follows. It is computationally hard (impossible?? – check with J) to compare two principal polarizations to check if they are isomorphic, so we compute the automorphism group of the Jacobian with respect to them, and if these automorphism groups are nonisomorphic, the principal polarizations are nonisomorphic as well. This allows us to compare and study the principal polarizations further.

Our method works as follows. Given any period matrix, we introduce new code to brute force compute many principal polarizations on the corresponding Jacobian. We then implement a modification of the psuedocode of Bruin-Jeroen-Sijsling [3] to compute for each found principal polarization, the automorphism group of the given Jacobian which fixes that polarization. If we make the swashbuckling conjecture 2 that our brute force method finds all principal polarizations, this then gives us a lower bound on the number of different principal polarizations for a given Jacobian.

The fact that our numerical computations with period matrices give rigorous results relies on the brilliant work of Costa-Mascot-Sijsling-Voight [2] (see Prop 6.1.1).

We demonstrate the power of our method with a variety of different period matrices found roaming in the wild. We exposit first a folk-lore method from low dimensional geometry which allows us to compute exact period matrices of cyclically branched covers of punctured spheres. Across the land in the realm of numerically computed(phrasing?) Galois representations and moduli spaces of elliptic curves, as codified by Mascot in a slight modification of [16], we also work with period matrices of modular curves. These modular curves are not cyclically branched covers, and thus give us a completely disjoint collection of curves to which we apply our methods. [We use our methods to give a different proof of ...., again making the assumption that the conjecture 2 is true.]

2. Period Matrices

ADD BACK IN INTRO TO CYCLICALLY BRANCHED...

2.1. Computing the Period Matrix on Cyclic Covers.

In this section, we use the flat structure of a surface to compute the period matrix. We will look at the simplest

[1]This is because End(Jac(I-WP)) $\simeq M_4(K)$, where K is imaginary quadratic. Thus, Jac(I-WP) is the product of elliptic curves with CM by K.

case where $n = 3$ and $d_1 = 1$. Then, since $\sum d_i = d$, a cone metric with cone angles $\frac{2\pi}{d}(d_1, d_2, d_3)$ is admissible. Y is topologically equivalent to a doubled triangle with angles $\frac{2\pi}{d}(d_1, d_2, d_3)$ so we construct X with d copies of Y, which yields a flat structure on X. Figure 1 shows the flat structure of Klein's quartic.

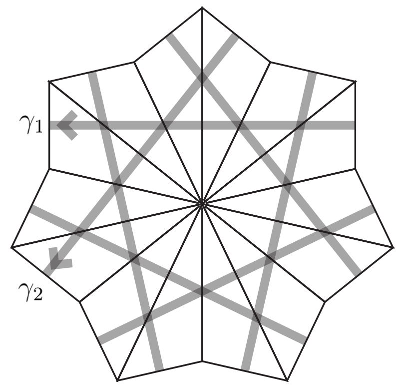

FIGURE 1. The flat fourteengon represents ω_1

The identification of edges are via parallel translations, which verifies that the cone metric is admissible. Identification of parallel edges yields closed cycles and the cyclicity gives away a homology basis with the following intersection matrix

$$\text{int} = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & -1 \\ -1 & 0 & 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 & 1 & 1 \\ 0 & 0 & -1 & -1 & 0 & 1 \\ 1 & 0 & 0 & -1 & -1 & 0 \end{pmatrix}.$$

Furthermore, we can produce flat structures that arise from ω_2 and ω_3 which are achieved from multipliers. The following period matrix is computed using the method from [9].

$$(\Pi) = (\int_\gamma \omega) = \begin{pmatrix} 1 & \zeta & \zeta^2 & \zeta^3 & \zeta^4 & \zeta^5 \\ 1 & \zeta^2 & \zeta^4 & \zeta^6 & \zeta^8 & \zeta^{10} \\ 1 & \zeta^4 & \zeta^8 & \zeta^{12} & \zeta^{16} & \zeta^{20} \end{pmatrix}$$

where ζ is the seventh root of unity.

2.2. A Non Cyclic Cover: The Modular Curve $X_0(63)$. ¡¡¡¡¡¡¡ HEAD

Recall that $SL_2(\mathbb{Z})$ acts transitively on the upper half plane $\mathfrak{h}$ by $\tau \mapsto \frac{a\tau+b}{c\tau+d}$. We quotient the upper half plane by subgroups Γ of $SL_2(\mathbb{Z})$ and metrize the quotient, however, this yields non-compact Riemann surfaces. To get a compact Riemann surface, we consider the extended upper half plane $\mathfrak{h}^+ := \mathfrak{h} \cup \mathbb{R} \cup \{\infty\}$ as a subset of $\mathbb{C}P^1$.

We are most interested in quotients of the upper half plane $\mathfrak{h}^+$ by the following subgroups of $SL_2(\mathbb{Z})$. These subgroups come up naturally in the study of modular forms associated to elliptic curves.

Definition 1.

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \;\middle|\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \mod N \right\}$$
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The automorphism groups of $X_0(N) := \mathfrak{h}^+/\Gamma_0(N)$ were calculated in [10] except for $N = 63$. The case of $X_0(63)$ was resolved 2 years later by Elkies in [4] by two different proofs: a conceptual one that uses enumerative geometry and the modular structure, and an explicit one that exhibits the modular equations. Our method would work for any N, we exposit the case of $N = 63$ due to its late blooming history.

Conjecture 1. *The program* `CullPB.m` *finds all principal polarizations on the curves we consider.*

If this conjecture is true, it would be a radically different proof than that of Elkies, since we approach by computing the automorphism group of the Jacobian of $X_0(63)$. Assuming this conjecture, we have

Theorem. $\mathrm{Aut}(X_0(63)) \simeq S_4 \times \mathbb{Z}/2$

Proof. Note the following theorem from [10]:

Lemma. $\mathrm{Aut}(X_0(63))$ *is either* $A_4 \times \mathbb{Z}/2$ *or* $S_4 \times \mathbb{Z}/2$.

Using the period matrix provided by Mascot, performed with 150 precisions, `autperio.sage` (using Conjecture 2) gives

Lemma. $\mathrm{Aut}(X_0(63))$ *is either* ${C_2}^4$ *or* $S_4 \times \mathbb{Z}/2$.

Therefore, it must be that $\mathrm{Aut}(X_0(63)) \simeq S_4 \times \mathbb{Z}/2$. □

The period matrix used in our calculation of $\mathrm{Aut}(\mathrm{Jac}(X_0(63)), p_i)$ was computed by Nicolas Mascot using an alteration of his personal code.

Remark. Mascot, in [16], discusses finding the period matrices for $X_1(N)$ by integrating cuspforms along modular symbols. His algorithm works for any compactified modular curve, but it works best when N is square-free. In the non-squarefree case, the coefficients in the q-expansions of the cuspforms and the j-invariant do not converge as quickly, thus they require more digits of precision.

Remark. In private correspondence, John Voight programmatically proved that $X_0(63)$ is not a cyclically branched cover of the sphere. Given that the genus of the quotient $X_0(63)/H$ is equal to the dimension of the H-invariant differentials, he shows that the list of dimensions of the space of H-invariant differentials on $X_0(63)$ (where H is a cyclic subgroup of $Aut(X_0(63))$) does not contain zero.

3. Studying Principal Polarizations via Automorphisms of the Jacobian

Definition 2. We say two principal polarizations p_1 and p_2 on A are **auto-equivalent** if and only if $\mathrm{Aut}(A, p_1) \simeq \mathrm{Aut}(A, p_2)$.

This program introduced in the next section produces many auto-equivalent principal polarizations. When this equivalent relation is taken, it greatly reduces the number of principal polarizations. Note that auto-equivalence is a weaker notion of equivalence than analytic equivalence, as discussed in section 6.

If the abelian variety is indeed a Jacobian, this method will in practice return at least enough polarizations to find the canonical principal polarization. As we discussed in the introduction, it is an unsolved problem to find all possible principal polarizations associated to a given abelian variety, called "explicit Narasimhan-Nori".

Recall that $SL_2(\mathbb{Z})$ acts transitively on the upper half plane $\mathfrak{h}$ by $\tau \mapsto \frac{a\tau+b}{c\tau+d}$. We quotient the upper half plane by subgroups Γ of $SL_2(\mathbb{Z})$ and metrize the quotient, however, this yields non-compact Riemann surfaces. To get a compact Riemann surface, we consider the extended upper half plane $\mathfrak{h}^+ := \mathfrak{h} \cup \mathbb{R} \cup \{\infty\}$ as a subset of $\mathbb{C}P^1$.

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¿¿¿¿¿¿¿ cd73ad6cdcc6853e440ebbb0835db6c53fda1d14 (maybe talk about relation to automorphism group?)

5. Programmatically Computing the Automorphism Group of Plane Curves and Abelian Varieties over $\mathbb{C}$

Remark. The first two subsections of the section are copied from section 4 of Bruin-Sijsling-Zotine [3] with lots of exposition and examples added for the readers' convenience. This sets us up to introduce the code for brute force calculating principal polarizations. All of our code is available at

`https://github.com/catherineray/aut-jac`

Let us examine abelian varieties represented as analytic groups $X := V/\Lambda$ and $X' := V'/\Lambda'$. They need not be Jacobians.

Theorem (BL 1.2.1)**.** *Let $X := V/\Lambda$ and $X' := V'/\Lambda'$ be abelian varieties. Under addition the set of homomorphisms* $\mathrm{Hom}(X, X')$ *forms an abelian group. There is an injective homomorphism of abelian groups:*

$$\rho : \mathrm{Hom}(X, X') \to \mathrm{Hom}(V, V')$$
$$f \mapsto F$$

The restriction to the lattice Λ is $\mathbb{Z}$-linear, thus we get an injective homomorphism:

$$\rho|_\Lambda : \mathrm{Hom}(X, X') \to \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \Lambda')$$
$$f \mapsto F|_\Lambda$$

We will namely use the representation $\rho|_\Lambda$ and find the basis of our set of maps in terms of this representation.

We work in the category of varieties equipped with principal polarizations, which we discuss in section 5.3. In this category, morphisms are morphisms of pairs. That is,

$$f : (X, c_1(\mathcal{L})) \to (Y, c_1(\mathcal{M}))$$

such that $f^*(Y, c_1(\mathcal{M})) = (X, c_1(\mathcal{L}))$ (for isomorphisms). We may represent polarizations as integral valued alternating forms.

Definition 5. Let a be a polarization of X. We call $\mathrm{Aut}(X, a)$ a **symplectic automorphism group** of X, as it respects the symplectic form a.

Let E_1 and E_2 be forms representing $c_1(\mathcal{L}_1)$ and $c_1(\mathcal{L}_2)$, respectively. Note that a map $\alpha : (X_1, c_1(\mathcal{L}_1)) \to (X_2, c_1(\mathcal{L}_2))$ such that

$$\alpha^*(c_1(\mathcal{L}_2)) = c_1(\mathcal{L}_1)$$

is equivalent to R in the image of $\rho|_\Lambda$ such that

$$R^t E_2 R = E_1$$

5.1. **Computing the Automorphism Group of Plane Curves.**

Remark. This section is on the algorithm used in `autplane.sage`.

In the case that our abelian variety is of the form $\mathrm{Jac}(C_i) =: J_i$, and we know the curve C_i, there is a special principal polarization E_i with respect to that curve C_i. This is programmatically found using Lemma 2.6 from [3].

Algorithm: Compute the set of isomorphisms between curves.
Input: Planar equations f_1, f_2 for curves C_1, C_2.
Output: The set of isomorphisms $C_1 \to C_2$, or the group $\mathrm{Aut}(C)$ if $C_1 = C_2$.

(1) Check if $g(C_1) = g(C_2)$; if not, return the empty set.
(2) Check if C_1 and C_2 are hyperelliptic; if so, use the methods in [15].
(3) Determine the period matrices P_1, P_2 of C_1, C_2 to the given precision.
(4) Determine a $\mathbb{Z}$-basis of $\mathrm{Hom}(J_1, J_2) \subset M_{2g\times 2g}(\mathbb{Z})$ represented by integral matrices $R \in M_{2g\times 2g}(\mathbb{Z})$. [Lemma 4.3 [3]]

(5) Using Fincke-Pohst[2], determine the finite set [from 5.1.9 BL]
$$S = \{R \in \mathrm{Hom}(J_1, J_2) \mid \mathrm{tr}((E_1^{-1} R^t E_2)R) = 2g\}$$
(6) Return the subset[3] of $R \in S$ which further satisfies $R^t E_2 R = E_1$. (These are the symplectic endomorphisms.)
(7) Look at the subset of R such that $\det(R) = \pm 1$. These are the symplectic automorphisms.
(8) If $J_1 = J_2$, find the group structure of this subset.

Note that if the curves C_1 and C_2 are non-hyperelliptic, by the precise Torelli theorem, we get $\mathrm{Hom}((J_1, E_1), (J_2, E_2)) \simeq \mathrm{Hom}(C_1, C_2) \sqcup \{\pm 1\}$ from this algorithm. So, we must remove the direct summand $\{\pm 1\}$.

Remark. Step 8 of the above algorithm was added by the second author to tame these unwieldy matrix groups, and is achieved as follows.

Algorithm: Compute the group structure of an underlying set of matrices.
Input: A set of matrices which are a group by multiplication.
Output: The group structure of the set.

(1) Check cardinality of the set. Call this N.
(2) Take first 15 elements of the set, use GAP to check if these generate a matrix group G of the correct order N. If not, it generates a group of order K, where $KM = N$. Take more elements of order dividing M until they generate a group of the correct order.
(3) Use `IdGroup(G)` in GAP.

5.2. **Computing the Automorphism Group of Abelian Varieties.**

Remark. This section is on the algorithm used in `autperio.sage`

Notation. Let $A := V/\Lambda$ be an abelian variety of dimension g. Let $e_1, ..., e_g$ be the chosen basis for V, and $\lambda_1, ..., \lambda_{2g}$ be a corresponding chosen basis for Λ. Let Π be the corresponding period matrix such that $A := \mathbb{C}^g / \Pi \mathbb{Z}^{2g}$.

Algorithm: Compute the group of isomorphisms between abelian varieties.
Input: Period matrices of abelian varieties J_1 and J_2, as Π_1 and Π_2 respectively.
Output: For each combination of principal polarizations (a_i, b_j), the set of isomorphisms between (J_1, a_i) and (J_2, b_j) (or the group, if they coincide).

(1) Check if $g_1 = g_2$; if not, return the empty set.
(2) Determine a $\mathbb{Z}$-basis of $\mathrm{Hom}(J_1, J_2) \subset M_{2g \times 2g}(\mathbb{Z})$ represented by integral matrices $R \in M_{2g \times 2g}(\mathbb{Z})$.
(3) Find many principal polarizations $\{a_i\}$ and $\{b_j\}$ for J_1 and J_2 respectively using `CullPB` (exposited in the next section).

[2] This is an algorithm for finding vectors of small norm. We use it here to solve for the finite set of solutions $R = \sum_{i=1}^{2g} \lambda_i B_i$, where B is the basis from step 4.

[3] The condition $R^t E_2 R = E_1$ (i.e., $E_1^{-1} R^t E_2 R = \mathrm{Id}$) implies that $\mathrm{tr}((E_1^{-1} R^t E_2)R) = 2g$. So we first solve for the latter to thin the results, then solve for the former from that set.

(4) Apply steps 5-8 of the previous section substituting each pair (a_i, b_j) for (E_1, E_2). For each pair, this will produce the set of isomorphisms between (J_1, a_i) and (J_2, b_j).
(5) If $(J_1, a_i) = (J_2, b_i)$, find the group structure of each set $\mathrm{Aut}(J_1, a_i)$ (using the algorithm in the previous section).

5.3. **Introduction to Polarizations: From Theory to Code.** The notion of a polarization of an abelian variety has many faces. If a complex torus has a polarization, it is an abelian variety.

Definition 6. A **polarization** of a complex torus X is an embedding $j : X \to \mathbb{P}^N$ for large enough N.

We can understand this embedding j as a map

$$p \mapsto [a_1(p) : \cdots : a_{N-1}(p)]$$

where a_i are a chosen generating set of global sections of a line bundle $\mathcal{L}$ on X.

Definition 7. A line bundle $\mathcal{L}$ is defined to be **very ample** on X if it defines a closed embedding into $\mathbb{P}^N$ for large enough N.

Definition 8. A line bundle is **ample** if a tensor power of the line bundle is very ample. Since the Chern class is additive, $c_1(\mathcal{L}^{\otimes k}) = kc_1(\mathcal{L})$, the ample bundle and its tensor power are equivalent datum.

Remark. In other words, $\mathcal{L}$ is defined to be ample if it (or a tensor power of it) specifies an embedding of X into projective space.

Definition 9. Line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ on X are **analytically equivalent** if there is a connected complex analytic space T, a line bundle $\mathcal{L}$ on $X \times T$, and points $t_1, t_2 \in T$ such that

$$\mathcal{L}|_{X \times \{t_i\}} \simeq \mathcal{L}_i$$

for $i = 1, 2$.

A line bundle $\mathcal{L}$ over X is specified up to analytic equivalence by its first Chern class $c_1(\mathcal{L}) \in H^2(X; \mathbb{Z})$. More precisely,

Theorem (2.5.3 BL)**.** *Let X be an abelian variety. For line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ over X, the following statements are equivalent:*
(1) $\mathcal{L}_1$ and $\mathcal{L}_2$ are analytically equivalent.
(2) $c_1(\mathcal{L}_1) = c_1(\mathcal{L}_2)$

Definition 10. The **first Chern class** of a line bundle $\mathcal{L}$ is the image of $\mathcal{L} \in \mathrm{Pic}(X) = H^1(\mathcal{O}_X^*)$ under the map c_1 on cohomology which arises as follows. Consider the exact sequence

$$0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0$$

and its long cohomology sequence:

$$\cdots \to H^1(\mathcal{O}_X^*) \xrightarrow{c_1} H^2(X, \mathbb{Z}) \to \cdots$$

We associate to every first Chern class an alternating form.

Theorem (BL 1.3.2 & 2.1.2)**.**

$$\psi : H^2(X; \mathbb{Z}) \simeq \mathrm{Alt}^2(\Lambda, \mathbb{Z})$$

Let S be the set of $c_1(\mathcal{L})$ where $\mathcal{L}$ ranges over all holomorphic line bundles on X. The image $\psi(S)$ is isomorphic to all Hermitian alternating forms.

Theorem (BL 2.1.6)**.** *Let $X := V/\Lambda$ be an abelian variety. For an alternating form $E : V \times V \to \mathbb{R}$, the following conditions are equivalent:*

(1) There is a holomorphic line bundle $\mathcal{L}$ on X such that $\psi(c_1(\mathcal{L})) = E$.
(2) $E(\Lambda, \Lambda) \subseteq \mathbb{Z}$, and

$$E(iv, iw) = E(v, w)$$

Remark. Note that from each element $\mathrm{Alt}^2(\Lambda, \mathbb{Z})$ we obtain via $\mathbb{R}$-linear extension an alternating form $\mathrm{Alt}^2(V, \mathbb{R})$ (as in rational versus analytic representation, see [BL 1.2.1]). We also have an isomorphism between real valued forms satisfying 2.1.6(2) and Hermitian forms.

It is important to emphasize that not all forms satisfying 2.1.6(2) correspond to Chern classes of ample line bundles. Ampleness is stronger than holomorphicity, hence we need a stronger condition.

Definition 11. A line bundle $\mathcal{L}$ on X is called **positive** if $c_1(\mathcal{L})$ is represented by a positive-definite Hermitian form.

Theorem. *Let X be a smooth complex projective variety. A line bundle $\mathcal{L}$ on X is ample if and only if it is positive.*

This is how we ask the computer to find polarizations of an abelian variety X, which are steps 1 and 2 of the following section.

However, there may be infinitely many polarizations. We are interested in a particular kind of polarization.

Definition 12. A polarization $c_1(\mathcal{L})$ of X is called **principal** if $\mathcal{L}$ has only one section up to constants, i.e. $\dim H^0(X, \mathcal{L}) = 1$.

As a motivational theorem:

Theorem (BL 4.1.2)**.** *Every polarization is induced by a principal polarization via an isogeny.*

By Narasimhan-Nori [17], there are only finitely many principal polarizations on a variety X, which is irreducible and smooth. And as a corollary, only finitely many curves may have the same Jacobian since each non-isomorphic curve gives a non-isomorphic principal polarization on its Jacobian.

5.4. **Finding Principal Polarizations.** We begin with a representation of our abelian variety as $A := \mathbb{C}^g / \Pi \mathbb{Z}^{2g}$.

Then Λ is the associated lattice spanned by the columns of Π. Thus, we have a distinguished basis for the homology of A, corresponding to the columns of Π.

Algorithm: Compute many principal polarizations on a given abelian variety A.

Input: An abelian variety $A := \mathbb{C}^g/\Pi\mathbb{Z}^{2g}$, where Λ is the associated lattice to the period matrix Π.

Output: Many principal polarizations on A.

(1) The magma function `FindPolarizationBasis` determines all integral alternating pairings E on the homology, i.e., $E \in \mathrm{Alt}^2(\mathbb{C}^g, \mathbb{Z})$, for whose real extension we have:

$$E(iv, iw) = E(v, w)$$

This is a basis of alternating forms $\{E_i\}$.

(2) Check that E is positive-definite.

(3) `CullPB.m` tries some small combinations and sees if E_i actually gives a pairing with determinant 1 indicating that E_i is a principal polarization. If so, it returns E_i. This gives us a set $\{E_k\}$ of integral pairings on the homology.

(4) For each i, we rewrite these pairings in a symplectic basis. That is, we find a basis of Λ in which

$$E_i = \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix}$$

where $D = \mathrm{diag}(d_1, ..., d_g)$ which we may do by the elementary divisor theorem (section 3.1, [1]).

Remark. This does nothing but modify the (homology) basis of Λ. Multiplying Π on the right with this integral matrix, we get a new period matrix Q whose columns span exactly the same lattice but for which the standard symplectic pairing E is actually the Chern class of a line bundle. This is often called the Frobenius form of the period matrix Π.

5.5. **Results of Applying Our Method.** ¡¡¡¡¡¡¡ HEAD

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Remark. Note that the University of Bristol's GroupNames database at the time of writing has groups up to order 500 with full names and structure description. In the cases where the order is greater than 500, we use the output of `StructureDescription(G);`

TABLE 1. Automorphism Groups wrt each of the Principal Polarizations

Curve C	Genus	# Principal Polarizations	$\mathrm{Aut}(\mathrm{Jac}(C), a_i)$	$\lvert \mathrm{Aut}(\mathrm{Jac}, a_i)\rvert$	GAPID
Klein	3	2	$S_4 \times C_2$	48	[48, 48]
			$GL_3(F_2) \times C_2$	336	[336, 209]
Fermat	3	2	$(C_4 \wr C_2) \times C_2$	64	[64, 101]
			$(C_4{}^2 \rtimes S_3) \times C_2$	192	[192, 944]
$12(1,5,6)$	3	3	D_6	12	[12, 4]
			$C_4 \times S_3$	24	[24, 5]
			$C_4 \times D_4$	32	[32, 25]
Bring	4	2	$C_2{}^2 \times D_4$	32	[32, 46]
			$C_2 \times S_5$	240	[240, 189]
I-WP	4	9	$C_2{}^4$	16	[16, 14]
			$C_2{}^2 \times C_6$	24	[24, 15]
			$C_2{}^2 \times D_4$	32	[32, 46]
			$C_2{}^3 \times C_6$	48	[48, 52]
			$C_2{}^2 \times S_4$	96	[96, 226]
			$C_6 \times S_4$	144	[144, 188]
			$(C_2 \times C_6) \times (C_3 \rtimes D_4)$	288	[288, 1002]
			$C_3 \times (((C_6 \times C_2) : C_2) \times D_8)$	576	[576, 7780]
			$C_6 \times (S_3 \times ((C_6 \times C_2) : C_2))$	864	[864, 4523]
$X_0(63)$	5	2	$C_2{}^5$	32	[32,51]
			$C_2{}^2 \times S_4$	96	[96, 226]

6. Questions and Answers on Abelian Varieties with Multiple Principal Polarizations

We speak here of polarizations up to auto-equivalence and ask natural questions on Jacobians with multiple principal polarizations, answering all but one of the questions using methods developed in our paper.

We fix some notation. Let θ_C be the canonical principal polarization of $\mathrm{Jac}(C)$ with respect to C. We call $\mathrm{Aut}(A, a_i)$ a symplectic automorphism group of A, as the automorphisms respect the principal polarization a_i, which is a symplectic form on A.

Question. $\mathrm{Aut}(\mathrm{Jac}(C), \theta_C)$ *will have the highest order of all symplectic automorphism groups of* $\mathrm{Jac}(C)$.

This is proven false by example $12(1,5,6)$, where $\lvert \mathrm{Aut}(\mathrm{Jac}(12(1,5,6)), \phi_{12(1,5,6)})\rvert = 24$, but $\lvert \mathrm{Aut}(\mathrm{Jac}(12(1,5,6)), a_i)\rvert = 32$ is achieved. It is more dramatically proven false by Schoen's I-WP Surface, where $\lvert \mathrm{Aut}(\mathrm{Jac}(\text{I-WP}), \phi_{\text{I-WP}})\rvert = 288$, but $\lvert \mathrm{Aut}(\mathrm{Jac}(\text{I-WP}), a_i)\rvert$ achieves 576 and 864.

Question. *Principal polarizations p_1 and p_2 are auto-equivalent if and only if they are analytically equivalent. In other words,*

$$\mathrm{Aut}(X, p_1) \simeq \mathrm{Aut}(X, p_2) \Leftrightarrow p_1 = p_2$$

The direction ($\Leftarrow$) is clear because $\mathcal{L}$ and $\mathcal{M}$ are analytically equivalent if and only if $c_1(\mathcal{L}) = c_1(\mathcal{M})$ by [BL 2.5.3]. The other direction ($\Rightarrow$) is false. This is proven false by applying our method to the the following two *non-isomorphic* curves with the same (unpolarized) Jacobian from Theorem 1 of [6]:

$$X : 3y^2 = (2x^2 - 2)(16x^4 + 28x^2 + 1)$$

$$X' : -y^2 = (2x^2 + 2)(16x^4 + 12x^2 + 1)$$

which both have $\mathrm{Aut}(\mathrm{Jac}(X), \theta_X) \simeq C_2 \times C_2 \simeq \mathrm{Aut}(\mathrm{Jac}(X'), \theta_{X'})$.

Question. *If* $\mathrm{Jac}(C) \simeq \mathrm{Jac}(C')$ *as complex varieties, then*

$$\mathrm{Aut}(\mathrm{Jac}(C), \theta_C) \simeq \mathrm{Aut}(\mathrm{Jac}(C'), \theta_{C'})$$

We checked this question on the family of hyperelliptic cases of genus 2 from [6] Theorem 1, where it is true. However, there is no reason to expect this to be true in general. Yet, we cannot disprove it easily.

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