

AUTOMORPHISMS OF THE JACOBIAN

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ABSTRACT. In this paper, we calculate and interpret the interaction between the automorphism group of a Riemann surface and the automorphism group of its Jacobian. We exposit and expand some calculational techniques from computational arithmetic geometry and hyperbolic geometry. Using these methods, we calculate in a new way the automorphism groups of well-known surfaces such as Klein’s quartic, Fermat’s quartic, and Bring’s curve. We exploit the fact that these surfaces can be described as cyclic branched covers over a punctured sphere to apply to further examples. We also use this approach on the modular curve $X_0(63)$, as an example that is not a cyclic cover over a sphere. We find several principal polarizations on many Jacobians of these Riemann surfaces, and the automorphism groups with respect to each of these polarizations. We discuss and answer questions on Jacobians with multiple principal polarizations.

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1. INTRODUCTION

We are interested in the interaction between the automorphism groups of Riemann surfaces and the automorphism groups of their Jacobian varieties. Hurwitz's theorem gives us an upper bound of the order of the automorphism group of a compact Riemann surface of genus greater than one. However, computing the automorphism group is in general difficult. We exposit and improve two complementary methods to calculate the automorphism group of a given Riemann surface.

One method is from computational arithmetic geometry. This program is based on the work of Bruin-Sijsling-Zotine introduced in [2]. The program takes an explicit plane curve and provides the automorphism group of its Jacobian with respect to the curve's canonical principal polarization. By Torelli's theorem, there is an abstract isomorphism between these automorphism groups if the curve is hyperelliptic. If the curve is not hyperelliptic, there is an involution that follows.

The other method is from hyperbolic geometry. In [13], the first author develops the tessellation-flag method to compute the automorphism group of Fermat's quartic. This method works particularly well on highly symmetric surfaces, e.g. those whose Weierstrass points have uniform weight. Lee presents Fermat's quartic as an eightfold cyclic branched cover over a thrice punctured sphere and finds a regular hyperbolic tessellation on the surface that is preserved under the automorphism group. Using this tessellation, Lee computes the automorphism group of Fermat's quartic in a generator-relation format. In this paper, we extend this to surfaces that have Weierstrass points of different weights.

These two methods clarify each other. On one hand, Lee's method of investigating curves as cyclic covers over spheres eases the computation. We discuss such covers in section 2 where further details can be found in [14]. These surfaces have many symmetries, hence one can identify the hyperbolic structure, flat structure, an explicit basis of holomorphic 1-forms, algebraic description, etc. The cyclicity on each surface makes it easy to compute period matrices which we use as inputs in our programs in section 4. On the other hand, our programs allow us to check the results obtained from Lee's method.

To show the range of the Jacobian program, we also treat the case of the modular curve $X_0(63)$, which is not a cyclic cover over a sphere. Modular curves give information about modular forms and provide friendly examples of moduli spaces. The compactified modular curves are Riemann surfaces, and therefore, we may apply our methods to them. The case of all $X_0(N)$ except $N = 63$ was completed by [9]. The case of $X_0(63)$ was done by Elkies in [3] soon after using two different proofs. We present a different proof via the Jacobian.

From the endomorphism ring of a Jacobian, we can directly understand the splitting of $\text{Jac}(C)$ into simple abelian varieties (Corollary 4.8, [2]). In the case that these varieties are elliptic curves, which occurs surprisingly often, we can write down the point count of $\text{Jac}(C)$ over any finite field, and therefore the L-function of $\text{Jac}(C)$. The endomorphism ring is calculated along the way to the automorphism group.

There are two main approaches to calculate automorphism groups of Jacobian varieties over \mathbb{C} . One approach is to study maps from a Riemann surface to 1-dimensional tori that are not quotient maps. Instead, we use the aforementioned program based on the work of Bruin-Sijssling-Zotine. This program finds principal polarizations of any given Jacobian¹ from scratch, and calculates the automorphism group of the Jacobian with respect to each of these found principal polarizations. This has the side effect of identifying different principal polarizations when their corresponding automorphism groups are different.

Understanding of the number of principal polarizations for a given Jacobian variety remains an unsolved problem, called “explicit Narasimhan-Nori” [18]. Only the simple case is understood [11], which we discuss in section 4.5. Finding examples which are non-simple is therefore desirable. We do this for several Jacobians, including the Jacobian of Schoen’s I-WP Surface.

The section rundown of this paper is as follows:

In section 2, we summarize the theory of cyclically branched covers over punctured spheres from the first author’s thesis [14] and discuss tools we use in finding the plane curve model, tessellation, period matrix, and the automorphism group.

In section 3, we discuss the precise Torelli theorem and exposit the outline of its proof.

In section 4, we exposit the workings of the two programs: One takes a plane curve representation and produces the automorphism group of the Jacobian with respect to the canonical principal polarization. The other, given only the period matrix, computes the automorphism group of the Jacobian with respect to many different principal polarizations. We finish by discussing and comparing prior work on finding multiple principal polarizations.

In section 5, we see our methods in action and find some interesting results. Focusing on lower genus surfaces ($g = 3, 4$) that are cyclic covers over spheres, we find multiple examples of abelian varieties with several isomorphism classes of principal polarizations. We finish by giving a new proof of the automorphism group of $X_0(63)$, which was the only remaining unresolved automorphism group of a compact modular curve until [3].

¹This program also works for all abelian varieties for which we know the period matrix.

In section 6, we discuss questions and answers about the nature of Jacobians with several principal polarizations.

2. GEOMETRIC COMPUTATION OF THE AUTOMORPHISM GROUP OF SURFACES

In this section, we will use the notion of cyclically branched covers over punctured spheres. These surfaces are particularly symmetric, thus one can find various equivalent descriptions of each surface. First, we will define and discuss tools that we use throughout this paper. Then we use cyclicity to create a fairly simple algorithm that yields the period matrices. This algorithm is the only alternative the authors know of to numerical approximation. Lastly, we seek a tessellation on each surface that exhibits the automorphism group.

We refer to Klein’s quartic as our leading example for each computation in the following subsections.

2.1. Cyclically Branched Covers over Punctured Spheres. We will discuss the topological construction of a cyclically branched cover over a sphere, then determine its conformal type. This section is extracted from chapter 3 of [14].

We define a covering over an n -punctured sphere in the following way. Let $Y := \mathbb{S}^2 \setminus \{p_1, \dots, p_n\}$ be an n -punctured sphere and let $q \in Y$ be a base point. Let γ_i be simple curves from q to p_i so that γ_i are mutually disjoint. These will serve as the branch cuts. We assign a positive integer d_i for each i that shows how the branching occurs at each branch cut. We call d_i the *branching index*. Let d be the degree of the covering map and use j to index Y_1, \dots, Y_d . For each i and j we identify the “left” side of γ_i of Y_j with the “right” side of γ_i of $Y_{j+d_i} \pmod{d}$.

Definition 1. We say that a surface S is a **d -fold cyclic branched cover over a sphere** if $S/(\mathbb{Z}/d\mathbb{Z})$ is a sphere.

The following theorem shows when such a covering is defined as a closed surface.

Theorem (Lee). *Given branching indices (d_1, \dots, d_n) , a d -fold cyclically branched cover over an n -punctured sphere is a closed surface if and only if $\sum_{i=1}^n d_i \equiv 0 \pmod{d}$.*

Moreover, the genus of the covering can be computed using the Riemann-Hurwitz formula.

Proposition. *Let X be a d -fold cyclically branched cover over an n -punctured sphere defined by branching indices (d_1, \dots, d_n) . Then*

$$\text{genus}(X) = \frac{d(n-2)}{2} + 1 - \frac{1}{2} \sum \gcd(d, d_i).$$

Now we pin down its conformal type. We will obtain cone metrics that arise from the branching indices. Let a cone metric on a compact Riemann surface be defined by cone angles $(\theta_1, \dots, \theta_n)$. Then, we have the following proposition that connects the topology (the Euler characteristic) and the geometry (total curvature) of the surface.

Proposition. *Given a compact Riemann surface of genus g with a cone metric, let p_1, \dots, p_n be distinguished points with respective cone angles θ_i . Then, $\sum \theta_i = 2\pi(2g - 2 + n)$.*

For example, a euclidean triangular pillowcase is topologically a sphere with a cone metric defined by angles $(\theta_1, \theta_2, \theta_3)$ where θ_i is the cone angle at each vertex.

Let X be a cyclic cover over Y . We say a cone metric on Y is *admissible* if its pullback yields a translational structure on X . Given a cone metric on Y , the following definition gives us a natural way of finding admissible cone metrics.

Definition 2. Given branching indices (d_1, \dots, d_n) , we say $a \in \{1, \dots, d-1\}$ is a *multiplier* if the cone metric given by cone angles $\frac{2\pi}{d} \cdot a(d_1, \dots, d_n) := \frac{2\pi}{d}(a \cdot d_1 \pmod{d}, \dots, a \cdot d_n \pmod{d})$ is admissible. For simplicity, we denote $a(d_1, \dots, d_3)$ as (a_1, \dots, a_3) where $a_i \in \{0, \dots, d-1\}$.

Remark. Given branching indices (d_1, \dots, d_n) and (a_1, \dots, a_n) where $a_i \equiv a \cdot d_i \pmod{d}$ for some $a \in \mathbb{Z}$, if $a_i > 0$ for all i and $\sum a_i = d(n-2)$, then the cone metric given by cone angles $\frac{2\pi}{d}(a_1, \dots, a_n)$ is admissible. This notion is particularly helpful when $n = 3$.

Theorem (Lee). *Let $X \rightarrow Y$ be a d -fold cyclically branched cover over a thrice-punctured sphere with branching indices (d_1, d_2, d_3) . Then there are exactly g admissible cone metrics that arise from multipliers.*

For example, in [8], Klein's quartic is identified as a sevenfold cyclic cover over a thrice punctured sphere. It is defined by the branching indices $(1, 2, 4)$. The admissible cone metrics arise from multipliers 1, 2, and 4.

Lastly, we will locate Weierstrass points on the coverings. This information will be used in subsection 2.3. These are points where the dimension-count of the Riemann-Roch theorem is not generic. For example, Klein's quartic has three holomorphic 1-forms with the following divisors

$$(\omega_1) = \tilde{p}_2 + 3\tilde{p}_3, \quad (\omega_2) = \tilde{p}_1 + 3\tilde{p}_2, \quad (\omega_3) = 3\tilde{p}_1 + \tilde{p}_3.$$

We define the weight of a point to measure how far the point is from being generic. A generic point is where the order of zeros form the sequence $0, 1, \dots, g-1$ given a basis. This sequence is preserved under the change of basis. For Klein's quartic, at \tilde{p}_i , this sequence is $0, 1, 3$. The weight at each \tilde{p}_i is computed by the difference of the two sequences: $(0-0) + (1-1) + (3-2) = 1$.

The following propositions are from [4].

Proposition. *Let S be a compact Riemann surface of genus ≥ 2 . Let wt_p be the weight of $p \in S$. Then $\sum_{p \in S} wt_p = (g-1)g(g+1)$.*

Proposition. *Let $W(S)$ be the finite set of Weierstrass points on S . If $\phi \in \text{Aut}(S)$, then $\phi(W(S)) = W(S)$. In fact, the sequences of the order of zeros at $p \in S$ and $\phi(p)$ are the same.*

However, in general, the action of $\text{Aut}(S)$ on $W(S)$ is not transitive [?].

Next, we use the Wronski metric to find all Weierstrass points that are not necessarily \tilde{p}_i .

Definition 3. Given a basis of holomorphic differentials $\omega_i = f_i dz$ on a Riemann surface X of genus g , the Wronskian defined by

$$\mathcal{W}(z) := \det \left(\frac{d^j f_k(z)}{dz^j} \right)_{j=0, \dots, g-1, k=1, \dots, g}$$

is a non-trivial holomorphic function on X that induces a metric which we call the *Wronski metric*.

By induction, one can show that a zero of the Wronskian is a Weierstrass point on X and the order of a zero at a point equals its weight. In our case, the Wronskian has simple zeros at the preimages of the midpoint of two p_i .

2.2. Computation of the Period Matrix on Cyclic Covers. In this section, we use the flat structure of a surface to compute the period matrix. We will look at the simplest case where $n = 3$ and $d_1 = 1$. Then, since $\sum d_i = d$, a cone metric with cone angles $\frac{2\pi}{d}(d_1, d_2, d_3)$ is admissible. Let Y be a sphere with such cone angles. We construct X with d copies of Y , which yields a flat structure on X . Figure 1 shows the flat structure of Klein's quartic.

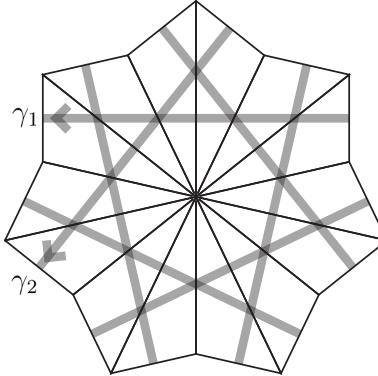


FIGURE 1. The flat fourteengon represents ω_1

The identification of edges are via parallel translations, which verifies that the cone metric is admissible. Identification of parallel edges yields closed cycles and the cyclicity gives away a homology basis with the following intersection matrix

$$\text{int} = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & -1 \\ -1 & 0 & 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 & 1 & 1 \\ 0 & 0 & -1 & -1 & 0 & 1 \\ 1 & 0 & 0 & -1 & -1 & 0 \end{pmatrix}.$$

Furthermore, we can produce flat structures that arise from ω_2 and ω_3 which are achieved from multipliers. The following period matrix is computed using the method from [8]

$$(\Pi) = \left(\int_{\gamma} \omega \right) = \begin{pmatrix} 1 & \zeta & \zeta^2 & \zeta^3 & \zeta^4 & \zeta^5 \\ 1 & \zeta^2 & \zeta^4 & \zeta^6 & \zeta^8 & \zeta^{10} \\ 1 & \zeta^4 & \zeta^8 & \zeta^{12} & \zeta^{16} & \zeta^{20} \end{pmatrix}$$

where ζ is the seventh root of unity.

2.3. The Automorphism Group via Tessellation. In [13], the first author computes the automorphism group of an eightfold cyclic cover over a thrice punctured sphere via a tessellation on the surface where the vertices correspond to the Weierstrass points of uniform weight. In this paper, we conjecturally generalize this algorithm to surfaces that have Weierstrass points of different weights.

In this section, we wish to find a tessellation Δ on a surface S which exhibits the automorphism group of S . Formally, we wish for $\text{Aut}(S)$ to act freely and transitively on the tiles of Δ ; we denote this with a slight abuse of notation:

$$\text{Aut}(\Delta) \simeq \text{Aut}(S)$$

This notation is partially justified by defining $\text{Aut}(\Delta)$ to be the group of automorphisms which are orientation-preserving, send vertices to vertices, edges to edges, and faces to faces.

Definition 4. A **tessellation** Δ is a polygonal decomposition of the surface where the polygons are either disjoint or share an edge or vertex, and their union is the entire surface. We say Δ' is a refinement of Δ if the tiles of Δ can be subdivided into tiles of Δ' .

Definition 5. Let S be a d -fold cyclic cover over an n -punctured sphere. We define the **base tessellation** of S as a tessellation tiled by n -gons with valency d at every vertex.

For $n \geq 3$ and $d > \frac{n}{2}$, there exists a unique base tessellation on a surface of genus $g > 1$. The base tessellation is tiled by N -many n -gons where $-2\pi(2-2g) = N(\pi - n \cdot \frac{2\pi}{d})$. The right-hand-side of the formula is simply the area of the surface and the rest follows from the Gauss-Bonnet formula.

Algorithm: A particular tessellation Δ_T on S .

Input: A surface S which is a cyclically branched cover of a sphere.

Output: A particular tessellation Δ_T .

- (1) Construct a base tessellation Δ of S .
- (2) Separately, find the Weierstrass points of S using Definition 3.
- (3) Refine Δ until all tiles are similar and all Weierstrass points of S occur as vertices. Call this new tessellation $\tilde{\Delta}$.
- (4) Let G_T be the orientation-preserving automorphism group of a tile T of $\tilde{\Delta}$ (recall that all tiles are similar²). Restrict our attention to a tile T of $\tilde{\Delta}$. On this tile T , add lines until any tile of the refinement T' is the fundamental domain of G_T

²Note that the automorphism group G_T encodes the weight of the vertices of T . As automorphisms preserve weights, vertices of different weights cannot be mapped to each other. Since all T are similar, all G_T are the same.

acting on T . Doing this to every tile T gives a new tessellation, which we call Δ_T .

Conjecture 1. *The output tessellation Δ_T of this algorithm always exists and*

$$\text{Aut}(\Delta_T) \simeq \text{Aut}(S)$$

By checking against the automorphism groups obtained via the `autplane` program described in section 4.1, we have shown this conjecture to be true for the cyclically branched surfaces of genus < 5 mentioned in section 5.

Example 2. We perform this algorithm for Klein's quartic. We begin with the base tessellation Figure 2(a). Both d and n play a role as we already know the existence of the d -fold map and the order- n map that permutes the branched values. Due to definition 5, the base tessellation of Klein's quartic is tiled by 56 hyperbolic $\frac{2\pi}{7}$ -triangles (Figure 2(a)). Next, we locate all Weierstrass points. On Klein's quartic, all vertices of the base tessellation correspond to Weierstrass points, that is $\Delta = \tilde{\Delta}$. We move to the last step. Since each tile T of $\tilde{\Delta}$ is a regular triangle with all vertices the same weight, we get $G_T = \mathbb{Z}/3$. Thus, we subdivide T so that T' is a fundamental domain for the action of G_T on T . (Figure 2(b)).

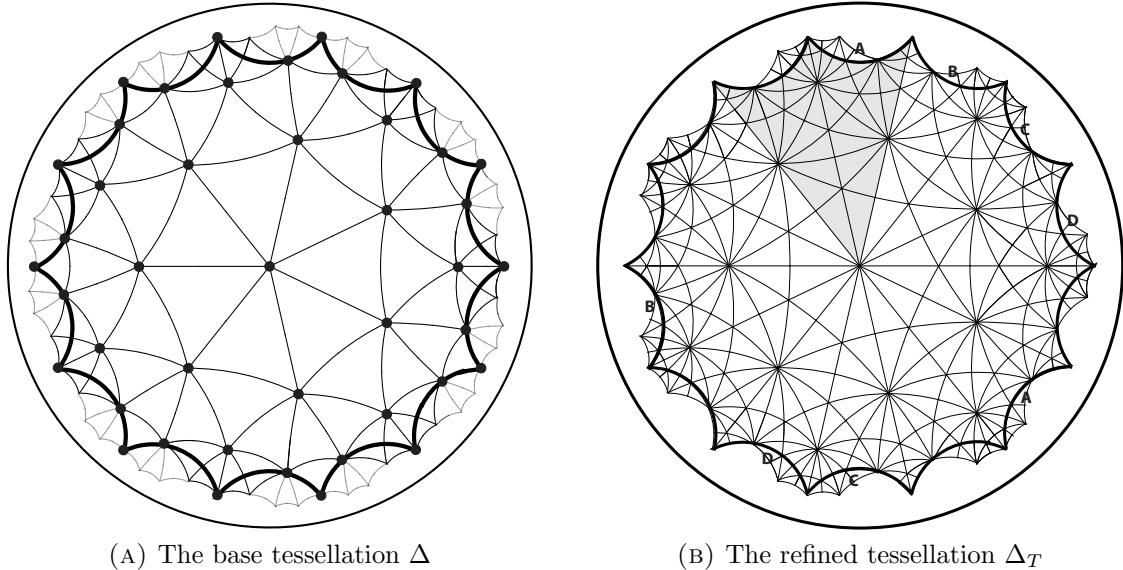


FIGURE 2. Klein's quartic

3. THE PRECISE TORELLI THEOREM

Remark. The first portion of this section is an English translation of the first two paragraphs of the Appendix [12], with slight modifications for the reader's convenience.

Let k be a field, and let X be a nice (smooth, projective and geometrically integral) curve over k of genus $g > 1$. Let $(\text{Jac}(X), a)$ denote the Jacobian of X together with the

Jacobian's canonical principal polarization a , which is of degree 1. Let X' be another nice curve over k . Any isomorphism $f : X \rightarrow X'$ defines by functoriality an isomorphism $f_J : (\text{Jac}(X), a) \rightarrow (\text{Jac}(X'), a')$.

Remark. The isomorphism $f : X \rightarrow X'$ induces an isomorphism between the divisor groups $\text{Div}^i(X)$ and $\text{Div}^i(X')$ for every i . This is because these groups are defined as formal combinations of codimension 1 subvarieties, and we can pull the inclusions of subvarieties of X' into X back via the isomorphism f . Note that the image of $\text{Div}^{g-1}(X)$ under the Abel-Jacobi map determines the canonical principal polarization of $\text{Jac}(X)$ with respect to X .

Torelli's theorem says that we get almost all of the isomorphisms $(\text{Jac}(X), a) \rightarrow (\text{Jac}(X'), a')$. More precisely:

Theorem. *Suppose X is hyperelliptic. Then for every isomorphism of polarized abelian varieties $F : (\text{Jac}(X), a) \xrightarrow{\sim} (\text{Jac}(X'), a')$, there exists a unique isomorphism $f : X \xrightarrow{\sim} X'$ such that $F = f_J$.*

Theorem. *Suppose X is not hyperelliptic. Then, for every isomorphism of polarized varieties $F : (\text{Jac}(X), a) \xrightarrow{\sim} (\text{Jac}(X'), a')$, there exists an isomorphism $f : X \xrightarrow{\sim} X'$ and $e \in \{\pm 1\}$ such that $F = e \cdot f_J$. Moreover, the pair (f, e) is uniquely determined by F .*

So, we conclude:

$$\text{Aut}(\text{Jac}(X), a) \simeq \begin{cases} \text{Aut}(X) & \text{if } X \text{ is hyperelliptic} \\ \text{Aut}(X) \oplus \mathbb{Z}/2 & \text{if } X \text{ is not hyperelliptic} \end{cases}$$

(Here we end quoting the Appendix). This precise form of Torelli's theorem is proved for algebraically closed fields by Weil [20]. A modern proof by Martens [16] proves Torelli's theorem as a combinatorial consequence of the Riemann-Roch theorem and Abel's theorem. Let ϕ be the Abel-Jacobi map $\phi : X \rightarrow \text{Jac}(X)$ (this map is determined up to translation by the torus $\text{Jac}(X)$). Let $\text{Div}^i(X)$ denote the degree $\leq i$ effective divisors on X , and W^i denote their image $\phi(\text{Div}^i(X))$.

Note that:

- $\text{Div}^1(X)$ is simply the points on X , so $\text{Div}^1(X) = X$.
- W^1 is birationally equivalent to X , that is, we can think of X as being embedded in its Jacobian.
- W^{g-1} determines the canonical polarization of $\text{Jac}(X) = W^g$ with respect to X .
- There is a map

$$\begin{aligned} \text{flip} : \text{Jac}(X) &\rightarrow \text{Jac}(X) \\ u &\mapsto -u + \phi(Z) \end{aligned}$$

where Z is a canonical divisor on X .

- By translation of U , we mean for every $p \in U \subset \text{Jac}(X)$, we have $p + a$ using the Jacobian group operation.

Let X and Y have the same Jacobian, and V^k denote $\phi(\text{Div}^k(Y))$. We finish by showing that if the canonical polarizations are the same, i.e. $V^{g-1} = W^{g-1}$, then V^1 is

a translate of W^1 or $\text{flip}(W^1)$. This is done by induction on the smallest r such that $V^1 \subseteq W^{r+1}$ or $V^1 \subseteq \text{flip}(W^{r+1})$, up to translation, showing that r must be 0.

In the case of hyperelliptic curves, $W^1 = \text{flip}(W^1)$, that is, they coincide as sets in $\text{Jac}(X)$. For non-hyperelliptic curves, $W^1 \neq \text{flip}(W^1)$, hence we have the $\mathbb{Z}/2$ factor.

4. PROGRAMMATICALLY COMPUTING THE AUTOMORPHISM GROUP OF PLANE CURVES AND ABELIAN VARIETIES OVER \mathbb{C}

Remark. This section is copied with from section 4 of [2] with exposition and examples added for the reader's convenience.

Let us examine abelian varieties represented as analytic groups $X := V/\Lambda$ and $X' := V'/\Lambda'$. They need not be Jacobians.

Theorem ([1] 1.2.1). *Let $X := V/\Lambda$ and $X' := V'/\Lambda'$ be abelian varieties. Under addition the set of homomorphisms $\text{Hom}(X, X')$ forms an abelian group. There is an injective homomorphism of abelian groups:*

$$\begin{aligned} \rho : \text{Hom}(X, X') &\rightarrow \text{Hom}(V, V') \\ f &\mapsto F \end{aligned}$$

The restriction to the lattice Λ is \mathbb{Z} -linear, thus we get an injective homomorphism:

$$\begin{aligned} \rho|_{\Lambda} : \text{Hom}(X, X') &\rightarrow \text{Hom}_{\mathbb{Z}}(\Lambda, \Lambda') \\ f &\mapsto F|_{\Lambda} \end{aligned}$$

We will namely use the representation $\rho|_{\Lambda}$ and find the basis of our set of maps in terms of this representation.

We work in the category of varieties equipped with principal polarizations, which we discuss in section 4.3. In this category, we consider morphisms of pairs. That is,

$$f : (X, c_1(\mathcal{L})) \rightarrow (Y, c_1(\mathcal{M}))$$

such that $f^*(Y, c_1(\mathcal{M})) = (X, c_1(\mathcal{L}))$ (for isomorphisms). We may represent polarizations as integral valued alternating forms.

Definition 6. Let a be a polarization of X . We call $\text{Aut}(X, a)$ a **symplectic automorphism group** of X , as it respects the symplectic form a .

Let E_1 and E_2 be forms representing $c_1(\mathcal{L}_1)$ and $c_1(\mathcal{L}_2)$, respectively. Note that a map $\alpha : (X_1, c_1(\mathcal{L}_1)) \rightarrow (X_2, c_1(\mathcal{L}_2))$ such that

$$\alpha^*(c_1(\mathcal{L}_2)) = c_1(\mathcal{L}_1)$$

is equivalent to R in the image of $\rho|_{\Lambda}$ such that

$$R^t E_2 R = E_1$$

4.1. Computing the Automorphism Group of Plane Curves.

Remark. This section is on the algorithm used in `autplane.sage`.

In the case that our abelian variety J_i is of the form $J_i = \text{Jac}(C_i)$, and we know C_i as a plane curve, then there is a special principal polarization E_i with respect to C_i . This is programmatically found using Lemma 2.6 from [2].

Algorithm: Compute the set of isomorphisms between curves.

Input: Planar equations f_1, f_2 for curves C_1, C_2 .

Output: The set of isomorphisms $C_1 \rightarrow C_2$, or the group $\text{Aut}(C)$ if $C_1 = C_2$.

- (1) Check if $g(C_1) = g(C_2)$; if not, return the empty set.
- (2) Check if C_1 and C_2 are hyperelliptic; if so, use the methods in [15].
- (3) Determine the period matrices P_1, P_2 of C_1, C_2 to the given precision.
- (4) Determine a \mathbb{Z} -basis of $\text{Hom}(J_1, J_2) \subset M_{2g \times 2g}(\mathbb{Z})$ represented by integral matrices $R \in M_{2g \times 2g}(\mathbb{Z})$. [Lemma 4.3 [2]]
- (5) Using Fincke-Pohst³, determine the finite set [from 5.1.9 [1]]

$$S = \{R \in \text{Hom}(J_1, J_2) \mid \text{tr}((E_1^{-1} R^t E_2) R) = 2g\}$$

- (6) Return the subset⁴ of S where $R^t E_2 R = E_1$. (These are the symplectic endomorphisms.)
- (7) The subset in which $\det(R) = \pm 1$ is the set of symplectic automorphisms.
- (8) If $J_1 = J_2$, find the group structure of the subset from step (7).

Note that if the curves C_1 and C_2 are non-hyperelliptic, we get $\text{Hom}((J_1, E_1), (J_2, E_2)) \simeq \text{Hom}(C_1, C_2) \sqcup \{\pm 1\}$ from this algorithm. By the precise Torelli theorem, we must remove the direct summand $\{\pm 1\}$.

Remark. Step 8 of the above algorithm was added by the second author to tame these unwieldy matrix groups, and is achieved as follows.

Algorithm: Compute the group structure of an underlying set of matrices.

Input: A set of matrices which is a group by multiplication.

Output: The group structure of the set.

- (1) Check cardinality of the set. Call this N .
- (2) Take first 15 elements of the set, use GAP to check if these generate a matrix group G of the correct order N . If not, it generates a group of order K , where $KM = N$. Take more elements of order dividing M until they generate a group of the correct order.
- (3) Use `IdGroup(G)` in GAP.

³This is an algorithm for finding vectors of small norm. We use it here to solve for the finite set of solutions $R = \sum_{i=1}^{2g} \lambda_i B_i$, where B is the basis from step (4).

⁴The condition $R^t E_2 R = E_1$ (i.e., $E_1^{-1} R^t E_2 R = \text{Id}$) implies that $\text{tr}((E_1^{-1} R^t E_2) R) = 2g$. So we first solve for the latter to thin the results, then solve for the former from that set.

4.2. Computing the Automorphism Group of Abelian Varieties.

Remark. This section is on the algorithm used in `autperio.sage`

Notation. Let $A := V/\Lambda$ be an abelian variety of dimension g . Let $\{e_1, \dots, e_g\}$ be the chosen basis for V , and $\{\lambda_1, \dots, \lambda_{2g}\}$ be a corresponding chosen basis for Λ . Let Π be the corresponding period matrix such that $A := \mathbb{C}^g / \Pi \mathbb{Z}^{2g}$.

Algorithm: Compute the group of isomorphisms between abelian varieties.

Input: Period matrices Π_1 and Π_2 of abelian varieties J_1 and J_2 , respectively.

Output: For each combination of principal polarizations (a_i, b_j) , the set of isomorphisms between (J_1, a_i) and (J_2, b_j) (or the group, if they coincide).

- (1) Check if $g_1 = g_2$; if not, return the empty set.
- (2) Determine a \mathbb{Z} -basis of $\text{Hom}(J_1, J_2) \subset M_{2g \times 2g}(\mathbb{Z})$ represented by integral matrices $R \in M_{2g \times 2g}(\mathbb{Z})$.
- (3) Find many principal polarizations $\{a_i\}$ and $\{b_j\}$ for J_1 and J_2 respectively using `CullPB` (exposed in the next section).
- (4) Apply steps (5)-(8) of the previous section substituting each pair (a_i, b_j) for (E_1, E_2) . For each pair, this will produce the set of isomorphisms between (J_1, a_i) and (J_2, b_j) .
- (5) If $(J_1, a_i) = (J_2, b_j)$, find the group structure of each set $\text{Aut}(J_1, a_i)$ (using the algorithm in the previous section).

4.3. Introduction to Polarizations: From Theory to Code. The notion of a polarization of an abelian variety has many faces. If a complex torus has a polarization, it is an abelian variety.

Definition 7. A **polarization** of a complex torus X is an embedding $j : X \rightarrow \mathbb{P}^N$ for large enough N .

We can understand this embedding j as a map

$$p \mapsto [a_1(p) : \dots : a_{N-1}(p)]$$

where a_i are a chosen generating set of global sections of a line bundle \mathcal{L} on X .

Definition 8. A line bundle \mathcal{L} is defined to be **very ample** on X if it defines a closed embedding into \mathbb{P}^N for large enough N .

Definition 9. A line bundle is **ample** if a tensor power of the line bundle is very ample. Since the Chern class is additive, $c_1(\mathcal{L}^{\otimes k}) = kc_1(\mathcal{L})$, the ample bundle and its tensor power are equivalent datum.

Remark. In other words, \mathcal{L} is defined to be ample if it (or a tensor power of it) specifies an embedding of X into projective space.

Definition 10. Line bundles \mathcal{L}_1 and \mathcal{L}_2 on X are **analytically equivalent** if there is a connected complex analytic space T , a line bundle \mathcal{L} on $X \times T$, and points $t_1, t_2 \in T$ such that

$$\mathcal{L}|_{X \times \{t_i\}} \simeq \mathcal{L}_i$$

for $i = 1, 2$.

A line bundle \mathcal{L} over X is specified up to analytic equivalence by its first Chern class $c_1(\mathcal{L}) \in H^2(X; \mathbb{Z})$. More precisely,

Theorem ([1] 2.5.3). *Let X be an abelian variety. For line bundles \mathcal{L}_1 and \mathcal{L}_2 over X , the following statements are equivalent:*

- (1) \mathcal{L}_1 and \mathcal{L}_2 are analytically equivalent.
- (2) $c_1(\mathcal{L}_1) = c_1(\mathcal{L}_2)$

Definition 11. The **first Chern class** of a line bundle \mathcal{L} is the image of $\mathcal{L} \in \text{Pic}(X) = H^1(\mathcal{O}_X^*)$ under the map c_1 on cohomology which arises as follows. Consider the exact sequence

$$0 \rightarrow \mathbb{Z} \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_X^* \rightarrow 0$$

and its long cohomology sequence:

$$\cdots \rightarrow H^1(\mathcal{O}_X^*) \xrightarrow{c_1} H^2(X; \mathbb{Z}) \rightarrow \cdots$$

We associate to every first Chern class an alternating form.

Theorem ([1] 1.3.2, 2.1.2).

$$\psi : H^2(X; \mathbb{Z}) \simeq \text{Alt}^2(\Lambda, \mathbb{Z})$$

Let S be the set of $c_1(\mathcal{L})$ where \mathcal{L} ranges over all holomorphic line bundles on X . The image $\psi(S)$ is isomorphic to all Hermitian alternating forms.

Theorem ([1] 2.1.6). *Let $X := V/\Lambda$ be an abelian variety. For an alternating form $E : V \times V \rightarrow \mathbb{R}$, the following conditions are equivalent:*

- (1) *There is a holomorphic line bundle \mathcal{L} on X such that $\psi(c_1(\mathcal{L})) = E$.*
- (2) *$E(\Lambda, \Lambda) \subseteq \mathbb{Z}$, and*

$$E(iv, iw) = E(v, w)$$

Remark. Note that from each element $\text{Alt}^2(\Lambda, \mathbb{Z})$ we obtain via \mathbb{R} -linear extension an alternating form $\text{Alt}^2(V, \mathbb{R})$ (as in rational versus analytic representation, see [[1] 1.2.1]). We also have an isomorphism between real valued forms satisfying 2.1.6(2) and Hermitian forms.

It is important to emphasize that not all forms satisfying 2.1.6(2) correspond to Chern classes of ample line bundles. Ampleness is stronger than holomorphicity, hence we need a stronger condition.

Definition 12. A line bundle \mathcal{L} on X is called **positive** if $c_1(\mathcal{L})$ is represented by a positive-definite Hermitian form.

Theorem. *Let X be a smooth complex projective variety. A line bundle \mathcal{L} on X is ample if and only if it is positive.*

This is how we ask the computer to find polarizations of an abelian variety X , which are steps 1 and 2 of the algorithm in the following section.

However, there may be infinitely many polarizations. We are interested in a particular kind of polarization.

Definition 13. A polarization $c_1(\mathcal{L})$ of X is called **principal** if \mathcal{L} has only one section up to constants, i.e., $\dim H^0(X, \mathcal{L}) = 1$.

As a motivational theorem:

Theorem ([1] 4.1.2). *Every polarization is induced by a principal polarization via an isogeny.*

By Narasimhan-Nori [18], there are only finitely many principal polarizations on an irreducible and smooth variety X . As a corollary, only finitely many curves may have the same Jacobian since each non-isomorphic curve gives a non-isomorphic principal polarization on its Jacobian.

4.4. Finding Principal Polarizations. We begin with a representation of our abelian variety as $A := \mathbb{C}^g/\Pi\mathbb{Z}^{2g}$.

Then Λ is the associated lattice spanned by the columns of Π . Thus, we have a distinguished basis for the homology of A , corresponding to the columns of Π .

Algorithm: Compute many principal polarizations on a given abelian variety A .

Input: An abelian variety $A := \mathbb{C}^g/\Pi\mathbb{Z}^{2g}$, where Λ is the associated lattice to the period matrix Π .

Output: Many principal polarizations on A .

- (1) The magma function `FindPolarizationBasis` determines all integral alternating pairings E on the homology, i.e., $E \in \text{Alt}^2(\mathbb{C}^g, \mathbb{Z})$, for whose real extension we have

$$E(iv, iw) = E(v, w)$$

This is a basis of alternating forms $\{E_i\}$.

- (2) Check that E is positive-definite.
- (3) `CullPB.m` tries some small combinations and sees if E_i actually gives a pairing with determinant 1 indicating that E_i is a principal polarization. If so, it returns E_i . This gives us a set $\{E_k\}$ of integral pairings on the homology.
- (4) For each i , we rewrite these pairings in a symplectic basis. That is, we find a basis of Λ in which

$$E_i = \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix}$$

where $D = \text{diag}(d_1, \dots, d_g)$ which we may do by the elementary divisor theorem ([1] section 3.1).

Remark. This does nothing but modify the (homology) basis of Λ . Multiplying Π on the right with this integral matrix, we get a new period matrix Q whose columns span exactly the same lattice but for which the standard symplectic pairing E is actually the Chern class of a line bundle. This is often called the Frobenius form of the period matrix Π .

Definition 14. We say two principal polarizations p_1 and p_2 on A are **auto-equivalent** if and only if $\text{Aut}(A, p_1) \simeq \text{Aut}(A, p_2)$.

This program produces many auto-equivalent principal polarizations. Via this equivalence, the number of principal polarizations is greatly reduced. Note that auto-equivalence is a weaker notion of equivalence than analytic equivalence, as discussed in section 6.

If the abelian variety is indeed a Jacobian, this method will in practice return at least enough polarizations to find the canonical principal polarization. It is an unsolved problem to find all possible principal polarizations associated to a given abelian variety, called “explicit Narasimhan-Nori”.

4.5. Abelian Varieties with Several Principal Polarizations. An unexpected result of this paper was finding multiple non-autoequivalent principal polarizations on many different Jacobians using a method not found in the literature. Before we state our result we summarize what is known about abelian varieties with several principal polarizations.

Definition 15. An abelian variety is **simple** if it is not isogenous to a product of abelian varieties of lower dimension.

Some previous work on finding multiple principal polarizations on abelian varieties has been done by finding two non-isomorphic curves with the same (unpolarized) Jacobian. By Torelli’s theorem, their associated canonical polarizations must be different. Otherwise, the curves would be isomorphic. There are results in which the non-simple case at genus two [7] and three [Brock, *Superspecial curves of genera two and three*] are discussed, though the authors was unable to find a copy of the latter.

Remark. There is only a canonical principal polarization on A if a curve C is specified so that $A = \text{Jac}(C)$. In other words, knowing that A is in the image of the functor Jac is not enough.

Another technique used in [5] and [6] gives examples of genus two curves that are non-isomorphic but give the same (simple) Jacobian. This is done in characteristic p with isogeny classes of abelian varieties which correspond to special Weil numbers. In [11] Theorem 1.5, Lange establishes that the order of $\text{Aut}(A)$ with certain restrictions and equivalence relations is equal to the number of principal polarizations on A up to isomorphism. This bijection between sets is induced by a principal polarization, so one must know that one exists to implement this theorem. One could compute the size of this specially carved out version of $\text{Aut}(A)$ by hand using Lange’s theorem. However, we compute directly the principal polarizations on A with the aid of a computer.

Another technique is used in [5] and [6] which gives examples of genus two curves that are non-isomorphic but give the same (simple) Jacobian. He does this in characteristic p by playing with isogeny classes of abelian varieties which correspond to special Weil numbers. In [11] Theorem 1.5, Lange establishes that the order of $\text{Aut}(A)$ with certain restriction conditions and equivalence relations is equal to the number of principal polarizations on A up to isomorphism, $\pi(A)$. This bijection between sets is induced by a principal polarization, so one must know that one exists to implement this theorem. One could compute the size of this specially carved out version of $\text{Aut}(A)$ by hand using Lange’s theorem. However, we compute directly the principal polarizations on A with the aid of a computer.

We use an entirely different technique, exposited in section 4.4, which treats both simple and non-simple cases. This technique, for example, gives us 9 not auto-equivalent principal polarizations on the Jacobian of Schoen's I-WP Surface. This is a very interesting result, especially since the variety $\text{Jac}(\text{I-WP})$ itself factors into a product of 4 elliptic curves⁵, so the remaining principal polarizations must come from interesting new cycles in the product of these elliptic curves. Other such results from this technique are shown in Table 2.

Remark. Since every abelian variety is isogenous to a product of simple abelian varieties

$$A \simeq A_1 \times \dots \times A_k$$

it is reasonable to ask how the numbers of principal polarizations on each A_i are related to that of A .

We establish some vocabulary to discuss this intuitively. Recall that we may also define a principal polarization on A as an isogeny which is also an isomorphism between $A \rightarrow A^\vee$, where A^\vee denotes the dual variety. Let A and B be arbitrary abelian varieties. Note that $\text{Corr}(A, B) \simeq \text{Hom}(B, A^\vee)$, where we take a correspondence from A to B to be a line bundle \mathcal{L} over the product $A \times B$ which is trivial when restricted to A or B .

We are interested in $\text{Aut}(A, A^\vee)$, which is isomorphic to $\text{Corr}(A, A)^\times$ but the problem of comparision arises immediately and obviously without having to pass to isomorphisms.

We wish to compare

$$\text{Corr}\left(\prod_{j=1}^k A_j, \prod_{i=1}^k A_i\right) \quad \text{and} \quad \prod_{i,j} \text{Corr}(A_j, A_i)$$

Let C and D be abelian varieties, and let $\pi(A)$ be the number of principal polarizations on A up to isomorphism. Given a line bundle on C and on D , we get a line bundle on $C \times D$, but not vice-versa. Intuitively, the product $C \times D$ may have many more interesting cycles than the product of the cycles of C and D , and may not necessarily restrict to a line bundle on C or D . Therefore, in general the number of principal polarizations of A is at least the product of the number of principal polarizations of the simple components A_i , that is,

$$\pi(A_1 \times \dots \times A_k) \geq \prod_{i=1}^k \pi(A_i)$$

as observed. We saw that $\text{Jac}(\text{I-WP})$ is isogenous to a product of 4 elliptic curves. Since these curves are all isomorphic, we only get one principal polarization from this decomposition, but we found at least 9 principal polarizations on their product.

5. EXAMPLE CALCULATIONS

This section is devoted to showing results from the methods discussed in sections 2 and 4.

⁵This is because $\text{End}(\text{Jac}(\text{I-WP})) \simeq M_4(K)$, where K is imaginary quadratic. Thus, $\text{Jac}(\text{I-WP})$ is the product of elliptic curves with CM by K .

We apply these methods to few genus 2, 3, and 4 surfaces that are cyclic branched covers of the sphere. We take examples from [14] which we expect to have large automorphism groups. We additionally investigate an interesting $X_0(63)$, a genus 5 surface that is not cyclically branched over a sphere.

The two tables highlight the different information revealed by the different methods. The first set of methods works given only a plane curve equation, and the second set works given only the period matrix. The tables are organized by genus and the order of the automorphism group.

The results in Table 1 are proved as follows. The genus and their (non)-hyperelliptic nature are calculated by tools from section 2. The automorphism groups and their orders are calculated by algorithms in section 4.1.

In the following individual sections, we derive the plane curve models of the cyclic covers of the sphere. We also derive the tessellations Δ_T to show that our examples of cyclically branched covers satisfy Conjecture 1, in section 2.3. Finally, we derive the period matrices using cyclicity (section 2.2) and use the results for calculations in Table 2.

Remark. The case of the modular curve $X_0(63)$ is calculated in a different way which does not require a plane curve model. See section 5.10.

The results in Table 2 are proved as follows. The principal polarizations, the corresponding symplectic automorphism groups, and their orders are found using `autperio.sage` (section 4.2).

Remark. Note that the University of Bristol's GroupNames database at the time of writing has groups up to order 500 with full names and structure description. In the cases where the order is greater than 500, we use the output of `StructureDescription(G)`;

TABLE 1. Plane Curve Automorphism Groups

Curve C	Plane Curve Model	Genus	Aut(C)	Aut(C)
*8(1, 3, 4)	$y^2 - (x^5 - x)$	2	$GL_2(F_3)$	48
*6(1, 1, 4)	$y^2 - (x^6 - 1)$	2	$S_3 \rtimes D_4$	24
7(1, 2, 4) (Klein's quartic)	$y^3x + x^3 + 1$	3	$GL_3(F_2)$	168
8(1, 2, 5) (Fermat's quartic)	$y^4 - x(x+1)(x-1)$	3	$C_4^2 \rtimes S_3$	96
12(1, 3, 8)	$y^3 - (x^4 - 1)$	3	$C_4 \cdot A_4$	48
*8(1, 1, 6)	$y^2 - (x^8 - 1)$	3	$D_4 \rtimes C_4$	32
*12(1, 5, 6)	$y^2 - (x^7 - x)$	3	$C_4 \times S_3$	24
*4(1, 3, 3, 1)	$y^4 - (x^2 - 1)(x^2 - a^2)^3$	3	$C_2 \times D_8$	16
5(1, 2, 4, 3) (Bring's curve)	$y^5 - x(x-1)^2(x+1)^3$	4	S_5	120
12(1, 4, 7) (I-WP)	$y^3 - (x^5 - x)$	4	$C_3 \times S_4$	72
$X_0(63)$?	5	$C_2 \times S_4$	48

An * indicates that the curve is hyperelliptic

TABLE 2. Automorphism Groups wrt each of the Principal Polarizations

Curve C	Genus	# Principal Polarizations	Aut(Jac(C), a_i)	Aut(Jac, a_i)	GAPID
Klein	3	2	$S_4 \times C_2$	48	[48, 48]
			$GL_3(F_2) \times C_2$	336	[336, 209]
Fermat	3	2	$(C_4 \wr C_2) \times C_2$	64	[64, 101]
			$(C_4^2 \rtimes S_3) \times C_2$	192	[192, 944]
12(1, 5, 6)	3	3	D_6	12	[12, 4]
			$C_4 \times S_3$	24	[24, 5]
			$C_4 \times D_4$	32	[32, 25]
Bring	4	2	$C_2^2 \times D_4$	32	[32, 46]
			$C_2 \times S_5$	240	[240, 189]
I-WP	4	9	C_2^4	16	[16, 14]
			$C_2^2 \times C_6$	24	[24, 15]
			$C_2^2 \times D_4$	32	[32, 46]
			$C_2^3 \times C_6$	48	[48, 52]
			$C_2^2 \times S_4$	96	[96, 226]
			$C_6 \times S_4$	144	[144, 188]
			$(C_2 \times C_6) \times (C_3 \rtimes D_4)$	288	[288, 1002]
			$C_3 \times (((C_6 \times C_2) : C_2) \times D_8)$	576	[576, 7780]
			$C_6 \times (S_3 \times ((C_6 \times C_2) : C_2))$	864	[864, 4523]
$X_0(63)$	5	2	C_2^5	32	[32, 51]
			$C_2^2 \times S_4$	96	[96, 226]

The number of principal polarizations refers to the number of found principal polarizations up to auto-equivalence.

5.1. 8(1, 3, 4). In this section, we look at a genus two surface defined as an eightfold cyclic cover over a thrice punctured sphere with branching indices (1, 3, 4). Since it is a genus two curve, it is hyperelliptic. We begin this section by discussing two different types of hyperelliptic curves that arise as cyclic branched covers over thrice punctured spheres, namely the $(1, \frac{d}{2} - 1, \frac{d}{2})$ -cover and the $(1, 1, d - 2)$ -cover. Recall that we let $d(d_1, \dots, d_n)$ denote the d -fold cyclic cover over an n -punctured sphere with branching indices (d_1, \dots, d_n) . The former is a genus- $\frac{d}{4}$ curve hence we will write it as a $4g(1, 2g - 1, 2g)$ cover. Similarly, the genus of the latter is $(\frac{d}{2} - 1)$ hence we denote it as $2g + 2(1, 1, 2g)$. The former type yields a basis of holomorphic 1-forms with divisors $(\omega_i) = (2i - 2)\tilde{p}_1 + (2g - 2i)\tilde{p}_2$, for $i = 1, \dots, g$. Given this basis, the Wronski metric tells us that all \tilde{p}_i are Weierstrass points. The quotient sphere under the d -fold cyclic map is a thrice punctured sphere, but the quotient under the hyperelliptic involution is a $(2g + 2)$ -punctured sphere. Note that the branched points are exactly the hyperelliptic points. We can think of this as a sphere with \tilde{p}_1 and \tilde{p}_2 placed at the North and South Pole, and $2g$ preimages of p_3 along the Equator. These points correspond to the zeros of its plane curve model $y^2 - x(x^{2g} - 1)$. For the $8(1, 3, 4)$ cover, we have $y^2 - (x^5 - x)$.

On the other hand, the $(2g + 2)$ -fold cover yields a basis of holomorphic 1-forms with divisors $(\omega_i) = (i - 1)\tilde{p}_1 + (i - 1)\tilde{p}_2 + (g - i)\tilde{p}_{31} + (g - i)\tilde{p}_{32}$, for $i = 1, \dots, g$. Then none of the \tilde{p}_i are Weierstrass points but \tilde{p}_* are, where p_* is a fixed point of an involution that fixes p_3 and interchanges p_1 and p_2 . Since it is not a branched point under the cyclic map, it has $2g + 2$ preimages which are all fixed under the hyperelliptic involution. The quotient sphere under the hyperelliptic involution can be viewed as a doubled $(2g + 2)$ -gon where all Weierstrass points are located at the vertices. The corresponding plane curve model is $y^2 - (x^{2g+2} - 1)$.

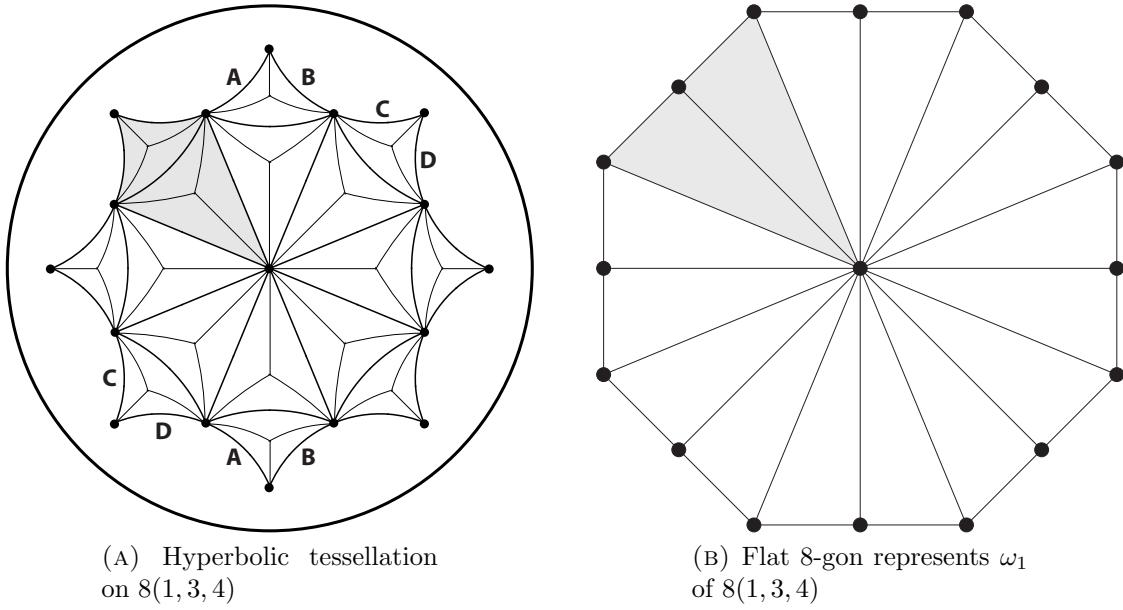
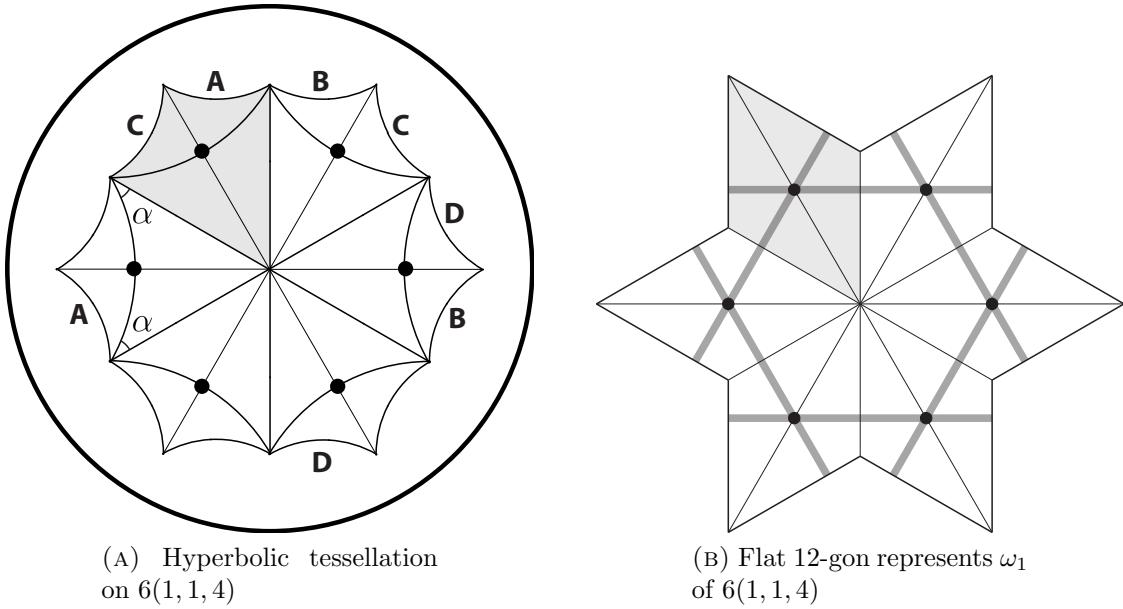
The $8(1, 3, 4)$ cover is a double cover over an octahedron, hence tiled by sixteen 8-valent triangles. We tile the hyperbolic disc via these triangles and mark the Weierstrass points by \bullet . Since these tiles are regular, we refine the tessellation to show the order-three symmetry on each tile. The shaded area in Figure 3(a) indicates a quotient sphere under the eightfold cyclic map.

In Figure 3(b) is the flat structure given by ω_1 . We take the closed cycles via the identification of edges and apply cyclicity. Then the period matrix is

$$(\Pi) := (A|B) = \begin{pmatrix} 1 & \frac{1+i}{\sqrt{2}} & | & i & \frac{-1+i}{\sqrt{2}} \\ 1 & \frac{-1+i}{\sqrt{2}} & | & -i & \frac{1+i}{\sqrt{2}} \end{pmatrix}.$$

5.2. 6(1, 1, 4). In this section, we look at another (hyperelliptic) genus two curve. It is a double cover over a doubled-hexagon and its plane curve model is $y^2 - (x^6 - 1)$. Note that we cannot apply the base tessellation (Definition 5) here⁶. Instead we use the fact that the quotient sphere (under the hyperelliptic involution) is a doubled-hexagon and tile the disc with twelve $(\frac{\pi}{3}, \frac{\pi}{6}, \frac{\pi}{6})$ -triangles. These angles come from Gauss-Bonnet formula $(-2\pi(2 - 2 \cdot 2) = 12(\pi - (2\alpha + \frac{\pi}{3})))$. We refine the tessellation so that the Weierstrass points are vertices of the tiling. Note that the center of Figure 4(a) is fixed under the sixfold map, not the hyperelliptic map.

⁶The conditions in Definition 5 were imposed because of this specific case.

FIGURE 3. $8(1, 3, 4)$ FIGURE 4. $6(1, 1, 4)$

The shaded area in Figure 4(a) indicates a quotient sphere (under the sixfold map) and the Weierstrass points are denoted by \bullet . The identification of edges yield closed cycles. Take the 1-forms achieved via multipliers, then the period matrix is

$$(\Pi) := (A|B) = \begin{pmatrix} 1 & \frac{1+\sqrt{3}i}{2} & | & \frac{-1+\sqrt{3}i}{2} & -1 \\ 1 & \frac{-1+\sqrt{3}i}{2} & | & \frac{-1-\sqrt{3}i}{2} & 1 \end{pmatrix}.$$

5.3. $8(1, 2, 5)$ Fermat's quartic. This example arises in [13] as an abstract quotient of a triply periodic polyhedral surface. The polyhedral surface is called the Octa-4 as it is the boundary of a polyhedron built by gluing four regular octahedra to a centered octahedron (Figure 5).

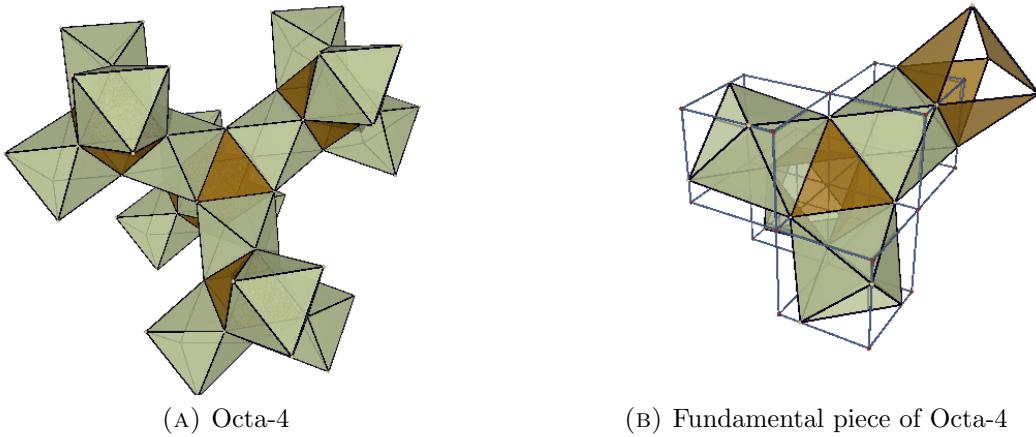


FIGURE 5. Octa-4

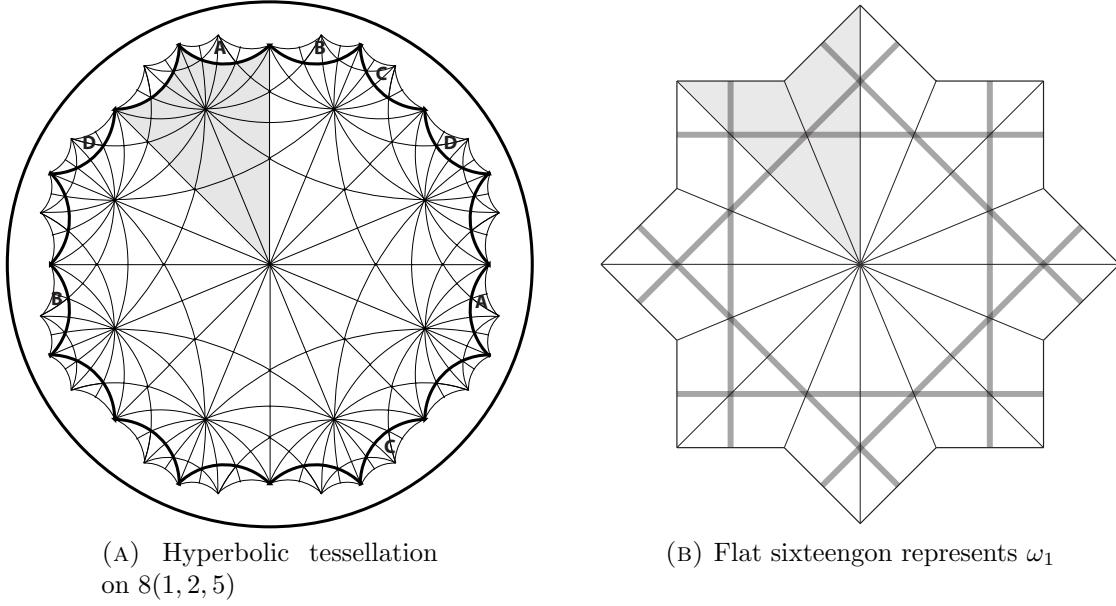
In [13], Lee shows that the polyhedral metric induces hyperbolic structures and various translation structures that are compatible with its conformal type. It is described as an eightfold cyclic cover over a sphere.

Furthermore, Lee also shows that the surface can be written as a fourfold cyclic cover over a 4-punctured sphere defined by branching indices $(1, 1, 1, 1)$. Then the two affine models $y^8 - x(x-1)^5$ are equivalent $y^4 - x(x-1)(x+1)$, and given the latter we achieve Fermat's quartic.

Remark. Using tessellations, the automorphism group of this Riemann surface is presented in [13] as $\text{Aut}(X) = \langle a, b \mid a^8 = b^3 = (ab)^2 = (a^2b^2)^3 = (a^4b^2)^3 = 1 \rangle$. This is equivalent to the group in Table 2 by the GAP small group identification function. We reprove this with `autplane.sage`.

Using the same method as earlier, that is, so that the intersection matrix is chosen as

$$\text{int}_1 = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 & 1 & 1 \\ 0 & 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & -1 & 0 \end{pmatrix},$$

FIGURE 6. $8(1, 2, 5)$

the period matrix Π_1 is computed as follows:

$$\Pi_1 = \begin{pmatrix} 1 & e^{\frac{\pi i}{4}} & i & e^{\frac{3\pi i}{4}} & -1 & e^{-\frac{3\pi i}{4}} \\ 1 & i & -1 & -i & 1 & i \\ 1 & e^{-\frac{3\pi i}{4}} & i & e^{-\frac{\pi i}{4}} & -1 & e^{\frac{\pi i}{4}} \end{pmatrix}$$

Remark. In [14], the homology basis is chosen so that the cycles appear as “handles” on the polyhedral surface. That is,

$$\text{int}_2 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}$$

This yields the following period matrix Π_2

$$\Pi_2 = (A|B) = \begin{pmatrix} 1-i & -((1+i)/(1+\sqrt{2})) & (1+i)/(1+\sqrt{2}) & | & 1+i & \sqrt{2} & 2-\sqrt{2} \\ -2i & 2i & 2i & | & 2i & -2 & -2 \\ -1-i & (1-i)(1+\sqrt{2}) & (-1+i)(1+\sqrt{2}) & | & 1-i & i\sqrt{2} & i(-2-\sqrt{2}) \end{pmatrix}$$

where

$$\tau = A^{-1}B = \begin{pmatrix} i & \frac{1+i}{2} & \frac{1+i}{2} \\ \frac{1+i}{2} & i & \frac{1+i}{2} \\ \frac{1+i}{2} & \frac{1+i}{2} & i \end{pmatrix}$$

5.4. 12(1, 3, 8). In this section, we look at an example that arises from the Shiga curve $y^3 = x^4 - 1$. This is a genus three non-hyperelliptic curve. We observe that there is an order-twelve map $x \mapsto ix$, $y \mapsto \zeta y$, where ζ is a third root of unity. As we have three branched values 0, 1, and ∞ , we rely on Appendix A of [14] and search for twelvefold cyclic covers over thrice punctured spheres where we find two possible candidates. One is defined as 12(1, 5, 6) and the other is defined as 12(1, 3, 8). The former is hyperelliptic whereas the latter is not. Therefore we conclude that the Shiga curve corresponds to the 12(1, 3, 8) cover.

The admissible cone metrics arise from multipliers and yield a basis of 1-forms with divisors $(\omega_1) = \tilde{p}_{31} + \tilde{p}_{32} + \tilde{p}_{33} + \tilde{p}_{34}$, $(\omega_2) = \tilde{p}_1 + \tilde{p}_{21} + \tilde{p}_{22} + \tilde{p}_{23}$, and $(\omega_3) = 4\tilde{p}_1$. The Wronski metric tells us that the Weierstrass points do not have equal weight: \tilde{p}_1 and \tilde{p}_{2i} have weight two, \tilde{p}_{3i} have weight one. Additionally we have $\text{wt}(\tilde{p}_{*i}) = 1$, where p_* is a fixed point of an involution that fixes p_3 and interchanges p_1 and p_2 . We begin with the base tessellation, then by Gauss-Bonnet formula, the surface is tiled by sixteen hyperbolic $(\frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{6})$ -triangles. The region bounded by the bold lines shows the tessellation of this surface (Figure 7(a)). We mark the Weierstrass points and subdivide the $(\frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{6})$ -triangles so that all triangles are congruent to each other. Vertices marked as \bullet are Weierstrass points of weight two, those marked as \circ are points of weight one.

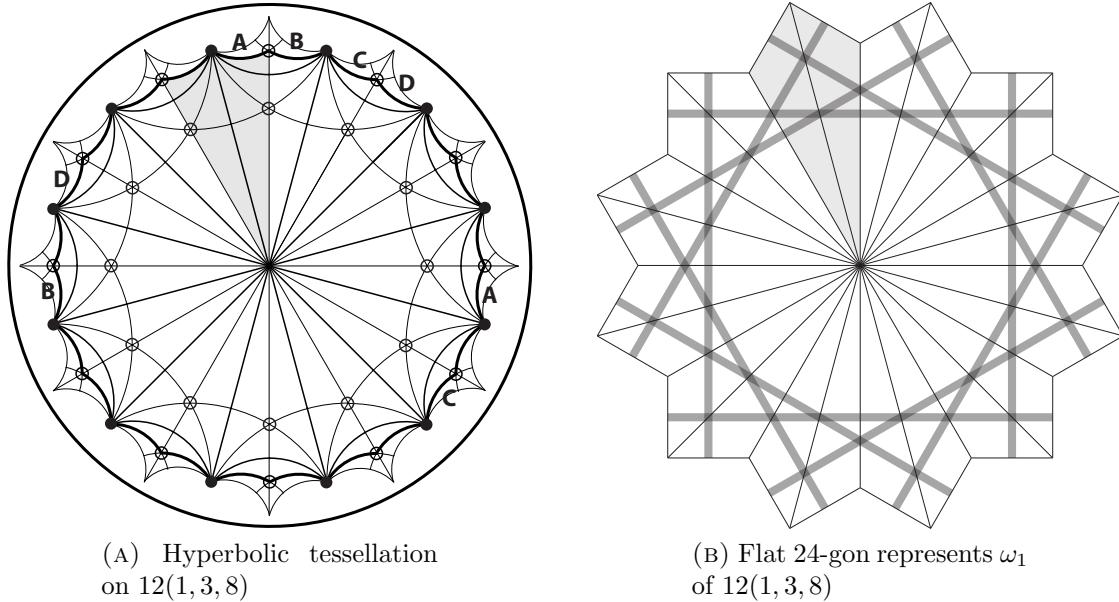


FIGURE 7. 12(1, 3, 8)

The period matrix of 12(1, 3, 8) is:

$$\Pi = \begin{pmatrix} 1 & e^{\frac{\pi i}{6}} & e^{\frac{\pi i}{3}} & i & e^{\frac{2\pi i}{3}} & e^{\frac{5\pi i}{6}} \\ 1 & e^{\frac{\pi i}{3}} & e^{\frac{2\pi i}{3}} & -1 & e^{-\frac{2\pi i}{3}} & e^{-\frac{\pi i}{3}} \\ 1 & e^{\frac{5\pi i}{6}} & e^{-\frac{\pi i}{3}} & i & e^{-\frac{2\pi i}{3}} & e^{\frac{\pi i}{6}} \end{pmatrix}$$

5.5. $8(1, 1, 6)$. In this section, we look at a genus three hyperelliptic curve defined as an eightfold cyclic cover with branching indices $(1, 1, 6)$. Its quotient sphere (under the hyperelliptic involution) is a doubled hexagon. We tile the hyperbolic disc by thirty two 8-valent triangles. Via the Wronski metric, we locate the Weierstrass points and refine the tessellation to show all symmetries (Figure 8(a)). Its plane curve model is $y^2 - (x^8 - 1)$ and the period matrix is

$$\Pi := \begin{pmatrix} 1 & e^{\frac{\pi i}{4}} & i & e^{\frac{3\pi i}{4}} & -1 & e^{-\frac{3\pi i}{4}} \\ 1 & i & -1 & -i & 1 & i \\ 1 & e^{\frac{3\pi i}{4}} & -i & e^{\frac{\pi i}{4}} & -1 & e^{-\frac{\pi i}{4}} \end{pmatrix}.$$

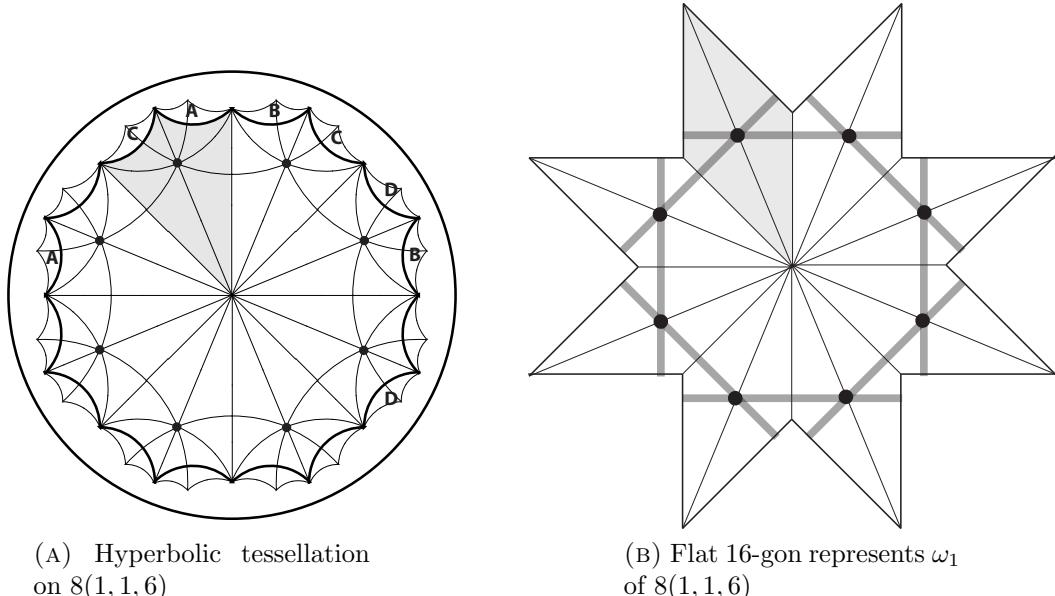
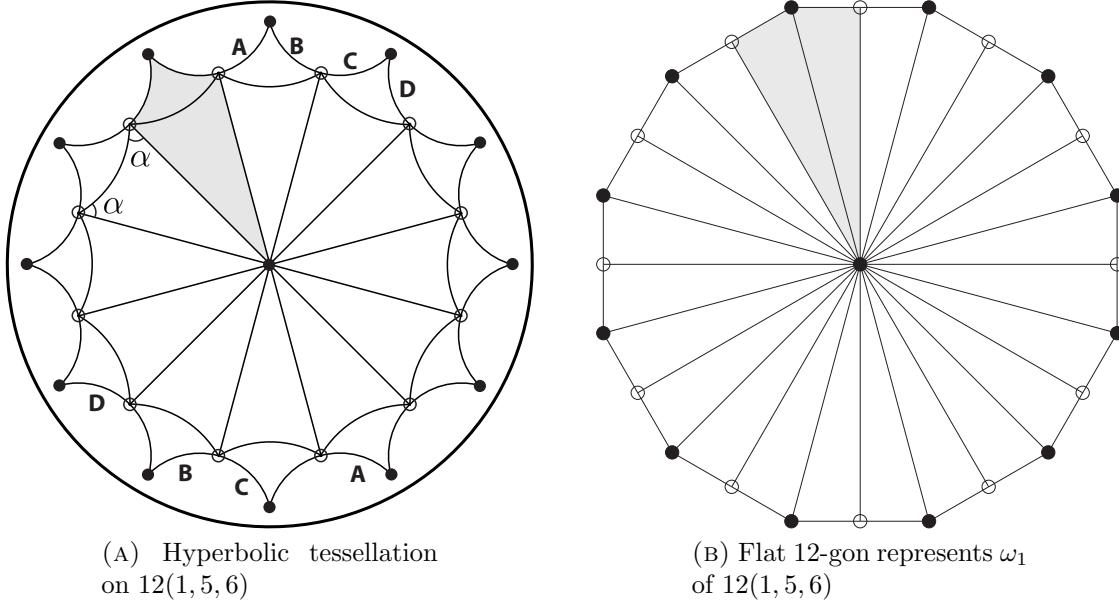


FIGURE 8. $8(1, 1, 6)$

5.6. $12(1, 5, 6)$. In this section, we look at another example of a genus three hyperelliptic surface which is defined as a twelvefold cyclic cover over a sphere with branching indices $(1, 5, 6)$. It is a double-cover over a sphere with branched points at the North and South Pole, and six equidistributed points on the Equator. Its plane curve model is $y^2 - x(x^6 - 1)$. We tile the hyperbolic disc by twenty four hyperbolic $(\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{4})$ -triangles $(-2\pi(2 - 2 \cdot 3) = 24(\pi - (2\alpha + \frac{\pi}{6}))$. All Weierstrass points are marked as either \bullet or \circ as they have different valency.

We use the identification of opposite edges of the 12-gon as closed cycles to form a homology basis and the 1-forms achieved via multipliers. Then the period matrix is

$$\Pi = \begin{pmatrix} 1 & e^{\frac{\pi i}{6}} & e^{\frac{2\pi i}{6}} & e^{\frac{3\pi i}{6}} & e^{\frac{4\pi i}{6}} & e^{\frac{5\pi i}{6}} \\ 1 & e^{\frac{\pi i}{2}} & e^{\frac{2\pi i}{2}} & e^{\frac{3\pi i}{2}} & e^{\frac{4\pi i}{2}} & e^{\frac{5\pi i}{2}} \\ 1 & e^{\frac{5\pi i}{6}} & e^{\frac{10\pi i}{6}} & e^{\frac{15\pi i}{6}} & e^{\frac{20\pi i}{6}} & e^{\frac{25\pi i}{6}} \end{pmatrix}$$

FIGURE 9. $12(1, 5, 6)$

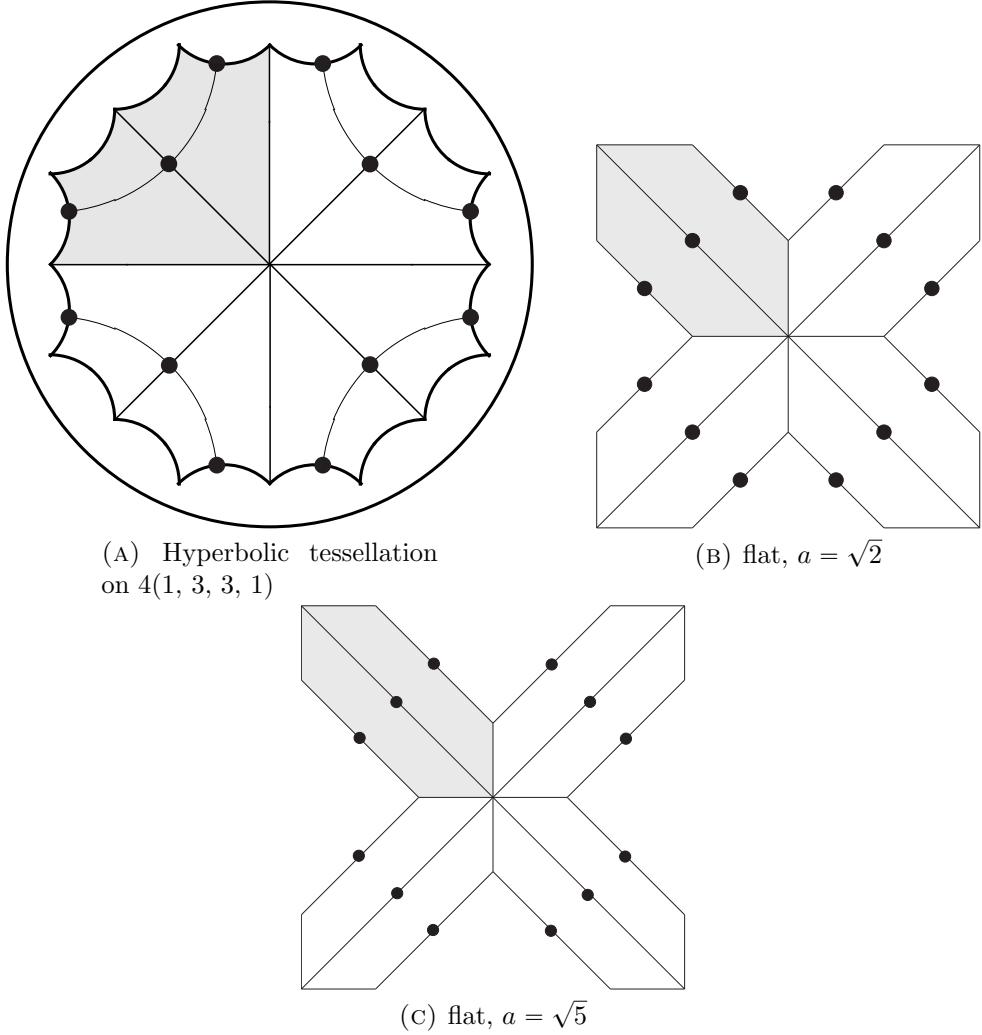
5.7. $4(1, 3, 3, 1)$. In this section, we look at a genus three hyperelliptic curve that is a cyclic cover over a 4-punctured sphere. In our case, the admissible cone metrics are achieved via multipliers, although this is not true for all cyclically branched covers over $n (\geq 4)$ -punctured spheres (See chapter 2). The admissible cone metrics yield a basis with divisors $(\omega_1) = 2\tilde{p}_2 + 2\tilde{p}_3$, $(\omega_2) = \tilde{p}_1 + \tilde{p}_2 + \tilde{p}_3 + \tilde{p}_4$, and $(\omega_3) = 2\tilde{p}_1 + 2\tilde{p}_4$. Given a chart where $(z(p_i)) = (-a, -1, 1, a)$ for some $a > 1$, the Wronskian tells us that the curve is hyperelliptic and the Weierstrass points are located at the preimages of q where $z(q) = 0$ or $z(q) = \infty$. The Weierstrass points are marked as \bullet in Figure 11(a).

Given the branching indices, that is, the appropriate cone angles, the four punctured sphere associated to ω_1 is a doubled trapezoid with angles $\frac{\pi}{4}$ and $\frac{3\pi}{4}$. However, the trapezoid cannot be pinned down uniquely (Figure 11). We parametrize the side lengths of the trapezoid so that $\overline{p_1 p_2} = 1$ and $\overline{p_2 p_3} = b$ for $b > 0$. Moreover, for ω_2 , we parametrize the side lengths to be 1 and c , and show that c can be arbitrary. Given $b, c > 0$, we calculate the period matrix to be:

$$\Pi = \begin{pmatrix} 1 & i & -1 & (b + \sqrt{2})e^{\frac{i\pi}{4}} & (b + \sqrt{2})e^{\frac{3i\pi}{4}} & (b + \sqrt{2})e^{-\frac{3i\pi}{4}} \\ 1 & -1 & 1 & ci & -ci & ci \\ 1 & -i & -1 & be^{\frac{3\pi i}{4}} & be^{\frac{9\pi i}{4}} & be^{\frac{15\pi i}{4}} \end{pmatrix}.$$

5.8. $5(1, 2, 4, 3)$ Bring's curve. In this section, we look at the genus four non-hyperelliptic Riemann surface associated to Kepler's small stellated dodecahedron. In [19], it is shown that the Riemann surface is a fivefold cyclic cover over a 4-punctured sphere defined by branching indices $(1, 2, 4, 3)$. Hence, our plane curve model is $y^5 - x(x-1)^2(x+1)^3$. It is also shown in [19] that it is biholomorphic to Bring's curve.

The period matrix of B , in its pure form, is calculated in [19] in Lemma 5.1.

FIGURE 10. $4(1, 3, 3, 1)$

5.9. $12(1, 4, 7)$ Schoen's I-WP Surface. In this section, we look at another genus four non-hyperelliptic surface. This surface appears in [14] as an abstract quotient of a triply periodic polyhedral surface where it is called the Octa-8 surface. It is also shown in [14] that this curve can be described as a threefold cyclic cover over a six-punctured sphere with branching indices $(1, 1, 1, 1, 1, 1)$. This map is the order-three Gauss map on the minimal surface in \mathbb{R}^3 . The branched points are located at the vertices of an octahedron hence the plane curve model is $y^3 - (x^5 - x)$. It is then shown that the cone metric yields a conformal structure that is compatible to Schoen's I-WP surface.

The base tessellation is tiled by twenty four $(\frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{6})$ -triangles. The branched points (under the twelvefold cyclic map) are Weierstrass points of weight four and the preimages of the midpoints of p_i and $p_{i'}$ are Weierstrass points of weight one. These are marked as \bullet and \circ , respectively in Figure 12(a).

The period matrix is

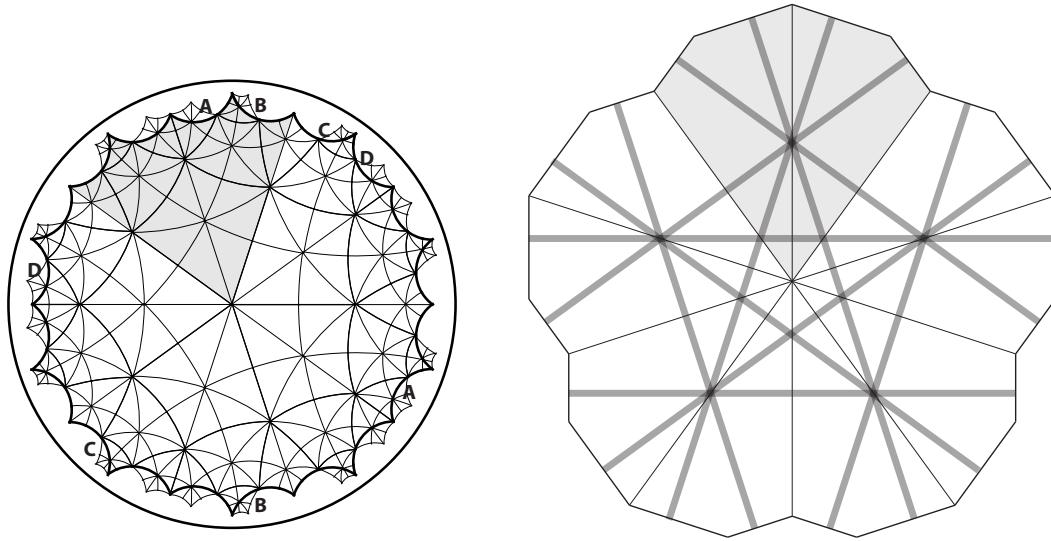


FIGURE 11. Bring's curve

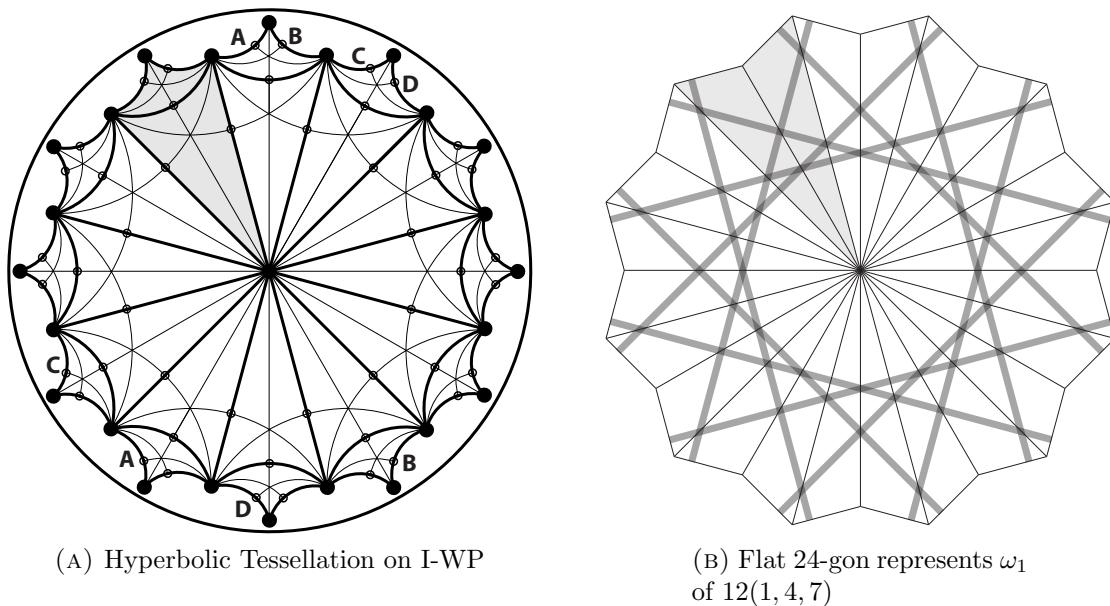


FIGURE 12. Schoen's I-WP surface

$$\Pi = \begin{pmatrix} 1 & e^{\frac{\pi i}{6}} & e^{\frac{\pi i}{3}} & i & e^{\frac{2\pi i}{3}} & e^{\frac{5\pi i}{6}} & -1 & e^{-\frac{5\pi i}{6}} \\ 1 & e^{\frac{\pi i}{3}} & e^{\frac{2\pi i}{3}} & -1 & e^{-\frac{2\pi i}{3}} & e^{-\frac{\pi i}{3}} & 1 & e^{\frac{\pi i}{3}} \\ 1 & e^{\frac{2\pi i}{3}} & e^{-\frac{2\pi i}{3}} & 1 & e^{\frac{2\pi i}{3}} & e^{-\frac{2\pi i}{3}} & 1 & e^{\frac{2\pi i}{3}} \\ 1 & e^{-\frac{5\pi i}{6}} & e^{\frac{\pi i}{3}} & -i & e^{\frac{2\pi i}{3}} & e^{-\frac{\pi i}{6}} & -1 & e^{\frac{\pi i}{6}} \end{pmatrix}$$

5.10. The Modular Curve $X_0(63)$. Recall that $SL_2(\mathbb{Z})$ acts transitively on the upper half plane \mathfrak{h} by $\tau \mapsto \frac{a\tau+b}{c\tau+d}$. We quotient the upper half plane by subgroups Γ of $SL_2(\mathbb{Z})$ and metrize the quotient, however, this yields non-compact Riemann surfaces. To get a compact Riemann surface, we consider the extended upper half plane $\mathfrak{h}^+ := \mathfrak{h} \cup \mathbb{R} \cup \{\infty\}$ as a subset of $\mathbb{C}P^1$.

We are most interested in quotients of the upper half plane \mathfrak{h}^+ by the following subgroups of $SL_2(\mathbb{Z})$. These subgroups come up naturally in the study of modular forms associated to elliptic curves.

Definition 16.

$$\begin{aligned}\Gamma_0(N) &:= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{N} \right\} \\ \Gamma_1(N) &:= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod{N} \right\} \\ \Gamma(N) &:= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}\end{aligned}$$

The automorphism groups of $X_0(N) := \mathfrak{h}^+/\Gamma_0(N)$ were calculated in [9] except for $N = 63$. The case of $X_0(63)$ was resolved 2 years later by Elkies in [3] by two different proofs: a conceptual one that uses enumerative geometry and the modular structure, and an explicit one that exhibits the modular equations.

Conjecture 3. *The program `CullPB.m` finds all principal polarizations on the curves we consider.*

If this conjecture is true, it would be a radically different proof than that of Elkies, since we approach by computing the automorphism group of the Jacobian of $X_0(63)$. Assuming this conjecture, we have

Theorem. $\text{Aut}(X_0(63)) \simeq S_4 \times \mathbb{Z}/2$

Proof. Note the following theorem from [9]:

Lemma. $\text{Aut}(X_0(63))$ is either $A_4 \times \mathbb{Z}/2$ or $S_4 \times \mathbb{Z}/2$.

Using the period matrix provided by Mascot, performed with 150 precisions, `autperio.sage` (using Conjecture 3) gives

Lemma. $\text{Aut}(X_0(63))$ is either C_2^4 or $S_4 \times \mathbb{Z}/2$.

Therefore, it must be that $\text{Aut}(X_0(63)) \simeq S_4 \times \mathbb{Z}/2$. □

The period matrix used in our calculation of $\text{Aut}(\text{Jac}(X_0(63)), p_i)$ was computed by Nicolas Mascot using an alteration of his personal code.

Remark. Mascot, in [17], discusses finding the period matrices for $X_1(N)$ by integrating cuspforms along modular symbols. His algorithm works for any compactified modular curve, but it works best when N is square-free. In the non-squarefree case, the coefficients in the q -expansions of the cuspforms and the j -invariant do not converge as quickly, thus they require more digits of precision.

Remark. In private correspondence, John Voight programmatically proved that $X_0(63)$ is not a cyclically branched cover of the sphere. Given that the genus of the quotient $X_0(63)/H$ is equal to the dimension of the H -invariant differentials, he shows that the list of dimensions of the space of H -invariant differentials on $X_0(63)$ (where H is a cyclic subgroup of $\text{Aut}(X_0(63))$) does not contain zero.

6. CLOSING REMARKS

The code used in this paper can all be found at <https://github.com/catherineray/aut-jac>.

We speak here of polarizations up to auto-equivalence and ask natural questions on Jacobians with multiple principle polarizations, answering all but one of the questions using methods developed in our paper.

We fix some notation. Let θ_C be the canonical principal polarization of $\text{Jac}(C)$ with respect to C . We call $\text{Aut}(A, a_i)$ a symplectic automorphism group of A , as the automorphisms respect the principal polarization a_i , which is a symplectic form on A .

Question. $\text{Aut}(\text{Jac}(C), \theta_C)$ will have the highest order of all symplectic automorphism groups of $\text{Jac}(C)$.

This is proven false by example $12(1, 5, 6)$, where $|\text{Aut}(\text{Jac}(12(1, 5, 6)), \theta_{12(1, 5, 6)})| = 24$, but $|\text{Aut}(\text{Jac}(12(1, 5, 6)), a_i)| = 32$ is achieved. It is more dramatically proven false by Schoen's I-WP Surface, where $|\text{Aut}(\text{Jac}(I\text{-WP}), \theta_{I\text{-WP}})| = 288$, but $|\text{Aut}(\text{Jac}(I\text{-WP}), a_i)|$ achieves 576 and 864.

Question. Principal polarizations p_1 and p_2 are auto-equivalent if and only if they are analytically equivalent. In other words,

$$\text{Aut}(X, p_1) \simeq \text{Aut}(X, p_2) \Leftrightarrow p_1 = p_2$$

The direction (\Leftarrow) is clear because \mathcal{L} and \mathcal{M} are analytically equivalent if and only if $c_1(\mathcal{L}) = c_1(\mathcal{M})$ by [[1] 2.5.3]. The other direction is false. This is proven false by the following two *non-isomorphic* curves with the same (unpolarized) Jacobian from Theorem 1 [5]:

$$X : 3y^2 = (2x^2 - 2)(16x^4 + 28x^2 + 1)$$

$$X' : -y^2 = (2x^2 + 2)(16x^4 + 12x^2 + 1)$$

which both have $\text{Aut}(\text{Jac}(X), \theta_X) \simeq C_2 \times C_2 \simeq \text{Aut}(\text{Jac}(X'), \theta_{X'})$.

Question. If $\text{Jac}(C) \simeq \text{Jac}(C')$ as complex varieties, then

$$\text{Aut}(\text{Jac}(C), \theta_C) \simeq \text{Aut}(\text{Jac}(C'), \theta_{C'})$$

We checked this question on the family of hyperelliptic cases of genus 2 from [5] Theorem 1, where it is true. However, there is no reason to expect this to be true in general.

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