AUTOMORPHISMS OF ABELIAN VARIETIES AND PRINCIPAL POLARIZATIONS

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Abstract. Abelian varieties with several principal polarizations are incredibly rare. The still unsolved Narasimhan-Nori conjecture asks for a closed formula for the number of non-isomorphic principal polarizations of any given abelian variety. We give lower bounds on the number of principal polarizations on any given abelian variety by introducing a new computer program. We show, for example, that the Jacobian of Schoen's Wrapping Package surface, from minimal surface theory, has at least 9 non-isomorphic principal polarizations. We also explore the Jacobians of Klein's quartic, Fermat's quartic, Bring's curve, the modular curve $X_0(63)$, and more.

1. Introduction

Our paper introduces a new computer program to find multiple non-isomorphic principal polarizations on abelian varieties in characteristic 0, the code for which is publically available. In general, abelian varieties with several principal polarizations are incredibly rare. In fact, abelian varieties with no principal polarizations are, in some sense, dense on the moduli stack of abelian varieties.

The fact that an abelian variety admits only a finite number of isomorphism classes of principal polarizations was established by Narasimhan-Nori in [17]. In this paper, they pose the problem of finding a closed formula for the number of principal polarizations of any given abelian variety over any field, which is still unsolved.

In this paper, we introduce an entirely different technique to find different principal polarizations on Jacobians, exposited in Section 4.5, which treats both simple and non-simple cases in characteristic 0. We use this to give lower bounds on the number of non-isomorphic principal polarizations. For example, we have the following result.

Theorem 1. *Let $\pi(X)$ denote the number of non-isomorphic principal polarizations on any given variety X. Let I-WP denote Schoen's I-WP surface, then*

$$\pi(\textit{I-WP}) \geqslant 9.$$

This is a charming result, especially since the variety Jac(I-WP) itself factors into a product of 4 elliptic curves[1], so the remaining principal polarizations must come from interesting new cycles in the product of these elliptic curves. Other such surprising results found by applying our technique are shown in Table 1, Section 5.1.

Our method works as follows. Given any period matrix, we introduce new code to compute many principal polarizations on the corresponding abelian variety. We then

[1] This is because $\mathrm{End}(\mathrm{Jac(I\text{-}WP)}) \simeq M_4(K)$, where K is imaginary quadratic. Thus, Jac(I-WP) is the product of elliptic curves with CM by K.

implement a modification of the psuedocode of Bruin-Jeroen-Sijsling [3] to compute, for each found principal polarization, the automorphism group of the given Jacobian which fixes that polarization. If the automorphism groups are different, then the principal polarizations are non-isomorphic. This gives us a lower bound on the number of different principal polarizations for a given Jacobian. The fact that numerical computations with period matrices, which we use for surfaces which are not cyclically branched covers of the punctured sphere, give rigorous results relies on the brilliant work of Costa-Mascot-Sijsling-Voight [2] (see Prop 6.1.1).

We demonstrate the power of our method with a variety of different period matrices. We exposit some low dimensional geometry which allows us to compute exact period matrices of cyclically branched covers of punctured spheres. We also work with period matrices of modular curves, as codified by Mascot in a slight modification of [16] toward computing Galois representations. These modular curves are not cyclically branched covers, and thus give us a completely disjoint collection of curves to which we apply our methods. Assuming that our program finds *all* principal polarizations of any given abelian variety, we give a new proof of the automorphism group of $X_0(63)$, which was the only remaining unresolved automorphism group of a compact modular curve until [5].

2. Summary of Prior Work

We summarize what is known about abelian varieties with several principal polarizations. The previous work toward the Narasimhan-Nori conjecture can be divided into the simple and non-simple cases (in which we can further indicate char 0 or char p cases). An abelian variety is simple if it is not isogenous to a product of abelian varieties of lower dimension.

A technique developed by Lange first for *simple* varieties in characteristic 0. In [11] Theorem 1.5, Lange establishes for simple varieties that the order of $\mathrm{Aut}(A)$ with certain restriction conditions and equivalence relations is equal to the number of principal polarizations on A up to isomorphism, $\pi(A)$. One could in principal compute the size of this specially carved out version of $\mathrm{Aut}(A)$ by hand using Lange's theorem, however, it is computationally infeasible. This is incredible, it gives us a a set equivalence between a slighly carved automorphism group of a variety, and its set of principal polarizations.

Remark. In Theorem 3.1, Lange further establishes bounds on $\pi(A)$ in terms of the class group of $End_{\mathbb{Q}}(A)$, if $End_{\mathbb{Q}}(A)$ is a *totally real* number field K (and thus the variety A is simple).

More recently, Lange treated the non-simple case of products of elliptic curves without complex multiplication in Theorem 3.5 [12]. He did so by giving an interpretation of the number of principal polarizations in terms of class numbers of definite Hermitian forms.

All other previous works known to the authors on finding multiple principal polarizations on abelian varieties have been done by finding two non-isomorphic curves with the same (unpolarized) Jacobian. Therefore, their associated canonical polarizations must be different by the Torelli theorem. Otherwise, the curves would be isomorphic. All papers that the we know of using this technique do so only in characteristic p.

Remark. There is only a canonical principal polarization on A if a curve C is specified so that $A = \mathrm{Jac}(C)$. If C is not specified, there is no canonical choice – knowing that A is in the image of the functor Jac is not enough. Thus, there can be several "canonical" principal polarizations on one Jacobian, its canonicality only refers to the fact that it comes from a curve.

The papers using this non-isomorphic curve technique discuss the case of *nonsimple* Jacobians of curves of genus two [8] and three [Brock, *Superspecial curves of genera two and three*], though we were unable to find a copy of the latter.

This technique is again used by E. Howe [6] and [7] which gives examples of non-isomorphic genus two curves with the same *simple* Jacobian. He finds such examples in characteristic p by playing with isogeny classes of abelian varieties which correspond to special Weil numbers. This summarizes all previous work known to the authors.

3. Background

In [14], Lee classifies curves with cone metrics that are realizable as a quotient of a triply periodic surface embedded in $\mathbb{R}^3$. By triply periodic, we mean that the surface is invariant under a rank-three lattice $\Lambda \subset \mathbb{R}^3$. Specifically, they consider curves that are cyclically branched over $\mathbb{CP}^1$, hence we will devote this section to providing the background. This section is a summary of chapter 3 from [14].

3.1. Cone metrics.

First, we discuss the topological construction of cyclic covers over $\mathbb{CP}^1$. This will naturally yield cone metrics on the curves.

Construction of cyclic covers over $\mathbb{CP}^1$.

Definition 1. We say that a curve C is a d-fold cyclic cover over $\mathbb{CP}^1$ if $C/(\mathbb{Z}/d\mathbb{Z}) = \mathbb{CP}^1$.

We construct such curves with the given data: let $p_i, \dots, p_n \in \mathbb{CP}^1$ be n distinct points. Let $Y := \mathbb{CP}^1 \backslash \{p_1, \dots, p_n\}$ and let γ_i be a branch cut from p_i to some $q \in Y$ so that γ_i are mutually disjoint. For each i, assign $d_i \in \{1, \dots, d-1\}$ and call it the **branching index at** p_i. Let d be the degree of the covering map and use j to label $Y_1, \dots, Y_d$. For each i and j, we identify the "left side" of γ_i of Y_j to the "right side" of γ_i of $Y_{j+d_i \pmod d}$. We denote such a covering C by a d-tuple $d(d_1, \dots, d_n)$.

Remark. A covering $d(d_1, \dots, d_n)$ is uniquely defined up to homeomorphism. That is, the construction only depends on d_i and is independent of p_i, γ_i, and q. We will assume that $\sum\limits_{i=1}^{n} d_i \equiv 0 \pmod d$ and also $\gcd(d_1, \dots, d_n) = 1$. The former guarantees that the covering is closed and the latter guarantees that the covering is connected. In fact, both are sufficient and necessary conditions. Then one can compute the genus of the curve by Riemann-Hurwitz formula and $g(C) = \frac{d(n-2)}{2} + 1 - \frac{1}{2}\sum\limits_{i=1}^{n} \gcd(d, d_i)$.

Branching indices on Octa-4. As a running example, we will look at the curve from [13]. It is there called Octa-4 due to the formation of the triply periodic surface. The underlying genus three curve C (Figure 1 (reprinted from [13])) is invariant under an order-eight rotational symmetry. The curve is an eightfold cyclic cover over $\mathbb{CP}^1$ denoted as $8(1, 2, 5)$.

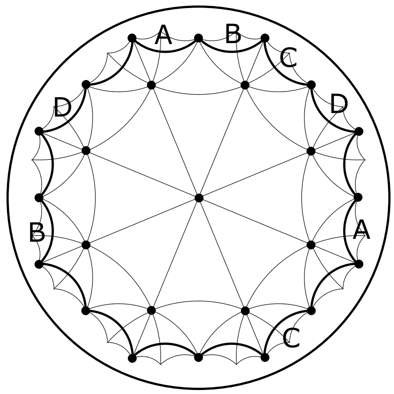

FIGURE 1. Hyperbolic tessellation on $8(1, 2, 5)$

For each i, there are $\gcd(d, d_i)$ preimages $\widetilde{p_i}$ of p_i on C. Hence, each $\widetilde{p_i}$ is a non-trivial cone point. To pin down a holomorphic 1-form on C, we find cone metrics on $\mathbb{CP}^1$ and pull-back to its covering. We say that a cone metric on $\mathbb{CP}^1$ is **admissible** if its pullback yields a flat structure on C. The following proposition is a version of Gauss-Bonnet theorem.

Proposition. *Given a compact Riemann surface of genus g with a cone metric, let $p_1, \dots, p_n$ be distinguished points with respective cone angles θ_i. Then* $\sum_{i=1}^{n} \theta_i = 2\pi(2g - 2 + n)$.

Specifically, the sum of cone angles on a genus zero curve is $2\pi(n-2)$. Given branching indices such that $\sum_{i=1}^{n} d_i = d(n-2)$, we get a cone metric on the Riemann sphere with cone angles $\frac{2\pi d_i}{d}$ at each p_i. For example, by putting a cone metric on the quotient sphere where the cone angles are $\frac{1\pi}{4}$, $\frac{2\pi}{4}$, and $\frac{5\pi}{4}$ as in Figure 2, one can see that the identification of edges are by translations. In other words, this gives rise to a translation structure on the eightfold cover of the sphere. Moreover, we obtain a holomorphic 1-form with one order-4 zero.

Our goal is to find other admissible cone metrics on the sphere that yield different translation structures (linearly independent 1-forms) on the same curve. We claim that the cone metric given by cone angles $\frac{2\pi a_i}{d}$ where $a_i \equiv d_i \pmod{d}$ is admissible (Theorem 3.4, [14]). An easier way of finding admissible cone metrics is given by the following notion of multipliers.

Definition 2. Given branching indices $d(d_1, \dots, d_n)$, we say $a \in \{1, \dots, d-1\}$ is a **multiplier** if the cone metric given by cone angles $\frac{2\pi}{d}(a \cdot d_1 \pmod{d}, \dots, a \cdot d_n \pmod{d})$ is admissible.

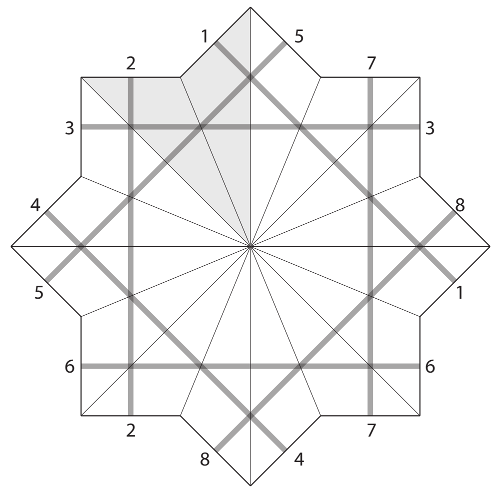

FIGURE 2. A flat structure ω_1 on $8(1,2,5)$

Theorem 3.5, [14] proves that for $n = 3$, there are exactly g multipliers. In other words, one achieves a basis of holomorphic 1-forms.

Admissible cone metrics on Octa-4. Given branching indices $8(1,2,5)$, multipliers 1, 2, and 5 give rise to cone metrics with cone angles $\frac{2\pi}{8}(1,2,5)$, $\frac{2\pi}{8}(2,4,2)$, and $\frac{2\pi}{8}(5,2,1)$, respectively. These cone metrics yield a basis of holomorphic 1-forms with the following divisors:

$$(\omega_1) = 4\widetilde{p_3}, \qquad (\omega_2) = \widetilde{p_1} + \widetilde{p_{2_1}} + \widetilde{p_{2_2}} + \widetilde{p_3}, \qquad (\omega_3) = 4\widetilde{p_1}.$$

Figure 3 represents ω_2 given by cone angles $\frac{2\pi}{8}(2,4,2)$. The four simple zeros are marked on the figure.

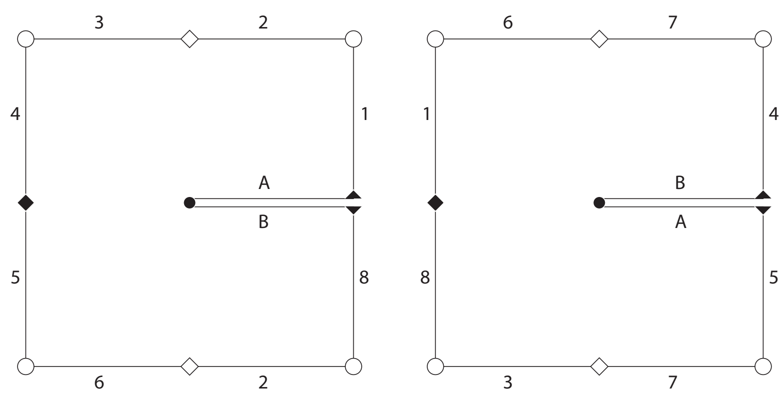

FIGURE 3. A flat structure ω_2 on $8(1,2,5)$

Remark. Note that the multipliers preserve the labeling of edges in both Figure 2 and Figure 3: 1,2,3,4,5,6,2,8,4,7,6,1,8,3,7,5.

Remark. Note that 3, 6, and 7 are not multipliers as their corresponding forms do not yield canonical divisors. On the other hand, 4 is not a multiplier as the cone metric derived from cone angles $\frac{2\pi}{8}(4,0,4)$ is not admissible. This induces a meromorphic form whose divisor is $3\widetilde{p_1} - \widetilde{p}_{21} - \widetilde{p}_{22} + 3\widetilde{p_3}$.

Remark. The geometric representation of ω_3 can also be achieved by reflecting $\frac{\pi}{8}(5,2,1)$ triangles. However, we omit the corresponding figure.

3.2. **Computing the Period Matrix on Cyclic Covers.** In this section, we use the flat structure of a surface to compute the period matrix of a given surface. We will look at the simplest case where $n=3$ and $d_1=1$. Then, since $\sum d_i = d$, a cone metric with cone angles $\frac{2\pi}{d}(d_1,d_2,d_3)$ is admissible. Y is topologically equivalent to a doubled triangle with angles $\frac{2\pi}{d}(d_1,d_2,d_3)$ so we construct X with d copies of Y, which yields a flat structure on X. We will follow the underlying surface of Octa-4 as our leading example.

Figure 2 shows the flat structures on the underlying surface of Octa-4 [13]. The identification of edges are via parallel translations, which verifies that the cone metric is admissible. Parallel translations yield closed cycles on the surface from which we get a homology basis with the following intersection matrix

$$\mathrm{int}_1 = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 & 1 & 1 \\ 0 & 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & -1 & 0 \end{pmatrix}.$$

Then the period matrix is computed as follows:

$$\Pi_1 = \begin{pmatrix} 1 & e^{\frac{\pi i}{4}} & i & e^{\frac{3\pi i}{4}} & -1 & e^{-\frac{3\pi i}{4}} \\ 1 & i & -1 & -i & 1 & i \\ 1 & e^{-\frac{3\pi i}{4}} & i & e^{-\frac{\pi i}{4}} & -1 & e^{\frac{\pi i}{4}} \end{pmatrix}.$$

In general, from cyclicity we get

$$\Pi = \begin{pmatrix} 1 & e^{\frac{2\alpha_1 \pi i}{d}} & e^{\frac{4\alpha_1 \pi i}{d}} & \cdots & e^{\frac{2(2g-1)\alpha_1 \pi i}{d}} \\ 1 & e^{\frac{2\alpha_2 \pi i}{d}} & e^{\frac{4\alpha_2 \pi i}{d}} & \cdots & e^{\frac{2(2g-1)\alpha_2 \pi i}{d}} \\ \vdots & & & & \\ 1 & e^{\frac{2\alpha_g \pi i}{d}} & e^{\frac{4\alpha_g \pi i}{d}} & \cdots & e^{\frac{2(2g-1)\alpha_g \pi i}{d}} \end{pmatrix}$$

where α_i are the multipliers.

Remark. In [14], the first author computes the period matrix by choosing a different homology basis. The closed cycles are images of the "handles" on the polyhedral surface embedded in $\mathbb{R}^3$. This yields the following intersection matrix

$$\mathrm{int}_2 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}$$

and the following period matrix

$$\Pi_2 = (A|B) = \begin{pmatrix} 1-i & -\frac{1+i}{1+\sqrt{2}} & \frac{1+i}{1+\sqrt{2}} & 1+i & \sqrt{2} & 2-\sqrt{2} \\ -2i & 2i & 2i & 2i & -2 & -2 \\ -1-i & (1-i)(1+\sqrt{2}) & (-1+i)(1+\sqrt{2}) & 1-i & i\sqrt{2} & i(-2-\sqrt{2}) \end{pmatrix}$$

and

$$\tau = (A^{-1}B) = \begin{pmatrix} i & \frac{1+i}{2} & \frac{1+i}{2} \\ \frac{1+i}{2} & i & \frac{1+i}{2} \\ \frac{1+i}{2} & \frac{1+i}{2} & i \end{pmatrix}.$$

3.3. **A Non Cyclic Cover: The Modular Curve $X_0(63)$.** Recall that $SL_2(\mathbb{Z})$ acts transitively on the upper half plane $\mathfrak{h}$ by $\tau \mapsto \frac{a\tau+b}{c\tau+d}$. We quotient the upper half plane by subgroups Γ of $SL_2(\mathbb{Z})$ and metrize the quotient, however, this yields non-compact Riemann surfaces. To get a compact Riemann surface, we consider the extended upper half plane $\mathfrak{h}^+ := \mathfrak{h} \cup \mathbb{R} \cup \{\infty\}$ as a subset of $\mathbb{C}P^1$.

We are most interested in quotients of the upper half plane $\mathfrak{h}^+$ by the following subgroups of $SL_2(\mathbb{Z})$. These subgroups come up naturally in the study of modular forms associated to elliptic curves.

Definition 3.

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \;\middle|\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \mod N \right\}$$
$$\Gamma_1(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \;\middle|\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \mod N \right\}$$
$$\Gamma(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \;\middle|\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod N \right\}$$

The automorphism groups of $X_0(N) := \mathfrak{h}^+/\Gamma_0(N)$ were calculated in [10] except for $N = 63$. The case of $X_0(63)$ was resolved 2 years later by Elkies in [5] by two different proofs: a conceptual one that uses enumerative geometry and the modular structure, and an explicit one that exhibits the modular equations. Our method would work for any N, we exposit the case of $N = 63$ due to its late blooming history.

Conjecture 2. *The program `CullPB.m` finds all principal polarizations on the curves we consider.*

If this conjecture is true, it would be a radically different proof than that of Elkies, since we approach by computing the automorphism group of the Jacobian of $X_0(63)$. Assuming this conjecture, we have

Theorem. $\mathrm{Aut}(X_0(63)) \simeq S_4 \times \mathbb{Z}/2$

Proof. Note the following theorem from [10]:

Lemma. $\mathrm{Aut}(X_0(63))$ *is either* $A_4 \times \mathbb{Z}/2$ *or* $S_4 \times \mathbb{Z}/2$.

Using the period matrix provided by Mascot, performed with 150 precisions, `autperio.sage` (using Conjecture 2) gives

Lemma. $\mathrm{Aut}(X_0(63))$ *is either* ${C_2}^4$ *or* $S_4 \times \mathbb{Z}/2$.

Therefore, it must be that $\mathrm{Aut}(X_0(63)) \simeq S_4 \times \mathbb{Z}/2$. □

The period matrix used in our calculation of $\mathrm{Aut}(\mathrm{Jac}(X_0(63)), p_i)$ was computed by Nicolas Mascot using an alteration of his personal code.

Remark. Mascot, in [16], discusses finding the period matrices for $X_1(N)$ by integrating cuspforms along modular symbols. His algorithm works for any compactified modular curve, but it works best when N is square-free. In the non-squarefree case, the coefficients in the q-expansions of the cuspforms and the j-invariant do not converge as quickly, thus they require more digits of precision.

Remark. In private correspondence, John Voight programmatically proved that $X_0(63)$ is not a cyclically branched cover of the sphere. Given that the genus of the quotient $X_0(63)/H$ is equal to the dimension of the H-invariant differentials, he shows that the list of dimensions of the space of H-invariant differentials on $X_0(63)$ (where H is a cyclic subgroup of $\mathrm{Aut}(X_0(63))$) does not contain zero.

4. Programmatically Computing the Automorphism Group of Plane Curves and Abelian Varieties over $\mathbb{C}$

Remark. The first two subsections of the section are copied from section 4 of Bruin-Sijsling-Zotine [3] with lots of exposition and examples added for the readers' convenience. This sets us up to introduce the code for brute force calculating principal polarizations. All of our code is available at

`https://github.com/catherineray/aut-jac`

We discuss the certification of these numerical results in Section 4.3.

Let us examine abelian varieties represented as analytic groups $X := V/\Lambda$ and $X' := V'/\Lambda'$. They need not be Jacobians.

Remark. We slightly abuse notation here, Λ represents both a matrix in $M_{g\times 2g}(\mathbb{Z})$, and the $2g$-integral-dimensional (i.e., g-complex-dimensional) lattice in $\mathbb{C}^g$ generated by the columns of that matrix.

Theorem (BL 1.2.1)**.** *Let* $X := V/\Lambda$ *and* $X' := V'/\Lambda'$ *be abelian varieties. Under addition the set of homomorphisms* $\mathrm{Hom}(X, X')$ *forms an abelian group. There is an injective homomorphism of abelian groups:*

$$\rho : \mathrm{Hom}(X, X') \to \mathrm{Hom}(V, V')$$
$$f \mapsto F$$

The restriction to the lattice Λ is $\mathbb{Z}$-linear, thus we get an injective homomorphism:

$$\rho|_\Lambda : \mathrm{Hom}(X, X') \to \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \Lambda')$$
$$f \mapsto F|_\Lambda$$

We will namely use the representation $\rho|_\Lambda$ and find the basis of our set of maps in terms of this representation.

We work in the category of varieties equipped with principal polarizations, which we discuss in section 4.4. In this category, morphisms are morphisms of pairs. That is,

$$f : (X, c_1(\mathcal{L})) \to (Y, c_1(\mathcal{M}))$$

such that $f^*(Y, c_1(\mathcal{M})) = (X, c_1(\mathcal{L}))$ (for isomorphisms). We may represent polarizations as integral valued alternating forms.

Definition 4. Let a be a polarization of X. We call $\mathrm{Aut}(X, a)$ a **symplectic automorphism group** of X, as it respects the symplectic form a.

Let E_1 and E_2 be forms representing $c_1(\mathcal{L}_1)$ and $c_1(\mathcal{L}_2)$, respectively. Note that a map $\alpha : (X_1, c_1(\mathcal{L}_1)) \to (X_2, c_1(\mathcal{L}_2))$ such that

$$\alpha^*(c_1(\mathcal{L}_2)) = c_1(\mathcal{L}_1)$$

is equivalent to a matrix $R \in M_{2g}(\mathbb{Z})$ in the image of $\rho|_\Lambda$ such that

$$R^t E_2 R = E_1$$

4.1. **Computing the Automorphism Group of Plane Curves.**

Remark. This section is on the algorithm used in `autplane.sage`.

In the case that our abelian variety is of the form $\mathrm{Jac}(C_i) =: J_i$, and we know the curve C_i, there is a special principal polarization E_i with respect to that curve C_i. This is programmatically found using Lemma 2.6 from [3].

Algorithm: Compute the set of isomorphisms between curves.
Input: Planar equations f_1, f_2 for curves C_1, C_2.
Output: The set of isomorphisms $C_1 \to C_2$, or the group $\mathrm{Aut}(C)$ if $C_1 = C_2$.

(1) Check if $g(C_1) = g(C_2)$; if not, return the empty set.
(2) Check if C_1 and C_2 are hyperelliptic; if so, use the methods in [15].
(3) Determine the period matrices Π_1, Π_2 of C_1, C_2 to the given precision.
(4) Determine a $\mathbb{Z}$-basis of $\mathrm{Hom}(J_1, J_2) \subset M_{2g\times 2g}(\mathbb{Z})$ represented by integral matrices $R \in M_{2g\times 2g}(\mathbb{Z})$. [Lemma 4.3 [3]]

(5) Using Fincke-Pohst[2], determine the finite set [from 5.1.9 BL]
$$S = \{R \in \mathrm{Hom}(J_1, J_2) \mid \mathrm{tr}((E_1^{-1} R^t E_2) R) = 2g\}$$
(6) Return the subset[3] of $R \in S$ which further satisfies $R^t E_2 R = E_1$. (These are the symplectic endomorphisms.)
(7) Look at the subset of R such that $\det(R) = \pm 1$. These are the symplectic automorphisms.
(8) If $J_1 = J_2$, find the group structure of this subset.

Note that if the curves C_1 and C_2 are non-hyperelliptic, by the precise Torelli theorem, we get $\mathrm{Hom}((J_1, E_1), (J_2, E_2)) \simeq \mathrm{Hom}(C_1, C_2) \sqcup \{\pm 1\}$ from this algorithm. So, we must remove the direct summand $\{\pm 1\}$.

Remark. Step 8 of the above algorithm was added by Ray to tame these unwieldy matrix groups, and is achieved as follows.

Algorithm: Compute the group structure of an underlying set of matrices.
Input: A set of matrices which are a group by multiplication.
Output: The group structure of the set.

(1) Check cardinality of the set. Call this N.
(2) Take first 15 elements of the set, use GAP to check if these generate a matrix group G of the correct order N. If not, it generates a group of order K, where $KM = N$. Take more elements of order dividing M until they generate a group of the correct order.
(3) Use `IdGroup(G)` in GAP.

4.2. **Computing the Automorphism Group of Abelian Varieties.**

Remark. This section is on the algorithm used in `autperio.sage`

Notation. Let $A := V/\Lambda$ be an abelian variety of dimension g. Let $e_1, ..., e_g$ be the chosen basis for V, and $\lambda_1, ..., \lambda_{2g}$ be a corresponding chosen basis for Λ. Let Π be the corresponding period matrix such that $A := \mathbb{C}^g/\Pi\mathbb{Z}^{2g}$.

Algorithm: Compute the group of isomorphisms between abelian varieties.
Input: Period matrices of abelian varieties J_1 and J_2, as Π_1 and Π_2 respectively.
Output: For each combination of principal polarizations (a_i, b_j), the set of isomorphisms between (J_1, a_i) and (J_2, b_j) (or the group, if they coincide).

(1) Check if $g_1 = g_2$; if not, return the empty set.
(2) Determine a $\mathbb{Z}$-basis of $\mathrm{Hom}(J_1, J_2) \subset M_{2g \times 2g}(\mathbb{Z})$ represented by integral matrices $R \in M_{2g \times 2g}(\mathbb{Z})$.
(3) Find many principal polarizations $\{a_i\}$ and $\{b_j\}$ for J_1 and J_2 respectively using `CullPB` (exposited in the next section).

[2]This is an algorithm for finding vectors of small norm. We use it here to solve for the finite set of solutions $R = \sum_{i=1}^{2g} \lambda_i B_i$, where B is the basis from step 4.

[3]The condition $R^t E_2 R = E_1$ (i.e., $E_1^{-1} R^t E_2 R = \mathrm{Id}$) implies that $\mathrm{tr}((E_1^{-1} R^t E_2) R) = 2g$. So we first solve for the latter to thin the results, then solve for the former from that set.

(4) Apply steps 5-8 of the previous section substituting each pair (a_i, b_j) for (E_1, E_2). For each pair, this will produce the set of isomorphisms between (J_1, a_i) and (J_2, b_j).
(5) If $(J_1, a_i) = (J_2, b_i)$, find the group structure of each set $\mathrm{Aut}(J_1, a_i)$ (using the algorithm in the previous section).

4.3. **Certifying Heuristic Methods.** We must certify that numerically computed endomorphisms of a Jacobian are infact endomorphisms of that Jacobian. This is extremely nontrivial because in an analytic sense, abelian varieties with no principal polarizations are dense in the moduli space.

We have two distinct cases, exact period matrices, and numerical ones. The period matrices associated to cyclically branched covers of punctured spheres via the method described in Section 3.2 are exact, as are the period matrices of varieties with complex multiplication.

If the entries of the period matrices are exact and algebraic over $\mathbb{Q}$, this extra certification step is unneccesary. We need only check that putative endomorphisms are correct via a simple linear-algebraic verification. This amounts to given a period matrix $\Pi \in M_{g,2g}(\overline{\mathbb{Q}})$ associated to an abelian variety C^g/Λ, find

$$M\Pi = \Pi R,$$

where $R \in M_{2g}(\overline{\mathbb{Q}})$ and $M \in M_g(\overline{\mathbb{Q}})$. We then know that R is an endomorphism of Λ and M is a representation of the endomorphism of the abelian variety on $\mathbb{C}^g$, the tangent space around the origin.

If the entries of the period matrix is are non-exact, then to certify that such numerical M and R found above are exact and not putative, we must use the fantastic work of Costa-Mascot-Sijsling-Voight [2], which we now quickly summarize.

A key conceptual part of their approach is relating prime divisors on $X \times Y$ to endomorphisms of the Jacobians $\mathrm{Jac}(X)$ and $\mathrm{Jac}(Y)$ via the theory of correspondences. This is exposited fully in Section 3.3 of [4]. All endomorphisms of Jacobians come from correspondences associated to prime divisors, i.e., prime correspondences (Lemma 3.3.11 [4]). Each correspondece on $X \times Y$ induces a homomorphism of the divisor groups of X and Y. These homomorphisms in turn induce homomorphisms of Jacobians. In our case, $X = Y$, but we will exposit the more general case. Let C be a prime divisor on $X \times Y$, then we denote the associated correspondence as follows.

$$\begin{tikzcd} & X \times Y \arrow[dl, "\pi_1^C"'] \arrow[dr, "\pi_2^C"] & \\ X & & Y \end{tikzcd}$$

$$\begin{tikzcd} & \mathrm{Div}(C) \arrow[dr, "(\pi_2^C)_*"] & \\ \mathrm{Div}(X) \arrow[ur, "(\pi_1^C)^*"] \arrow[rr, dashed] & & \mathrm{Div}(Y) \end{tikzcd}$$

$$\begin{tikzcd} & \mathrm{Jac}(C) \arrow[dr, "(\pi_2^C)_*"] & \\ \mathrm{Jac}(X) \arrow[ur, "(\pi_1^C)^*"] \arrow[rr, dashed] & & \mathrm{Jac}(Y) \end{tikzcd}$$

A putative endomorphism is α a potential element of $\mathrm{Aut}(\mathrm{Jac}(X))$ we compute numerically as a matrix R via the methods in Section 4 . Their method of certifying putative endomorphisms are actual one is along the lines of the following. We go through the map α_X, defined as the composite of the Abel-Jacobi Map, the putative morphism α, and the Mumford birational map. These maps are discussed in detail on page 6 of [3].

$$\begin{tikzcd} X \arrow[r, "AJ"] \arrow[rrr, bend right, "\alpha_X"] & \mathrm{Jac}(X) \arrow[r, "\alpha"] & \mathrm{Jac}(X) \arrow[r, dashed, "\mathrm{Mum}"] & \mathrm{Sym}^g(X). \end{tikzcd}$$

Their method is then along the lines of the following.

- Given α, we now use the Puiseux lifting described in Section 5 [2] to write down the image $\{\tilde{Q}_j\}_j$ of any $x \in X$ under the above composition M_α.
- We then check if there is a divisor E which fits the points $\{\tilde{Q}_j\}_j$. If so, we then modify the divisor E by removing all of its nilpotent components, we call this new divisor Y. This ensures that Y is a prime divisor, and which confirms that α is an actual endomorphism of J. This is because of the above bijection between prime correspondences $X \leftarrow X \times X \rightarrow X$, and endomorphisms of $\mathrm{Jac}(X)$.
- If we indeed find such a Y, this ensures that α is an honest endomorphism.

4.3.1. *From Simple to Non-simple.* While we are on the topic of correspondences, we wish to make a remark on how to use techniques for simple varieties to bound below the number of principal polarizations of nonsimple varieties. Our program does not use this, but it is useful for context.

Remark. Since every abelian variety is isogenous to a product of simple abelian varieties

$$A \simeq A_1 \times ... \times A_k,$$

it is reasonable to ask how the numbers of principal polarizations on A_i are related to that of A.

Let's quickly establish some vocabulary to discuss this intuitively. Recall that we may also define a principal polarization on A as an isogeny which is also an isomorphism between $A \to A^\vee$, where $A^\vee$ denotes the dual variety. Let A and B be arbitrary abelian varieties. Note that $\mathrm{Corr}(A, B) \simeq \mathrm{Hom}(B, A^\vee)$, where we take a correspondence from A to B to be a line bundle $\mathcal{L}$ over the product $A \times B$ which is trivial when restricted to A or B.

We are interested in $\mathrm{Aut}(A, A^\vee)$, which is isomorphic to $\mathrm{Corr}(A, A)^\times$ but the problem of comparision arises immediately and obviously without having to pass to isomorphisms. We wish to compare

$$\mathrm{Corr}\left(\prod_{j=1}^{k} A_j, \prod_{i=1}^{k} A_i\right) \qquad \text{and} \qquad \prod_{i,j} \mathrm{Corr}(A_j, A_i).$$

Let C and D be abelian varieties. Given a line bundle on C and on D, we get a line bundle on $C \times D$, but not vice-versa. Intuitively, the product $C \times D$ may have many more interesting cycles than the product of the cycles of C and D, and may not necessarily restrict to a line bundle on C or D. Therefore, in general the number of principal polarizations of A is at least the product of the principal polarizations of the simple components A_i, that is,

$$\pi(A_1 \times ... \times A_k) \geqslant \prod_{i=1}^{k} \pi(A_i)$$

as observed. For example, Jac(I-WP) is isogenous to a product of 4 elliptic curves. Since these curves are all isomorphic, we only get one principal polarization from this decomposition, but we found at least 9 principal polarizations on their product.

4.4. **Introduction to Polarizations: From Theory to Code.** The notion of a polarization of an abelian variety has many faces. If a complex torus has a polarization, it is an abelian variety.

Definition 5. A **polarization** of a complex torus X is an embedding $j : X \to \mathbb{P}^N$ for large enough N.

We can understand this embedding j as a map

$$p \mapsto [a_1(p) : \cdots : a_{N-1}(p)]$$

where a_i are a chosen generating set of global sections of a line bundle $\mathcal{L}$ on X.

Definition 6. A line bundle $\mathcal{L}$ is defined to be **very ample** on X if it defines a closed embedding into $\mathbb{P}^N$ for large enough N.

Definition 7. A line bundle is **ample** if a tensor power of the line bundle is very ample. Since the Chern class is additive, $c_1(\mathcal{L}^{\otimes k}) = kc_1(\mathcal{L})$, the ample bundle and its tensor power are equivalent datum.

Remark. In other words, $\mathcal{L}$ is defined to be ample if it (or a tensor power of it) specifies an embedding of X into projective space.

Definition 8. Line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ on X are **analytically equivalent** if there is a connected complex analytic space T, a line bundle $\mathcal{L}$ on $X \times T$, and points $t_1, t_2 \in T$ such that

$$\mathcal{L}|_{X \times \{t_i\}} \simeq \mathcal{L}_i$$

for $i = 1, 2$.

A line bundle $\mathcal{L}$ over X is specified up to analytic equivalence by its first Chern class $c_1(\mathcal{L}) \in H^2(X; \mathbb{Z})$. More precisely,

Theorem (2.5.3 BL)**.** *Let X be an abelian variety. For line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ over X, the following statements are equivalent:*

(1) $\mathcal{L}_1$ *and* $\mathcal{L}_2$ *are analytically equivalent.*
(2) $c_1(\mathcal{L}_1) = c_1(\mathcal{L}_2)$

Definition 9. The **first Chern class** of a line bundle $\mathcal{L}$ is the image of $\mathcal{L} \in \mathrm{Pic}(X) = H^1(\mathcal{O}_X^*)$ under the map c_1 on cohomology which arises as follows. Consider the exact sequence

$$0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0$$

and its long cohomology sequence:

$$\cdots \to H^1(\mathcal{O}_X^*) \xrightarrow{c_1} H^2(X, \mathbb{Z}) \to \cdots$$

We associate to every first Chern class an alternating form.

Theorem (BL 1.3.2 & 2.1.2)**.**

$$\psi : H^2(X; \mathbb{Z}) \simeq \mathrm{Alt}^2(\Lambda, \mathbb{Z})$$

Let S be the set of $c_1(\mathcal{L})$ where $\mathcal{L}$ ranges over all holomorphic line bundles on X. The image $\psi(S)$ is isomorphic to all Hermitian alternating forms.

Theorem (BL 2.1.6)**.** *Let* $X := V/\Lambda$ *be an abelian variety. For an alternating form* $E : V \times V \to \mathbb{R}$*, the following conditions are equivalent:*
(1) *There is a holomorphic line bundle* $\mathcal{L}$ *on* X *such that* $\psi(c_1(\mathcal{L})) = E$*.*
(2) $E(\Lambda, \Lambda) \subseteq \mathbb{Z}$*, and*

$$E(iv, iw) = E(v, w)$$

Remark. Note that from each element $\mathrm{Alt}^2(\Lambda, \mathbb{Z})$ we obtain via $\mathbb{R}$-linear extension an alternating form $\mathrm{Alt}^2(V, \mathbb{R})$ (as in rational versus analytic representation, see [BL 1.2.1]). We also have an isomorphism between real valued forms satisfying 2.1.6(2) and Hermitian forms.

It is important to emphasize that not all forms satisfying 2.1.6(2) correspond to Chern classes of ample line bundles. Ampleness is stronger than holomorphicity, hence we need a stronger condition.

Definition 10. A line bundle $\mathcal{L}$ on X is called **positive** if $c_1(\mathcal{L})$ is represented by a positive-definite Hermitian form.

Theorem. *Let* X *be a smooth complex projective variety. A line bundle* $\mathcal{L}$ *on* X *is ample if and only if it is positive.*

This is how we ask the computer to find polarizations of an abelian variety X, which are steps 1 and 2 of the following section.

However, there may be infinitely many polarizations. We are interested in a particular kind of polarization.

Definition 11. A polarization $c_1(\mathcal{L})$ of X is called **principal** if $\mathcal{L}$ has only one section up to constants, i.e. $\dim H^0(X, \mathcal{L}) = 1$.

As a motivational theorem:

Theorem (BL 4.1.2)**.** *Every polarization is induced by a principal polarization via an isogeny.*

By Narasimhan-Nori [17], there are only finitely many principal polarizations on a variety X, which is irreducible and smooth. And as a corollary, only finitely many curves may have the same Jacobian since each non-isomorphic curve gives a non-isomorphic principal polarization on its Jacobian.

4.5. **Finding Principal Polarizations.** We begin with a representation of our abelian variety as $A := \mathbb{C}^g/\Pi\mathbb{Z}^{2g}$.

Then Λ is the associated lattice spanned by the columns of Π. Thus, we have a distinguished basis for the homology of A, corresponding to the columns of Π.

Algorithm: Compute many principal polarizations on a given abelian variety A.
Input: An abelian variety $A := \mathbb{C}^g/\Pi\mathbb{Z}^{2g}$, where Λ is the associated lattice to the period matrix Π.
Output: Many principal polarizations on A.

(1) The magma function `FindPolarizationBasis` determines all integral alternating pairings E on the homology, i.e., $E \in \mathrm{Alt}^2(\mathbb{C}^g, \mathbb{Z})$, for whose real extension we have:
$$E(iv, iw) = E(v, w)$$
This is a basis of alternating forms $\{E_i\}$.
(2) Check that E is positive-definite.
(3) `CullPB.m` tries some small combinations and sees if E_i actually gives a pairing with determinant 1 indicating that E_i is a principal polarization. If so, it returns E_i. This gives us a set $\{E_k\}$ of integral pairings on the homology.
(4) For each i, we rewrite these pairings in a symplectic basis. That is, we find a basis of Λ in which
$$E_i = \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix}$$
where $D = \mathrm{diag}(d_1, ..., d_g)$ which we may do by the elementary divisor theorem (section 3.1, [1]).

Remark. This does nothing but modify the (homology) basis of Λ. Multiplying Π on the right with this integral matrix, we get a new period matrix Q whose columns span exactly the same lattice but for which the standard symplectic pairing E is actually the Chern class of a line bundle. This is often called the Frobenius form of the period matrix Π.

5. Studying Principal Polarizations via Automorphisms of the Jacobian

Definition 12. We say two principal polarizations p_1 and p_2 on A are **auto-equivalent** if and only if $\mathrm{Aut}(A, p_1) \simeq \mathrm{Aut}(A, p_2)$.

Our program produces many auto-equivalent principal polarizations. Note that auto-equivalence is a weaker notion of equivalence than analytic equivalence, as discussed in section 5.2.

If the abelian variety is indeed a Jacobian, this method will in practice return at least enough polarizations to find the canonical principal polarization. As we discussed in the introduction, it is an unsolved problem to find all possible principal polarizations associated to a given abelian variety, called "explicit Narasimhan-Nori".

5.1. **Examples.** With the exception of the modular curve $X_0(63)$, all curves on which we apply our method are cyclic covers over $\mathbb{CP}^1$.

Here we describe our examples in terms of their branching indices over $\mathbb{CP}^1$. For each case, we will show explicit flat structures that are induced from the branching indices. The flat structures give rise to holomorphic 1-forms that are used in computing period matrices. We will also describe each case via their plane curve models. In general, we choose $p_i = 0, 1, \infty$ (for covers over thrice punctured spheres) and describe each curve algebraically as $y^d = x^{d_1}(x-1)^{d_2}$.

The computation of the period matrices can be found in `github/...`. The general version of the period matrix is shown in Section 3.2.

Table 1. Automorphism Groups wrt each of the Principal Polarizations

Curve C	Genus	# Principal Polarizations	$\mathrm{Aut}(\mathrm{Jac}(C), a_i)$	$\lvert \mathrm{Aut}(\mathrm{Jac}, a_i) \rvert$	GAPID
Klein	3	2	$S_4 \times C_2$	48	[48, 48]
			$GL_3(F_2) \times C_2$	336	[336, 209]
Fermat	3	2	$(C_4 \wr C_2) \times C_2$	64	[64, 101]
			$(C_4^2 \rtimes S_3) \times C_2$	192	[192, 944]
$12(1,5,6)$	3	3	D_6	12	[12, 4]
			$C_4 \times S_3$	24	[24, 5]
			$C_4 \times D_4$	32	[32, 25]
Bring	4	2	$C_2^2 \times D_4$	32	[32, 46]
			$C_2 \times S_5$	240	[240, 189]
I-WP	4	9	C_2^4	16	[16, 14]
			$C_2^2 \times C_6$	24	[24, 15]
			$C_2^2 \times D_4$	32	[32, 46]
			$C_2^3 \times C_6$	48	[48, 52]
			$C_2^2 \times S_4$	96	[96, 226]
			$C_6 \times S_4$	144	[144, 188]
			$(C_2 \times C_6) \times (C_3 \rtimes D_4)$	288	[288, 1002]
			$C_3 \times (((C_6 \times C_2) : C_2) \times D_8)$	576	[576, 7780]
			$C_6 \times (S_3 \times ((C_6 \times C_2) : C_2))$	864	[864, 4523]
$X_0(63)$	5	2	C_2^5	32	[32,51]
			$C_2^2 \times S_4$	96	[96, 226]

Remark. Note that the University of Bristol's GroupNames database at the time of writing has groups up to order 500 with full names and structure description. In the cases where the order is greater than 500, we use the output of `StructureDescription(G);`

5.1.1. *Genus three Klein's quartic.* Klein's quartic is a genus three non-hyperelliptic curve defined by branching indices 7(1,2,4) [9]. Its multipliers give rise to cone metrics defined by cone angles $\frac{\pi}{7}(1,2,4), \frac{\pi}{7}(2,4,1)$, and $\frac{\pi}{7}(4,1,2)$. Figure 4 represents ω_1 and ω_2.

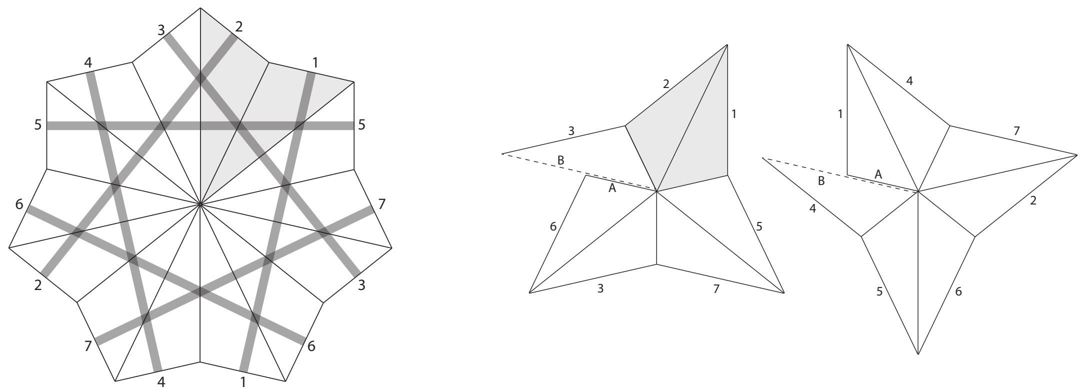

FIGURE 4. Two different flat structures on Klein's quartic.

Along with ω_3 one can compute the period matrix as in Section 3.2.

5.1.2. *Genus three Fermat's quartic.* Fermat's quartic is another genus three non-hyperelliptic curve defined by 8(1,2,5) [13]. The flat structures are described in Figure 2 and Figure 3. The period matrices can be found in Section 3.2. One can write the curve model as $y^8 = x(x-1)^2$. However, one can also consider a fourfold covering over $\mathbb{CP}^1$ branched over four points and write it as $y^4 = x(x-1)(x+1)$. See [13] for details.

5.1.3. *Genus three hyperelliptic curve.* As an example of a hyperelliptic curve, we look into 12(1,5,6). Its quotient under the hyperelliptic involution is $\mathbb{CP}^1$ where the eight hyperelliptic points are located at the North Pole, South Pole, and at six equidistributed points along the Equator. That is, the curve can be described as $y^2 = x(x^6-1)$.

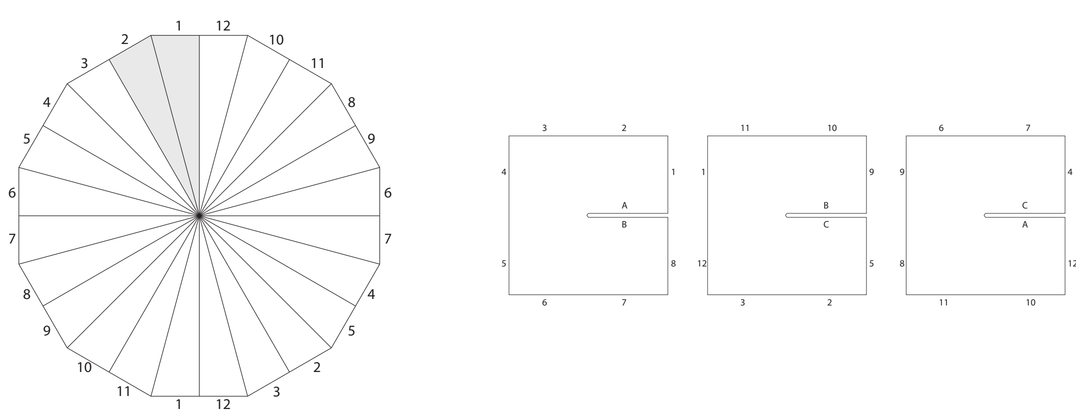

FIGURE 5. Two different flat structures on 12(1,5,6).

5.1.4. *Genus four Bring's curve.* Bring's curve is a genus four non-hyperelliptic curve denoted by 5(1,2,4,3). In [18], this curve as $y^5 = (x+1)x^2(x+1)^4$ by choosing $p_i = -1, 0, 1, \infty$.

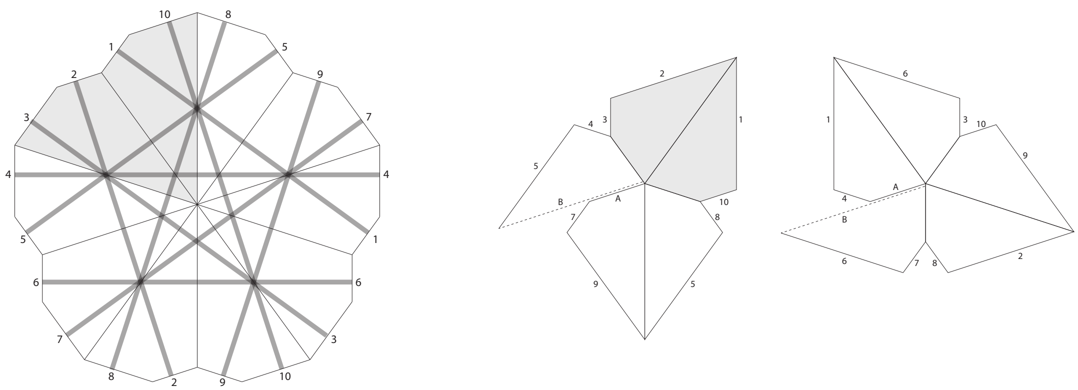

Figure 6. Two different flat structures on Bring's curve.

5.1.5. *Schoen's I-WP minimal surface.* Lastly, I-WP is another genus four non-hyperelliptic curve defined by 12(1,4,7). This notation to this covering is due to [**?**], where it is shown that Schoen's I-WP surface is equipped with cone metrics that are compatible with the twelvefold cyclic cover over $\mathbb{CP}^1$. There, it is also shown that the curve is a threefold cover over the octahedron, hence one can write it as $y^3 = x(x^4 - 1)$.

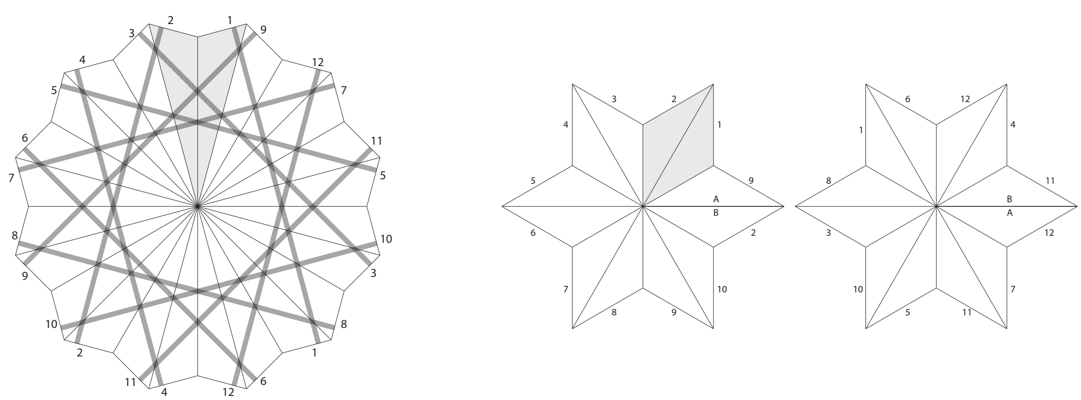

Figure 7. Two different flat structures on Schoen's I-WP.

5.2. **Questions and Answers on Abelian Varieties with Multiple Principal Polarizations.** We speak here of polarizations up to auto-equivalence and ask natural questions on Jacobians with multiple principal polarizations, answering all but one of the questions using methods developed in our paper.

We fix some notation. Let θ_C be the canonical principal polarization of $\mathrm{Jac}(C)$ with respect to C. We call $\mathrm{Aut}(A, a_i)$ a symplectic automorphism group of A, as the automorphisms respect the principal polarization a_i, which is a symplectic form on A.

Question. $\mathrm{Aut}(\mathrm{Jac}(C), \theta_C)$ *will have the highest order of all symplectic automorphism groups of* $\mathrm{Jac}(C)$.

This is proven false by example $12(1,5,6)$, where $|\,\mathrm{Aut}(\mathrm{Jac}(12(1,5,6)), \phi_{12(1,5,6)})| = 24$, but $|\,\mathrm{Aut}(\mathrm{Jac}(12(1,5,6)), a_i)| = 32$ is achieved. It is more dramatically proven false by Schoen's I-WP Surface, where $|\operatorname{Aut}(\operatorname{Jac}(\text{I-WP}), \phi_{\text{I-WP}})| = 288$, but $|\operatorname{Aut}(\operatorname{Jac}(\text{I-WP}), a_i)|$ achieves 576 and 864.

Question. *Principal polarizations p_1 and p_2 are auto-equivalent if and only if they are analytically equivalent. In other words,*

$$\operatorname{Aut}(X, p_1) \simeq \operatorname{Aut}(X, p_2) \Leftrightarrow p_1 = p_2$$

The direction ($\Leftarrow$) is clear because $\mathcal{L}$ and $\mathcal{M}$ are analytically equivalent if and only if $c_1(\mathcal{L}) = c_1(\mathcal{M})$ by [BL 2.5.3]. The other direction ($\Rightarrow$) is false. This is proven false by applying our method to the the following two *non-isomorphic* curves with the same (unpolarized) Jacobian from Theorem 1 of [6]:

$$X : 3y^2 = (2x^2 - 2)(16x^4 + 28x^2 + 1)$$

$$X' : -y^2 = (2x^2 + 2)(16x^4 + 12x^2 + 1)$$

which both have $\operatorname{Aut}(\operatorname{Jac}(X), \theta_X) \simeq C_2 \times C_2 \simeq \operatorname{Aut}(\operatorname{Jac}(X'), \theta_{X'})$.

Question. *If* $\operatorname{Jac}(C) \simeq \operatorname{Jac}(C')$ *as complex varieties, then*

$$\operatorname{Aut}(\operatorname{Jac}(C), \theta_C) \simeq \operatorname{Aut}(\operatorname{Jac}(C'), \theta_{C'})$$

We checked this question on the family of hyperelliptic cases of genus 2 from [6] Theorem 1, where it is true. However, there is no reason to expect this to be true in general. Yet, we cannot disprove it easily.

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