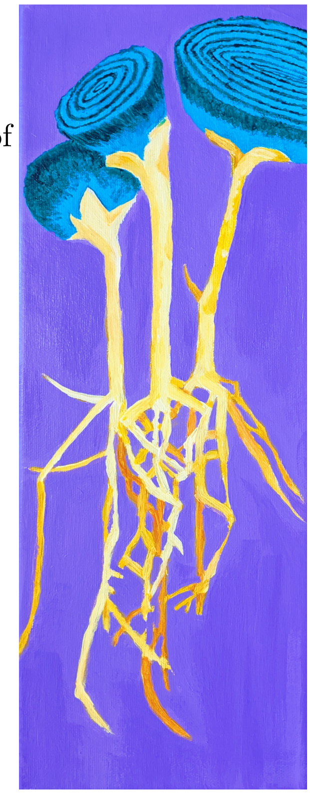
The fundamental concepts of number and space are engaged in an intense discourse that we have only begun to tap into. The relationship between the two has been explored in depth since the earliest records of mathematics. The more recent approaches toward unification have failed due to their over reliance on abstraction and a lack of hands-on understanding. My ongoing research suggests that a different language, focusing on the most elementary building blocks of geometric construction, may provide an alternative framework for unifying our understanding of spaces and numbers.

To understand the whole, study the parts. The topological theory of spaces is rather like the chemical theory of molecules, with spheres playing the role of atoms. Just as all molecules are built from atoms, bound together in various ways, so are all spaces built from spheres, bound together in various ways. Topology examines how the parts of a space are bound together to produce the whole, and its success presumes a detailed and comprehensive understanding of the parts. And this understanding has undergone dramatic change over the past few decades; for spheres, like atoms, have proven to be far more intricate than originally thought. We now have learned to speak of "prime" spheres, whose relation to ordinary spheres is perhaps a little like the relation of modern quarks to the original atoms of Democritus. And just as we understand quarks using the theory of quantum chromodynamics, so do we understand "prime" spheres using a similarly recondite innovation called chromatic homotopy theory. This theory may perhaps be likened usefully to nuclear spectroscopy: it exposes a formerly inaccessible "sphere spectrum," whose various component "spectral lines" are distinguished not by characteristic frequencies, nor by excitation energies, but rather by an abstract arithmetical label called "height." Decomposing a sphere into its spectrum of prime spheres of different heights is perhaps rather like decomposing a nucleon and its various excited states into quarks in different conditions of relative motion. In this way, chromatic homotopy theory exposes profound, far-reaching patterns that would otherwise be invisible – patterns that have revolutionized our understanding of the spherical parts that make up the spatial wholes that are the subject matter of topology.

The Periodic Table of the Spheres. As Mendeleev deconstructed and organized chemicals into the periodic table of elements, so does the deconstruction and understanding of spaces lead us to a table of prime spheres. The rows of our prime spheres table correspond to each prime number, and the columns of our table are a measure of the amount of complexity, which we call ***height***. Previous work in chromatic homotopy theory attempted to compute the properties of each prime sphere in the table by computing one height and prime at a time.

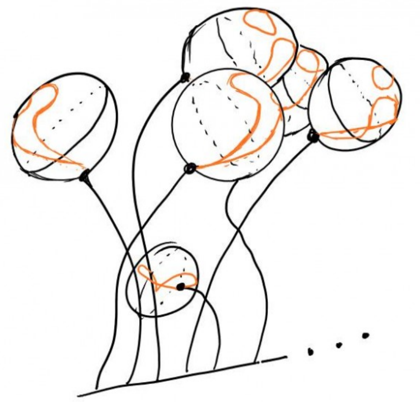
Unfortunately, new discovery slowed considerably after the 1990's due to a lack of computational tools. We knew qualitative information on the periodicity of the prime spheres from the work of [Hopkins-Smith], but not any quantitative information. Our understanding of the table of prime spheres was restricted entirely to height 1 and 2 [Shimomura]. It is as if for 40 years, the periodic table consisted only of those two columns headed by hydrogen and beryllium. As though every element beyond this maddeningly narrow strip lay shrouded in inaccessible mystery. As though even oxygen and nitrogen, without which air and water are alike incomprehensible, were as remote and obscure as the Higgs boson, decades before the Large Hadron Collider.

Special Families of Curves Shed New Light on Height. The sticking point, of course, was the steadily increasing complexity of those prime spheres of higher height, which made them ever more difficult to imagine, which made them ever more computationally inaccessible. In other words, a large part of the blockage of computing properties of the table's other columns originated in a lack of a hands-on understanding of objects of *height* greater than 2. Height is a measure of complexity, and objects with unusually high symmetry have higher height. My research constructs families of curves with unusually high symmetry which, I discovered, model the behavior of the prime spheres.



The families of curves I constructed are an onion with a prime-power number of layers all pinched together at one point. These families of curves have an elegant fractal structure that reflects their construction from a nested infinite sequence of finite arithmetics, one for each power of some fixed prime number. Thus, from one simple construction, we get a whole zoo of spaces. The more layers of onions one adds, the higher in complexity the curve becomes. Further, their interesting and fractal complexity information is entirely captured from just looking around their pinch point, that is, at the roots of the onion.

My attached research sample [Lee-Ray] initiates the explicit construction one of these unusually symmetrical families of curves, building on the work of [Gorbounov-Mahowald]. What makes the resulting family computationally useful is not only that it is unusually symmetrical, but also that its construction is inductive, and that all its constituent curves can be written down and understood quite explicitly. These latter features permit the investigation of prime spheres via computations no longer restricted to one prime and one height at a time – computations that can suddenly expose patterns extending across a wide range of primes and heights. My work provides new methods that reveal the elementary properties of prime spheres, and that expose the overarching algebraic structure of much of the periodic table to which they belong. And, remarkably enough, my results suggest that the essential complexity of a prime sphere is ultimately discernible in the

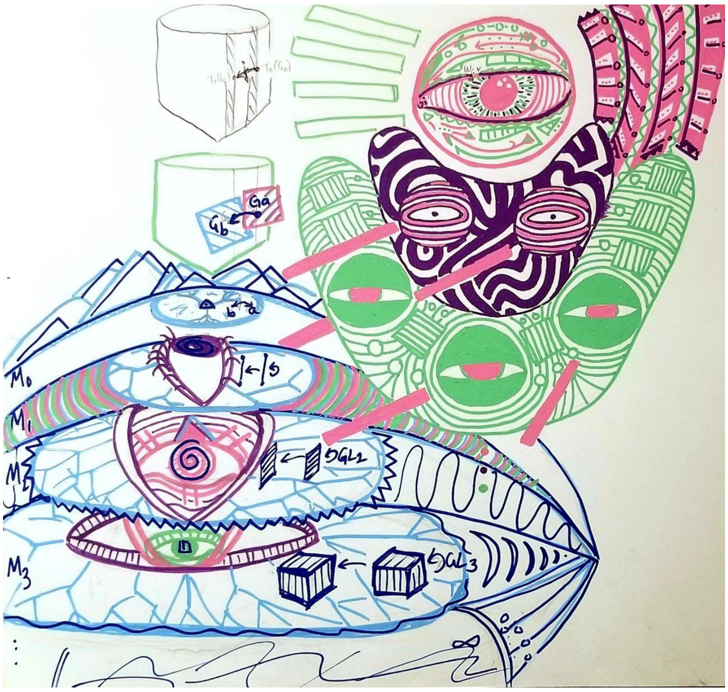


finite arithmetic of a clock with a prime- power number of distinct, equally-spaced marks on the periphery of its dial.

My results suggest that the complexity of the sphere is primarily captured by a simple representation of the modular arithmetic of a prime-powered clock as vectors in a vector space, which we call *regular*. Through this process one goes from a piecemeal confusion to a system of vectors that, in their logical order, harmony, and brilliance, appear as an organic whole, like a painting before the mind's eye. This allows us to capture massive amounts of algebraic structure previously thought to be impossibly complex in a simple family of vectors.

Plans and Ambitions. The new computational methods and tools my work has exposed and begun to exploit have already proven powerful enough to overcome (some of) the long-standing challenges posed by the considerable arithmetic-topological complexity of prime spheres with height greater than 2; at last, the maddening stalemate of the past forty years seems to be nearing its end. And the early results to which my methods and tools have led are tantalizingly suggestive; already it is clear that the newly exposed data cohere in elegant and attractive ways. But these early results are obviously just the tip of an iceberg.

Closing Speculations. I very much doubt that the striking developments in topology I have foreseen above will reach their conclusion without conferring upon arithmetic some similarly striking reciprocal enrichment; perhaps the prime spheres will eventually make some profound statement about the prime numbers.



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