Discrete Fourier Transform 离散傅立叶变化

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j2\pi kn/N} \qquad k=0, \dots N-1$$

$\underline{x[n]}$: input signal 输入信号

$\underline{x[n] \cdot e^{-j2\pi kn/N}}$: project input signal to complex plane 将输入信号投射到复数坐标上.

n: discrete time index

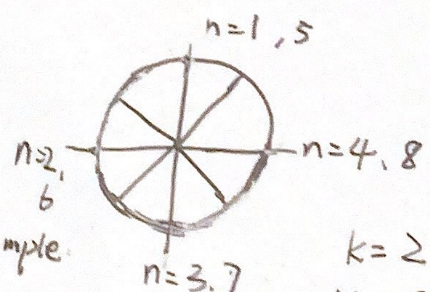
k: discrete frequecy index

$w_k = 2\pi k/N$ frequency in radians per second

$f_k = f_s \cdot k/N$ frequency in Hz (f_s : sampling rate)

requirement for periodicity:

$$w_0 = \frac{k}{N} 2\pi$$



$$w_0 = \frac{2}{8} \cdot 2\pi = \frac{\pi}{2} \text{ rad/sample}$$

$k=2$
$N=8$

period in 4 samples

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complex exponential

$$S_k^* = e^{-j2\pi kn/N} = \cos(2\pi kn/N) - j\sin(2\pi kn/N)$$

for $N=4$, thus for $n=0,1,2,3$ samples
$k=0,1,2,3$ frequencies

$$S_0^* = \cos(2\pi * 0 * n/4) - j\sin(2\pi * 0 * n/4) = [1,1,1,1]$$

$$S_1^* = \cos(2\pi * 1 * n/4) - j\sin(2\pi * 1 * n/4) = [1,-j,-1,j]$$

$$S_2^* = \cos(2\pi * 2 * n/4) - j\sin(2\pi * 2 * n/4) = [1,-1,1,-1]$$

$$S_3^* = \cos(2\pi * 3 * n/4) - j\sin(2\pi * 3 * n/4) = [1,j,-1,-j]$$

Scalar product

$$\langle x, S_k \rangle = \sum_{n=0}^{N-1} x[n] \cdot S_k^*[n] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N}$$

example:

$x[n] = [1,-1,1,-1]$; $N=4$

$$\langle x, S_0 \rangle = 1*1 + (-1)*1 + 1*1 + (-1)*1 = 0$$

$$\langle x, S_1 \rangle = 1*1 + (-1)*(-j) + 1*(-1) + (-1)*j = 0$$

$$\langle x, S_2 \rangle = 1*1 + (-1)*(-1) + 1*1 + (-1)*(-1) = 4$$

$$\langle x, S_3 \rangle = 1*1 + (-1)*j + 1*(-1) + (-1)*(-j) = 0$$

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DFT of complex exponential

$$x_1[n] = e^{j2\pi k_0 n/N} \quad \text{for } n=0,\dots,N-1$$

$$X_1[k] = \sum_{n=0}^{N-1} x_1[n]\, e^{-j2\pi kn/N}$$

(k = integer)

$$= \sum_{n=0}^{N-1} e^{j2\pi k_0 n/N}\, e^{-j2\pi kn/N}$$

$$= \sum_{n=0}^{N-1} e^{-j2\pi (k-k_0) n/N}$$

$$= \frac{1-e^{-j2\pi(k-k_0)}}{1-e^{-j2\pi(k-k_0)/N}}$$

(sum of geometric series)

if $k \neq k_0$, denominator $\neq 0$ and numerator $= 0$

thus $X_1[k] = N$ for $k = k_0$ and $X_1[k] = 0$ for $k \neq k_0$

DFT of any complex sinusoid

$$x_2[n] = e^{j(2\pi f_0 n + \varphi)} \quad \text{for } n=0,\dots,N-1$$

没有固定 <u>period</u>

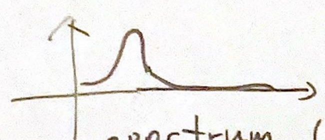
$k \neq$ integer

$$X_2[k] = \sum_{n=0}^{N-1} x_2[n]\, e^{-j2\pi kn/N}$$

$$= \sum_{n=0}^{N-1} e^{j(2\pi f_0 + \varphi)}\, e^{-j2\pi kn/N}$$

$$= e^{j\varphi} \sum_{n=0}^{N-1} e^{-j2\pi (k/N - f_0) n}$$

$$= e^{j\varphi}\, \frac{1-e^{-j2\pi(k/N - f_0)/N}}{1-e^{-j2\pi(k/N - f_0)}}$$



spectrum (abs(x))

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DFT of real sinusoids

$$X_3[k] = \sum_{n=-\frac{N}{2}}^{N/2-1} x_3[n] e^{-j2\pi kn/N}$$

$$= \sum_{n=-\frac{N}{2}}^{N/2-1} \left(\frac{A_0}{2} e^{j2\pi k_0 n/N} + \frac{A_0}{2} e^{-j2\pi k_0 n/N}\right) e^{-j2\pi kn/N}$$

$$= \sum_{n=-\frac{N}{2}}^{N/2-1} \frac{A_0}{2} e^{j2\pi k_0 n/N} \cdot e^{-j2\pi kn/N} +$$

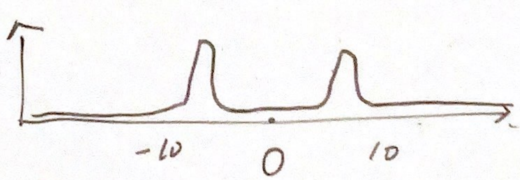
$$\sum_{n=-\frac{N}{2}}^{N/2-1} \frac{A_0}{2} e^{-j2\pi k_0 n/N} \cdot e^{-j2\pi kn/N}$$

$$= \sum_{n=-\frac{N}{2}}^{N/2-1} \frac{A_0}{2} e^{-j2\pi(k-k_0)n/N} + \sum_{n=-\frac{N}{2}}^{N/2-1} \frac{A_0}{2} e^{-j2\pi(k+k_0)n/N}$$

$= N\frac{A_0}{2}$ for $k=k_0, -k_0$; 0 for rest of k

2 peaks for real sinusoids.

1 peak for complex sinusoids.



inverse DFT

正

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]S_k[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\,e^{\underline{\underline{j2\pi kn/N}}} \qquad n=0,1,\ldots,N-1$$

example

$X[k]=[0,4,0,0]$; $N=4$

$x[0] = \frac{1}{4}(X*S)[n=0] = \frac{1}{4}(0*1+4*1+0*1+0*1) = 1$

$x[1] = \frac{1}{4}(X*S)[n=1] = \frac{1}{4}(0*1+4*j+0*(-1)+0*(-j))) = j$

$x[2] = \frac{1}{4}(X*S)[n=2] = \frac{1}{4}(0*1+4*(-1)+0*1+0*(-1) = -1$

$x[3] = \frac{1}{4}(X*S)[n=3] = \frac{1}{4}(0*1+4*(-j)+0*(-1)+0*j) = -j$