Homework Set 4

(Due Mar. 21)

Assignment

1. Every working day, John arrives at the bus stop exactly at 7:00am and takes the first bus after his arrival. Suppose the arrival time of the first bus is an exponential random variable with mean 20 minutes. Meanwhile, every working day and independently, Mary arrives at the same bus stop at a random time, which is uniformly distributed between 7:00am and 7:30am.
   (a) What is the probability that John will wait for more than 30 minutes tomorrow?
   (b) Consider Mary being late if she arrives at the bus stop after 7:20am. Suppose that Mary's arrival times among different days are independent to each other. What is the probability that Mary will be late on 2 or more working days among the next 10 working days?
   (c) What is the probability that John and Mary will meet at the station tomorrow?

2. Imagine a process which at each trial produces a digit from 0 to 9, and imagine that after each trial the process can be adjusted, if needed, in such a manner that regardless of the past outcomes all 10 possibilities are equally likely in your opinion. The process described above can be thought of as generating an infinite decimal $0.a_1a_2a_3\ldots$, where a_i is the outcome of trial i for $i = 1, 2, \ldots$
   (a) What is the probability that the process generates an infinite decimal that falls in the interval $[0, b]$ for given $b \in [0, 1]$?
   (b) Let b itself be expressible as an infinite decimal $0.b_1b_2b_3\ldots$ Show that the probability that it will take more than n trials to determine whether the generated number will be in $[0, b]$ or not is $1/10^n$.

3. A production process is adjusted daily and then tested. The output of the process in a given day acts like a Bernoulli process where the proportion p of defective items depends on the quality of two critical adjustments A and B according to the following table:

State	Probability of the state	Resulting process p
A good, B good	.6	.00
A good, B bad	.2	.05
A bad, B good	.1	.10
A bad, B bad	.1	1.00
	1.0	

For this process a sample of $n = 3$ items are taken; let $\tilde{r}$ denote the number of defective items in the sample.

(a) Give the conditional mf of $\tilde{r}$ for each of the four values of p.

(b) Give the conditional mf of $\tilde{p}$ given $\tilde{r} = 0$. Repeat for $\tilde{r} = 1$, 2, and 3.

4. For any two numbers x_1 and x_2 let $\max\{x_1, x_2\}$ denote the maximum of the two numbers and $\min\{x_1, x_2\}$ the minimum of the two numbers.

   Let $\tilde{d}$ be the random variable designating the unknown demand for a product, let $\tilde{d}$ have the range and mass function given in the table below, and suppose the decision maker stocks 4 units.

	Possible value of demand							
d:	0	1	2	3	4	5	6	7
$P(\tilde{d} = d)$:	.05	.10	.15	.25	.20	.10	.10	.05

   (a) Interpret the r.v. $\tilde{y} := \max\{\tilde{d} - 4, 0\}$ and give its mf.

   (b) Interpret the r.v. $\tilde{z} := \max\{4 - \tilde{d}, 0\}$ and gives its mf.

   (c) Define the r.v. $\tilde{w} := \max\{-4 + 2\tilde{d}, \tilde{d}\}$ and gives its mf.

5. The paired random variable $(\tilde{x}, \tilde{y})$ is said to have a bivariate normal distribution if their joint density function is given by

$$f(x, y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)} \times \left(\left(\frac{x-\mu_x}{\sigma_x} \right)^2 - \frac{2\rho(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y} + \left(\frac{y-\mu_y}{\sigma_y} \right)^2 \right)},$$

   for $-\infty < x < +\infty$, $-\infty < y < +\infty$, where σ_x, σ_y, μ_x, μ_y, and ρ are constants such that $-1 < \rho < 1$, $\sigma_x > 0$, $\sigma_y > 0$, $-\infty < \mu_x < +\infty$, and $-\infty < \mu_y < +\infty$.

   (a) Show that $\tilde{x}$ is normally distributed with mean μ_x and variance σ_x^2.

   (b) Show that the conditional density function of $\tilde{x}$ given that $\tilde{y} = y$ is normally distributed with mean $\mu_x + \rho\sigma_x(y - \mu_y)/\sigma_y$ and variance $\sigma_x^2(1 - \rho^2)$.

   (c) Show that $\rho(\tilde{x}, \tilde{y}) = \rho$ (hint: use the formula $E[\tilde{x}\tilde{y}] = E_Y[\tilde{y}E_{X|y}[\tilde{x}]]$).

6. A manuscript is sent to a typing firm consisting of typists A, B, and C. If it is typed by A, then the number of errors made is a Poisson random variable with mean 2.6; if typed by B, then the number of errors is a Poisson random variable with mean 3; and if typed by C, then it is a Poisson random variable with mean 3.4. Let $\tilde{x}$ denote the number of errors in the typed manuscript. Assume that each typist is equally likely to do the work.

   (a) Calculate $E[\tilde{x}]$.

   (b) Calculate $Var(\tilde{x})$.