TITLE

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DATE

1 Alternating Data Automata

1.1 $Th(\mathcal{D})$

Given a possibly infinite data domain $\mathcal{D}$, we denote by $Th(\mathcal{D}) = \langle \mathcal{D}, f_1, f_2, ..., f_m \rangle$ the set of syntactically correct first-order formulae with function symbols $f_1, f_2, ..., f_m$.

1.2 $\mathcal{B}^{\mathcal{Q}}$

Denote by the symbol $\mathcal{B}$ the two-element Boolean algebra $\mathcal{B} = (\{0, 1\}, \vee, \wedge, \neg, 0, 1)$.

Let $\mathcal{Q}$ be a set, then $\mathcal{B}^{\mathcal{Q}}$ is the set of all mappings from $\mathcal{Q}$ to $\mathcal{B}$.

1.3 Definition

An *Alternating Data Automaton* (ADA) is a tuple $\mathcal{A} = \langle \mathcal{D}, \mathcal{X}, \Sigma, \mathcal{Q}, \boldsymbol{i}, \mathcal{F}, \boldsymbol{g} \rangle$ where [1]:

- $\mathcal{D}$ is an initial data domain;
- $\mathcal{X} = \{x_1, x_2, ..., x_n\}$ is a set of variables;
- Σ is a finite alphabet of input events;
- $\mathcal{Q}$ is a finite set of states;
- $\boldsymbol{i} \in \mathcal{Q}$ is an initial state;
- $\mathcal{F} \subseteq \mathcal{Q}$ is a set of final states;
- $\boldsymbol{g}$ is a function of $\mathcal{Q}$ into the set of all functions of Σ into functions $\mathcal{D}^{\mathcal{X}} \times \mathcal{D}^{\mathcal{X}} \to (\mathcal{B}^{\mathcal{Q}} \to \mathcal{B})$.

We define $\boldsymbol{f} \in \mathcal{B}^{\mathcal{Q}}$ by the condition: $\boldsymbol{f}(q) = 1$ iff $q \in \mathcal{F}$.

1.4 Function $\boldsymbol{g}$

For each $q \in \mathcal{Q}$, $a \in \Sigma$, $v \in \mathcal{D}^{\mathcal{X}}$, $v' \in \mathcal{D}^{\mathcal{X}}$, $u \in \mathcal{B}^{\mathcal{Q}}$, $\boldsymbol{g}(q)(a)(v, v')(u)$ is a Boolean combination of $u(q_t)$ and $\phi_t(v, v')$ where $q_t \in \mathcal{Q}$ is the successor of q with symbol a and $\phi_t \in Th(\mathcal{D})$.

In the cases where v' is not used, we write $\boldsymbol{g}(q)(a)(v)(u)$ instead of $\boldsymbol{g}(q)(a)(v, v')(u)$.

For example, we define an ADA $\mathcal{A} = \langle \mathcal{D}_{\mathcal{A}}, \mathcal{X}_{\mathcal{A}}, \Sigma_{\mathcal{A}}, \mathcal{Q}_{\mathcal{A}}, \boldsymbol{i}_{\mathcal{A}}, \mathcal{F}_{\mathcal{A}}, \boldsymbol{g}_{\mathcal{A}} \rangle$, where:

- $\mathcal{D}_{\mathcal{A}} = \mathcal{N}$;
- $\mathcal{X}_{\mathcal{A}} = \{x\}$;
- $\Sigma_{\mathcal{A}} = \{a, b\}$;
- $\mathcal{Q}_{\mathcal{A}} = \{q_0, q_1, q_2, q_3\}$;
- $\boldsymbol{i}_{\mathcal{A}} = q_0$;
- $\mathcal{F}_{\mathcal{A}} = \{q_0\}$;
- $\boldsymbol{g}_{\mathcal{A}}$ is given by:

	a	b
q_0	$u(q_1) \wedge (x > 3 \wedge x' = x + 1) \vee$ $u(q_2) \wedge (x < 3 \wedge x' = x + 1) \vee$ $u(q_3) \wedge (x < 1 \wedge x' = x + 1)$	0
q_1	0	$u(q_0) \wedge x < 5$
q_2	0	$u(q_0) \wedge x = 3$
q_3	0	$u(q_1) \wedge x = 0$

$$\begin{aligned}
& \boldsymbol{g}(q_0)(a)(2,3)(\boldsymbol{f}) \\
= \; & \boldsymbol{f}(q_1) \wedge (2 > 3 \wedge 3 = 2 + 1) \vee \boldsymbol{f}(q_2) \wedge (2 < 3 \wedge 3 = 2 + 1) \vee \boldsymbol{f}(q_3) \wedge (2 < 1 \wedge 3 = 2 + 1) \\
= \; & 0 \wedge 0 \vee 0 \wedge 1 \vee 0 \wedge 0 \\
= \; & 0 \vee 0 \vee 0 \\
= \; & 0
\end{aligned}$$

$$\begin{aligned}
& \boldsymbol{g}(q_1)(b)(4)(\boldsymbol{f}) \\
= \; & \boldsymbol{f}(q_0) \wedge 4 < 5 \\
= \; & 1 \wedge 1 \\
= \; & 1
\end{aligned}$$

1.5 Extension of Function $\boldsymbol{g}$

Now we extend $\boldsymbol{g}$ to $\mathcal{Q} \to ((\mathcal{D}^{\mathcal{X}} \times \Sigma)^* \times \mathcal{B}^{\mathcal{Q}} \to \mathcal{B})$. Giving $q \in \mathcal{Q}$, $w \in (\mathcal{D}^{\mathcal{X}} \times \Sigma)^*$, $u \in \mathcal{B}^{\mathcal{Q}}$, we have:

- If $w = \lambda$, then $\boldsymbol{g}(q)(w, u) = u(q)$;
- If $w = \langle v, a\rangle$ with $v \in \mathcal{D}^{\mathcal{X}}$ and $a \in \Sigma$, then $\boldsymbol{g}(q)(w, u) = \boldsymbol{g}(q)(a)(v)(u)$;
- If $w = \langle v, a\rangle\langle v', b\rangle w'$ with $v, v' \in \mathcal{D}^{\mathcal{X}}$ and $a, b \in \Sigma$ and $w' \in (\mathcal{D}^{\mathcal{X}} \times \Sigma)^*$, then $\boldsymbol{g}(q)(w, u) = \boldsymbol{g}(q)(a)(v, v')(u)_{[u(q_t)/\boldsymbol{g}(q_t)(\langle v', b\rangle w', u)]}$ where $q_t \in \mathcal{Q}$ is the successor of q with symbol a.

1.6 Acceptance of a Word

Let $\mathcal{A} = \langle \mathcal{D}, \mathcal{X}, \Sigma, \mathcal{Q}, \boldsymbol{i}, \mathcal{F}, \boldsymbol{g}\rangle$ be an ADA, a word $w \in (\mathcal{D}^{\mathcal{X}} \times \Sigma)^*$ is accepted by $\mathcal{A}$ iff $\boldsymbol{g}(\boldsymbol{i})(w, \boldsymbol{f})$.

Let's take the example in last page, and we try the following words:

$$
\begin{aligned}
& \boldsymbol{g}(q_0)(\langle v_0, a\rangle\langle v_1, b\rangle, \boldsymbol{f}) \\
= \; & \boldsymbol{g}(q_1)(\langle v_1, b\rangle, \boldsymbol{f}) \wedge (v_0 > 3 \wedge v_1 = v_0 + 1) \\
\vee \; & \boldsymbol{g}(q_2)(\langle v_1, b\rangle, \boldsymbol{f}) \wedge (v_0 < 3 \wedge v_1 = v_0 + 1) \\
\vee \; & \boldsymbol{g}(q_3)(\langle v_1, b\rangle, \boldsymbol{f}) \wedge (v_0 < 1 \wedge v_1 = v_0 + 1) \\
= \; & \boldsymbol{f}(q_0) \wedge v_1 < 5 \wedge v_0 > 3 \wedge v_1 = v_0 + 1 \\
\vee \; & \boldsymbol{f}(q_0) \wedge v_1 = 3 \wedge v_0 < 3 \wedge v_1 = v_0 + 1 \\
\vee \; & \boldsymbol{f}(q_1) \wedge v_1 = 0 \wedge v_0 < 1 \wedge v_1 = v_0 + 1 \\
= \; & 1 \wedge v_1 < 5 \wedge v_0 > 3 \wedge v_1 = v_0 + 1 \\
\vee \; & 1 \wedge v_1 = 3 \wedge v_0 < 3 \wedge v_1 = v_0 + 1 \\
\vee \; & 0 \wedge v_1 = 0 \wedge v_0 < 1 \wedge v_1 = v_0 + 1 \\
= \; & v_1 < 5 \wedge v_0 > 3 \wedge v_1 = v_0 + 1 \vee v_1 = 3 \wedge v_0 < 3 \wedge v_1 = v_0 + 1 \\
= \; & v_1 = 3 \wedge v_0 < 3 \wedge v_1 = v_0 + 1 \\
= \; & v_0 = 2 \wedge v_1 = 3
\end{aligned}
$$

Therefore, the automaton $\mathcal{A}$ accepts the word $\langle 2, a\rangle\langle 3, b\rangle$.

1.7 Language

The language accepted by an alternating automaton $\mathcal{A} = \langle \mathcal{D}, \mathcal{X}, \Sigma, \mathcal{Q}, \boldsymbol{i}, \mathcal{F}, \boldsymbol{g}\rangle$ is the set $\mathcal{L}(\mathcal{A}) = \{w \in (\mathcal{D}^{\mathcal{X}} \times \Sigma)^* \mid \boldsymbol{g}(\boldsymbol{i})(w, \boldsymbol{f}) = 1\}$.

1.8 Complementation

Let $\mathcal{A} = \langle \mathcal{D}, \mathcal{X}, \Sigma, \mathcal{Q}, \boldsymbol{i}, \mathcal{F}, \boldsymbol{g}\rangle$ be an ADA, we can now construct another ADA $\overline{\mathcal{A}} = \langle \mathcal{D}', \mathcal{X}', \Sigma', \mathcal{Q}', \boldsymbol{i}', \mathcal{F}', \boldsymbol{g}'\rangle$ such that $\mathcal{L}(\overline{\mathcal{A}}) = \overline{\mathcal{L}(\mathcal{A})}$:

- $\mathcal{D}' = \mathcal{D}$;
- $\mathcal{X}' = \mathcal{X}$;
- $\Sigma' = \Sigma$;
- $\mathcal{Q}' = \mathcal{Q}$;
- $\boldsymbol{i}' = \boldsymbol{i}$;
- $\mathcal{F}' = \mathcal{Q} - \mathcal{F}$ therefore for each state $q \in \mathcal{Q}$, $\boldsymbol{f}'(q) = 1$ iff $q \in \mathcal{F}'$;
- For each $q \in \mathcal{Q}$, $a \in \Sigma$, $v \in \mathcal{D}^{\mathcal{X}}$, $v' \in \mathcal{D}^{\mathcal{X}}$, $u \in \mathcal{B}^{\mathcal{Q}}$, $u' \in \mathcal{B}^{\mathcal{Q}}$ and $u'(q) = \overline{u(q)}$, we have $\boldsymbol{g}'(q)(a)(v,v')(u') = \overline{\boldsymbol{g}(q)(a)(v,v')(u)_{[u(q_t)/\overline{u'(q_t)}]}}$ where $q_t \in \mathcal{Q}$ is the successor of q with symbol a.

Hence, for the extension of function $\boldsymbol{g}'$, we have:

- If $w = \lambda$, then:
  $\boldsymbol{g}'(q)(w,u') = u'(q)$;
- If $w = \langle v, a\rangle$ with $v \in \mathcal{D}^{\mathcal{X}}$ and $a \in \Sigma$, then:
  $\boldsymbol{g}'(q)(w,u') = \boldsymbol{g}'(q)(a)(v)(u') = \overline{\boldsymbol{g}(q)(a)(v)(u)_{[u(q_t)/\overline{u'(q_t)}]}}$ where $q_t \in \mathcal{Q}$ is a successor of q with symbol a;
- If $w = \langle v, a\rangle\langle v', b\rangle w'$ with $v \in \mathcal{D}^{\mathcal{X}}$, $v' \in \mathcal{D}^{\mathcal{X}}$, $a \in \Sigma$, $b \in \Sigma$ and $w' \in (\mathcal{D}^{\mathcal{X}} \times \Sigma)^*$, then:
  $\boldsymbol{g}'(q)(w,u') = \boldsymbol{g}'(q)(a)(v,v')(u')_{[u'(q_t)/\boldsymbol{g}'(q_t)(\langle v',b\rangle w',u')]} = \overline{\boldsymbol{g}(q)(a)(v,v')(u)_{[u(q_t)/\overline{\boldsymbol{g}'(q_t)(\langle v',b\rangle w',u')}]}}$
  where $q_t \in \mathcal{Q}$ is the successor of q with symbol a.

Let's still take the example in previous pages, now we construct the $\overline{\mathcal{A}} = \langle \mathcal{D}', \mathcal{X}', \Sigma', \mathcal{Q}', \boldsymbol{i}', \mathcal{F}', \boldsymbol{g}'\rangle$ such that $\mathcal{L}(\overline{\mathcal{A}}) = \overline{\mathcal{L}(\mathcal{A})}$:

- $\mathcal{D}' = \mathcal{N}$;
- $\mathcal{X}' = \{x\}$;
- $\Sigma' = \{a, b\}$;
- $\mathcal{Q}' = \{q_0, q_1, q_2, q_3\}$;
- $\boldsymbol{i}' = q_0$;
- $\mathcal{F}' = \{q_1, q_2, q_3\}$;
- $\boldsymbol{g}'$ is given by:

	a	b
q_0	$(u(q_1) \vee (x \leq 3 \vee x' \neq x+1)) \wedge$ $(u(q_2) \vee (x \geq 3 \vee x' \neq x+1)) \wedge$ $(u(q_3) \vee (x \geq 1 \vee x' \neq x+1))$	1
q_1	1	$u(q_0) \vee x \geq 5$
q_2	1	$u(q_0) \vee x \neq 3$
q_3	1	$u(q_1) \vee x \neq 0$

$$
\begin{aligned}
& \boldsymbol{g}'(q_0)(\langle v_0, a\rangle\langle v_1, b\rangle, \boldsymbol{f}') \\
= \; & (\boldsymbol{g}'(q_1)(\langle v_1, b\rangle, \boldsymbol{f}') \vee (v_0 \leq 3 \vee v_1 \neq v_0 + 1)) \\
\wedge \; & (\boldsymbol{g}'(q_2)(\langle v_1, b\rangle, \boldsymbol{f}') \vee (v_0 \geq 3 \vee v_1 \neq v_0 + 1)) \\
\wedge \; & (\boldsymbol{g}'(q_3)(\langle v_1, b\rangle, \boldsymbol{f}') \vee (v_0 \geq 1 \vee v_1 \neq v_0 + 1)) \\
= \; & (\boldsymbol{f}'(q_0) \vee v_1 \geq 5 \vee v_0 \leq 3 \vee v_1 \neq v_0 + 1) \\
\wedge \; & (\boldsymbol{f}'(q_0) \vee v_1 \neq 3 \vee v_0 \geq 3 \vee v_1 \neq v_0 + 1) \\
\wedge \; & (\boldsymbol{f}'(q_1) \vee v_1 \neq 0 \vee v_0 \geq 1 \vee v_1 \neq v_0 + 1) \\
= \; & (v_1 \geq 5 \vee v_0 \leq 3 \vee v_1 \neq v_0 + 1) \wedge (v_1 \neq 3 \vee v_0 \geq 3 \vee v_1 \neq v_0 + 1) \\
= \; & v_1 \geq 5 \vee v_0 \leq 3 \wedge v_1 \neq 3 \vee v_0 = 3 \vee v_1 \neq v_0 + 1 \\
= \; & v_0 \neq 2 \vee v_1 \neq 3
\end{aligned}
$$

Therefore, the automaton $\overline{\mathcal{A}}$ does not accept the word $\langle 2, a\rangle\langle 3, b\rangle$.

$$\boldsymbol{g}'(q_0)(\lambda, \boldsymbol{f}') = \boldsymbol{f}'(q_0) = 0$$

Therefore, the automaton $\overline{\mathcal{A}}$ accepts the empty word.

References

[1] Grzegorz Rozenberg and Arto Salomaa. *Handbook of Formal Languages*. 1996.

A Length of a Word

The length of a word w is defined as below:
$length(w) = k$ iff $w \in (\mathcal{D}^{\mathcal{X}} \times \Sigma)^k$

Hence, $length(\lambda) = 0$.

B Proof of the Correctness of Complementation

It is same to prove $\boldsymbol{g}(\boldsymbol{i})(w, \boldsymbol{f}) = \overline{\boldsymbol{g}'(\boldsymbol{i})(w, \boldsymbol{f}')}$ with $w \in (\mathcal{D}^{\mathcal{X}} \times \Sigma)^*$:

- If $w = \lambda$, then for each $q \in \mathcal{Q}$:

$$\begin{aligned} & \boldsymbol{g}(q)(w, \boldsymbol{f}) \\ = \; & \boldsymbol{f}(q) \\ = \; & q \in \mathcal{F} \\ = \; & \overline{q \in \mathcal{F}'} \\ = \; & \overline{\boldsymbol{f}'(q)} \\ = \; & \overline{\boldsymbol{g}'(q)(w, \boldsymbol{f}')} \end{aligned}$$

- If $w = \langle v, a \rangle$ with $v \in \mathcal{D}^{\mathcal{X}}$ and $a \in \Sigma$, then for each $q \in \mathcal{Q}$:

$$\begin{aligned} & \boldsymbol{g}(q)(w, \boldsymbol{f}) \\ = \; & \boldsymbol{g}(q)(a)(v)(\boldsymbol{f}) \\ = \; & \boldsymbol{g}(q)(a)(v)(\boldsymbol{f})_{[\boldsymbol{f}(q_t)/\overline{\boldsymbol{f}'(q_t)}]} \quad \diamondsuit \\ = \; & \overline{\overline{\boldsymbol{g}(q)(a)(v)(\boldsymbol{f})_{[\boldsymbol{f}(q_t)/\overline{\boldsymbol{f}'(q_t)}]}}} \\ = \; & \overline{\boldsymbol{g}'(q)(a)(v)(\boldsymbol{f}')} \\ = \; & \overline{\boldsymbol{g}'(q)(w, \boldsymbol{f}')} \end{aligned}$$

$\diamondsuit$ $q_t \in \mathcal{Q}$ is the successor of q with symbol a.

Now, let's ***suppose*** that for each k-length non-empty($k \geq 1$) word $w' = \langle v', a' \rangle w''$ with $v' \in \mathcal{D}^{\mathcal{X}}$ and $a' \in \Sigma$ and $w'' \in (\mathcal{D}^{\mathcal{X}} \times \Sigma)^{k-1}$, we always have $\boldsymbol{g}(q)(w', \boldsymbol{f}) = \overline{\boldsymbol{g}'(q)(w', \boldsymbol{f}')}$, then for any $(k+1)$-length word $w = \langle v, a \rangle w' = \langle v, a \rangle \langle v', a' \rangle w''$ with $v \in \mathcal{D}^{\mathcal{X}}$ and $a \in \Sigma$, we can have:

$$\begin{aligned} & \boldsymbol{g}(q)(w, \boldsymbol{f}) \\ = \; & \boldsymbol{g}(q)(\langle v, a \rangle \langle v', a' \rangle w'', \boldsymbol{f}) \\ = \; & \boldsymbol{g}(q)(a)(v, v')(\boldsymbol{f})_{[\boldsymbol{f}(q_t)/\boldsymbol{g}(q_t)(\langle v', a' \rangle w'', \boldsymbol{f})]} \quad \clubsuit \\ = \; & \boldsymbol{g}(q)(a)(v, v')(\boldsymbol{f})_{[\boldsymbol{f}(q_t)/\boldsymbol{g}(q_t)(w', \boldsymbol{f})]} \\ = \; & \boldsymbol{g}(q)(a)(v, v')(\boldsymbol{f})_{[\boldsymbol{f}(q_t)/\overline{\boldsymbol{g}'(q_t)(w', \boldsymbol{f}')}]} \\ = \; & \overline{\overline{\boldsymbol{g}(q)(a)(v, v')(\boldsymbol{f})_{[\boldsymbol{f}(q_t)/\overline{\boldsymbol{g}'(q_t)(w', \boldsymbol{f}')}]}}} \\ = \; & \overline{\boldsymbol{g}'(q)(a)(v, v')(\boldsymbol{f}')_{[\boldsymbol{f}'(q_t)/\boldsymbol{g}'(q_t)(w', \boldsymbol{f}')]}} \\ = \; & \overline{\boldsymbol{g}'(q)(w, \boldsymbol{f}')} \end{aligned}$$

$\clubsuit$ $q_t \in \mathcal{Q}$ is the successor of q with symbol a.

We already have $\boldsymbol{g}(q)(w, \boldsymbol{f}) = \overline{\boldsymbol{g}'(q)(w, \boldsymbol{f}')}$ when the length is 1. Therefore, we can have it for all the length $k \geq 1$.

Hence, $\boldsymbol{g}(\boldsymbol{i})(w, \boldsymbol{f}) = \overline{\boldsymbol{g}'(\boldsymbol{i})(w, \boldsymbol{f}')}$ with $w \in (\mathcal{D}^{\mathcal{X}} \times \Sigma)^*$.