OQS Final Project

Aaron Augustine - 20171207
Athreya Chandramouli - 2018121002

November 2020

1 Problem Statement

a. Given the following map

$$\Lambda_T(\rho) = \begin{pmatrix} \rho_{33} & \rho_{23} & \rho_{13} \\ \rho_{32} & \rho_{22} & \rho_{12} \\ \rho_{31} & \rho_{21} & \rho_{11} \end{pmatrix}$$

where,

$$\rho = \begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{13} \\ \rho_{21} & \rho_{22} & \rho_{23} \\ \rho_{31} & \rho_{32} & \rho_{33} \end{pmatrix}$$

Prove the map is positive, but not completely positive.

b. Now consider the state:

$$\rho = p\frac{\mathbb{I}}{9} + (1-p)\left|\Psi\right\rangle\left\langle\Psi\right|$$

where $|\Psi\rangle$ is a maximally entangled state:

$$\Psi = \frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle)$$

$$|0\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad |2\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

by applying map: $(\mathbb{I} \otimes \Lambda_T)$, find the threshold value of p, under which the state is entangled.

2 Solution

2.1 Problem a:

$$\Lambda_T(\rho) = \begin{pmatrix} \rho_{33} & \rho_{23} & \rho_{13} \\ \rho_{32} & \rho_{22} & \rho_{12} \\ \rho_{31} & \rho_{21} & \rho_{11} \end{pmatrix} \tag{1}$$

We know that

$$\rho = \sum_i p_i \left|\Psi_i\right\rangle\left\langle\Psi_i\right| \tag{2}$$

If $\Lambda_T(\left|\Psi_i\right\rangle\left\langle\Psi_i\right|) \geq 0$ whenever $\left|\Psi_i\right\rangle\left\langle\Psi_i\right| \geq 0 \quad \forall i$

then, $\Lambda_T(\rho) \geq 0$

Now given ρ is PSD, by **Sylvester's criterion**, 1st order principal minors are:

$$\rho_{11} \geq 0,\ \rho_{22} \geq 0,\ \rho_{33} \geq 0 \tag{3}$$

2nd order principal minors are:

$$\begin{vmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{vmatrix} \geq 0 \quad \Rightarrow \ \rho_{11}\rho_{22} - \rho_{12}\rho_{21} \geq 0 \tag{4}$$

$$\begin{vmatrix} \rho_{11} & \rho_{13} \\ \rho_{31} & \rho_{33} \end{vmatrix} \geq 0 \quad \Rightarrow \ \rho_{11}\rho_{33} - \rho_{13}\rho_{31} \geq 0 \tag{5}$$

$$\begin{vmatrix} \rho_{22} & \rho_{23} \\ \rho_{32} & \rho_{33} \end{vmatrix} \geq 0 \quad \Rightarrow \ \rho_{22}\rho_{33} - \rho_{23}\rho_{32} \geq 0 \tag{6}$$

3rd and highest order principal minor is:

$$\begin{vmatrix} \rho_{11} & \rho_{12} & \rho_{13} \\ \rho_{21} & \rho_{22} & \rho_{23} \\ \rho_{31} & \rho_{32} & \rho_{33} \end{vmatrix} \geq 0 \quad \Rightarrow \ \rho_{11}(\rho_{22}\rho_{33} - \rho_{23}\rho_{32}) - \rho_{12}(\rho_{21}\rho_{33} - \rho_{23}\rho_{31}) + \rho_{13}(\rho_{21}\rho_{32} - \rho_{22}\rho_{31}) \geq 0 \tag{7}$$

$$= \rho_{11}\rho_{22}\rho_{33} - \rho_{11}\rho_{23}\rho_{32} - \rho_{12}\rho_{21}\rho_{33} + \rho_{12}\rho_{23}\rho_{31} + \rho_{13}\rho_{21}\rho_{32} - \rho_{13}\rho_{22}\rho_{31} \geq 0 \tag{8}$$

Now, we do not know if ρ was formed from a pure state. However, for any combination of mixed states such that $|\Psi\rangle\langle\Psi| \geq 0$, for a map Λ, using **Eq. 2** we get:

$$\Lambda(\rho) = \sum_i p_i\ \Lambda(|\Psi_i\rangle\langle\Psi_i|) \tag{9}$$

But if $\Lambda(|\Psi_i\rangle\langle\Psi_i|) \geq 0 \quad \forall i,$ then $\Lambda(\rho)$ becomes a linear combination of PSD matrices.

($\because \forall i, \quad p_i \geq 0$ as p_i represents probability)

Hence, $\Lambda(\rho)$ becomes a linear combination of PSD matrices with non-negative scalar coefficients.

$$\Rightarrow \Lambda(\rho) \geq 0 \tag{10}$$

Therefore, we can prove positivity of the map for a density matrix of arbitrary pure states and that is sufficient to prove positivity of the map on density matrices across all states.
So let ρ be a density matrix formed from an aribitrary pure state

Consider the density matrix after applying the map as given in **Eq. 1**,

1st order principal minors are:

$$\rho_{33}, \rho_{22}, \rho_{11} \geq 0 \tag{11}$$

[from (**3**)]

2nd order principal minors are:

$$\begin{vmatrix} \rho_{33} & \rho_{23} \\ \rho_{32} & \rho_{22} \end{vmatrix} \geq 0 \quad \Rightarrow \rho_{33}\rho_{22} - \rho_{23}\rho_{32} \geq 0 \tag{12}$$

[from (**6**)]

$$\begin{vmatrix} \rho_{33} & \rho_{13} \\ \rho_{31} & \rho_{11} \end{vmatrix} \geq 0 \quad \Rightarrow \rho_{33}\rho_{11} - \rho_{13}\rho_{31} \geq 0 \tag{13}$$

[from (**5**)]

$$\begin{vmatrix} \rho_{22} & \rho_{12} \\ \rho_{21} & \rho_{11} \end{vmatrix} \geq 0 \quad \Rightarrow \rho_{22}\rho_{11} - \rho_{12}\rho_{21} \geq 0 \tag{14}$$

[from (**4**)]

3rd and highest order principal minor is:

$$\begin{vmatrix} \rho_{33} & \rho_{23} & \rho_{13} \\ \rho_{32} & \rho_{22} & \rho_{12} \\ \rho_{31} & \rho_{21} & \rho_{11} \end{vmatrix} \geq 0 \quad \Rightarrow \rho_{33}(\rho_{22}\rho_{11} - \rho_{12}\rho_{21}) - \rho_{23}(\rho_{32}\rho_{11} - \rho_{12}\rho_{31}) + \rho_{13}(\rho_{32}\rho_{21} - \rho_{22}\rho_{31}) \tag{15}$$

$$= \rho_{33}\rho_{22}\rho_{11} - \rho_{33}\rho_{12}\rho_{21} - \rho_{23}\rho_{32}\rho_{11} + \rho_{23}\rho_{12}\rho_{31} + \rho_{13}\rho_{32}\rho_{21} - \rho_{13}\rho_{22}\rho_{31} \geq 0 \tag{16}$$

[from (**8**)]

Hence by using **Sylvester's criterion**,

$$\Lambda_T(\rho) \geq 0, \text{ and the map is positive} \tag{17}$$

According to **Choi-Jamilkoski theorem**, the map will be completely positive if when it acts on one side of a $d \times d$ dimensional maximally entangled state, we get a positive output.
Here d = 3 and hence $d \times d = 9$.

Consider the maximally entangled state given in **Problem b**:

$$|\Psi\rangle = \frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle) \tag{18}$$

where,

$$0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad 1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad 2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \tag{19}$$

Thus,

$$|\Psi\rangle = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ 0 \\ 0 \\ 0 \\ \frac{1}{\sqrt{3}} \\ 0 \\ 0 \\ 0 \\ \frac{1}{\sqrt{3}} \end{pmatrix} \tag{20}$$

And,

$$|\Psi\rangle\langle\Psi| = \begin{pmatrix} \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} \end{pmatrix} \tag{21}$$

Applying the partial map on this matrix,

$$\mathbb{I} \otimes \Lambda_T(\rho) \quad \Rightarrow \begin{pmatrix} \Lambda_T \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \Lambda_T \begin{pmatrix} 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \Lambda_T \begin{pmatrix} 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \Lambda_T \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \Lambda_T \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 \end{pmatrix} & \Lambda_T \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 \end{pmatrix} \\ \Lambda_T \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 \end{pmatrix} & \Lambda_T \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \end{pmatrix} & \Lambda_T \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix} \end{pmatrix} \tag{22}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{23}$$

Let this matrix be A. To calculate its eigenvalues,

$Av = \lambda v$
$\Rightarrow (A - \mathbf{I}\lambda)v = 0$
$\Rightarrow |A - \mathbf{I}\lambda| = 0$

$$\Rightarrow \begin{vmatrix} -\lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} \\ 0 & -\lambda & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3}-\lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\lambda & 0 & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{3}-\lambda & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 & -\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3}-\lambda & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & -\lambda & 0 \\ \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\lambda \end{vmatrix} = 0 \tag{24}$$

On running this with sympy using the provided code, we get a negative root (-0.3333330951). Hence, Λ_T is not a completely positive map.

Code:

```
from sympy.abc import x,p
from sympy import poly, roots, Matrix, eye
import numpy as np

A = Matrix([[-x,0,0,0,0,0,0,0,1/3],
            [0,-x,0,0,0,1/3,0,0,0],
            [0,0,1/3-x,0,0,0,0,0,0],
            [0,0,0,-x,0,0,0,1/3,0],
            [0,0,0,0,1/3-x,0,0,0,0],
            [0,1/3,0,0,0,-x,0,0,0],
            [0,0,0,0,0,0,1/3-x,0,0],
            [0,0,0,1/3,0,0,0,-x,0],
            [1/3,0,0,0,0,0,0,0,-x]])


print(A.det())

det = poly(A.det())
r = roots(det)

print(r)
```

2.2 Problem b:

Positive maps can detect entangled states. If we have separable states given by $\sigma_{AB} = \sum_i p_i \sigma_i^A \otimes \sigma_i^B$, and a positive map Λ is acting on the state space of σ_i^B, then:

$$\Lambda(\sigma_i^B) \geq 0 \quad \forall i \tag{25}$$

Therefore,

$$\mathbb{I} \otimes \Lambda(\sigma_{AB}) = \mathbb{I} \otimes \Lambda(\sum_i p_i \sigma_i^A \otimes \sigma_i^B) \tag{26}$$

$$= \sum_i p_i \sigma_i^A \otimes \Lambda(\sigma_i^B) \tag{27}$$

[Linearity property]

$$\sum_i p_i \sigma_i^A \otimes \Lambda(\sigma_i^B) \geq 0 \tag{28}$$

[from (**25**)]

Hence, a positive map acted upon any separable states will always yield positive output. This isn't the case for entangled states. So we can apply the map in a similar fashion to **Eq. 21** and check for complete positivity while varying p, to determine the threshold of entanglement. We get the matrix as:

$$\rho = p\frac{\mathbb{I}}{9} + (1-p)\left|\Psi\right\rangle\left\langle\Psi\right| \tag{29}$$

$$\Rightarrow \begin{pmatrix} \frac{p}{9} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{p}{9} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{p}{9} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{p}{9} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{p}{9} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{p}{9} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{p}{9} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{p}{9} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{p}{9} \end{pmatrix} + \begin{pmatrix} \frac{1-p}{3} & 0 & 0 & 0 & \frac{1-p}{3} & 0 & 0 & 0 & \frac{1-p}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1-p}{3} & 0 & 0 & 0 & \frac{1-p}{3} & 0 & 0 & 0 & \frac{1-p}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1-p}{3} & 0 & 0 & 0 & \frac{1-p}{3} & 0 & 0 & 0 & \frac{1-p}{3} \end{pmatrix} \tag{30}$$

$$= \begin{pmatrix} \frac{p}{9}+\frac{1-p}{3} & 0 & 0 & 0 & \frac{1-p}{3} & 0 & 0 & 0 & \frac{1-p}{3} \\ 0 & \frac{p}{9} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{p}{9} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{p}{9} & 0 & 0 & 0 & 0 & 0 \\ \frac{1-p}{3} & 0 & 0 & 0 & \frac{p}{9}+\frac{1-p}{3} & 0 & 0 & 0 & \frac{1-p}{3} \\ 0 & 0 & 0 & 0 & 0 & \frac{p}{9} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{p}{9} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{p}{9} & 0 \\ \frac{1-p}{3} & 0 & 0 & 0 & \frac{1-p}{3} & 0 & 0 & 0 & \frac{p}{9}+\frac{1-p}{3} \end{pmatrix} \tag{31}$$

Applying the partial map, we obtain:

$$\begin{pmatrix} \frac{p}{9} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1-p}{3} \\ 0 & \frac{p}{9} & 0 & 0 & 0 & \frac{1-p}{3} & 0 & 0 & 0 \\ 0 & 0 & \frac{p}{9}+\frac{1-p}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{p}{9} & 0 & 0 & 0 & \frac{1-p}{3} & 0 \\ 0 & 0 & 0 & 0 & \frac{p}{9}+\frac{1-p}{3} & 0 & 0 & 0 & 0 \\ 0 & \frac{1-p}{3} & 0 & 0 & 0 & \frac{p}{9} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{p}{9}+\frac{1-p}{3} & 0 & 0 \\ 0 & 0 & 0 & \frac{1-p}{3} & 0 & 0 & 0 & \frac{p}{9} & 0 \\ \frac{1-p}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{p}{9} \end{pmatrix} \tag{32}$$

The point at which atleast one eigenvalue of this matrix turns negative, we can say that the state is entangled. So we use sympy to get the eigenvalues of this matrix:

Code:

```
from sympy.abc import x,p
from sympy import poly, roots, Matrix, eye
import numpy as np

B = Matrix([[p/9,0,0,0,0,0,0,0,(1-p)/3],
            [0,p/9,0,0,0,(1-p)/3,0,0,0],
            [0,0,p/9+(1-p)/3,0,0,0,0,0,0],
            [0,0,0,p/9,0,0,0,(1-p)/3,0],
            [0,0,0,0,p/9+(1-p)/3,0,0,0,0],
            [0,(1-p)/3,0,0,0,p/9,0,0,0],
            [0,0,0,0,0,0,p/9+(1-p)/3,0,0],
            [0,0,0,(1-p)/3,0,0,0,p/9,0],
            [(1-p)/3,0,0,0,0,0,0,0,p/9]])

lamda = eye(9)*x
```

```
    B = B - lamda


    charEqn = poly(B.det())

    pvals = []
    results = []

    for i in np.linspace(0,1,100):
        pvals.append(i)
        charEqni = charEqn.eval(p,i)
        ri = roots(charEqni)

    if any(t < 0 for t in ri.keys()):
        results.append(0)
    else:
        results.append(1)
```

And then we follow it up with a binary search on the value of p such that at least on root returns a negative value:

Code:

```
l = 0
r = 1

for i in range(1000):
    m = (l+r)/2
    print(m)
    charEqni = charEqn.eval(p,m)
    ri = roots(charEqni)

    if any(t < 0 for t in ri.keys()):
         l = m
    else:
         r = m

print(l)
```

This provides us with the value 0.749999874999875 (≈ 0.75) for p (assuming p lies between 0 and 1 as p represents probability), the threshold below which the state becomes entangled.

All our code can be found in this repository: `https://github.com/cathreya/OQS-Project`

3 Learnings and Research Applications:

We learned about the applications of positive maps in detecting entangled states in quantum information science. We dealt with Sylvester's criterion and the Choi-Jamilkoski theorem, we even explored the Peres-Horodecki (PPT) criterion and its use in the field. We realized our method was mostly viable for only density matrices upto 3×3 dimensions and such a method would not be sufficient to detect entanglement for higher dimension density matrices.

Recent papers we have briefly gone through tackle ideas such as the use of k-positive maps instead of positivity and complete-positivity. Such a map is shown to be able to catch entangled states missed by partial positive map application and would be more effective for higher dimension density matrices.