Discrete Fourier Transform

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Discrete Time Fourier Transform (DTFT)

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

Inverse Discrete Time Fourier Transform (IDTFT)

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}\, d\omega$$

Why Discrete Fourier Transform?

- DTFT is not a computationally convenient representation of the sequence.
- DFT is the representation of a sequence $x(n)$ by samples of its spectrum $X(\omega)$.
- DFT is a powerful computational tool for performing frequency analysis of discrete time signals.

DFT

$$X(\frac{2\pi}{N}k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}kn} \qquad ; \quad k = 0,1,......., N-1$$

IDFT

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(\frac{2\pi}{N}k) e^{j\frac{2\pi}{N}kn} \qquad ; \quad n = 0,1,......., N-1$$

DFT

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}kn} \qquad ; \quad k = 0,1,......., N-1$$

IDFT

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k) e^{j\frac{2\pi}{N}kn} \qquad ; \quad n = 0,1,......., N-1$$

Sampling in Frequency domain

Let us first consider the sampling the Fourier transform of an aperiodic discrete time sequence.

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

or

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

We sample X(ω) periodically in frequency at a spacing of 2π/N radians between successive samples. Let N be the number of samples in the frequency domain.

At ω = 2π/N, X(ω) becomes,

$$X\left(\frac{2\pi}{N}k\right) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\frac{2\pi}{N}kn} \qquad ; \quad k = 0,1,\ldots\ldots, N-1$$

Subdividing the previous expression into

$$X\left(\frac{2\pi}{N}k\right) = \ldots\ldots\ldots\ldots + \sum_{n=-N}^{-1} x(n) e^{-j\frac{2\pi}{N}kn} + \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}kn}$$

$$+ \sum_{n=N}^{2N-1} x(n) e^{-j\frac{2\pi}{N}kn} + \ldots\ldots\ldots\ldots$$

$$= \sum_{l=-\infty}^{\infty} \sum_{n=lN}^{lN+N-1} x(n) e^{-j\frac{2\pi}{N}kn}$$

changing n to n-lN and interchanging the summation we get,

$$= \sum_{n=0}^{N-1} \sum_{l=-\infty}^{\infty} x(n-lN)\, e^{-j\frac{2\pi}{N}kn}$$

$$= \sum_{n=0}^{N-1} x_p(n)\, e^{-j\frac{2\pi}{N}kn}$$

when no aliasing, $x(n) = x_p(n)$; $0 \le n \le N-1$

therefore

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}kn} \qquad ; \quad k = 0,1,\ldots\ldots,N-1$$

Multiply on both sides by $e^{j\frac{2\pi}{N}mk}$ and sum the product from k = 0 to N-1

$$\sum_{k=0}^{N-1} X(k)e^{j\frac{2\pi}{N}km} = \sum_{k=0}^{N-1}\sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi}{N}kn}e^{j\frac{2\pi}{N}km}$$

$$\sum_{k=0}^{N-1} X(k)e^{j\frac{2\pi}{N}km} = \sum_{k=0}^{N-1}\sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi}{N}k(n-m)}$$

$$\sum_{k=0}^{N-1} X(k)e^{j\frac{2\pi}{N}km} = \sum_{n=0}^{N-1} x(n)\sum_{k=0}^{N-1} e^{-j\frac{2\pi}{N}k(n-m)}$$

The inner summation can be given as,

$$\sum_{k=0}^{N-1} e^{-j\frac{2\pi}{N}k(n-m)} = \begin{cases} N & ; \quad n-m = 0, \pm N, \pm 2N, \ldots\ldots\ldots \\ 0 & ; \quad otherwise \end{cases}$$

Therefore,

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, e^{j\frac{2\pi}{N}kn} \quad ; \quad n = 0,1,2,\ldots\ldots\ldots N-1$$

The no. of complex multiplications = N^2

The no. of complex additions = N(N-1)

Twiddle Factor:

- Let us define a term, $W_N = e^{-j2\pi/N}$

  which is know as twiddle factor.

- The magnitude of the twiddle factor is given by $|e^{-j2\pi/N}| = 1$

  and the phase angle is given by $\angle e^{-j2\pi/N} = -2\pi/N$

- From the magnitude and phase angle values of W_N, we find that the twiddle factor is a vector on the unit circle and it represents N equally spaced samples.

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- Let us consider the term W_N^{kn} where kn=r.

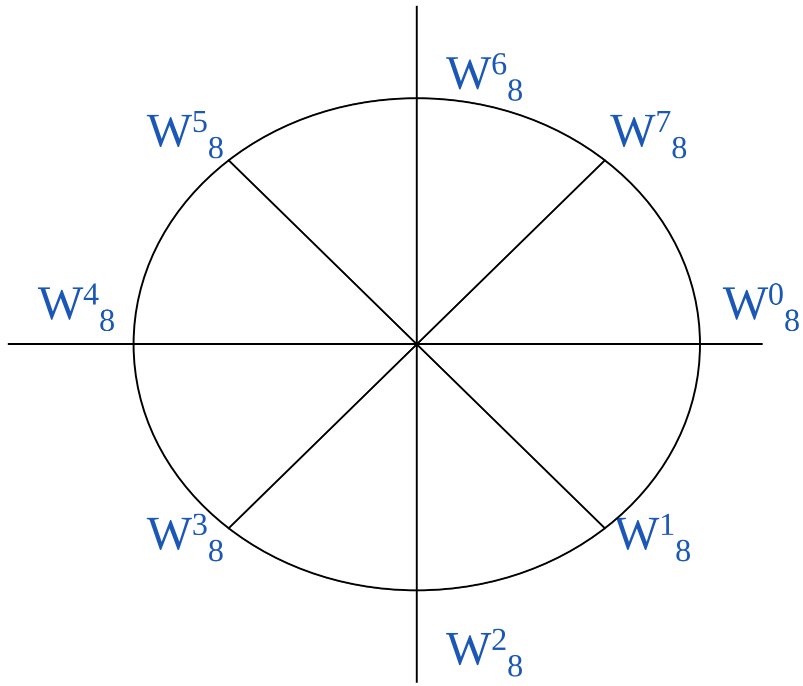
- i.e., $W_N^{kn} = W_N^{r}$

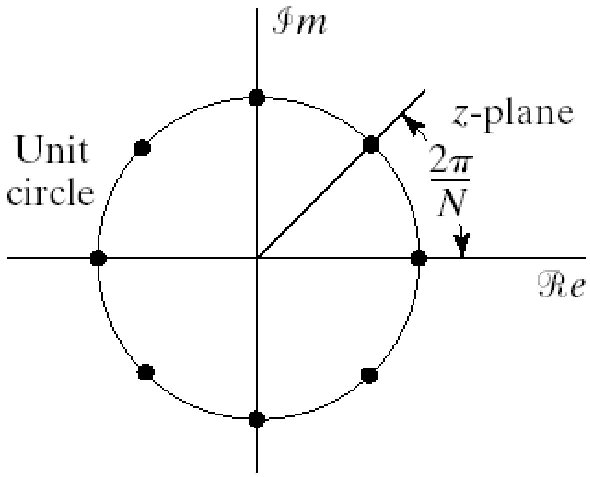
- For N=8, let r = 1, 2,…….., 16

$$W_8^{r} = e^{-j2\pi kn/N} = e^{-j\pi r/4}$$

- Compute

kn=r	W^r_8	$e^{-j\pi r/4}$	Magnitude	Phase angle
0	W^0_8	1	1	0
1	W^1_8	$(1/\sqrt{2})-j(1/\sqrt{2})$	1	$-\pi/4$
2	W^2_8	$-j$	1	$-\pi/2$
3	W^3_8	$-(1/\sqrt{2})-j(1/\sqrt{2})$	1	$-3\pi/4$
4	W^4_8	-1	1	$-\pi$
5	W^5_8	$-(1/\sqrt{2})+j(1/\sqrt{2})$	1	$-5\pi/4$
6	W^6_8	j	1	$-3\pi/2$
7	W^7_8	$(1/\sqrt{2})+j(1/\sqrt{2})$	1	$-7\pi/4$
8	W^8_8	1	1	-2π
9	W^9_8	$(1/\sqrt{2})-j(1/\sqrt{2})$	1	$-9\pi/4$
10	W^{10}_8	$-j$	1	$-5\pi/2$
11	W^{11}_8	$-(1/\sqrt{2})-j(1/\sqrt{2})$	1	$-11\pi/4$
12	W^{12}_8	-1	1	-3π
13	W^{13}_8	$-(1/\sqrt{2})+j(1/\sqrt{2})$	1	$-13\pi/4$
14	W^{14}_8	j	1	$-7\pi/2$
15	W^{15}_8	$(1/\sqrt{2})+j(1/\sqrt{2})$	1	$-15\pi/4$
16	W^{16}_8	1	1	-4π





- From the above figure we can find that W_N^r is a periodic function of r with period N, which is known as periodicity property of twiddle factor, i.e., $W^r = W^{r \pm N} = W^{r \pm 2N} = \ldots\ldots$

- From the table we find the symmetry property of twiddle factor.

$$W^r = -W^{r \pm \frac{N}{2}}$$