Linear Phase FIR Filters

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The main advantages of the FIR filter over IIR filter:

1. FIR filters are always stable.
2. FIR filters with exactly linear phase can easily be designed.
3. FIR filters can be realized in both recursive and non-recursive structures.
4. FIR filters are free of limit cycle oscillations, when implemented on the finite word length digital system.
5. Excellent design methods are available for various kinds of FIR filters.

The disadvantages of FIR filter are:

1. The implementation of narrow transition band FIR filters are very costly, as it requires considerably more arithmetic operations and hardware components such as multipliers, adders and delay elements.
2. Memory requirement and execution time are very high.

Linear Phase FIR filters

The transfer function of a FIR causal filter is given by

$$H(z) = \sum_{n=0}^{N-1} h(n)\, z^{-n} \qquad \text{..........................(1)}$$

where h(n) is the impulse response of the filter.

The Fourier Transform of h(n) is

$$H(z) = \sum_{n=0}^{N-1} h(n)\, e^{-j\omega n} = \pm \left| H(e^{j\omega}) \right| e^{j\theta(\omega)} \qquad \text{..........................(2)}$$

where $|H(e^{j\omega})|$ is magnitude response and $\theta(\omega)$ is phase response.

The Phase delay is $\tau_p = -\theta(\omega) / \omega$ ....(3)

& Group delay is $\tau_g = -d(\theta(\omega)) / d\omega$ ....(4)

For Fir filters with linear phase,

$\theta(\omega) = -\alpha\omega; \; -\pi \leq \omega \leq \pi$ ....(5)

where α is a constant phase delay in samples.

Substituting (5) in (3) and (4) we have $\tau_p = \tau_g = \alpha$, which means that α is independent of frequency.

We can write

$$\sum_{n=0}^{N-1} h(n) e^{-j\omega n} = \pm \left| H(e^{j\omega}) \right| e^{j\theta(\omega)} \qquad \text{..................(6)}$$

Expanding we get,

$$\sum_{n=0}^{N-1} h(n) \cos \omega n = \pm \left| H(e^{j\omega}) \right| \cos \theta(\omega) \qquad \text{..................(7)}$$

$$-\sum_{n=0}^{N-1} h(n) \sin \omega n = \pm \left| H(e^{j\omega}) \right| \sin \theta(\omega) \qquad \text{..................(8)}$$

Dividing (7) by (8) and substituting $\theta(\omega) = -\alpha\omega$, we get

$$\frac{\sum_{n=0}^{N-1} h(n)\cos\omega n}{\sum_{n=0}^{N-1} h(n)\sin\omega n} = \frac{\sin\alpha\omega}{\cos\alpha\omega}$$

rearranging,

$$\sum_{n=0}^{N-1} h(n)\sin(\alpha - n)\omega = 0 \qquad \text{..................(9)}$$

Equation (9) will be zero only when,

$h(n) = h(N-1-n)$ ....(10)

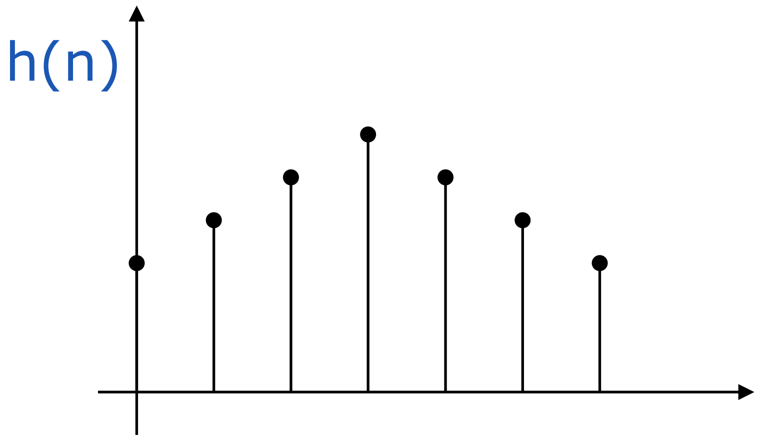
and $\alpha = \dfrac{N-1}{2}$ ....(11)

Therefore, FIR filters will have constant phase and group delays when the impulse response is symmetrical about,

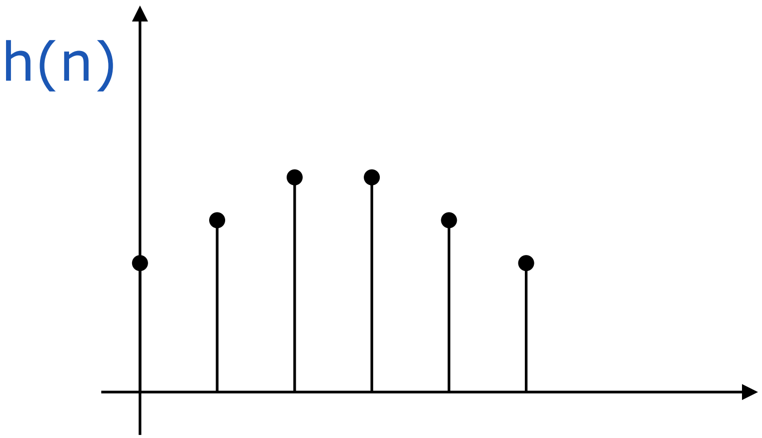
$$\alpha = \frac{N-1}{2}$$

The impulse response satisfying (10) and (11) for odd and even values of N is shown below.

N=7



N=6



If only constant group delay is required, and not the phase delay, we can write,

$$\theta(\omega) = \beta - \alpha\omega \qquad \text{..............(12)}$$

We have

$$H(z) = \pm \left| H(e^{j\omega}) \right| e^{j(\beta - \alpha\omega)} \qquad \text{......................(13)}$$

$$\sum_{n=0}^{N-1} h(n)\, e^{-j\omega n} = \pm \left| H(e^{j\omega}) \right| e^{j(\beta - \alpha\omega)} \qquad \text{......................(14)}$$

Expanding we get,

$$\sum_{n=0}^{N-1} h(n) \cos \omega n = \pm \left| H(e^{j\omega}) \right| \cos(\beta - \alpha\omega) \qquad \text{..................(15)}$$

$$-\sum_{n=0}^{N-1} h(n) \sin \omega n = \pm \left| H(e^{j\omega}) \right| \sin(\beta - \alpha\omega) \qquad \text{..................(16)}$$

Dividing (15) by (16) and rearranging, we get

$$\sum_{n=0}^{N-1} h(n) \sin[\beta - (\alpha - n)\omega] = 0 \qquad \text{..................(17)}$$

If $\beta = \pi/2$, we get,

$$\sum_{n=0}^{N-1} h(n)\cos(\alpha - n)\omega = 0 \qquad \text{..................(18)}$$

Equation (18) will be zero only when,

$h(n) = - h(N-1-n)$ ....(19)

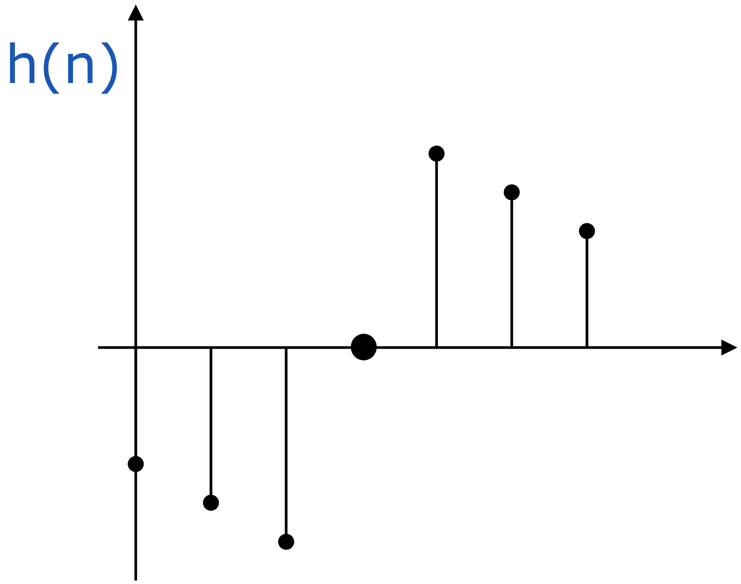
and $$\alpha = \frac{N-1}{2}$$ ....(20)

Therefore, FIR filters will have constant group delay and not constant phase delay when the impulse response is antisymmetrical about,

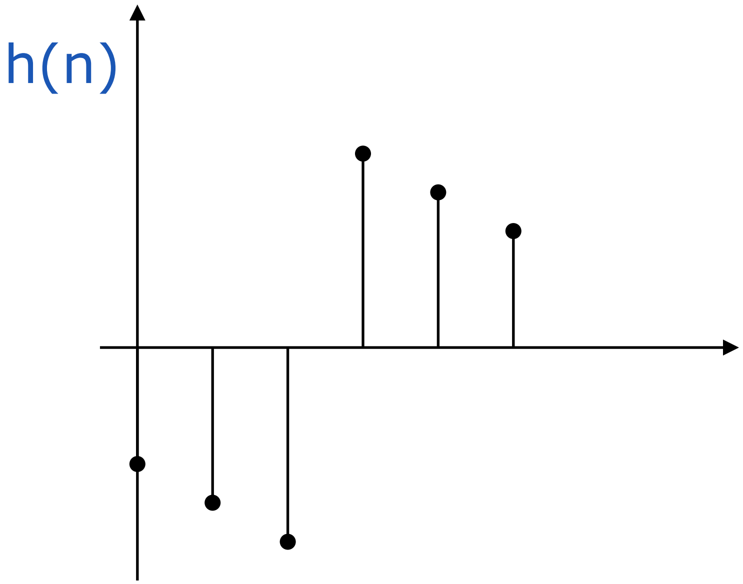
$$\alpha = \frac{N-1}{2}$$

The impulse response satisfying (19) and (20) for odd and even values of N is shown below.

N=7



N=6



Symmetrical FIR Filters

Frequency response of linear phase FIR filters

Case I: Symmetrical impulse response, N odd

The frequency response of impulse response is given by,

$$H(e^{j\omega}) = \sum_{n=0}^{N-1} h(n)\, e^{-j\omega n}$$

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{N-3}{2}} h(n)\, e^{-j\omega n} + h\left(\frac{N-1}{2}\right) e^{-j\omega\frac{N-1}{2}} + \sum_{n=\frac{N+1}{2}}^{N-1} h(n)\, e^{-j\omega n}$$

let n = N-1-n, we have,

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{N-3}{2}} h(n) e^{-j\omega n} + h\left(\frac{N-1}{2}\right) e^{-j\omega\frac{N-1}{2}} + \sum_{n=0}^{\frac{N-3}{2}} h(N-1-n) e^{-j\omega(N-1-n)}$$

For symmetrical response, h(n) = h(N-1-n)

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{N-3}{2}} h(n) e^{-j\omega n} + h\left(\frac{N-1}{2}\right) e^{-j\omega\frac{N-1}{2}} + \sum_{n=0}^{\frac{N-3}{2}} h(n) e^{-j\omega(N-1-n)}$$

$$H(e^{j\omega}) = e^{-j\omega\frac{N-1}{2}} \left[\sum_{n=0}^{\frac{N-3}{2}} h(n) e^{j\omega(\frac{N-1}{2}-n)} + h\left(\frac{N-1}{2}\right) + \sum_{n=0}^{\frac{N-3}{2}} h(n) e^{-j\omega(\frac{N-1}{2}-n)} \right]$$

$$H(e^{j\omega}) = e^{-j\omega\frac{N-1}{2}} \left[\sum_{n=0}^{\frac{N-3}{2}} 2h(n) \cos\omega\left(\frac{N-1}{2} - n\right) + h\left(\frac{N-1}{2}\right) \right]$$

let $\frac{N-1}{2} - n = n$

then

$$H(e^{j\omega}) = e^{-j\omega\frac{N-1}{2}} \left[\sum_{n=1}^{\frac{N-1}{2}} 2h\left(\frac{N-1}{2} - n\right) \cos\omega n + h\left(\frac{N-1}{2}\right) \right]$$

$$H(e^{j\omega}) = e^{-j\omega\frac{N-1}{2}} \sum_{n=0}^{\frac{N-1}{2}} a(n) \cos \omega n$$

where

$$a(n) = 2h\left(\frac{N-1}{2} - n\right) \qquad and \qquad a(0) = h\left(\frac{N-1}{2}\right)$$

We can write

$$H(e^{j\omega}) = e^{-j\omega\frac{N-1}{2}}\ \overline{H}(e^{j\omega}) = \overline{H}(e^{j\omega}) e^{j\theta(\omega)}$$

where

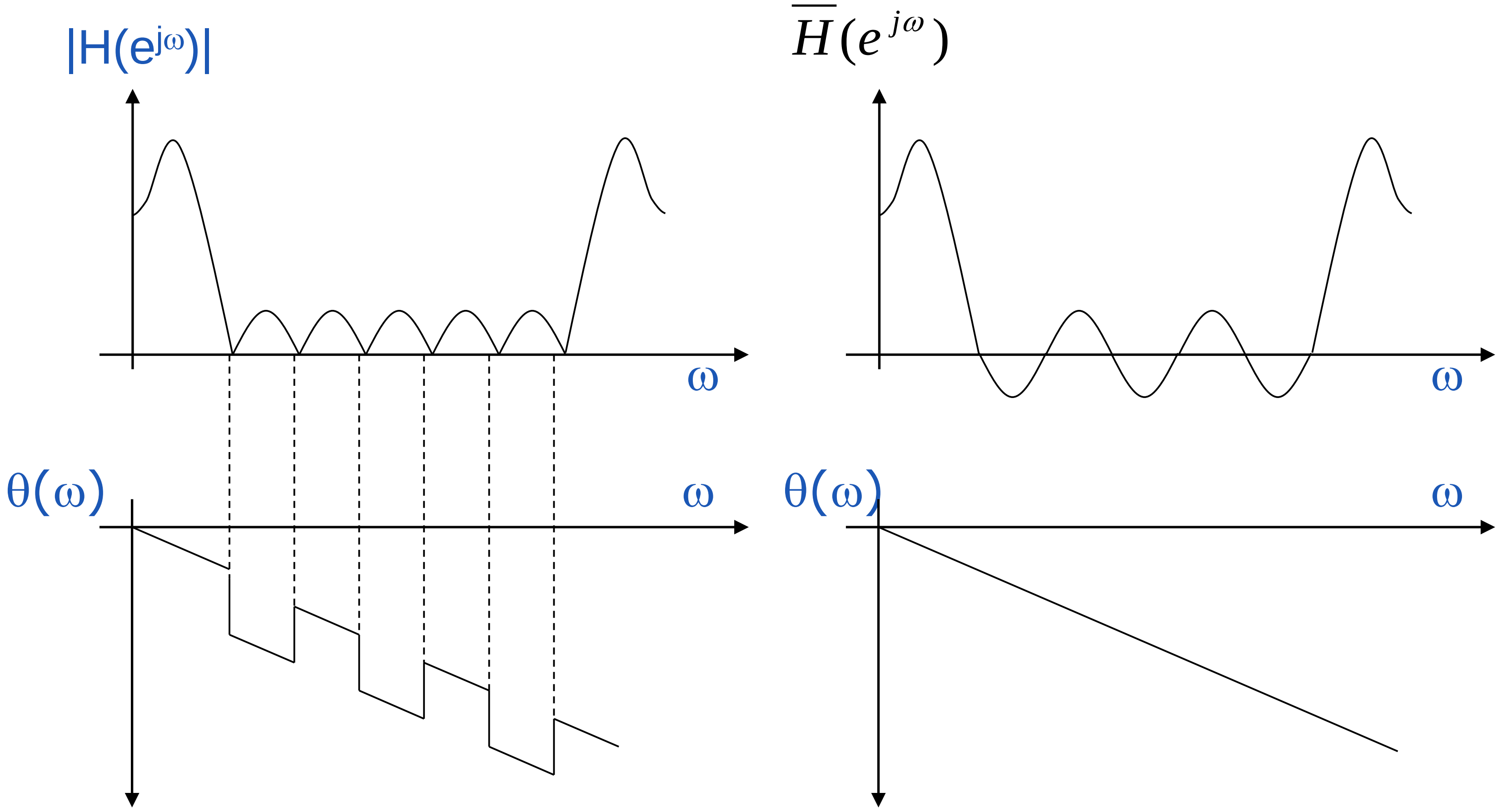
$$\overline{H}(e^{j\omega}) = \sum_{n=0}^{\frac{N-1}{2}} a(n) \cos \omega n$$

and

$$\theta(\omega) = -\alpha\omega = -\left(\frac{N-1}{2}\right)\omega$$

$\overline{H}(e^{j\omega})$ is called as zero – phase frequency response, which takes both positive and negative values, whereas the magnitude response is strictly nonnegative.

The frequency response of symmetric impulse response are shown below



Case II: Symmetrical impulse response, N even

The frequency response of impulse response is given by,

$$H(e^{j\omega}) = \sum_{n=0}^{N-1} h(n)\, e^{-j\omega n}$$

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{N-2}{2}} h(n)\, e^{-j\omega n} + \sum_{n=\frac{N}{2}}^{N-1} h(n)\, e^{-j\omega n}$$

Let n = N-1-n, we have,

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{N-2}{2}} h(n)\, e^{-j\omega n} + \sum_{n=0}^{\frac{N-2}{2}} h(N-1-n)\, e^{-j\omega(N-1-n)}$$

For symmetrical response, h(n) = h(N-1-n)

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{N-2}{2}} h(n)\, e^{-j\omega n} + \sum_{n=0}^{\frac{N-2}{2}} h(n)\, e^{-j\omega(N-1-n)}$$

$$H(e^{j\omega})=e^{-j\omega\frac{N-1}{2}}\left[\sum_{n=0}^{\frac{N-2}{2}}h(n)e^{j\omega(\frac{N-1}{2}-n)}+\sum_{n=0}^{\frac{N-2}{2}}h(n)e^{-j\omega(\frac{N-1}{2}-n)}\right]$$

$$H(e^{j\omega})=e^{-j\omega\frac{N-1}{2}}\left[\sum_{n=0}^{\frac{N-2}{2}}2h(n)\cos\omega\left(\frac{N-1}{2}-n\right)\right]$$

let $$\frac{N-1}{2}-n=n-\frac{1}{2}$$

then $$H(e^{j\omega})=e^{-j\omega\frac{N-1}{2}}\left[\sum_{n=1}^{\frac{N}{2}}2h\left(\frac{N}{2}-n\right)\cos\left(n-\frac{1}{2}\right)\omega\right]$$

$$H(e^{j\omega})=e^{-j\omega\frac{N-1}{2}}\sum_{n=1}^{\frac{N}{2}}b(n)\cos\left(n-\frac{1}{2}\right)\omega$$

where $$b(n)=2h\left(\frac{N}{2}-n\right)$$

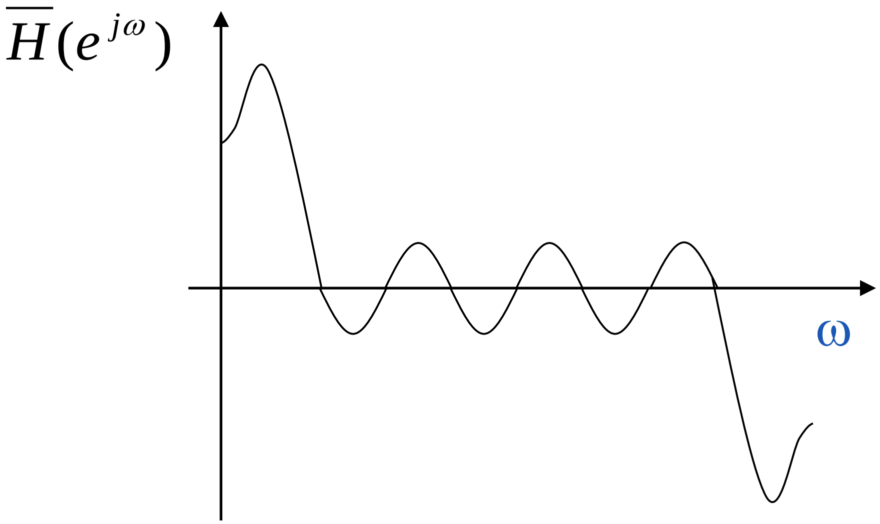
We can write

$$H(e^{j\omega}) = e^{-j\omega\frac{N-1}{2}}\,\overline{H}(e^{j\omega}) = \overline{H}(e^{j\omega})e^{j\theta(\omega)}$$

where
$$\overline{H}(e^{j\omega}) = \sum_{n=0}^{\frac{N}{2}} b(n)\cos\left(n-\frac{1}{2}\right)\omega$$

and
$$\theta(\omega) = -\alpha\omega = -\left(\frac{N-1}{2}\right)\omega$$

The frequency response of symmetric impulse response for N even is shown below



Antisymmetrical FIR Filters

Case III: Antisymmetrical impulse response, N odd

The frequency response of impulse response is given by,

$$H(e^{j\omega}) = \sum_{n=0}^{N-1} h(n) e^{-j\omega n}$$

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{N-3}{2}} h(n) e^{-j\omega n} + h\left(\frac{N-1}{2}\right) e^{-j\omega\frac{N-1}{2}} + \sum_{n=\frac{N+1}{2}}^{N-1} h(n) e^{-j\omega n}$$

Let n = N-1-n, we have,

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{N-3}{2}} h(n) e^{-j\omega n} + h\left(\frac{N-1}{2}\right) e^{-j\omega\frac{N-1}{2}} + \sum_{n=0}^{\frac{N-3}{2}} h(N-1-n) e^{-j\omega(N-1-n)}$$

For antisymmetrical response, h(n) = - h(N-1-n) and h((N-1)/2) = 0

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{N-3}{2}} h(n) e^{-j\omega n} - \sum_{n=0}^{\frac{N-3}{2}} h(n) e^{-j\omega(N-1-n)}$$

$$H(e^{j\omega}) = e^{-j\omega\frac{N-1}{2}} \left[\sum_{n=0}^{\frac{N-3}{2}} h(n) e^{j\omega(\frac{N-1}{2}-n)} - \sum_{n=0}^{\frac{N-3}{2}} h(n) e^{-j\omega(\frac{N-1}{2}-n)} \right]$$

$$H(e^{j\omega}) = e^{-j\omega\frac{N-1}{2}} j \left[\sum_{n=0}^{\frac{N-3}{2}} 2h(n) \sin \omega\left(\frac{N-1}{2} - n \right) \right]$$

let $\frac{N-1}{2} - n = n$

then

$$H(e^{j\omega}) = e^{-j\omega\frac{N-1}{2}} \; e^{j\frac{\pi}{2}} \left[\sum_{n=1}^{\frac{N-1}{2}} 2h\left(\frac{N-1}{2} - n \right) \sin \omega n \right]$$

$$H(e^{j\omega}) = e^{j\left(\frac{\pi}{2} - \frac{N-1}{2}\omega\right)} \sum_{n=1}^{\frac{N-1}{2}} c(n) \sin \omega n$$

where $c(n) = 2h\left(\frac{N-1}{2} - n\right)$

We can write

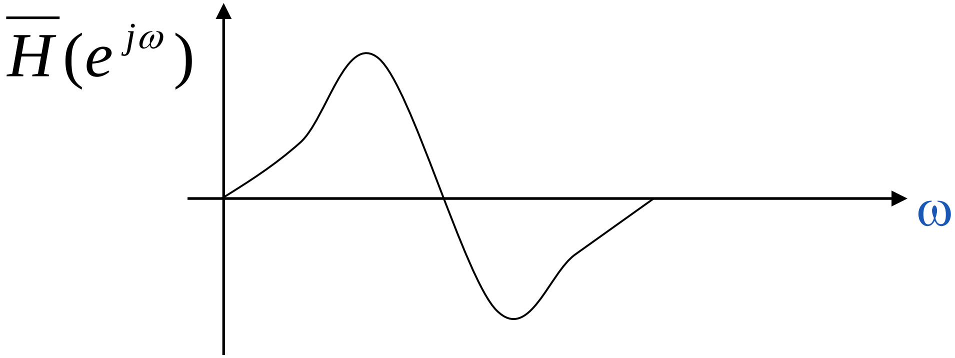
$$H(e^{j\omega}) = e^{j\left(\frac{\pi}{2} - \frac{N-1}{2}\omega\right)} \overline{H}(e^{j\omega}) = \overline{H}(e^{j\omega}) e^{j\theta(\omega)}$$

where

$$\overline{H}(e^{j\omega}) = \sum_{n=0}^{\frac{N-1}{2}} c(n) \sin \omega n$$

and $\theta(\omega) = \beta - \alpha\omega = \frac{\pi}{2} - \frac{N-1}{2}\omega$

The frequency response of antisymmetric impulse response is shown below



Case IV: Antisymmetrical impulse response, N even

The frequency response of impulse response is given by,

$$H(e^{j\omega}) = \sum_{n=0}^{N-1} h(n) e^{-j\omega n}$$

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{N-2}{2}} h(n) e^{-j\omega n} + \sum_{n=\frac{N}{2}}^{N-1} h(n) e^{-j\omega n}$$

let n = N-1-n, we have,

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{N-2}{2}} h(n) e^{-j\omega n} + \sum_{n=0}^{\frac{N-2}{2}} h(N-1-n) e^{-j\omega(N-1-n)}$$

For symmetrical response, h(n) = - h(N-1-n)

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{N-2}{2}} h(n) e^{-j\omega n} - \sum_{n=0}^{\frac{N-2}{2}} h(n) e^{-j\omega(N-1-n)}$$

$$H(e^{j\omega}) = e^{-j\omega\frac{N-1}{2}} \left[\sum_{n=0}^{\frac{N-2}{2}} h(n) e^{j\omega(\frac{N-1}{2}-n)} - \sum_{n=0}^{\frac{N-2}{2}} h(n) e^{-j\omega(\frac{N-1}{2}-n)} \right]$$

$$H(e^{j\omega}) = e^{-j\omega\frac{N-1}{2}} \; j \left[\sum_{n=0}^{\frac{N-2}{2}} 2h(n) \sin \omega\left(\frac{N-1}{2} - n \right) \right]$$

Let $\frac{N-1}{2} - n = n - \frac{1}{2}$

then

$$H(e^{j\omega}) = e^{-j\omega\frac{N-1}{2}} \; e^{j\frac{\pi}{2}} \left[\sum_{n=1}^{\frac{N}{2}} 2h\left(\frac{N}{2} - n\right) \sin\left(n - \frac{1}{2}\right)\omega \right]$$

$$H(e^{j\omega}) = e^{j\left(\frac{\pi}{2} - \frac{N-1}{2}\omega\right)} \sum_{n=1}^{\frac{N}{2}} d(n) \sin\left(n - \frac{1}{2}\right)\omega$$

where
$$d(n)=2h\left(\frac{N}{2}-n\right)$$

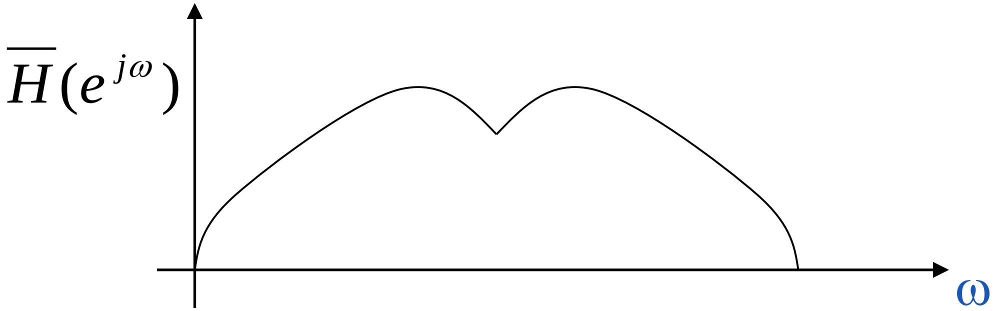
We can write

$$H(e^{j\omega})=e^{j\left(\frac{\pi}{2}-\frac{N-1}{2}\omega\right)}\,\overline{H}(e^{j\omega})=\overline{H}(e^{j\omega})e^{j\theta(\omega)}$$

where
$$\overline{H}(e^{j\omega})=\sum_{n=0}^{\frac{N}{2}}d(n)\,\sin\left(n-\frac{1}{2}\right)\omega$$

and
$$\theta(\omega)=\beta-\alpha\,\omega=\frac{\pi}{2}-\frac{N-1}{2}\omega$$

The frequency response of antisymmetric impulse response for N even is shown below



Note:

1. The impulse response of symmetric with odd number of samples can be used to design all types of filters.
2. The symmetric impulse response having even number of samples cannot be used to design highpass filters.
3. The frequency response of antisymmetric impulse is imaginary and these types of filters are most suitable for Hilbert transformers and differentiators.

Location of the zeros

Location of the zeros of linear phase FIR filters

The transfer function of a linear phase FIR filter is given by

$$H(z) = \sum_{n=0}^{N-1} h(n)\, z^{-n}$$

If $z_0 \neq 0$ is an finite zero of $H(z_0)$, then

$$H(z)\Big|_{z=z_0} = H(z_0) = \sum_{n=0}^{N-1} h(n)\, z_0^{-n} = 0$$

For a linear phase filter, h(n) = h(N-1-n)

$$h(0) + h(1)\, z_0^{-1} + h(2)\, z_0^{-2} + \dots\dots\dots\dots + h(N-1)\, z_0^{-(N-1)} = 0$$

Then $$h(N-1) + h(N-2)\, z_0^{-1} + h(N-3)\, z_0^{-2} + \dots\dots\dots + h(0)\, z_0^{-(N-1)} = 0$$

$$z_0^{-(N-1)} \left[h(N-1)\, z_0^{N-1} + h(N-2)\, z_0^{N-2} + h(N-3)\, z_0^{N-3} + \dots\dots\dots + h(0) \right] = 0$$

$$z_0^{-(N-1)} \sum_{n=0}^{N-1} h(n)\, z_0^{n} = 0$$

$$\sum_{n=0}^{N-1} h(n) [z_0^{-1}]^{-n} = 0$$

$$H(z_0^{-1}) = 0$$

Therefore, we can say that if z_0 is a zero of H(z), then z_0^{-1} is also a zero.

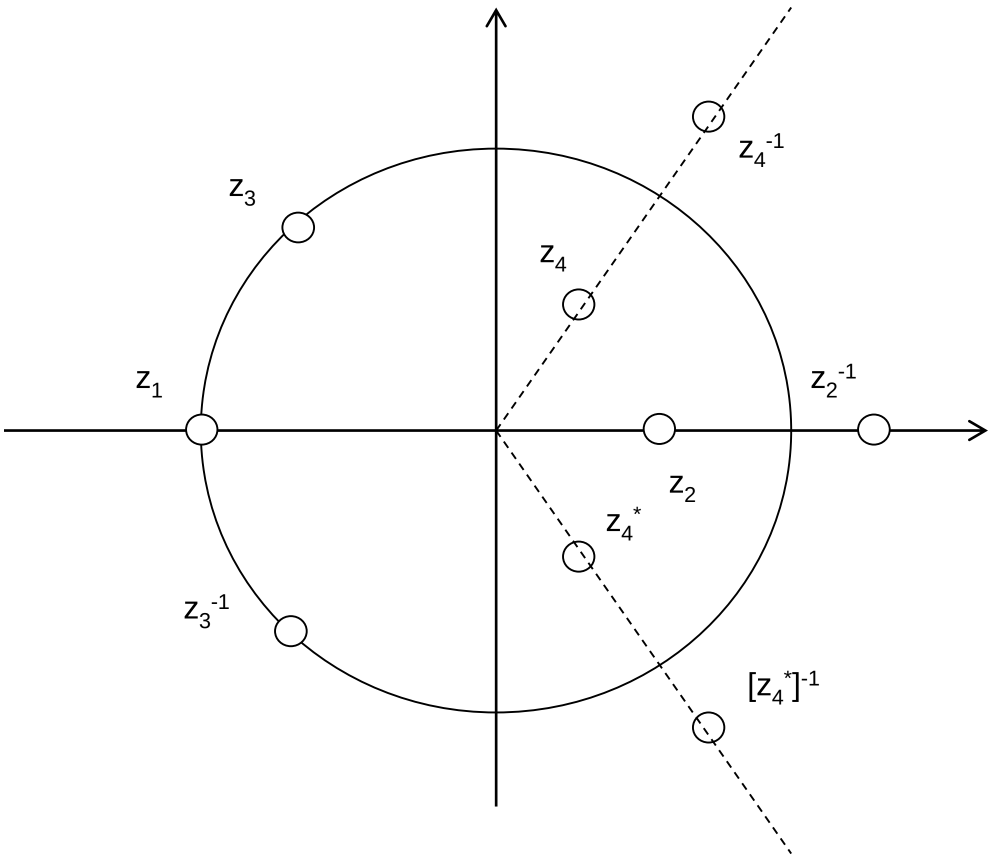
Note:

If $z_1 = -1$, then $z_1^{-1} = z_1$, the zero lies at $z_1 = -1$, this group contains only one zero on the unit circle.

If z_2 is real zero with $|z_2|<1$ then z_2^{-1} is also a real zero and there are two one zeros in this group.

If z_3 is a complex zero with $|z_3| = 1$ then $z_3^{-1} = z_3^*$ and there are two zeros in this group.

If z_4 is a complex zero with $|z_4| \neq 1$, then this group contains four zeros z_4, z_4^{-1}, z_4^* and $(z_4^*)^{-1}$.


z_4^{-1}
z_3
z_4
z_1
z_2^{-1}
z_2
z_4^*
z_3^{-1}
$[z_4^*]^{-1}$