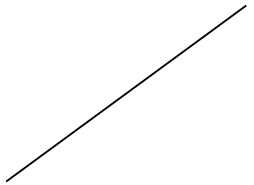
BEIZER CURVES

Polynomial Functions

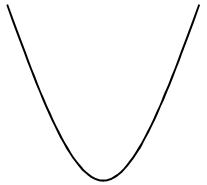
- Linear:

$$f(t) = at + b$$



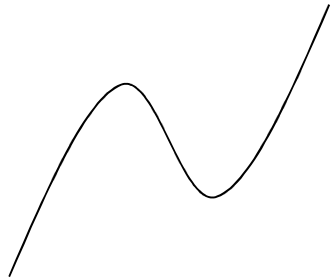
- Quadratic:

$$f(t) = at^2 + bt + c$$



- Cubic:

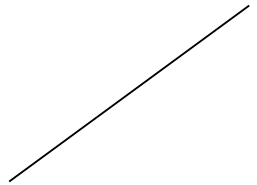
$$f(t) = at^3 + bt^2 + ct + d$$



Vector Polynomials (Curves)

- Linear:

$$\mathbf{f}(t) = \mathbf{a}t + \mathbf{b}$$

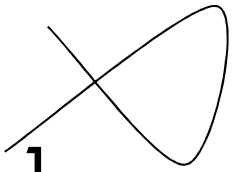


- Quadratic:

$$\mathbf{f}(t) = \mathbf{a}t^2 + \mathbf{b}t + \mathbf{c}$$

- Cubic:

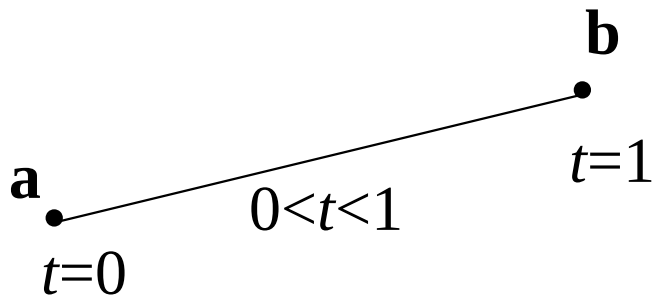
$$\mathbf{f}(t) = \mathbf{a}t^3 + \mathbf{b}t^2 + \mathbf{c}t + \mathbf{d}$$



We usually define the curve for $0 \leq t \leq 1$

Linear Interpolation

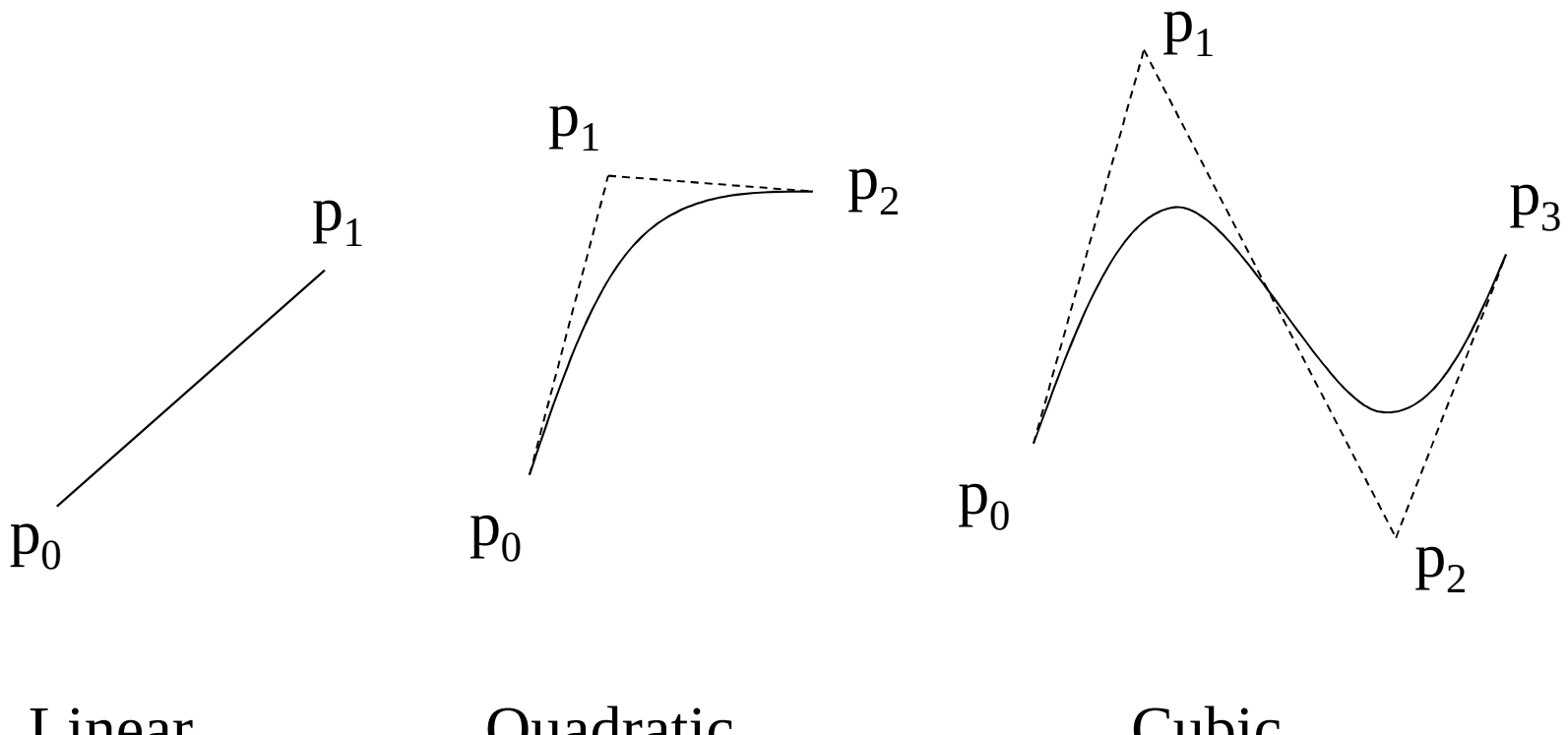
- Linear interpolation (Lerp) is a common technique for generating a new value that is somewhere in between two other values
- A 'value' could be a number, vector, color, or even something more complex like an entire 3D object…
- Consider interpolating between two points **a** and **b** by some parameter *t*



$$Lerp(t, \mathbf{a}, \mathbf{b}) = (1-t)\mathbf{a} + t\mathbf{b}$$

Bezier Curves

- Bezier curves can be thought of as a higher order extension of linear interpolation



Bezier Curves

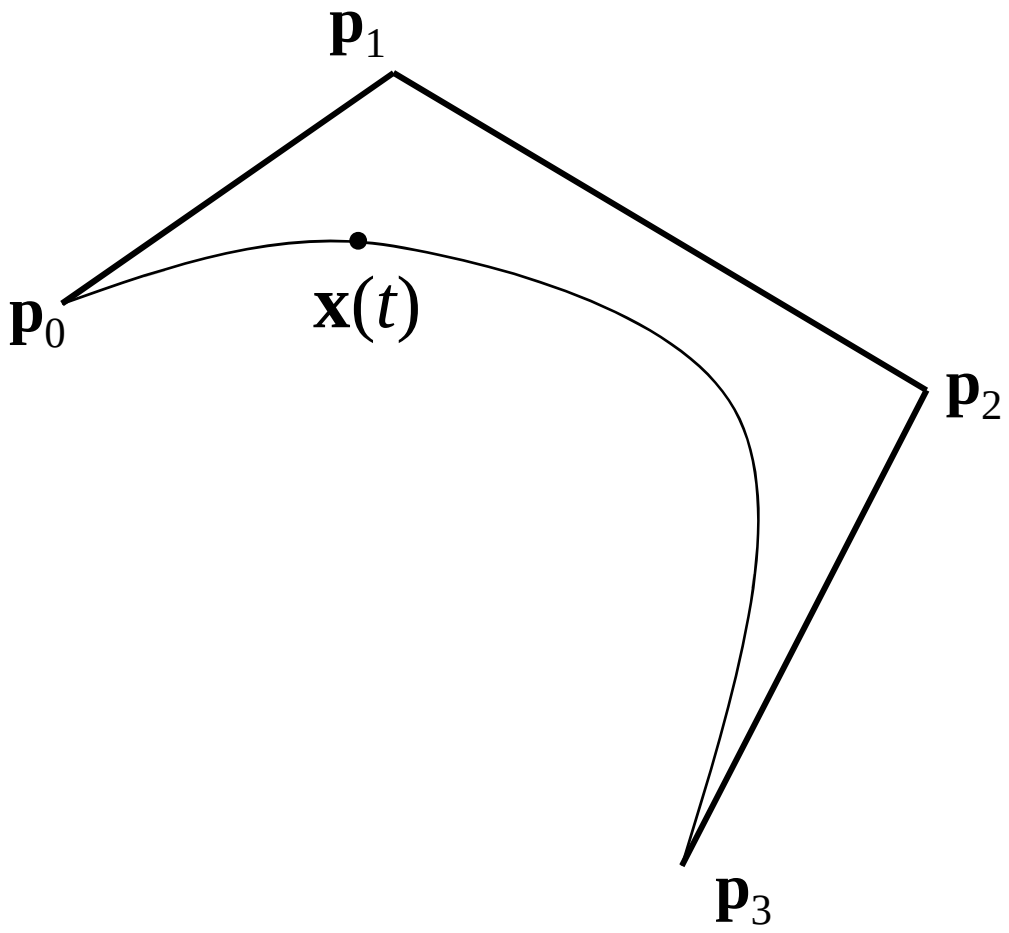
- Spline Approximation method
- Easy to implement
- Curve can be fitted to any number of control points.
- Degree of Bezier polynomial – depends on the no. of ctrl points and their relative position.

Bezier Curve Formulation

- There are lots of ways to formulate Bezier curves mathematically. Some of these include:
  - de Castlejau (recursive linear interpolations)
  - Bernstein polynomials (functions that define the influence of each control point as a function of t)
  - Cubic equations (general cubic equation of t)
  - Matrix form
- We will briefly examine each of these

Bezier Curve

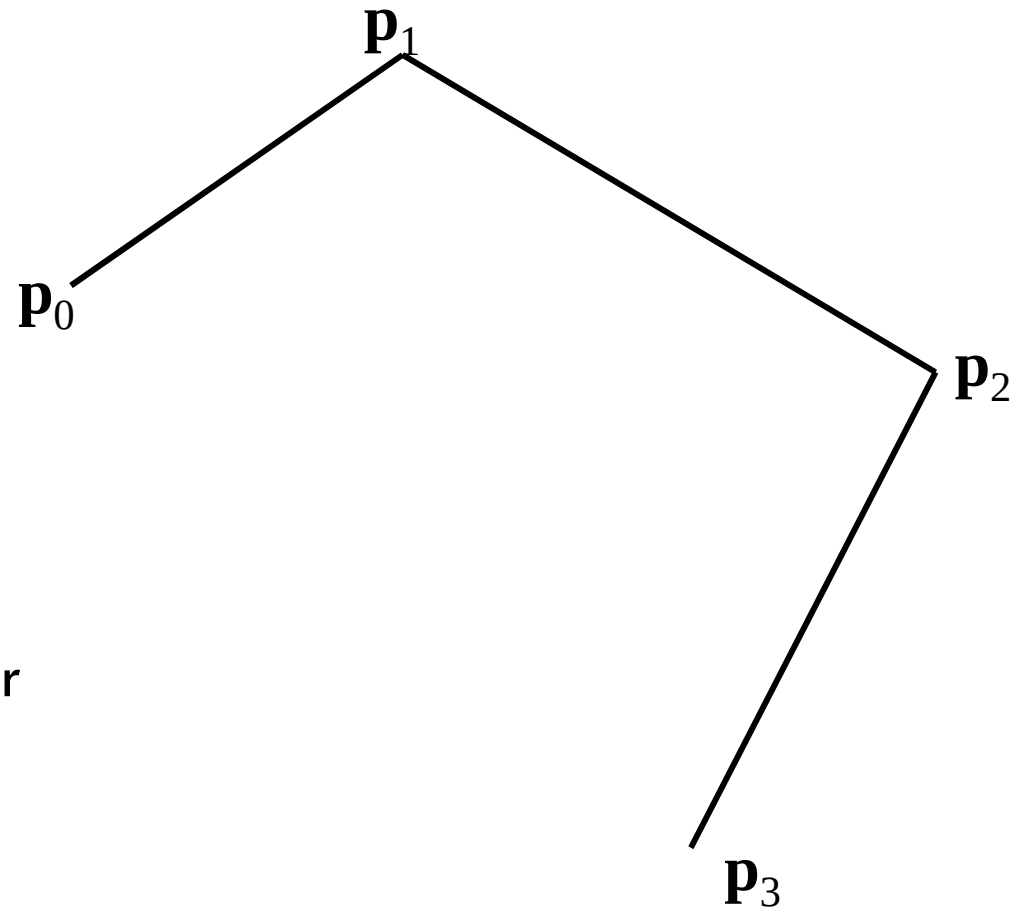
- Find the point **x** on the curve as a function of parameter *t*:



de Casteljau Algorithm

- The de Casteljau algorithm describes the curve as a recursive series of linear interpolations
- This form is useful for providing an intuitive understanding of the geometry involved, but it is not the most efficient form

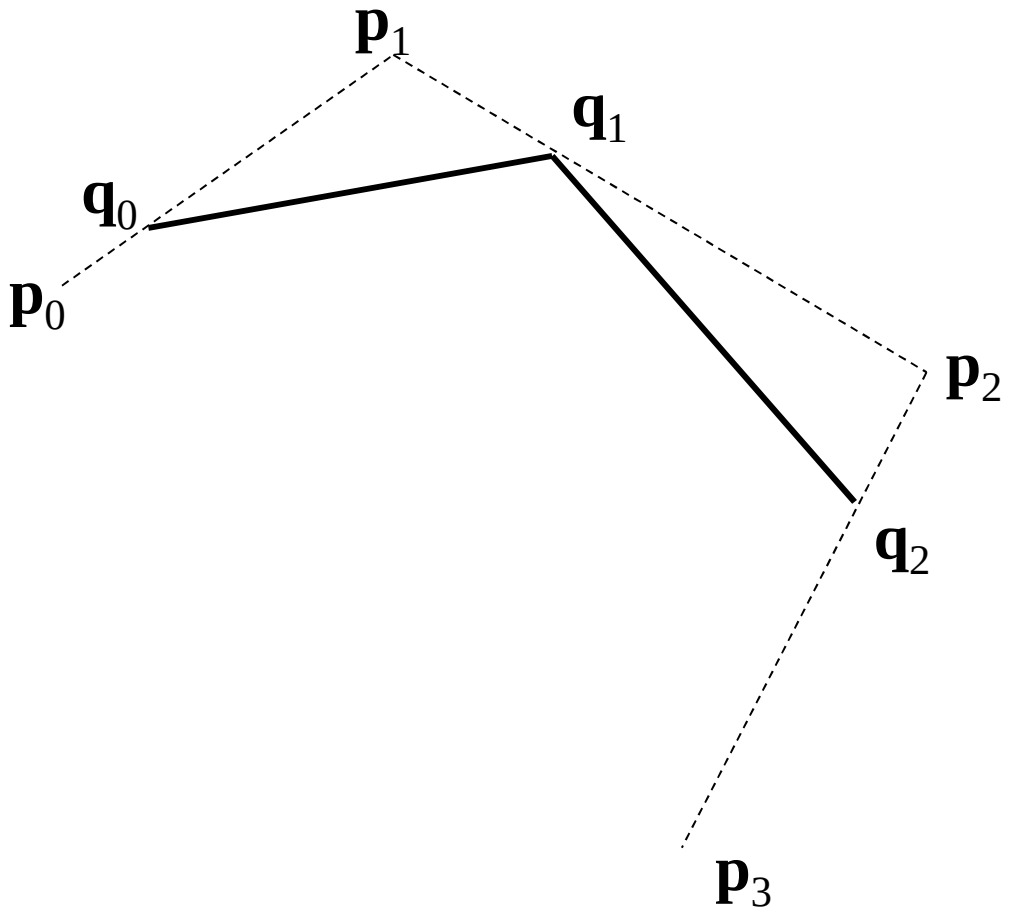
de Casteljau Algorithm



- We start with our original set of points
- In the case of a cubic Bezier curve, we start with four points

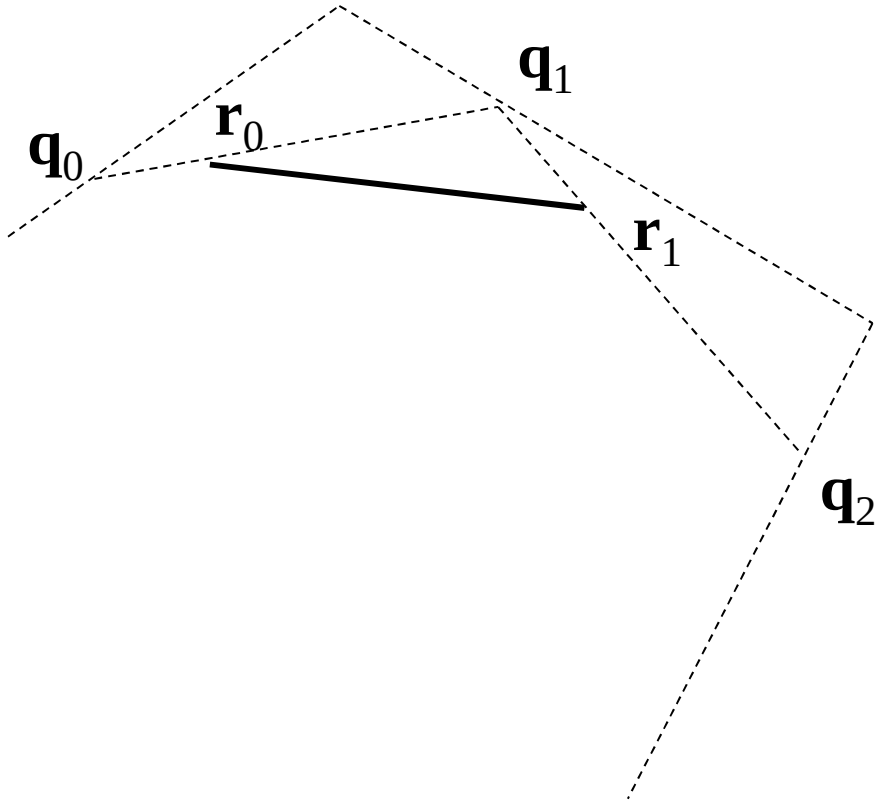
de Casteljau Algorithm

$$\mathbf{q}_0 = Lerp(t, \mathbf{p}_0, \mathbf{p}_1)$$
$$\mathbf{q}_1 = Lerp(t, \mathbf{p}_1, \mathbf{p}_2)$$
$$\mathbf{q}_2 = Lerp(t, \mathbf{p}_2, \mathbf{p}_3)$$



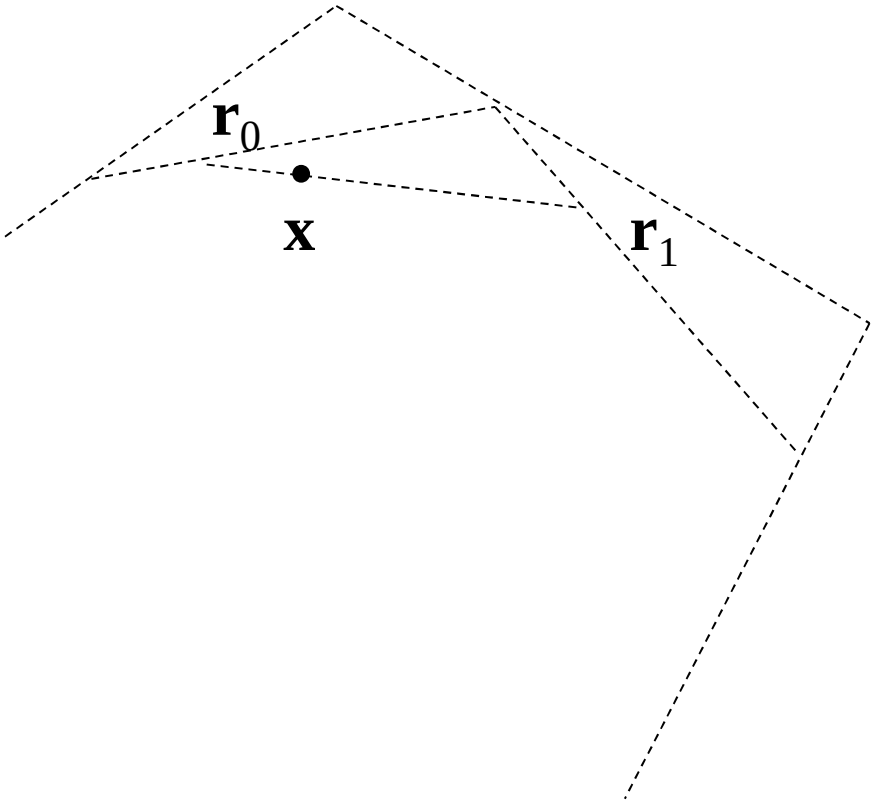
de Casteljau Algorithm

$$\mathbf{r}_0 = Lerp(t, \mathbf{q}_0, \mathbf{q}_1)$$
$$\mathbf{r}_1 = Lerp(t, \mathbf{q}_1, \mathbf{q}_2)$$

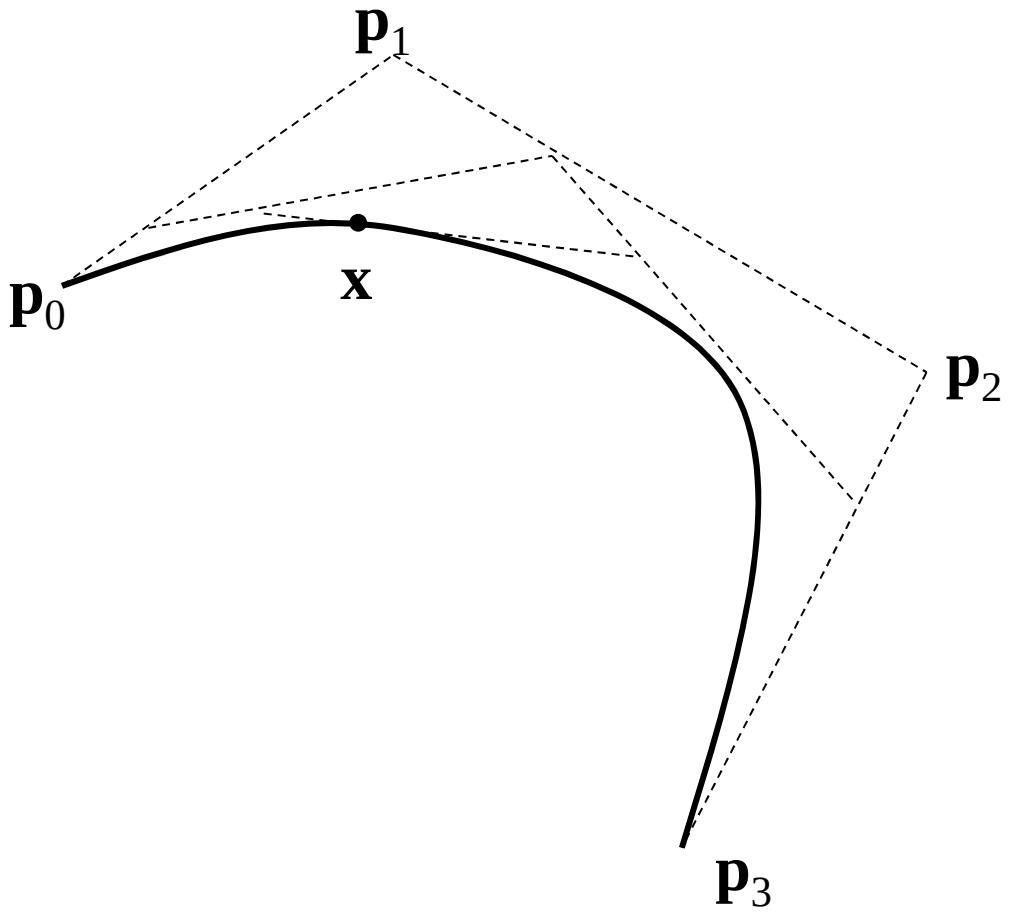


de Casteljau Algorithm

$$\mathbf{x} = Lerp(t, \mathbf{r}_0, \mathbf{r}_1)$$

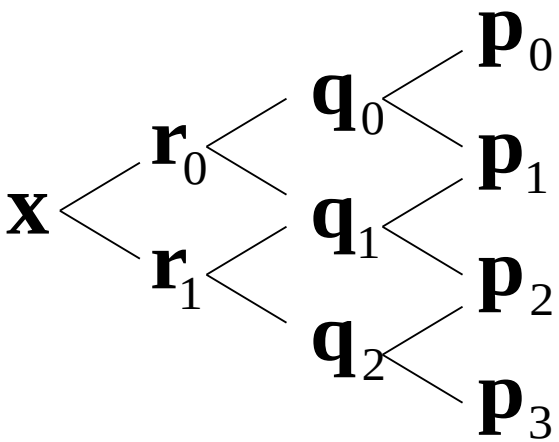


Bezier Curve



Recursive Linear Interpolation

$$\mathbf{x} = Lerp(t, \mathbf{r}_0, \mathbf{r}_1) \quad \begin{matrix} \mathbf{r}_0 = Lerp(t, \mathbf{q}_0, \mathbf{q}_1) \\ \mathbf{r}_1 = Lerp(t, \mathbf{q}_1, \mathbf{q}_2) \end{matrix} \quad \begin{matrix} \mathbf{q}_0 = Lerp(t, \mathbf{p}_0, \mathbf{p}_1) \\ \mathbf{q}_1 = Lerp(t, \mathbf{p}_1, \mathbf{p}_2) \\ \mathbf{q}_2 = Lerp(t, \mathbf{p}_2, \mathbf{p}_3) \end{matrix} \quad \begin{matrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{matrix}$$



Expanding the Lerps

$$\mathbf{q}_0 = Lerp(t, \mathbf{p}_0, \mathbf{p}_1) = (1-t)\mathbf{p}_0 + t\mathbf{p}_1$$
$$\mathbf{q}_1 = Lerp(t, \mathbf{p}_1, \mathbf{p}_2) = (1-t)\mathbf{p}_1 + t\mathbf{p}_2$$
$$\mathbf{q}_2 = Lerp(t, \mathbf{p}_2, \mathbf{p}_3) = (1-t)\mathbf{p}_2 + t\mathbf{p}_3$$

$$\mathbf{r}_0 = Lerp(t, \mathbf{q}_0, \mathbf{q}_1) = (1-t)((1-t)\mathbf{p}_0 + t\mathbf{p}_1) + t((1-t)\mathbf{p}_1 + t\mathbf{p}_2)$$
$$\mathbf{r}_1 = Lerp(t, \mathbf{q}_1, \mathbf{q}_2) = (1-t)((1-t)\mathbf{p}_1 + t\mathbf{p}_2) + t((1-t)\mathbf{p}_2 + t\mathbf{p}_3)$$

$$\mathbf{x} = Lerp(t, \mathbf{r}_0, \mathbf{r}_1) = (1-t)((1-t)((1-t)\mathbf{p}_0 + t\mathbf{p}_1) + t((1-t)\mathbf{p}_1 + t\mathbf{p}_2))$$
$$+ t((1-t)((1-t)\mathbf{p}_1 + t\mathbf{p}_2) + t((1-t)\mathbf{p}_2 + t\mathbf{p}_3))$$

Bernstein Polynomial Form

$$\mathbf{x} = (1-t)((1-t)((1-t)\mathbf{p}_0 + t\mathbf{p}_1) + t((1-t)\mathbf{p}_1 + t\mathbf{p}_2)) + t((1-t)((1-t)\mathbf{p}_1 + t\mathbf{p}_2) + t((1-t)\mathbf{p}_2 + t\mathbf{p}_3))$$

$$\mathbf{x} = (1-t)^3\mathbf{p}_0 + 3(1-t)^2 t\mathbf{p}_1 + 3(1-t)t^2\mathbf{p}_2 + t^3\mathbf{p}_3$$

$$\mathbf{x} = (-t^3 + 3t^2 - 3t + 1)\mathbf{p}_0 + (3t^3 - 6t^2 + 3t)\mathbf{p}_1 + (-3t^3 + 3t^2)\mathbf{p}_2 + (t^3)\mathbf{p}_3$$

Cubic Bernstein Polynomials

$$\mathbf{x} = \left(-t^3 + 3t^2 - 3t + 1\right)\mathbf{p}_0 + \left(3t^3 - 6t^2 + 3t\right)\mathbf{p}_1 + \left(-3t^3 + 3t^2\right)\mathbf{p}_2 + \left(t^3\right)\mathbf{p}_3$$

$$\mathbf{x} = B_0^3(t)\mathbf{p}_0 + B_1^3(t)\mathbf{p}_1 + B_2^3(t)\mathbf{p}_2 + B_3^3(t)\mathbf{p}_3$$

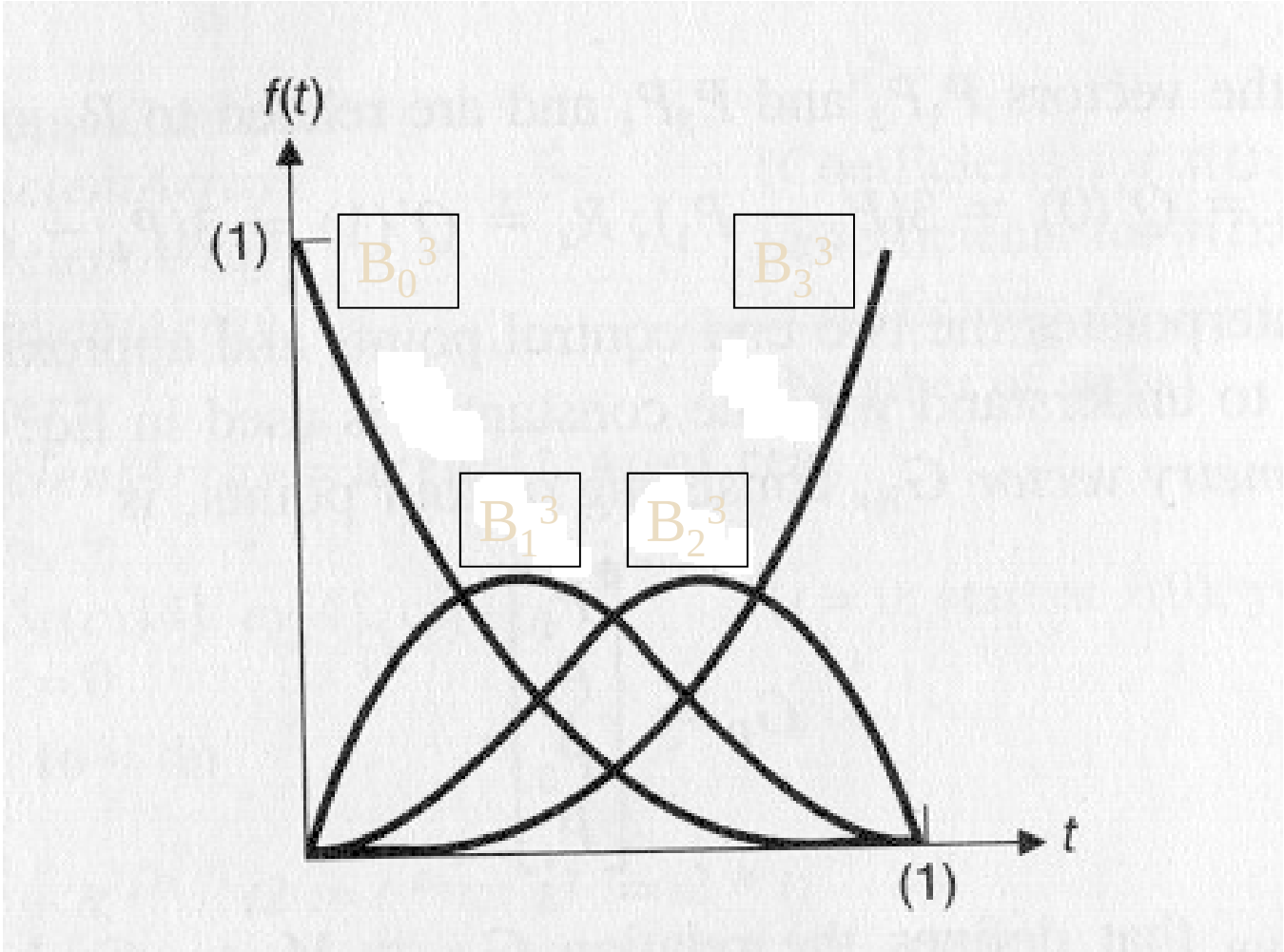
$$B_0^3(t) = -t^3 + 3t^2 - 3t + 1$$

$$B_1^3(t) = 3t^3 - 6t^2 + 3t$$

$$B_2^3(t) = -3t^3 + 3t^2$$

$$B_3^3(t) = t^3$$

Bernstein Polynomials



Bernstein Polynomials

$$B_0^3(t) = -t^3 + 3t^2 - 3t + 1$$
$$B_1^3(t) = 3t^3 - 6t^2 + 3t$$
$$B_2^3(t) = -3t^3 + 3t^2$$
$$B_3^3(t) = t^3$$

$$B_0^2(t) = t^2 - 2t + 1$$
$$B_1^2(t) = -2t^2 + 2t$$
$$B_2^2(t) = t^2$$

$$B_0^1(t) = -t + 1$$
$$B_1^1(t) = t$$

$$B_i^n(t) = \binom{n}{i}(1-t)^{n-i}(t)^i \qquad \binom{n}{i} = \frac{n!}{i!(n-i)!}$$

$$\sum B_i^n(t) = 1$$

Bernstein Polynomials

- No.of ctrl points = n+1
- $P_k = (x_k, y_k, z_k)$ $\quad 0 \le k \le n$
- Bernstein polynomial form of a Bezier curve:
- Pk are control points bernstein basis function is $B_i^n(t)$

$$B_i^n(t) = \binom{n}{i}(1-t)^{n-i}(t)^i$$

$$P(t) = \sum_{i=0}^{n} B_i^n(t)\mathbf{p}_i$$

Bezier Curves

- Individual coordinates are

$$x(u) = \sum_{i=0}^{n} B_i^n(u) x_i$$

$$y(u) = \sum_{i=0}^{n} B_i^n(u) y_i$$

$$z(u) = \sum_{i=0}^{n} B_i^n(u) z_i$$

- Curve is polynomial of degree one less than the no. of ctrl points.

Bezier Curves

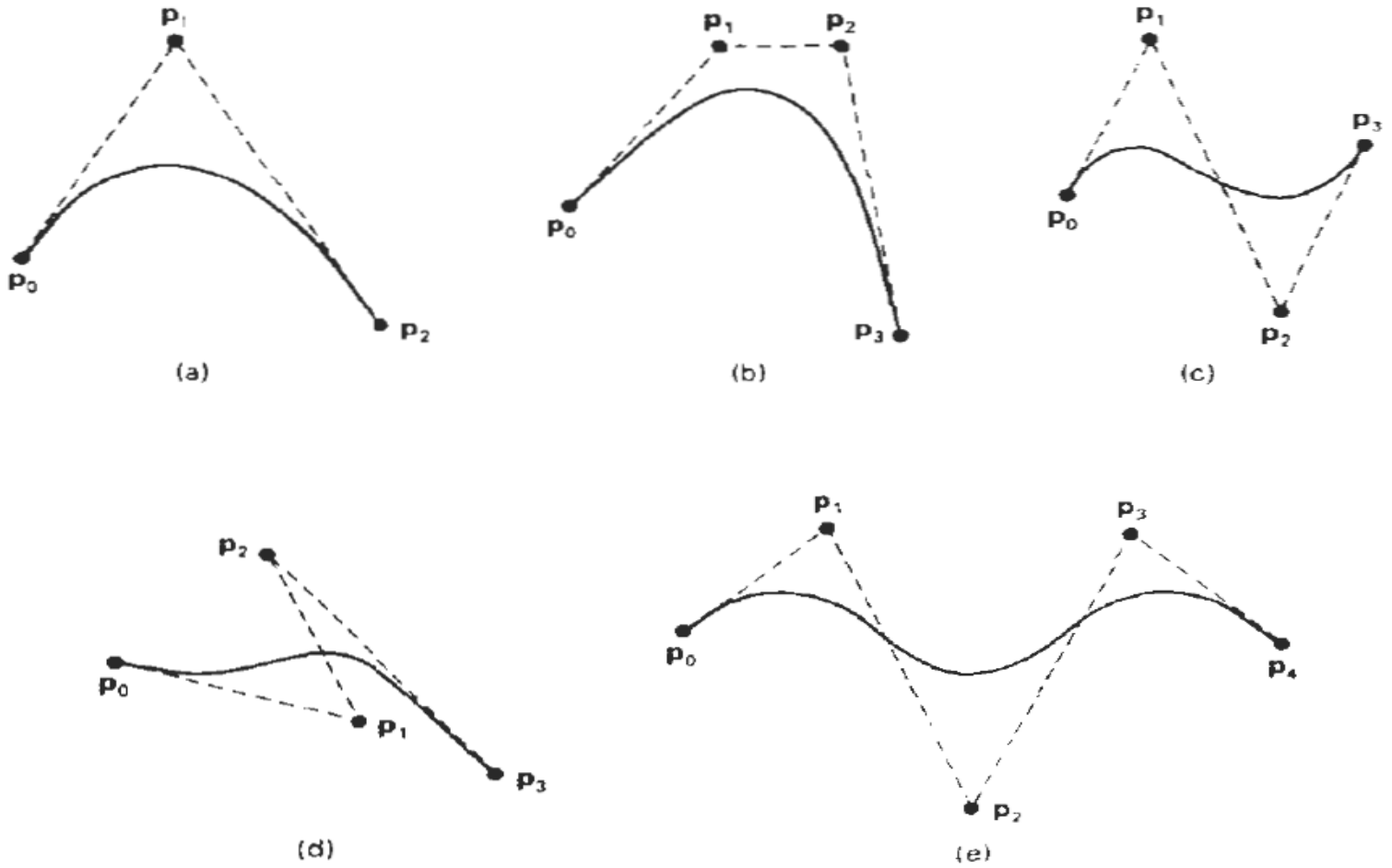


Figure 10-34
Examples of two-dimensional Bézier curves generated from three, four, and five control points. Dashed lines connect the control-point positions.

Cubic Equation Form

$$\mathbf{x} = \left(-t^3 + 3t^2 - 3t + 1\right)\mathbf{p}_0 + \left(3t^3 - 6t^2 + 3t\right)\mathbf{p}_1 + \left(-3t^3 + 3t^2\right)\mathbf{p}_2 + \left(t^3\right)\mathbf{p}_3$$

$$\mathbf{x} = \left(-\mathbf{p}_0 + 3\mathbf{p}_1 - 3\mathbf{p}_2 + \mathbf{p}_3\right)t^3 + \left(3\mathbf{p}_0 - 6\mathbf{p}_1 + 3\mathbf{p}_2\right)t^2 + \left(-3\mathbf{p}_0 + 3\mathbf{p}_1\right)t + \left(\mathbf{p}_0\right)1$$

Cubic Equation Form

$$\mathbf{x} = \left(-\mathbf{p}_0 + 3\mathbf{p}_1 - 3\mathbf{p}_2 + \mathbf{p}_3\right)t^3 + \left(3\mathbf{p}_0 - 6\mathbf{p}_1 + 3\mathbf{p}_2\right)t^2 + \left(-3\mathbf{p}_0 + 3\mathbf{p}_1\right)t + \left(\mathbf{p}_0\right)1$$

$$\mathbf{x} = \mathbf{a}t^3 + \mathbf{b}t^2 + \mathbf{c}t + \mathbf{d}$$

$$\mathbf{a} = \left(-\mathbf{p}_0 + 3\mathbf{p}_1 - 3\mathbf{p}_2 + \mathbf{p}_3\right)$$

$$\mathbf{b} = \left(3\mathbf{p}_0 - 6\mathbf{p}_1 + 3\mathbf{p}_2\right)$$

$$\mathbf{c} = \left(-3\mathbf{p}_0 + 3\mathbf{p}_1\right)$$

$$\mathbf{d} = \left(\mathbf{p}_0\right)$$

Cubic Equation Form

- If we regroup the equation by terms of exponents of *t*, we get it in the standard cubic form
- This form is very good for fast evaluation, as all of the constant terms (**a**,**b**,**c**,**d**) can be precomputed
- The cubic equation form obscures the input geometry, but there is a one-to-one mapping between the two and so the geometry can always be extracted out of the cubic coefficients

Cubic Matrix Form

$$\mathbf{x} = \mathbf{a}t^3 + \mathbf{b}t^2 + \mathbf{c}t + \mathbf{d}$$

$$\mathbf{a} = \left(-\mathbf{p}_0 + 3\mathbf{p}_1 - 3\mathbf{p}_2 + \mathbf{p}_3\right)$$
$$\mathbf{b} = \left(3\mathbf{p}_0 - 6\mathbf{p}_1 + 3\mathbf{p}_2\right)$$
$$\mathbf{c} = \left(-3\mathbf{p}_0 + 3\mathbf{p}_1\right)$$
$$\mathbf{d} = \left(\mathbf{p}_0\right)$$

$$\mathbf{x} = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \cdot \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \\ \mathbf{d} \end{bmatrix} \qquad \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \\ \mathbf{d} \end{bmatrix} = \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{bmatrix}$$

Cubic Matrix Form

$$\mathbf{x} = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} p_{0x} & p_{0y} & p_{0z} \\ p_{1x} & p_{1y} & p_{1z} \\ p_{2x} & p_{2y} & p_{2z} \\ p_{3x} & p_{3y} & p_{3z} \end{bmatrix}$$

Matrix Form

$$\mathbf{x} = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} p_{0x} & p_{0y} & p_{0z} \\ p_{1x} & p_{1y} & p_{1z} \\ p_{2x} & p_{2y} & p_{2z} \\ p_{3x} & p_{3y} & p_{3z} \end{bmatrix}$$

$$\mathbf{x} = \mathbf{t} \cdot \mathbf{B}_{Bez} \cdot \mathbf{G}_{Bez}$$

$$\mathbf{x} = \mathbf{t} \cdot \mathbf{C}$$

Cubic Bezier Curves

- Cubic Bezier function in matrix form

$$P(u) = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} . \mathbf{M}_{\mathbf{BEZ}} . \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix}$$

- where the Bezier Matrix is

$$M_{BEZ} = \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Matrix Form

- We can rewrite the equations in matrix form
- This gives us a compact notation and shows how different forms of cubic curves can be related
- It also is a very efficient form as it can take advantage of existing 4x4 matrix hardware support…

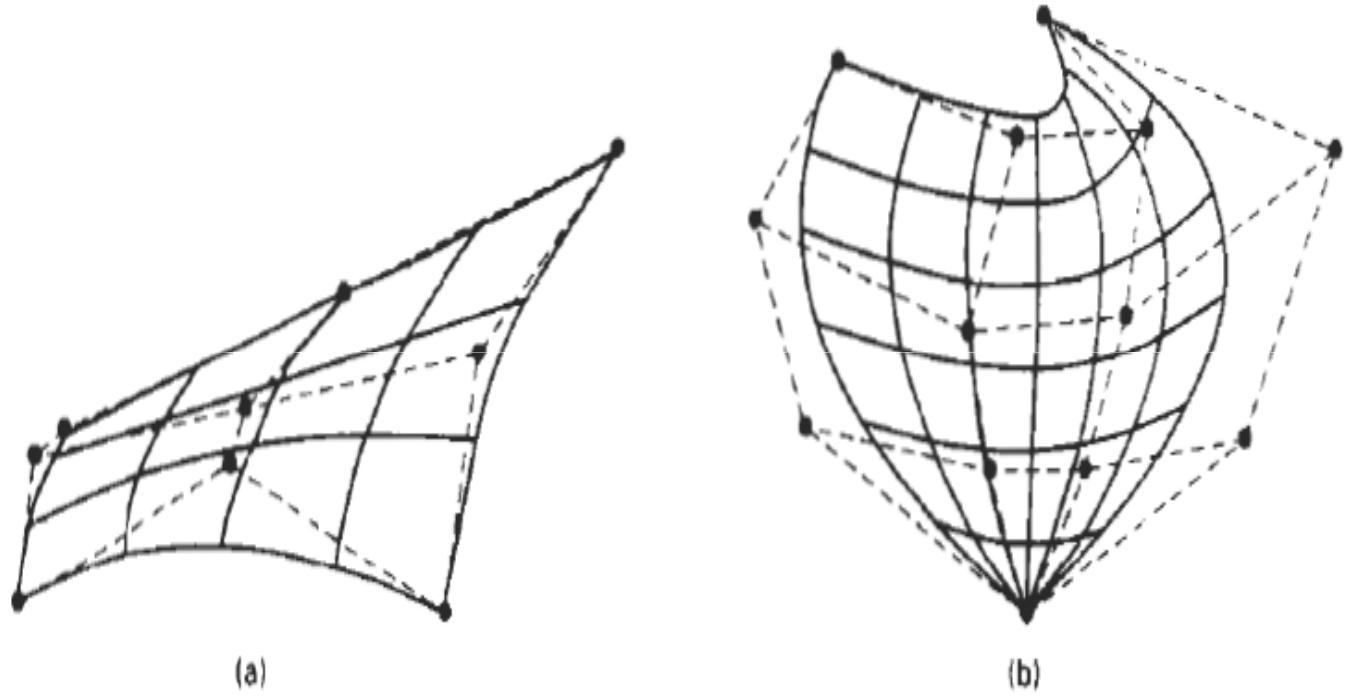
Bezier Surfaces

- Two sets of orthogonal Bezier curves can be used to design an object surface by an input mesh of ctrl pts.
- Bezier surface function = Cartesian product of Bezier blending functions.

$$\mathbf{P(u,\ v)} = \sum_{\mathbf{j=0}}^{\mathbf{m}} \sum_{\mathbf{k=0}}^{\mathbf{n}} \mathbf{p}_{\mathbf{j,k}} \mathbf{BEZ}_{\mathbf{j,m}}(\mathbf{v}) \mathbf{BEZ}_{\mathbf{k,n}}(\mathbf{u})$$

with $p_{j,k}$ specifying the location of the (m+1) by (n+1) ctrl points.

Bezier Surfaces



a) m=3,n=3 and b) m=4,n=4 Dashed lines connect the control points.

Properties of Bezier Curves

Properties of Bezier Curves

- Curve passes thro the first and last control point (interpolation)
  - Boundary conditions at endpoints are

$$P(0) = p_0$$

$$P(1) = p_n$$

  - Degree of polynomial is one less than the control points
- First derivatives are

$$P'(0) = -np_0 + np_1$$

$$P'(1) = -np_{n-1} + np_n$$

- The tangent(slope) to the curve at the first control point is along the line joining the first and second control points
- The tangent at the last control point is along the line joining the second last and last control points

Properties of Bezier Curves

- Second derivatives are

$$\mathbf{P}''(0) = \mathbf{n}(\mathbf{n}-1)[(\mathbf{p}_2 - \mathbf{p}_1) - (\mathbf{p}_1 - \mathbf{p}_0)]$$

$$\mathbf{P}''(1) = \mathbf{n}(\mathbf{n}-1)[(\mathbf{p}_{n-2} - \mathbf{p}_{n-1}) - (\mathbf{p}_{n-1} - \mathbf{p}_n)]$$

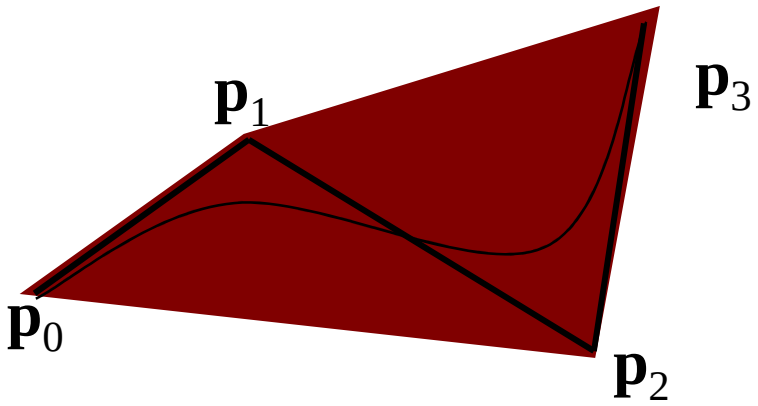
- The curve lies entirely within the convex hull of its control points
  - The Bernstein polynomials sum to 1 and are everywhere positive

$$\sum_{k=0}^{n} BEZ_{k,n}(u) = 1$$

Curve is invariant under any affine transformations.

Convex Hull Property

- If we take all of the control points for a Bezier curve and construct a convex polygon around them, we have the *convex hull* of the curve
- An important property of Bezier curves is that every point on the curve itself will be somewhere within the convex hull of the control points

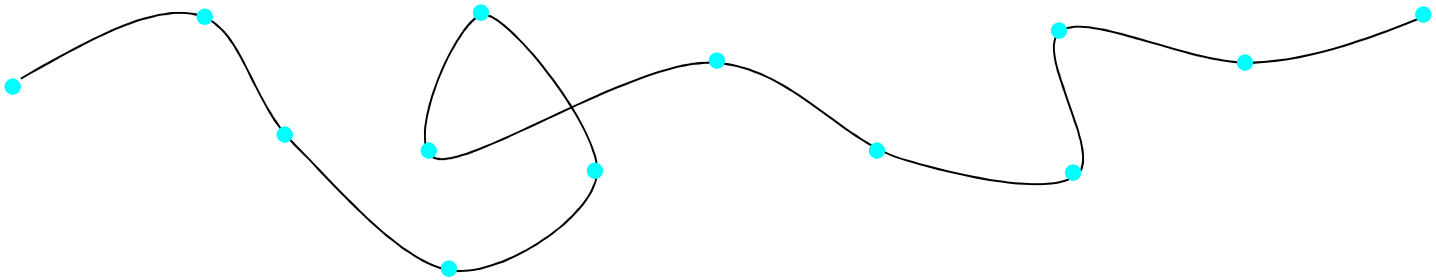


Continuity

- A cubic curve defined for *t* ranging from 0 to 1 will form a single continuous curve and not have any gaps
- We say that it has *geometric continuity*, or C^0 *continuity*
- We can easily see that the first derivative will be a continuous quadratic function and the second derivative will be a continuous linear function
- The third derivative (and all others) are continuous as well, but in a trivial way (constant), so we generally just say that a cubic curve has *second derivative continuity* or C^2 *continuity*
- In general, the higher the continuity value, the 'smoother' the curve will be, although it's actually a little more complicated than that…

Interpolation / Approximation

- We say that cubic Bezier curves *interpolate* the two endpoints ($\mathbf{p}_0$ & $\mathbf{p}_3$), but only *approximate* the interior points ($\mathbf{p}_1$ & $\mathbf{p}_2$)
- In geometric design applications, it is often desirable to be able to make a single curve that interpolates through several points

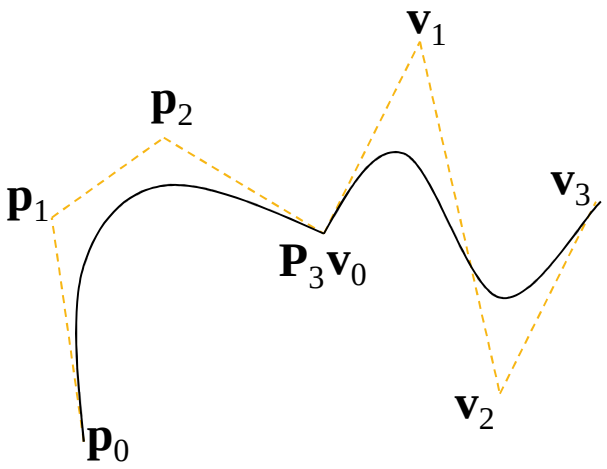


Piecewise Curves

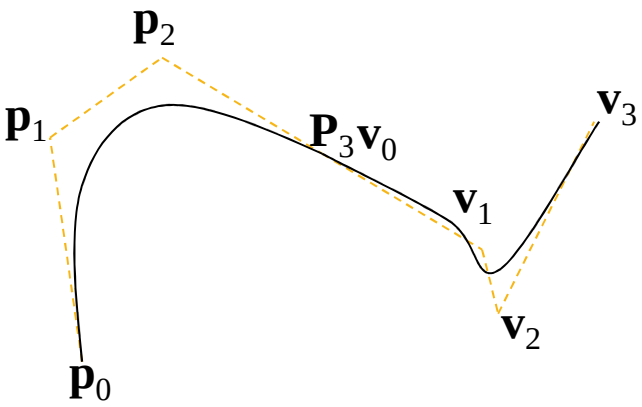
- Rather than use a very high degree curve to interpolate a large number of points, it is more common to break the curve up into several simple curves
- For example, a large complex curve could be broken into cubic curves, and would therefore be a *piecewise cubic curve*
- For the entire curve to look smooth and continuous, it is necessary to maintain C^1 continuity across segments, meaning that the position and tangents must match at the endpoints
- For smoother looking curves, it is best to maintain the C^2 continuity as well

Connecting Bezier Curves

- A simple way to make larger curves is to connect up Bezier curves
- Consider two Bezier curves defined by $\mathbf{p}_0 \ldots \mathbf{p}_3$ and $\mathbf{v}_0 \ldots \mathbf{v}_3$
- If $\mathbf{p}_3 = \mathbf{v}_0$, then they will have C^0 continuity
- If $(\mathbf{p}_3 - \mathbf{p}_2) = (\mathbf{v}_1 - \mathbf{v}_0)$, then they will have C^1 continuity
- C^2 continuity is more difficult…



C^0 continuity



C^1 continuity