Closure Properties of Regular Sets

Beulah A.
AP/CSE

Definition of Regular Expression

- Let Σ be an alphabet. The regular expressions over Σ and the sets that they denote are defined recursively as follows:

1. φ is a regular expression and denotes the empty set.
2. ε is a regular expression and denotes the set {ε}
3. For each a ∈ Σ, 'a' is a regular expression and denotes the set {a}.
4. If r and s are regular expressions denoting the languages R and S respectively then (r + s), (rs), (r)* are regular expressions that denotes the sets RUS, RS and R* respectively.

Theorem 1

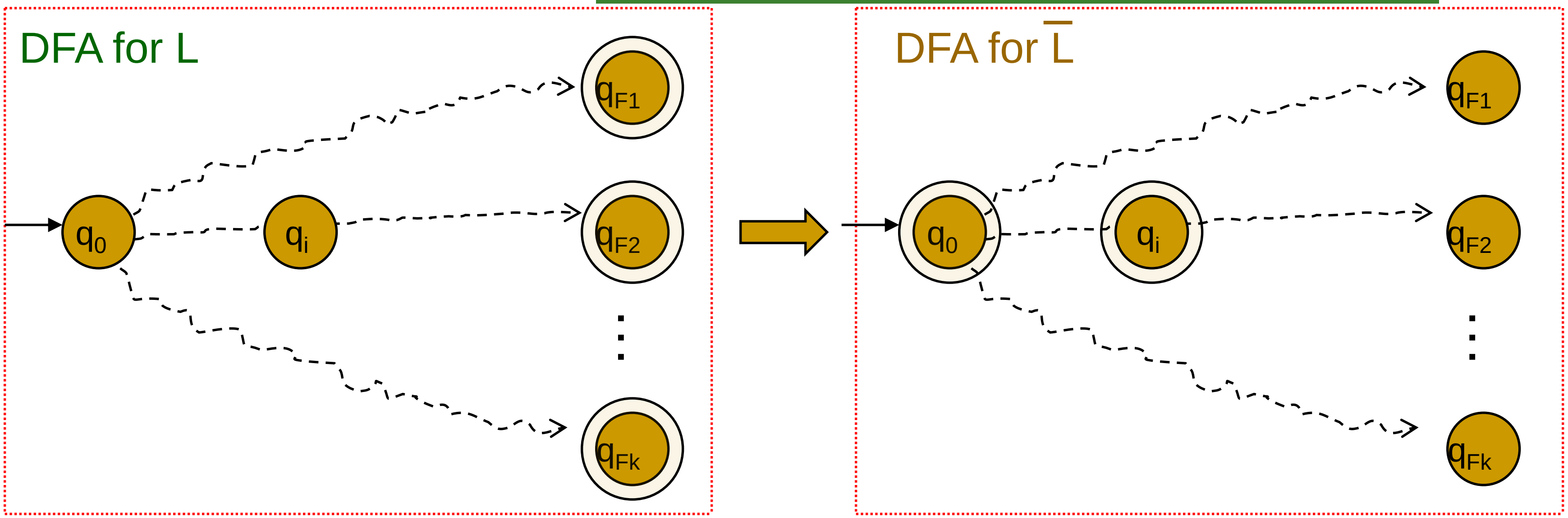
- Regular sets are closed under union, concatenation, and closure
- Proof
  - If L_1 and L_2 are regular, then there are regular expressions r_1 and r_2 denoting the languages L_1 and L_2 ,respectively.
  - By definition of RE "If r and s are regular expressions denoting the languages R and S respectively then (r + s), (rs), (r)* are regular expressions that denotes the sets RUS, RS and R* respectively"
  - Therefore $(r_1 + r_2)$, $(r_1 . r_2)$ and (r_1^*) are regular expressions denoting the languages $L_1 \cup L_2$, $L_1 . L_2$ and L_1^*

Theorem 2

- The class of regular sets is closed under complementation. ie If L is a regular set over $\sum$, then $L=\sum^{*}-L$ is a regular set
- Proof
  - Suppose that L is a regular over an alphabet $\sum$.
  - There is a DFA $M=(Q, \sum, \delta, q_0, F)$ accepting L
  - Design a DFA $M' = (Q, \sum, \delta, q_0, F')$, where $F' = Q – F$
  - Now, we have that $L(M') = \sum^{*} - L$.
  - Hence, the complement of L is regular

Theorem 2

Convert every final state into non-final, and every non-final state into a final state



Assumes q0 is a non-final state. If q0 is a final state, invert it.

Theorem 3

- Regular sets are closed under intersection
- Proof
  - By DeMorgan's law:

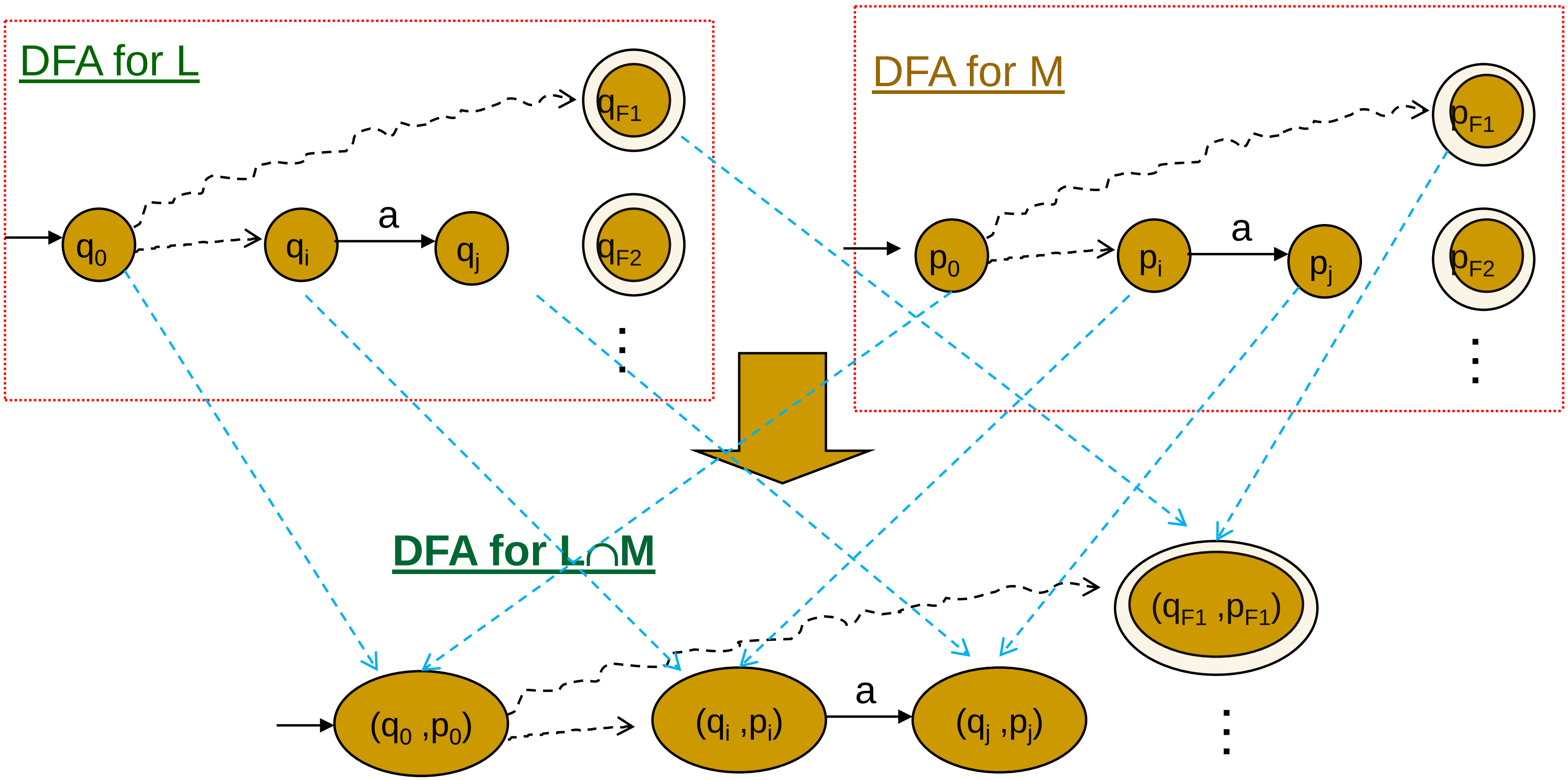
$$L \cap M = \overline{(\overline{L} \cup \overline{M})}$$

  - Since Regular Sets are closed under union and complementation, they are also closed under intersection

Theorem 3

- A_L = DFA for L = $\{Q_L, \Sigma, q_L, F_L, \delta_L\}$
- A_M = DFA for M = $\{Q_M, \Sigma, q_M, F_M, \delta_M\}$
- Build $A_{L \cap M} = \{Q_L x\ Q_M, \Sigma, (q_L,q_M), F_L\ x\ F_M, \delta\}$ such that:
  - $\delta((p,q),a) = (\delta_L(p,a), \delta_M(q,a))$, where p in Q_L, and q in Q_M
- This construction ensures that a string w will be accepted if and only if w reaches an accepting state in both input DFAs.

Theorem 3



DFA for L
q0
qi
a
qj
qF1
qF2
DFA for M
p0
pi
a
pj
pF1
pF2
DFA for L∩M
(q0 ,p0)
(qi ,pi)
a
(qj ,pj)
(qF1 ,pF1)

Theorem 4

- Regular sets are closed under set difference
- Proof
  - $L - M = L \cap \overline{M}$

Closed under intersection

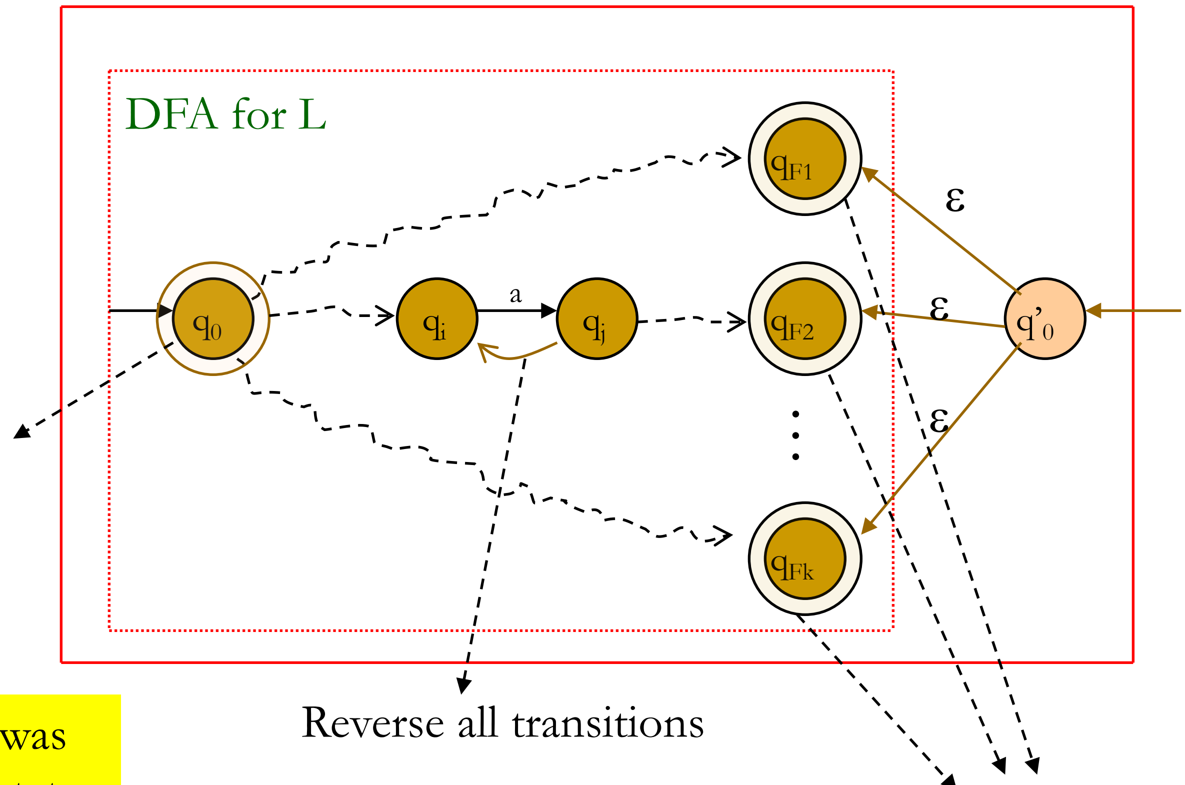
Closed under complementation

- Therefore, L - M is also regular

Theorem 5

- If L is regular then L^R is also regular

New ε-NFA for L^R



New start state

Make the old start state as the only new final state

Reverse all transitions

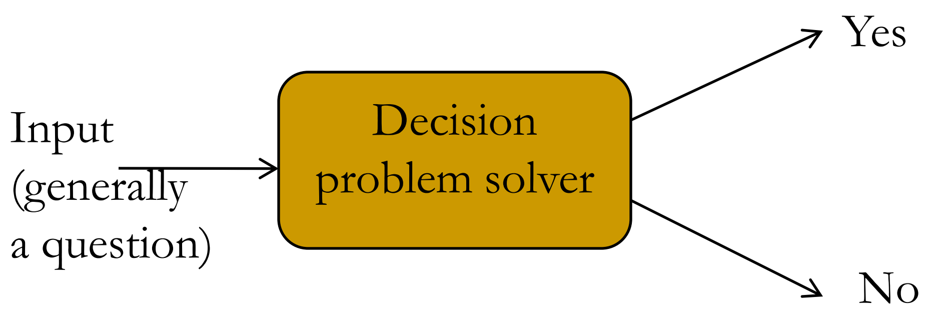
Convert the old set of final states into <u>non-final</u> states

What to do if q_0 was one of the final states in the input DFA?

Decision Properties of Regular Sets

Introduction

- <u>Any "decision problem" looks like this</u>



Membership Question

- Decision Problem: Given L, is w in L?
- Possible answers: Yes or No
- Approach:
  1. Build a DFA for L
  2. Input w to the DFA
  3. If the DFA ends in an accepting state, then yes; otherwise no.

Emptiness Test

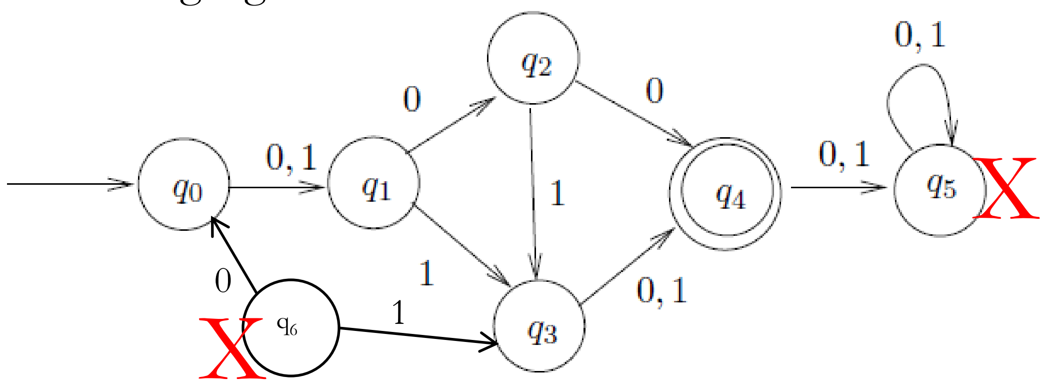
- Decision Problem: Is L=Ø ?
- Approach:
  1. Build a DFA for L
  2. From the start state, run a *reachability* test, which returns:
     1. success: if there is at least one final state that is reachable from the start state
     2. failure: otherwise
  3. L=Ø if and only if the reachability test fails

Finiteness

- Decision Problem: Is L finite or infinite?
- Approach:
  1. Build DFA for L
  2. Remove all states unreachable from the start state
  3. Remove all states that cannot lead to any accepting state.
  4. After removal, check for cycles in the resulting FA
  5. L is finite if there are no cycles; otherwise it is infinite

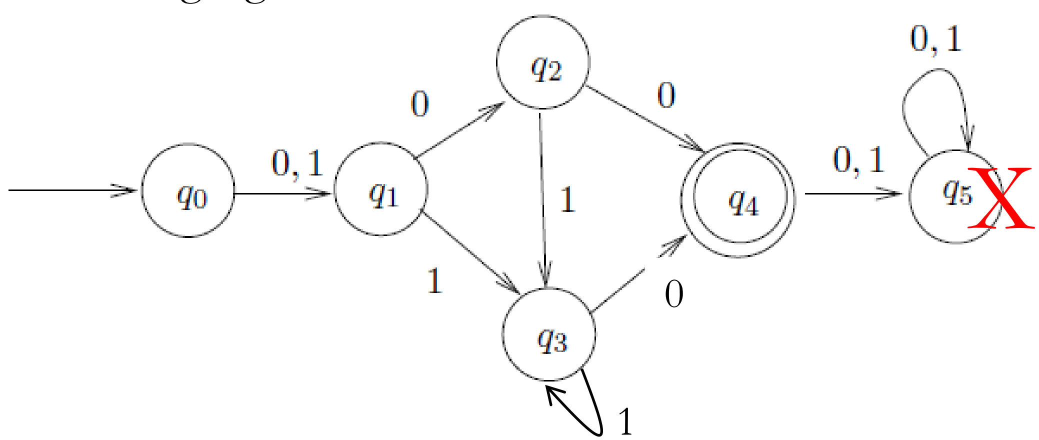
Finiteness Test - Examples

Ex 1) Is the language of this DFA finite or infinite?



FINITE

Ex 2) Is the language of this DFA finite or infinite?



INFINITE

Summary

- Closure Properties of Regular Sets
- Decision Properties of Regular Sets

Test Your Knowledge

- Regular sets are closed under union,concatenation and kleene closure.
  a) True
  b) False
  c) Depends on regular set
  d) Can’t say
- If L1 and L2 are regular sets then intersection of these two will be
  a) Regular
  b) Non Regular
  c) Recursive
  d) Non Recursive

Test Your Knowledge

- Reverse of a DFA can be formed by

  a) using PDA

  b) making final state as non-final

  c) making final as starting state and starting state as final state

  d) None of the mentioned

- Complement of (a + b)* will be

  a) phi

  b) null

  c) a

  d) b

Reference

- Hopcroft J.E., Motwani R. and Ullman J.D, "Introduction to Automata Theory, Languages and Computations", Second Edition, Pearson Education, 2008