SPLINE REPRSENTATION

Splines

- Possible to repersent a curve by single polynominal function

$$\mathbf{Q}(\bar{u}) = (X(\bar{u}), Y(\bar{u}))$$

  - Are single valued functions of u yield x and coordinates.
  - Difficult to represent a satisfactory curve using single polynomial.
  - Customary to break the curve into n segments by defining polynomial functions for each segment
  - Hook the segments to form piecewise polynomial curve.
  - The values of u that corresponds to the joints between segments are called knots.

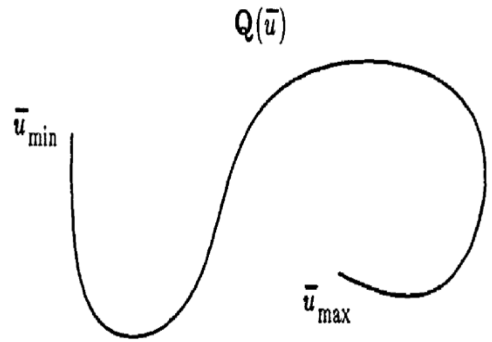


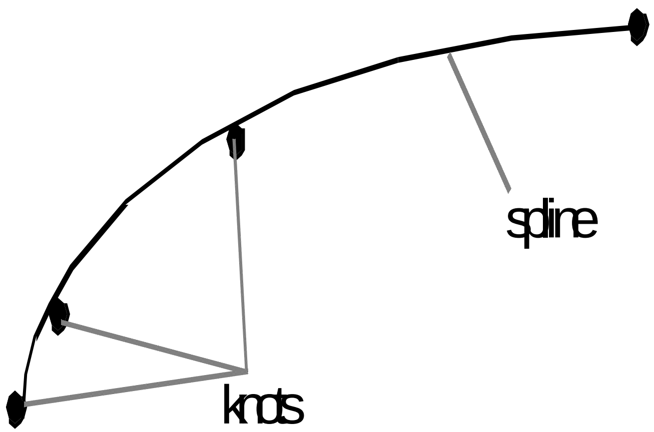
Figure 1. A parametrically defined curve.

Splines

- Spline curve is a curve described with a piecewise cubic polynomial functions whose first order or second derivatives are continuous across various curve cross sections.

- In graphics it refers to any composite curve formed with polynomial sections ,satisfying specified continuity conditions at the boundary of the pieces.

- Used in graphics applications
    - to design curve and surface shapes,
    - to digitize drawings
    - specify animation paths for the objects.



Splines

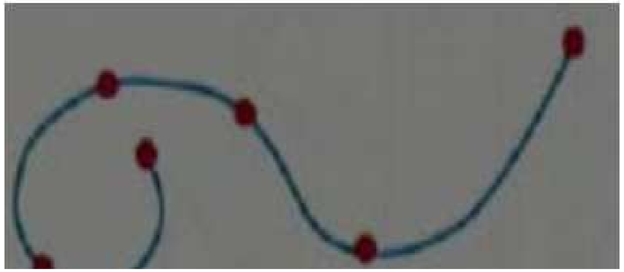
- The spline curve is reprsented using set of coordinate positions called control points.

- A spline curve is defined, modified and manipulated with operations on control points.

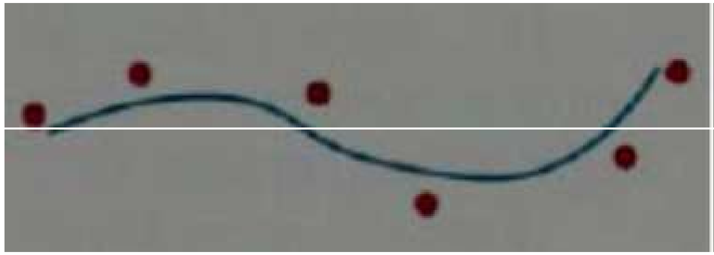
- The control points are fitted with piecewise continous parmetric polynominal functions by two ways
    - Interpolation
    - Approximation

Splines

- Interpolation Method:
  - When the curve passes through each control points, the resulting curve is said to interpolate the set of control points.
  - Used to digitize drawings, animation path



Approximation Method:
  - The curve not necessarily passes through each control points.
  - Eg: Bezier curves or B-Splinecurves
  - Used to design tools to structure object surfaces.



Spline Manipulation

- Initial curve is designed and then manipulated using control points.
  - Designer can set up an initial curve,
  - After the polynomial fit is displayed for a given set of control points
  - The designer can reposition some of the control points to restructure the shape of the curve.
- Convex Hull – Convex polygon boundary that encloses a set of control points

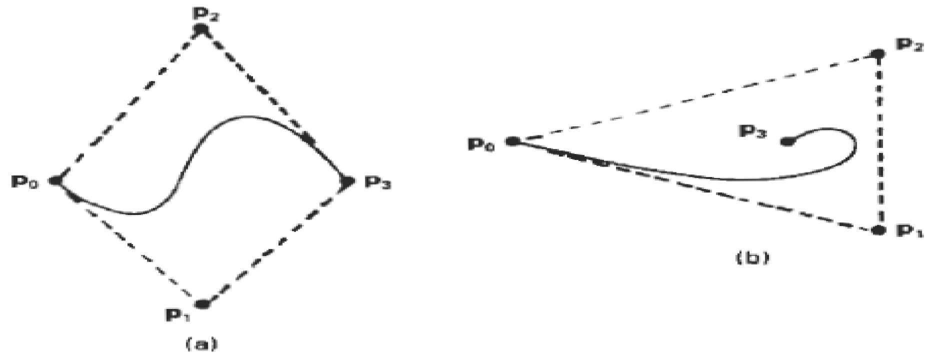


Figure 10-22
Convex-hull shapes (dashed lines) for two sets of control points.

Splines

- Control Graph – Polyline connecting a sequence of control points for an approximation spline.

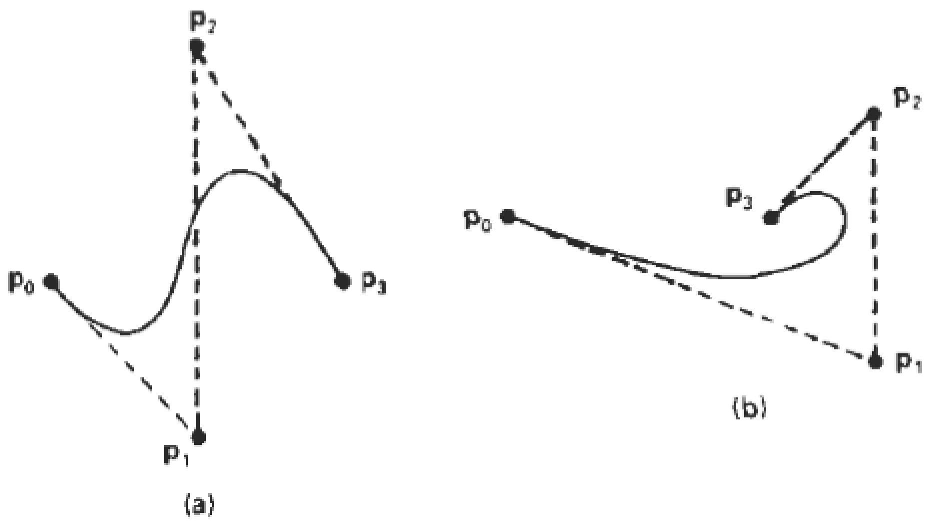


Figure 10-23
Control-graph shapes (dashed lines) for two different sets of control points.

Parametric Continuity Conditions

- To ensure smooth transition between one section of piecewise parametric curve to the next, various continuity

conditions can be imposed at the connection points.

- Each section of the spline is described with a a set of parametric coordinate functions

$$X = x(u) \quad y = y(u) \; z = z(u) \quad u1 <= u <= u2$$

- We set parametric continuity by matching the parametric derivatives of adjoining curve sections at the common boundary.

Parametric Continuity Cx

Zero-order parametric continuity:C0

The values of x,y,z at u2 for the first curve section are equal to the values of x,y,z at u1 for the next curve.

First-order parametric continuity:C1

The first parametric derivatives of the coordinate functions for two successive curve sections are equal at their joining point

(b)

Second-order parametric continuity:C2

The first and second parametric derivatives of the coordinate functions for two successive curve sections are equal at their joining point

(c)

Geometric Continuity Gx

- Alternate method for joining two successive curve sections is to specify conditions for Geometric continuity
- Parametric derivates should be proportional to each other at their common boundary.
- Zero Order (G0):
  - Two sections must have the same coordinate position at boundary point
- First Order G1:
  - Parametric first derivatives are proportional at the intersection
- Second order G2:
  - 1st and 2nd parametric derivatives of the curve sections are proportional at their boundary.

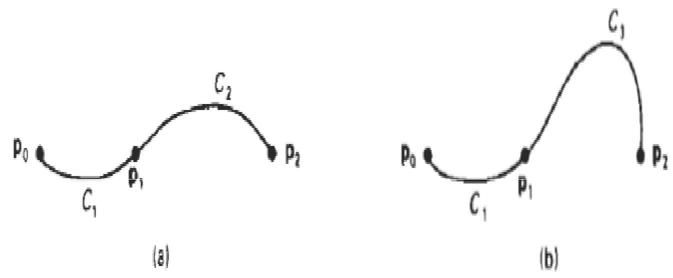
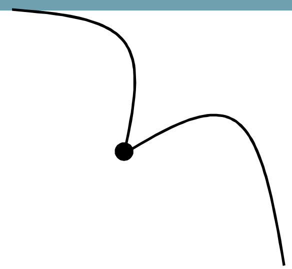


Figure 10-25
Three control points fitted with two curve sections joined with (a) parametric continuity and (b) geometric continuity, where the tangent vector of curve C_3 at point p_1 has a greater magnitude than the tangent vector of curve C_1 at p_1.

Order of continuity

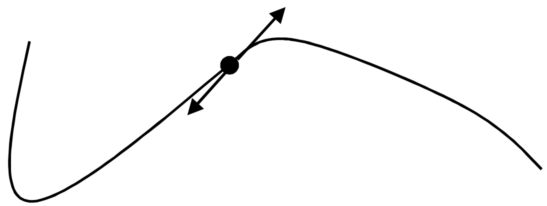
$G^{(0)}$



$C^{(0)}$ 0^{th} order continuity

(a)

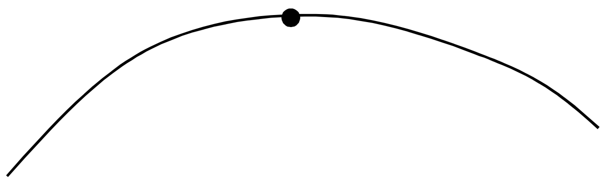
$G^{(1)}$



$C^{(1)}$ 1^{st} order continuity

(b)

$G^{(2)}$



$C^{(2)}$ 2^{nd} order continuity

(c)

Spline Specifications

- 3 Methods for specifying spline representation
  - State set of boundary conditions imposed on the spline
  - State the matrix of the spline
  - State the set of blending functions that determine how specified geometric constraints are combined to calculate positions along the curve path.

  - Suppose, the x coordinate of a spline section has
    - $x(u) = a_x u^3 + b_x u^2 + c_x u + d_x \quad 0 <= u <= 1$

      Boundary conditions may be set on endpoints x(0) and x(1)

      And the parametric first dervatives at the endpoints x'(0) and x'(1).

  - Using that determine a_x, b_x, c_x, d_x.

- From the boundary conditions, obtain the matrix

$$x(u) = [u^3\ u^2\ u\ 1] \begin{bmatrix} a_x \\ b_x \\ c_x \\ d_x \end{bmatrix}$$

$$= \mathbf{U} \cdot \mathbf{C}$$

We can write the boundary conditions in the matrix form

$$\mathbf{C} = \mathbf{M}_{spline} \cdot \mathbf{M}_{geom}$$

Where M_{geom} is a four element col. Matrix containing the geometric constraints(boundary conditions)

M_{spline} 4 * 4 matrix that provides a characterization for spline curves

- So we can say,

$$x(u) = \mathbf{U} \cdot \mathbf{M}_{\text{spline}} \cdot \mathbf{M}_{\text{geom}}$$

We can expand this to obtain a polynomial representation for
Coordinate x in terms of geometric constraint parameters.

$$x(u) = \sum_{k=0}^{3} g_k \cdot BF_k(u)$$

gk – control point coordinates and slope of the curve at control points.
BFk(u) – Polynominal blending functions.

Cubic Splines - Intro

- This is a class of splines often used to
  - Set paths for object motions.
  - Representation of an object or drawing

- Cubic spline requires less calculations and memory
- They are more stable
- More flexible for modeling arbitrary curve shapes.

Cubic Splines

- Given control pts cubic splines are obtained by fitting the input points with piecewise cubic polynomial curves that passes thro every ctrl pt.
- No. of ctrl pts = n+1 specified with coordinates

  $P = (x_k, y_k, z_k) \qquad k = 0,1,2,\ldots n$

- The equations describes the parametric cubic polynomial to be fitted between each pair of ctrl pts by

$$x(u) = a_x u^3 + b_x u^2 + c_x u + d_x$$

$$y(u) = a_y u^3 + b_y u^2 + c_y u + d_y, \qquad (0 \le u \le 1)$$

$$z(u) = a_z u^3 + b_z u^2 + c_z u + d_z$$

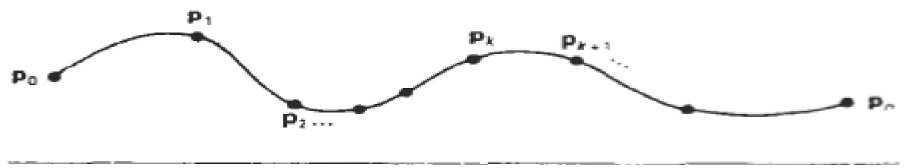


Figure 10-26
A piecewise continuous cubic-spline interpolation of $n + 1$ control points.

Cubic Splines

- For each of these eqns.
  - Determine the values of a,b,c,d for the polynomial representation for each n curve between n+1 ctrl pts.
  - Done by setting enough boundary conditions at the joints.
- Common Methods for setting boundary conditions
  - Natural Cubic splines
  - Hermite Interpolation
  - Cardinal Splines
  - Kochanek-Bartel Splines

Natural Cubic Spline

- First curve developed in graphics based on interpolation method
- Given n + 1 points
  - Generate a curve with n segments
  - 4n polynominal coefficients to be determined
  - Curves passes through points
  - Curve is C^2 continuous
  - For each interior n-1 ctrl points we have four boundary conditions so 4n-4 equations are to be satisfied by 4n polynomials
  - To get 4 eqns,
    - Obtain one equation from p_0.
    - Obtain one equation from p_1
    - To get two more, follow either of the following
      - Set 2nd derivative of p_0 and p_n to 0.
      - Add two dummy ctrl pts p-1 and pn+1 – so we have 4n eqns from interior n points.

Natural Cubic Spline

- Disadvantages:
  - Allows for no local control – if one ctrl pt is altered, then entire curve is affected.
  - Part of the curve not been restructured.

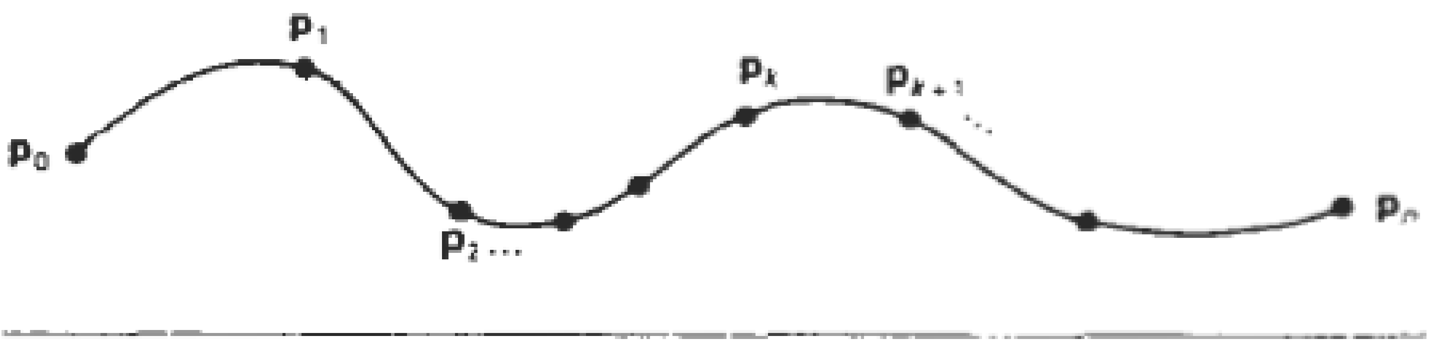


Figure 10-26
A piecewise continuous cubic-spline interpolation of $n + 1$ control .points.

Hermite Interpolation

- It is an interpolating piecewise cubic polynomial with a specified tangent at each control point.
- Can be adjusted locally.
- If p(u) – parametric cubic function of curve section between p_k and p_{k+1}
- Boundary conditions that define curve sections are

$$\mathbf{P}(0) = \mathbf{p}_k$$
$$\mathbf{P}(1) = \mathbf{p}_{k+1}$$
$$\mathbf{P}'(0) = \mathbf{D}\mathbf{p}_k$$
$$\mathbf{P}'(1) = \mathbf{D}\mathbf{p}_{k+1}$$

- Vector equivalent for the Hermite-curve section as

$$\mathbf{P}(u) = \mathbf{a}u^3 + \mathbf{b}u^2 + \mathbf{c}u + \mathbf{d}, \qquad 0 \le u \le 1$$

- The Matrix equivalent as

$$\mathbf{P}(u) = [u^3 \ u^2 \ u \ 1] \cdot \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \\ \mathbf{d} \end{bmatrix}$$

The derivative of point function as

$$\mathbf{P}'(u) = [3u^2 \ 2u \ 1 \ 0] \cdot \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \\ \mathbf{d} \end{bmatrix}$$

Hermite Interpolation

$$\begin{bmatrix} \mathbf{p}_k \\ \mathbf{p}_{k-1} \\ D\mathbf{p}_k \\ D\mathbf{p}_{k+1} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \\ \mathbf{d} \end{bmatrix}$$

Substituting endpoints 0 and 1 for parameter u

$$\begin{bmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \\ \mathbf{d} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix}^{-1} \cdot \begin{bmatrix} \mathbf{p}_k \\ \mathbf{p}_{k+1} \\ D\mathbf{p}_k \\ D\mathbf{p}_{k-1} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \mathbf{p}_k \\ \mathbf{p}_{k+1} \\ D\mathbf{p}_k \\ D\mathbf{p}_{k+1} \end{bmatrix}$$

$$= \mathbf{M}_H \cdot \begin{bmatrix} \mathbf{p}_k \\ \mathbf{p}_{k+1} \\ D\mathbf{p}_k \\ D\mathbf{p}_{k+1} \end{bmatrix}$$

Heremite boundary conditions

Hermite Interpolation

$$\mathbf{P}(u) = [u^3\ u^2\ u\ 1] \cdot \mathbf{M}_H \cdot \begin{bmatrix} \mathbf{p}_k \\ \mathbf{p}_{k+1} \\ \mathbf{Dp}_k \\ \mathbf{Dp}_{k+1} \end{bmatrix}$$

- Where M_H is the Hermite Matrix
- Multiplying the matrices in the eqn. we get

$$\begin{aligned} \mathbf{P}(u) &= \mathbf{p}_k(2u^3 - 3u^2 + 1) + \mathbf{p}_{k+1}(-2u^3 + 3u^2) - \mathbf{Dp}_k(u^3 - 2u^2 + u) \\ &\quad + \mathbf{Dp}_{k+1}(u^3 - u^2) \\ &= \mathbf{p}_k H_0(u) + \mathbf{p}_{k+1} H_1(u) + \mathbf{Dp}_k H_2(u) + \mathbf{Dp}_{k+1} H_3(u) \end{aligned}$$

- where H (u) for k=0,1,2,3 are referred as blending functions because they blend the boundary constraint values to obtain each coordinate position along the curve.

Cardinal Splines

- Cardinal splines are interpolating piecewise cubic polynomial with specified tangents at the boundary of each curve section.
- Value for the slope at a ctrl pt is calculated from the coordinates of the two adjacent control points.
- A cardinal spline section is specified with four consecutive control points –
  - Middle two control points are the section endpoints
  - other two are used in calculation of endpoint slopes.

Cardinal Splines

- If P(u) is the parametric rep. of a section between pk and pk+1 then four ctrl points p_{k-1} to p_{k+1} are used to set the boundary conditions.

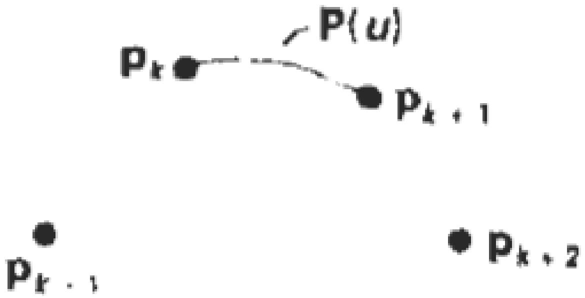


Figure 10-29
Parametric point function $\mathbf{P}(u)$ for a cardinal-spline section between control points $\mathbf{p}_k$ and $\mathbf{p}_{k+1}$.

$$\mathbf{P}(0) = \mathbf{p}_k$$

$$\mathbf{P}(1) = \mathbf{p}_{k+1}$$

$$\mathbf{P}'(0) = \frac{1}{2}(1 - t)(\mathbf{p}_{k+1} - \mathbf{p}_{k-1})$$

$$\mathbf{P}'(1) = \frac{1}{2}(1 - t)(\mathbf{p}_{k+2} - \mathbf{p}_k)$$

Cardinal Splines

- Slopes at p_k and p_{k+1} are taken proportional to chords p_{k-1},p_{k+1} ,p_k and p_{k+2}
- Parameter 't' is called the tension parameter – how tightly or loosely the curve fits the ctrl pts.
- When t = 0 , curve is called Catmull-Rom splines or Overhauser splines

Figure 10-31
Effect of the tension parameter on the shape of a cardinal spline section.

Cardinal Splines

$$\mathbf{P}(u) = [u^3\ u^2\ u\ 1] \cdot \mathbf{M}_C \cdot \begin{bmatrix} \mathbf{p}_{k-1} \\ \mathbf{p}_k \\ \mathbf{p}_{k+1} \\ \mathbf{p}_{k+2} \end{bmatrix}$$

where the cardinal matrix is

$$\mathbf{M}_C = \begin{bmatrix} -s & 2-s & s-2 & s \\ 2s & s-3 & 3-2s & -s \\ -s & 0 & s & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

with $s = (1 - t)/2$.

$$\begin{aligned} \mathbf{P}(u) &= \mathbf{p}_{k-1}(-su^3 + 2su^2 - su) + \mathbf{p}_k[(2-s)u^3 + (s-3)u^2 + 1] \\ &\quad + \mathbf{p}_{k+1}[(s-2)u^3 + (3-2s)u^2 + su] + \mathbf{p}_{k+2}(su^3 - su^2) \\ &= \mathbf{p}_{k-1}CAR_0(u) + \mathbf{p}_kCAR_1(u) + \mathbf{p}_{k+1}CAR_2(u) + \mathbf{p}_{k+2}CAR_3(u) \end{aligned}$$

Kochanek-Bartels Splines

- Two additional parameters are introduced to provide further flexibility in adjusting the curve.
- Boundary conditions for p_{k-1}, p_k, p_{k+1}, p_{k+2}

$$\mathbf{P}(0) = \mathbf{p}_k$$

$$\mathbf{P}(1) = \mathbf{p}_{k+1}$$

$$\mathbf{P}'(0)_{\text{in}} = \tfrac{1}{2}(1 - t)[(1 + b)(1 - c)(\mathbf{p}_k - \mathbf{p}_{k-1}) + (1 - b)(1 + c)(\mathbf{p}_{k+1} - \mathbf{p}_k)]$$

$$\mathbf{P}'(1)_{\text{out}} = \tfrac{1}{2}(1 - t)[(1 + b)(1 + c)(\mathbf{p}_{k+1} - \mathbf{p}_k) + (1 - b)(1 - c)(\mathbf{p}_{k+2} - \mathbf{p}_{k+1})]$$

Kochanek-Bartels Splines

- Where t = tension parameter, b = bias and c = continuity parameter.
- 'b' is used to adjust the amount that the curve bends at each end of the section so that the curve section can be skewed toward one end or the other.
- C – controls the continuity of tangent vector across boundaries of sections.

Figure 10-33
Effect of the bias parameter on the shape of a Kochanek–Bartels spline section.

Kochanek-Bartels Splines

- If c is assigned a non-zero value, there is a discontinuity in the slope of the curve across section boundaries.
- Applications:
  - designed to model animation paths – especially abrupt changes in motion of a object can be simulated with $c \neq 0$.