Infinite Impulse Response Filter

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Design of Digital filters from Analog filters

- For the given specifications of a digital filter, the derivation of the digital filter transfer function requires three steps.

1. Map the desired digital filter specifications into those for an equivalent analog filter.

2. Derive the analog transfer function for the analog prototype.

3. Transform the transfer function of the analog prototype into an equivalent digital filter transfer function.

Sl. No.	Analog filter	Digital filter
1	It processes analog inputs and generates analog outputs.	It processes and generates digital data.
2	Analog filters are constructed from active and passive electronic components.	A digital filter consists of elements like adder, multiplier and delay unit.
3	Analog filter is described by a differential equation.	Digital filter is described by difference equation.
4	The frequency response of an analog filter can be modified by changing the components.	The frequency response of an digital filter can be modified by changing the filter coefficients.

Advantages of Digital filter:

1. Unlike analog filter, the digital filter performance is not influenced by component ageing, temperature and power supply variations.

2. A digital filter is highly immune to noise and possesses considerable parameter stability.

3. Digital filters afford a wide variety of shapes for the amplitude and phase responses.

4. There are no problems of input and output impedance matching with digital filters.

5. Digital filters can be operated over a wide range of frequencies.

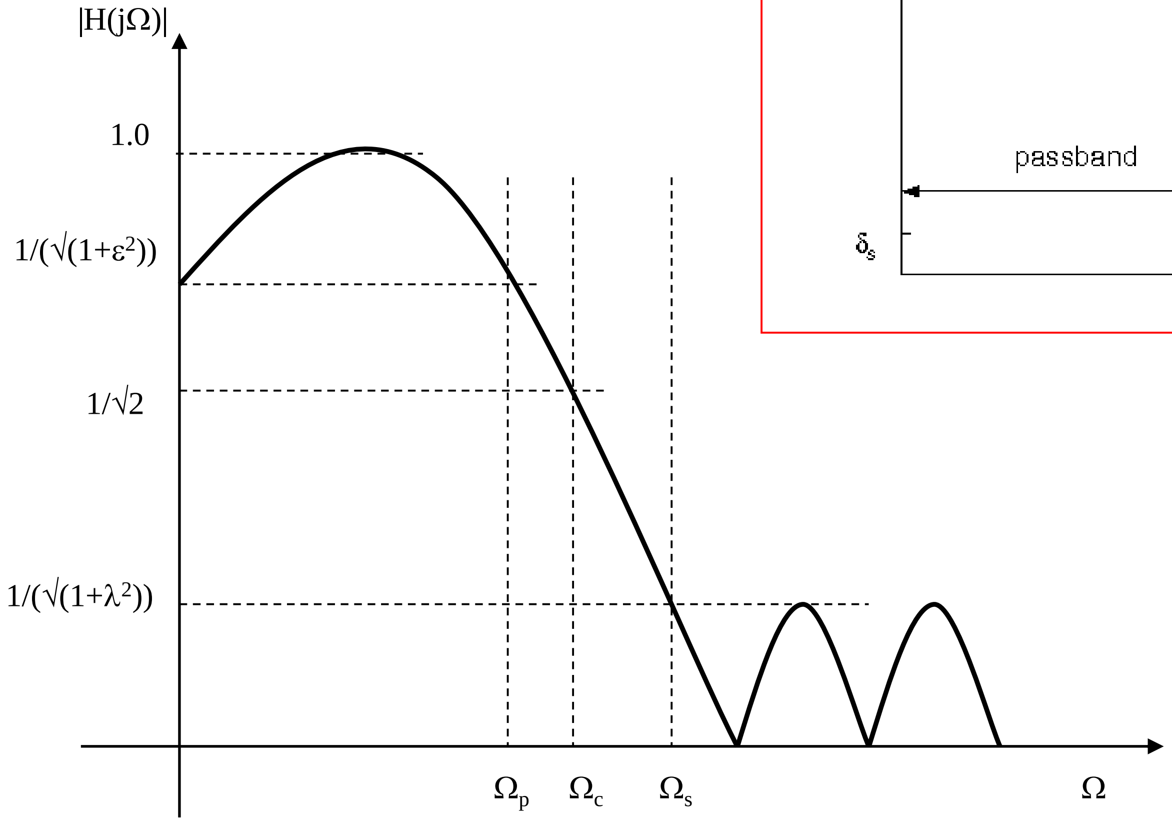
6. The coefficients of digital filter can be programmed and altered any time to obtain the desired characteristics.

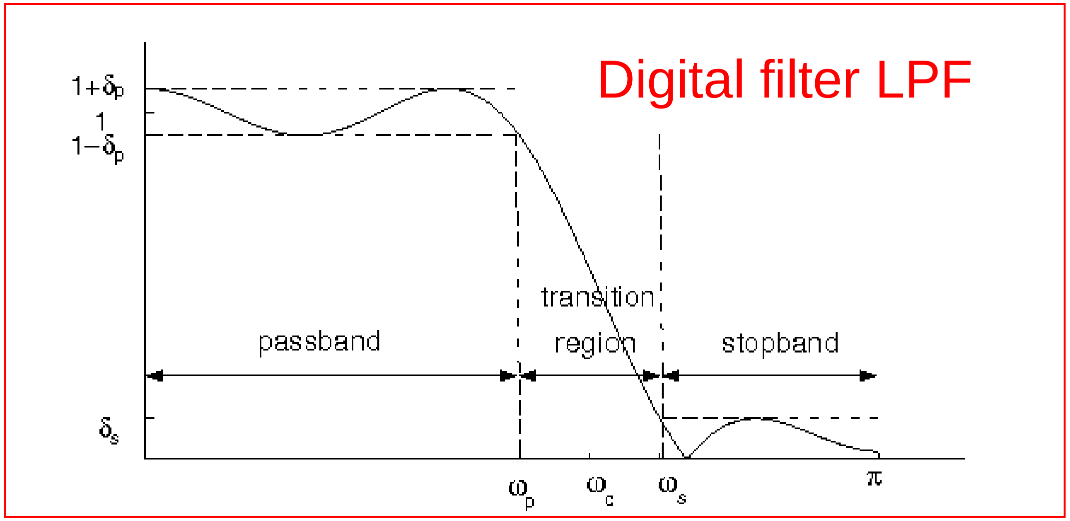
7. Multiple filtering is possible only in digital filter.

Disadvantage of Digital filter:

1. The quantization error arises due to finite word length in the representation of signals and parameters.

Analog filter LPF:





where ε - parameter specifying allowable pass band

λ - parameter specifying allowable stop band

Analog Low pass Butterworth filter:

The magnitude function of the Butterworth low pass filter is given by

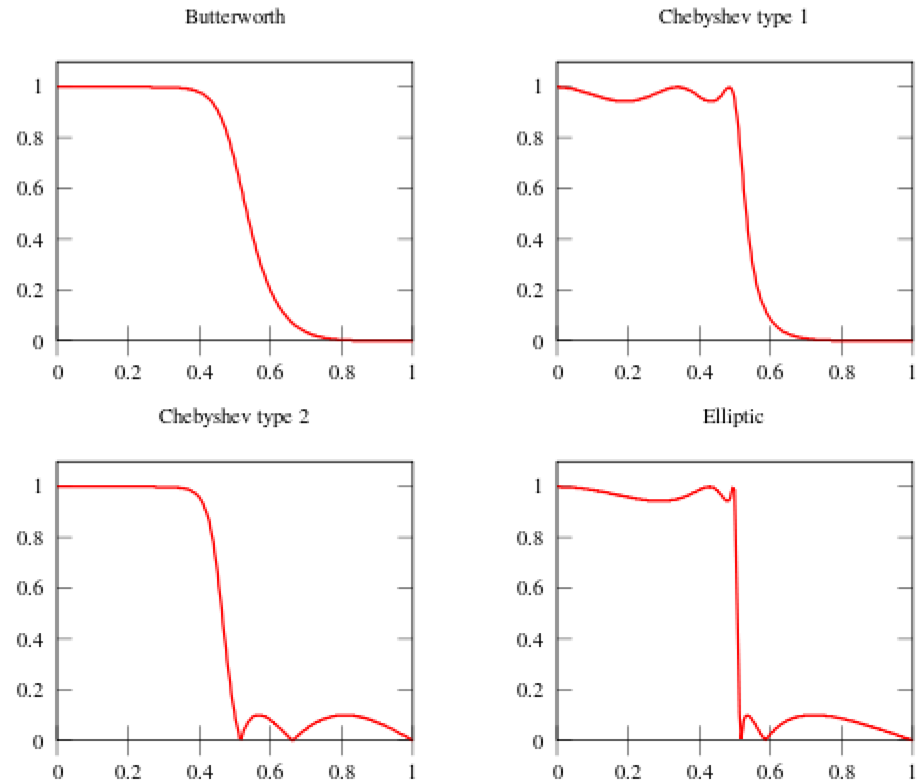
$$|H(j\Omega)| = \frac{1}{\left[1+(\Omega/\Omega_C)^{2N}\right]^{1/2}} \qquad N = 1,2,3,\ldots\ldots$$

where N is the order of the filter and Ω_c is the cutoff frequency.

It can be seen that the magnitude response approaches the ideal low pass characteristics as the order N increases.

The magnitude square function of a normalized Butterworth filter (to 1 rad/sec cutoff frequency) is given by

$$|H(j\Omega)|^2 = \frac{1}{1+(\Omega)^{2N}} \qquad N = 1,2,3,\ldots\ldots$$



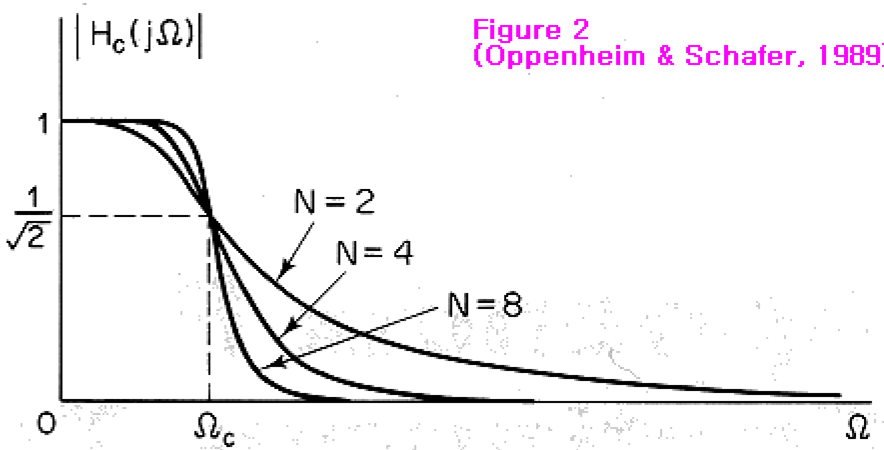


Figure 2
(Oppenheim & Schafer, 1989)

Now let us derive the transfer function of a stable filter.

Substituting $\Omega = s / j$, we get,

$|H(j\Omega)|^2 = |H(s)|^2 = H(s)\,H(-s)$

$$H(s)H(-s) = \frac{1}{1+\left(\frac{s}{j}\right)^{2N}} = \frac{1}{1+(-s^2)^N}$$

The above relation tells that this function has poles in the Left Half as well as Right Half of s-plane, because of the presence of two factors H(s) and H(-s). If H(s) has roots in the LHP then H(-s) has the corresponding roots in the RHP. The roots are obtained by equating the denominator to zero, i.e., $1+(-s^2)^N = 0$.

For N odd,

$$s^{2N} = 1 = e^{j2\pi k}$$

$$s_k = e^{j2\pi k/2N} = e^{j\pi k/N} \qquad k=1,2,3,\ldots,2N$$

For N even,

$$s^{2N} = -1 = e^{j(2k-1)\pi}$$

$$s_k = e^{j(2k-1)\pi/2N} \qquad k=1,2,3,\ldots,2N$$

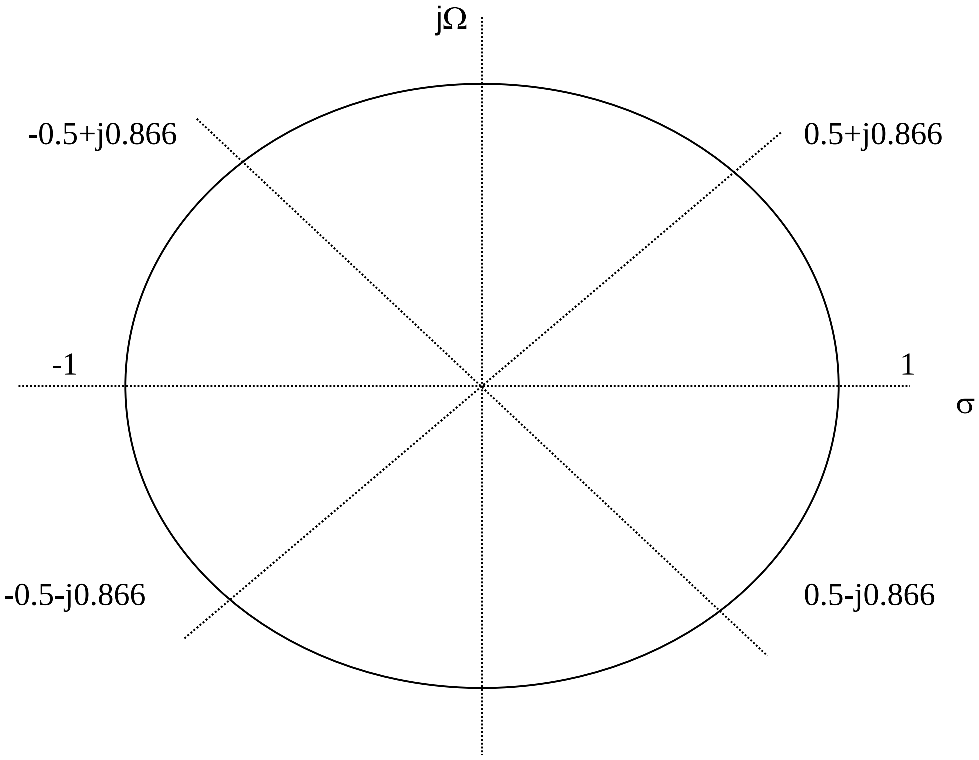
Example: Let N=3

Therefore, k = 1,2,3,4,5,6

Since N is odd,

$$s_1 = 0.5+j0.866$$
$$s_2 = -0.5+j0.866$$
$$s_3 = -1$$
$$s_4 = -0.5-j0.866$$
$$s_5 = 0.5-j0.866$$
$$s_6 = 1$$

All the poles are located in the s-plane as shown below. It is found that the angular separation between the poles is given by 360°/2N. In this case, it is 60°.



To ensure stability and considering only the poles that lie in the LHP, we can write the denominator of the transfer function H(s) as,

Denominator of H(s) $= (s+1)\,((s+0.5)^2 + (0.866)^2)$
$= (s+1)\,(s^2+s+1)$

Therefore the transfer function of a third order Butterworth filter for cutoff frequency Ω_c = 1rad/sec is,

$$H(s) = \frac{1}{(s+1)(s^2+s+1)}$$

The poles, which lie in the LHP, can be found using the formula

$$s_k = e^{j\phi_k}$$

where

$$\phi_k = \frac{\pi}{2} + \frac{(2k-1)\pi}{2N} \qquad k = 1,2,\ldots..N$$

The following table gives Butterworth polynomials for various values of N for Ω_c = 1 rad/sec.

N	Denominator of H(s)
1	$(s+1)$
2	$(s^2 + 1.4142s + 1)$
3	$(s + 1)(s^2 + s + 1)$
4	$(s^2 + 0.7654s + 1)(s^2 + 1.8478s + 1)$
5	$(s + 1)(s^2 + 0.6180s + 1)(s^2 + 1.6180s + 1)$
6	$(s^2 + 0.5176s + 1)(s^2 + 1.4142s + 1)(s^2 + 1.9319s + 1)$
7	$(s + 1)(s^2 + 0.4450s + 1)(s^2 + 1.2470s + 1)(s^2 + 1.8019s + 1)$
8	$(s^2 +0.3902s+1)(s^2 +1.1111s +1)(s^2 +1.6629s+1)(s^2 +1.9616s +1)$

In general, the unnormalized poles are derived from normalized poles by,

$$s_k' = \Omega_c \, s_k$$

The transfer function of such type of Butterworth filter can be obtained by substituting s by s/Ω_c in the transfer function of the Butterworth filter.

Order of the Filter

Now let us consider the maximum pass band attenuation in positive dB is α_p (< 3 dB) at pass band frequency Ω_p and α_s is the minimum stop band attenuation in positive dB at the stop band frequency Ωs.

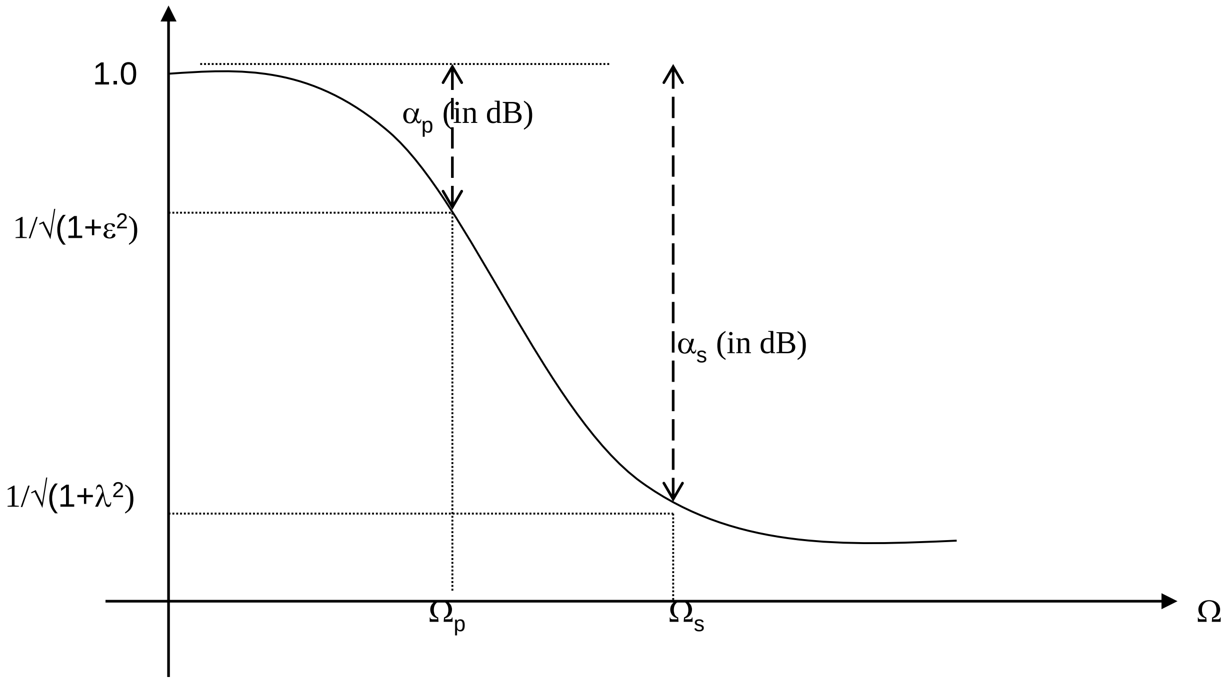
Now the magnitude function can be written as

$$|H(j\Omega)| = \frac{1}{\left[1+\varepsilon^2\left(\Omega/\Omega_p\right)^{2N}\right]^{1/2}}$$

$$|H(j\Omega)|^2 = \frac{1}{1+\varepsilon^2\left(\Omega/\Omega_p\right)^{2N}}$$

Taking logarithm on both sides we have,

$$20\log|H(j\Omega)| = 10\log 1 - 10\log\left[1+\varepsilon^2\left(\Omega/\Omega_p\right)^{2N}\right]$$



at $\Omega = \Omega_p$, i.e., $\alpha_p = 10 \log (1+\varepsilon^2)$

therefore, $\varepsilon = (10^{0.1\alpha_p} - 1)^{1/2}$

at $\Omega = \Omega_s$, i.e., $\alpha_s = 10 \log (1+\varepsilon^2 (\Omega_s/\Omega_p)^{2N})$

therefore,

$$\left(\frac{\Omega_s}{\Omega_p}\right)^N = \sqrt{\frac{10^{0.1\alpha_p} - 1}{10^{0.1\alpha_s} - 1}}$$

$$N = \frac{\log\left(\sqrt{\dfrac{10^{0.1\alpha_p} - 1}{10^{0.1\alpha_s} - 1}}\right)}{\log\left(\dfrac{\Omega_s}{\Omega_p}\right)}$$

Since it does not results in an integer value, normally it is round off to the nearest integer.

$$N \geq \frac{\log\left(\sqrt{\dfrac{10^{0.1\alpha_p} - 1}{10^{0.1\alpha_s} - 1}}\right)}{\log\left(\dfrac{\Omega_s}{\Omega_p}\right)}$$

or

$$N \geq \frac{\log\left(\dfrac{\lambda}{\varepsilon}\right)}{\log\left(\dfrac{\Omega_s}{\Omega_p}\right)}$$

where $\lambda = \left(10^{0.1\alpha_s} - 1\right)^{1/2}$ and $\varepsilon = \left(10^{0.1\alpha_p} - 1\right)^{1/2}$

for simplicity, let, $A=\lambda/\varepsilon$ and $k=\Omega_p/\Omega_s$ (k is transition ratio)

Therefore, $$N \geq \frac{\log(A)}{\log(1/k)}$$

Steps to design an analog Butterworth low pass filter:

1. From the given specifications, find the order of the filter N.

2. Round off it to the next higher integer.

3. Find the transfer function H(s) for $\Omega_c = 1$ rad/sec for the value of N.

4. Calculate the value of cutoff frequency Ω_c.

5. Find the transfer function $H_a(s)$ for the above value of Ω_c by substituting $s = s/\Omega_c$ in H(s).

Design of IIR digital filters from analog filters

- If the conversion technique is to be effective, it should possess the following desirable properties.

  i. The $j\Omega$ axis in the s-plane should map into the unit circle in the z-plane. Thus there will be a direct relationship between the two frequency variables in the two domain.

  ii. The LHP of the s-plane should map into the inside of the unit circle in the z-plane. Thus a stable analog filter will be converted to a stable digital filter.