Sampling Theorem

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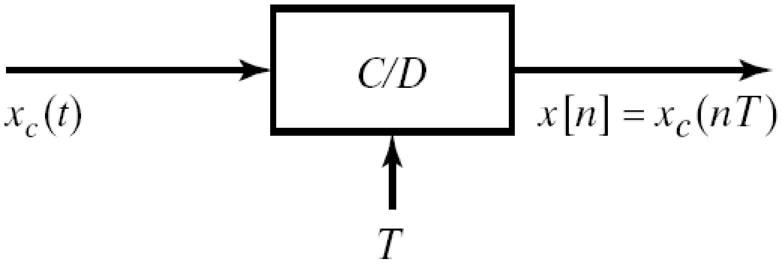
Periodic Sampling

In this method x[n] obtain from $x_c(t)$ according to t relation :



$$x[n] = x_c(nT) \quad -\infty < n < \infty$$

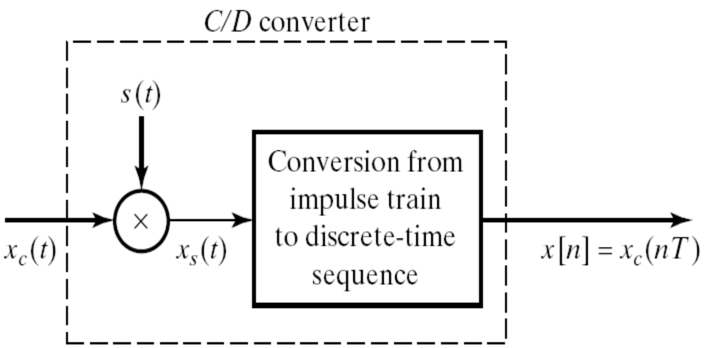
$$T \rightarrow sampling \ \ period \qquad f_s = 1/T \rightarrow sampling \ \ frequency$$

- The sampling operation is generally not invertible i.e., given the output x[n] it is not possible in general to reconstruct $x_c(t)$. Although we remove this ambiguity by restricting $x_c(t)$.

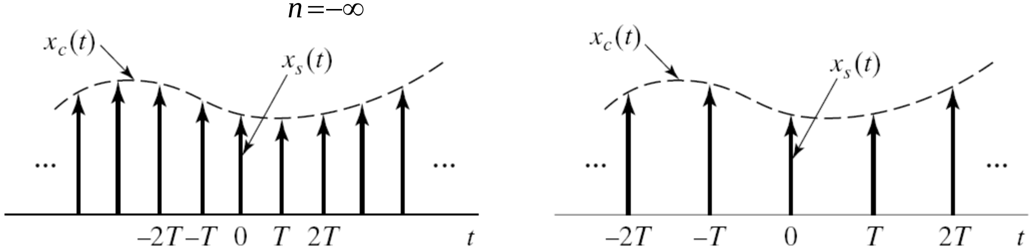
Sampling with a Periodic Impulse Train

- Figure(a) is not a representation of any physical circuits, but it is convenient for gaining insight in both the time and frequency domain.

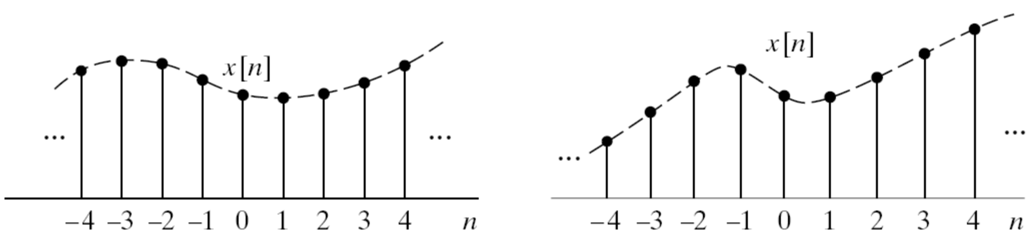
$$s(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT)$$



(a)



(b)



(c)

(a) Overall system

(b) $x_s(t)$ for two sampling rates

(c) Output for two sampling rates

Frequency Domain Representation of Sampling

$$x_s(t) = x_c(t)s(t) = x_c(t)\sum_{n=-\infty}^{+\infty}\delta(t-nT) \quad (Modulation)$$

$$x_s(t) = \sum_{n=-\infty}^{+\infty} x_c(nT)\delta(t-nT) \quad (Shifting\ \ property)$$

Frequency Domain Representation of Sampling

- By applying the continuous-time Fourier transform to equation

$$x_s(t) = \sum_{n=-\infty}^{+\infty} x_c(nT)\delta(t - nT)$$

We obtain

$$X_S(j\Omega) = \sum_{n=-\infty}^{+\infty} x_c(nT)e^{-j\Omega Tn}$$

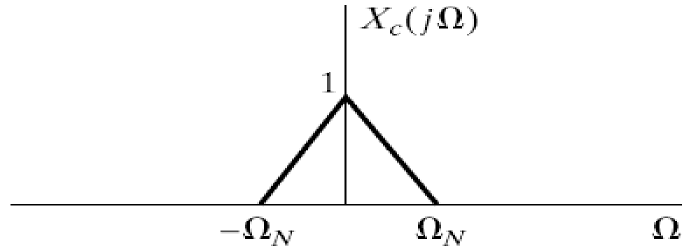
$$x[n] = x_c(nT) \quad and \quad X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$$

consequently

$$X_s(j\Omega) = X(e^{j\omega})\Big|_{\omega=\Omega T} = X(e^{j\Omega T}) \Rightarrow X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right)$$

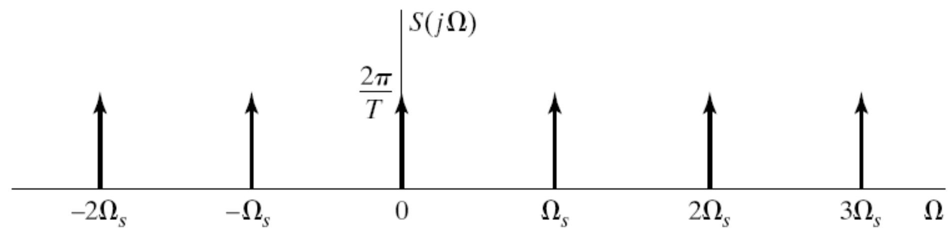
Exact Recovery of Continuous-Time from Its Samples

- (a) represents a band limited Fourier transform of $x_c(t)$ Whose highest nonzero frequency is Ω_N



(a)

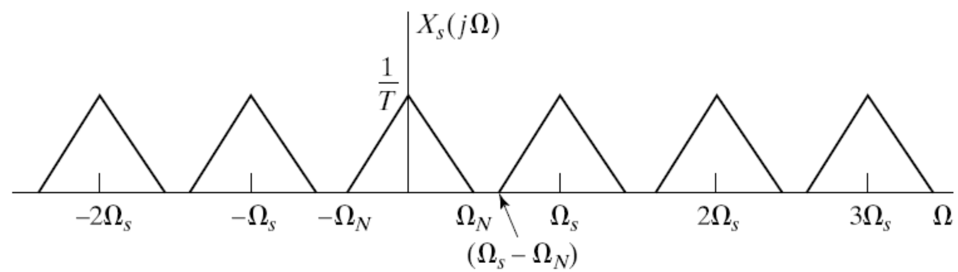
- (b) represents a periodic impulse train with Ω_S frequency.



(b)

- (c) shows the output of impulse modulator in the case

$$\Omega_S - \Omega_N > \Omega_N \Rightarrow \Omega_S > 2\Omega_N$$



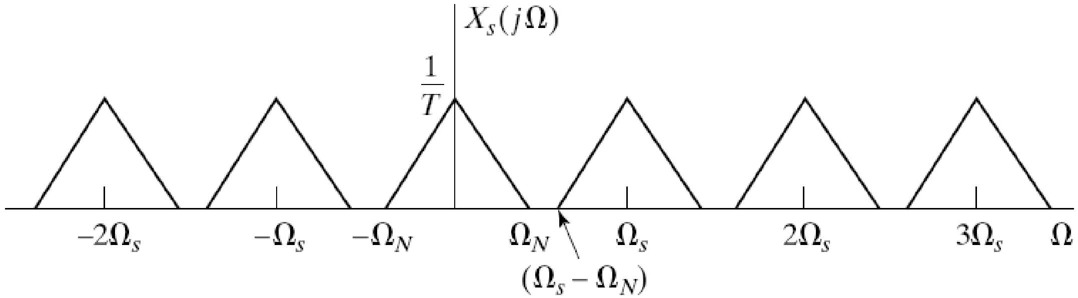
(c)

Exact Recovery of Continuous-Time from Its Samples

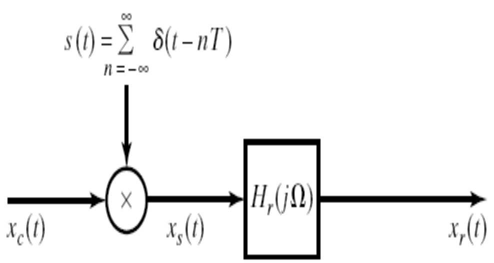
- In this case $X_C(j\Omega)$ don't overlap
- therefore $x_c(t)$ can be recovered from $x_s(t)$ with an ideal low pass filter $H_r(j\Omega)$ with gain T and cutoff frequency

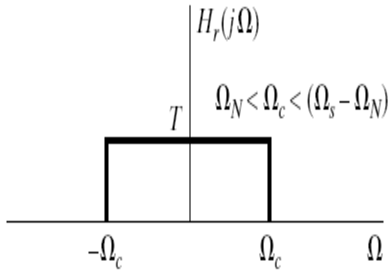
$$X_r(j\Omega) = X_C(j\Omega)$$

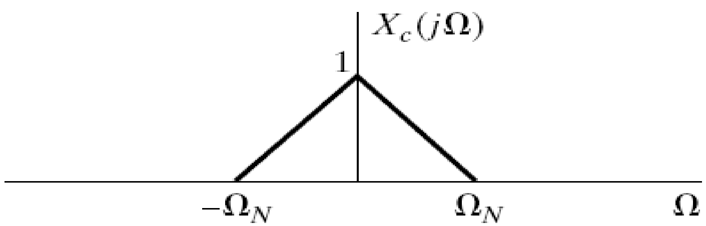
- It means



(c)







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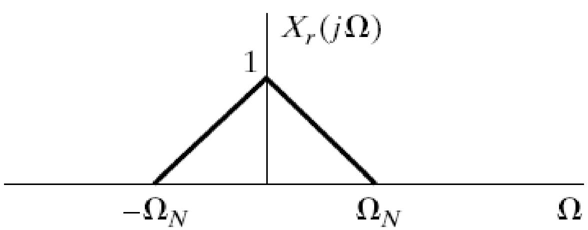
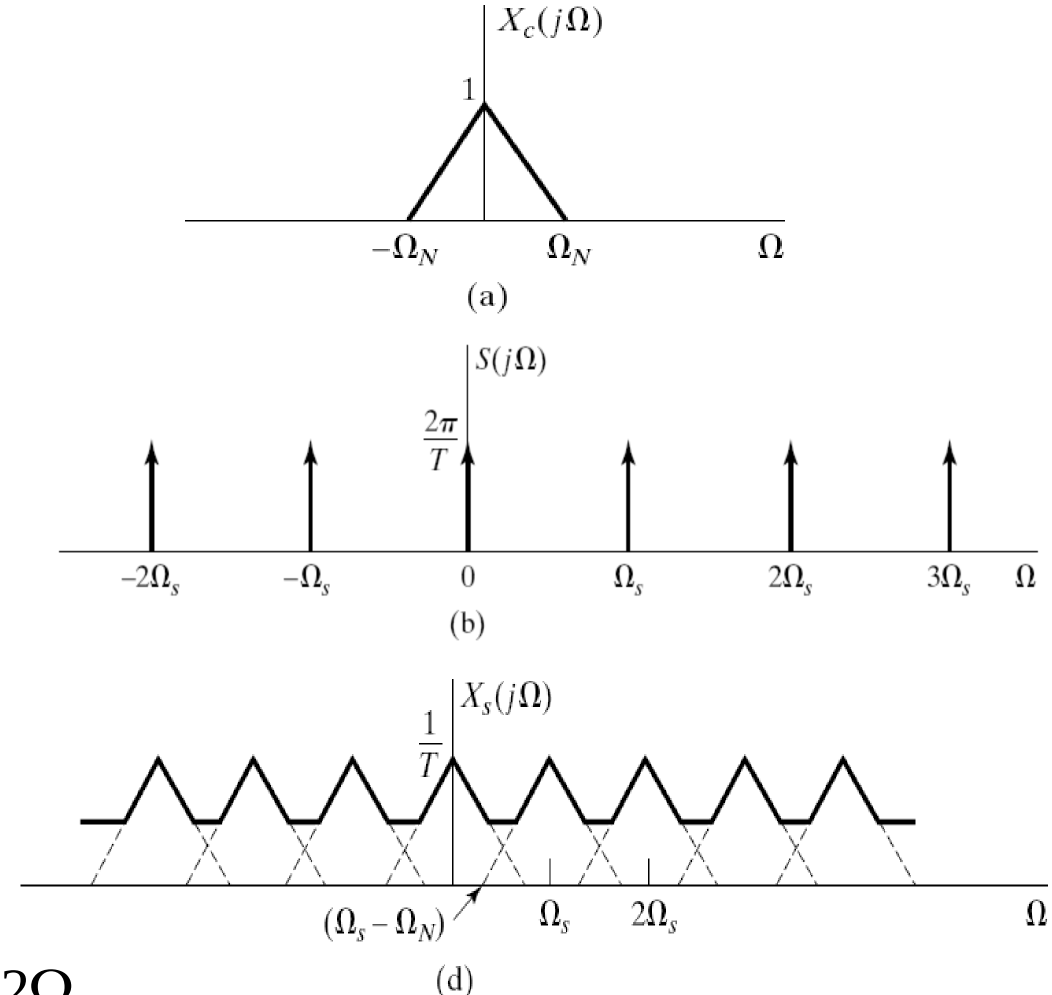


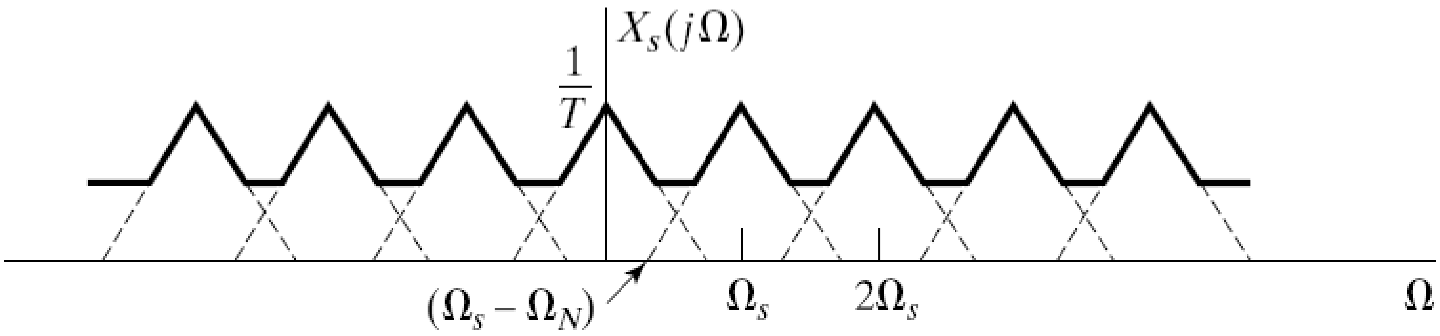
Aliasing Distortion

- (a) represents a band limited Fourier transform of $x_c(t)$ Whose highest nonzero frequency is Ω_N .

- (b) represents a periodic impulse train with frequency. Ω_S
- (c) shows the output of impulse modulator in the case $\Omega_S - \Omega_N < \Omega_N \Rightarrow \Omega_S < 2\Omega_N$



Aliasing Distortion



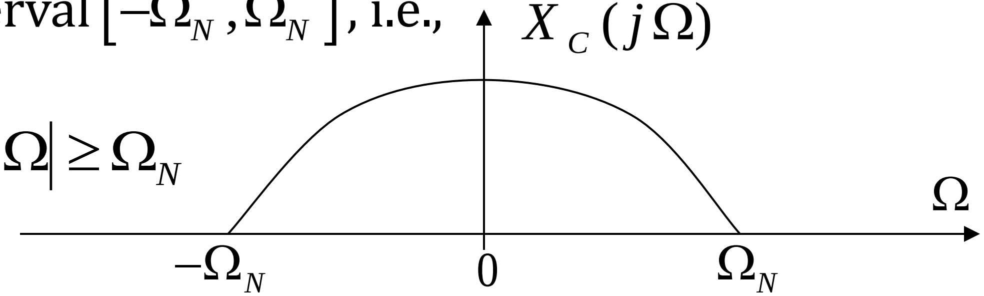
(d)

- In this case the copies of $X_c(j\Omega)$ overlap and is not longer recoverable by lowpass filtering therefore the reconstructed signal is related to original continuous-time signal through a distortion referred to as aliasing distortion.

Nyquist Sampling Theorem

- **Sampling theorem** describes precisely how much information is retained when a function is sampled, or whether a band-limited function can be exactly reconstructed from its samples.
- **Sampling Theorem**: Suppose that $x_c(t) \leftrightarrow X_C(j\Omega)$ is band-limited to a frequency interval $[-\Omega_N, \Omega_N]$, i.e.,

$$X_C(j\Omega) = 0 \quad \text{for} \quad |\Omega| \geq \Omega_N$$

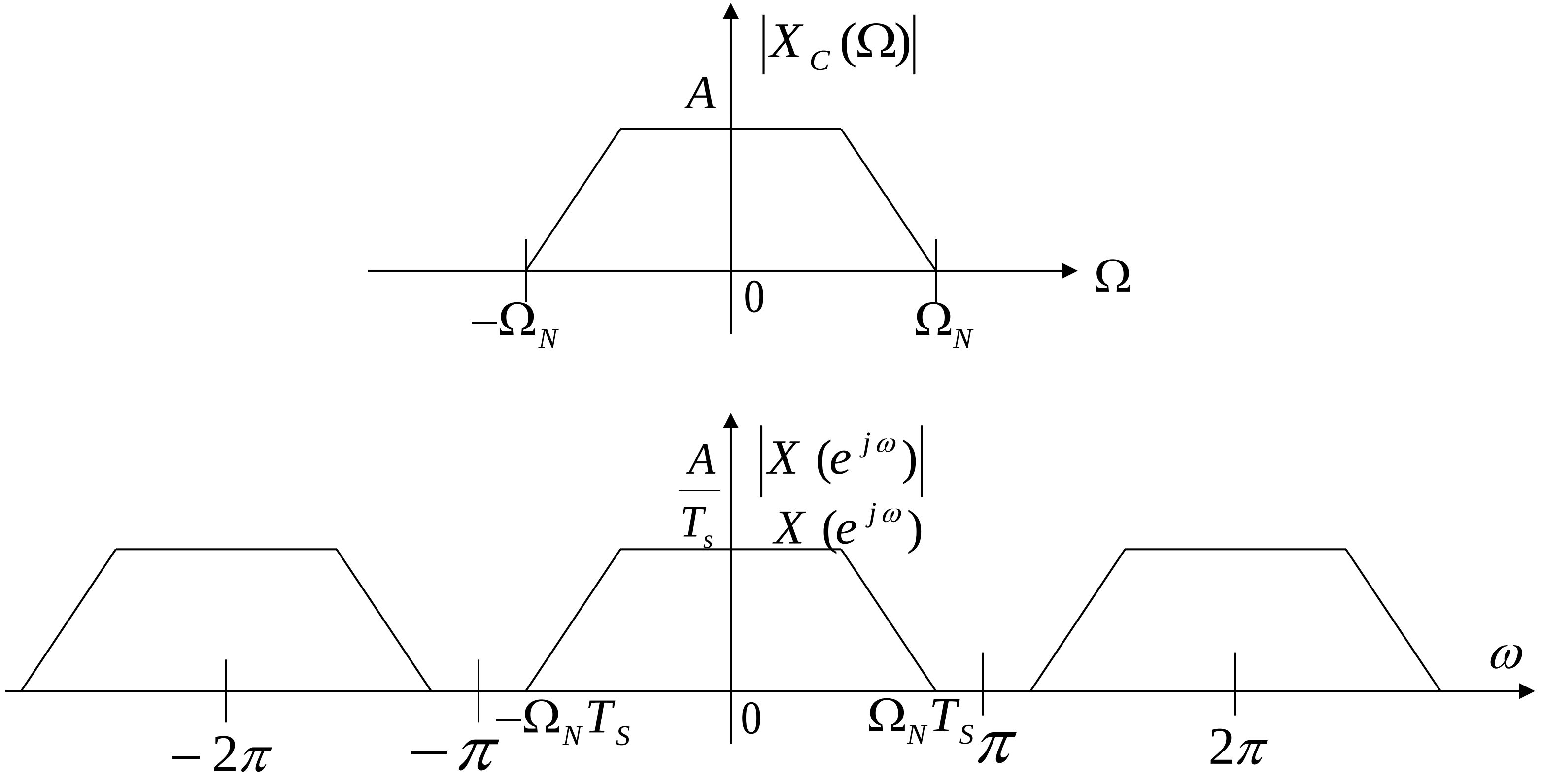


Then $x_c(t)$ can be exactly reconstructed from equidistant samples

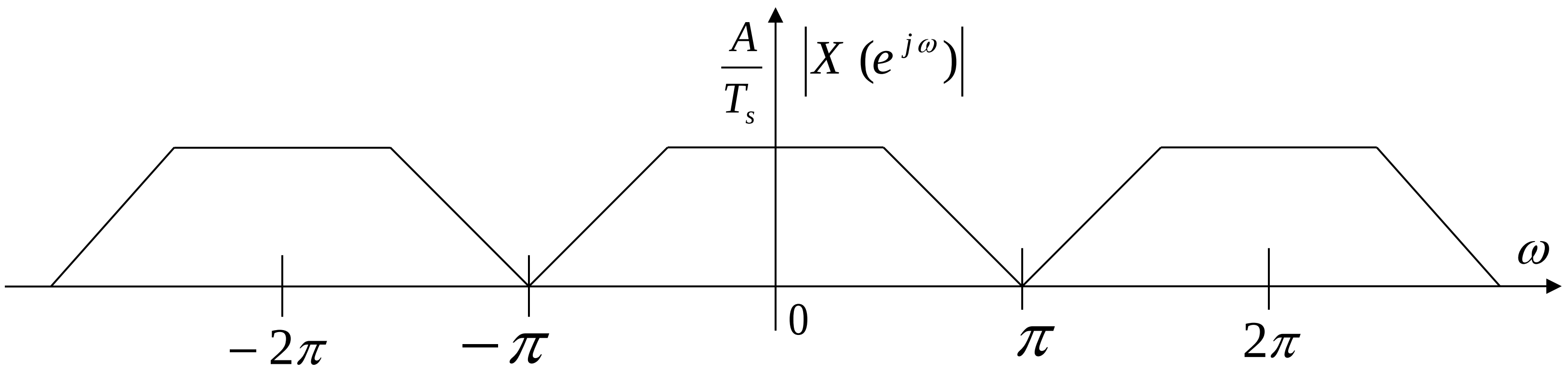
$$x[n] = x_c(nT_s) = x_c(2\pi n / \Omega_s) \quad \Omega_s > 2\Omega_N$$

where $T_s = 2\pi / \Omega_s$ is the sampling period, $f_s = 1/T_s$ is the sampling frequency (samples/second), $\Omega_s = 2\pi / T_s$ is for radians/second.

Oversampled



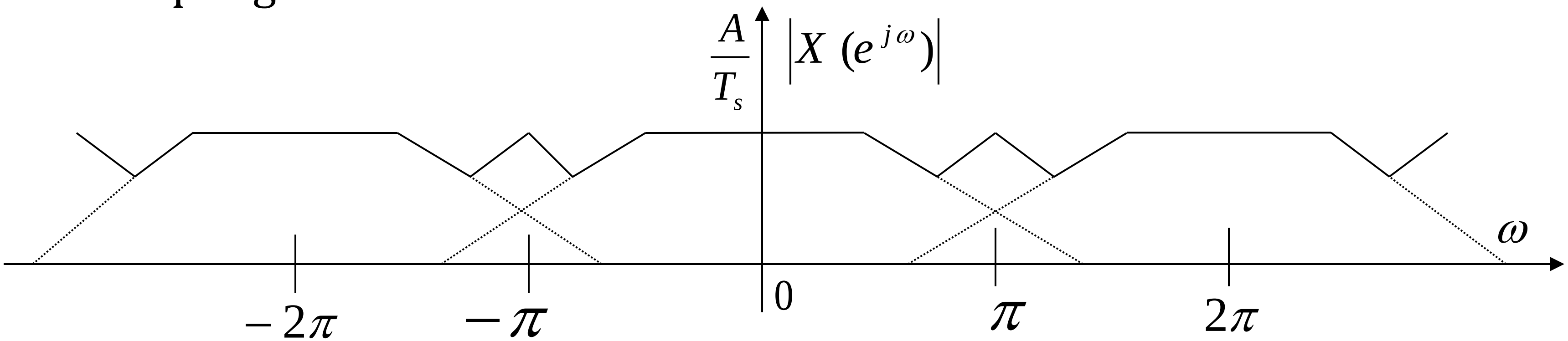
Critically Sampled



According to the Sampling Theorem, in general the signal cannot be reconstructed from samples at the rate $T_S = \pi / \Omega_N$.
This is because of errors will occur if $\omega = \pi.$, the folded frequencies will add at $X_c(\Omega_N) \neq 0$

Undersampled (aliased)

If sampling theorem condition is **not** satisfied



- The frequencies are **folded - summed**. This **changes** the shape of the spectrum. There is no process whereby the added frequencies can be **discriminated** - so the process is not reversible.
- Thus, the original (continuous) signal cannot be reconstructed exactly. Information is lost, and false (alias) information is created.
- If a signal is not **strictly band-limited**, sampling can still be done at twice the **effective band-limited**.

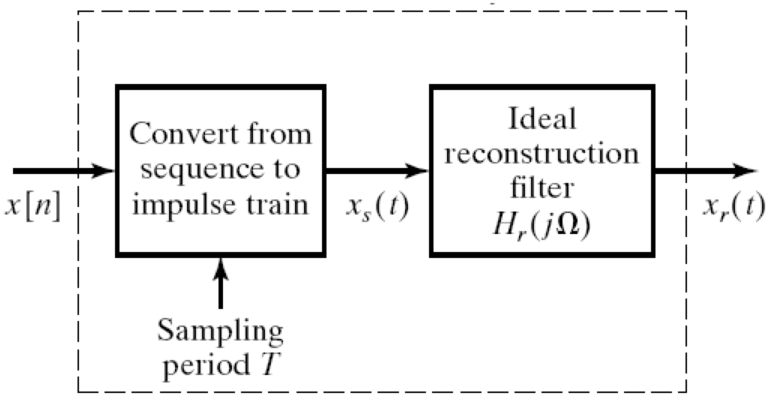
Reconstruction of a Bandlimited Signal from Its Samples

- Figure(a) represents an ideal reconstruction system.
- Ideal reconstruction filter has the gain of T and cutoff frequency Ω_c

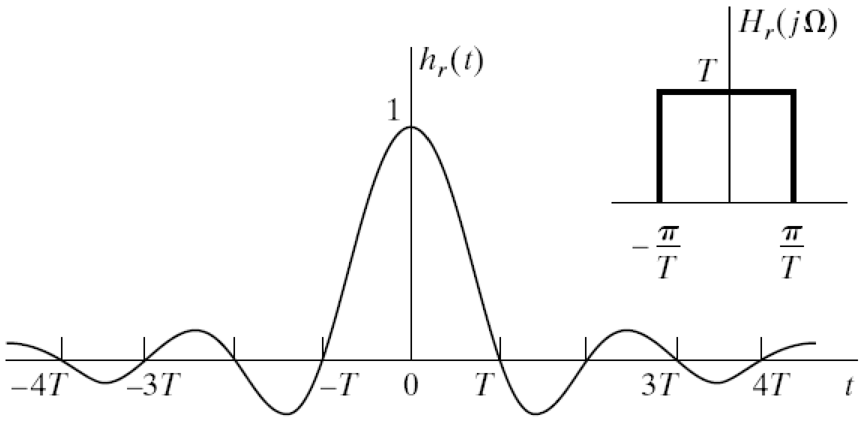
$$\Omega_N < \Omega_C < \Omega_S - \Omega_N$$

we choice $\Omega_C = \Omega_S / 2 = \pi / T$.

This choice is appropriate for any relationship between Ω_S and Ω_N.



(a)



(b)

Reconstruction of a Bandlimited Signal from Its Samples

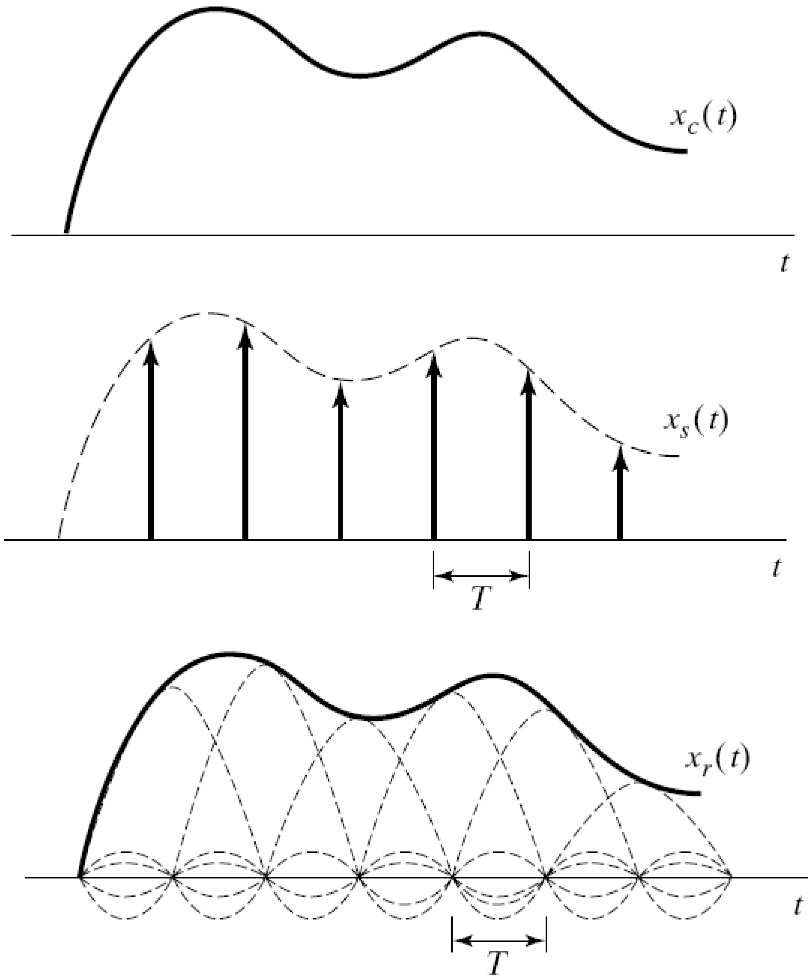
- Therefore

$$x_s(t) = \sum_{n=-\infty}^{+\infty} x[n]\delta(t-nT)$$

$$x_r(t) = \sum_{n=-\infty}^{+\infty} x[n]h_r(t-nT)$$

$$h_r(t) = \frac{\sin(\pi t/T)}{\pi t/T}$$

$$x_r(t) = \sum_{n=-\infty}^{+\infty} x[n]\frac{\sin(\pi(t-nT)/T)}{\pi(t-nT)/T}$$



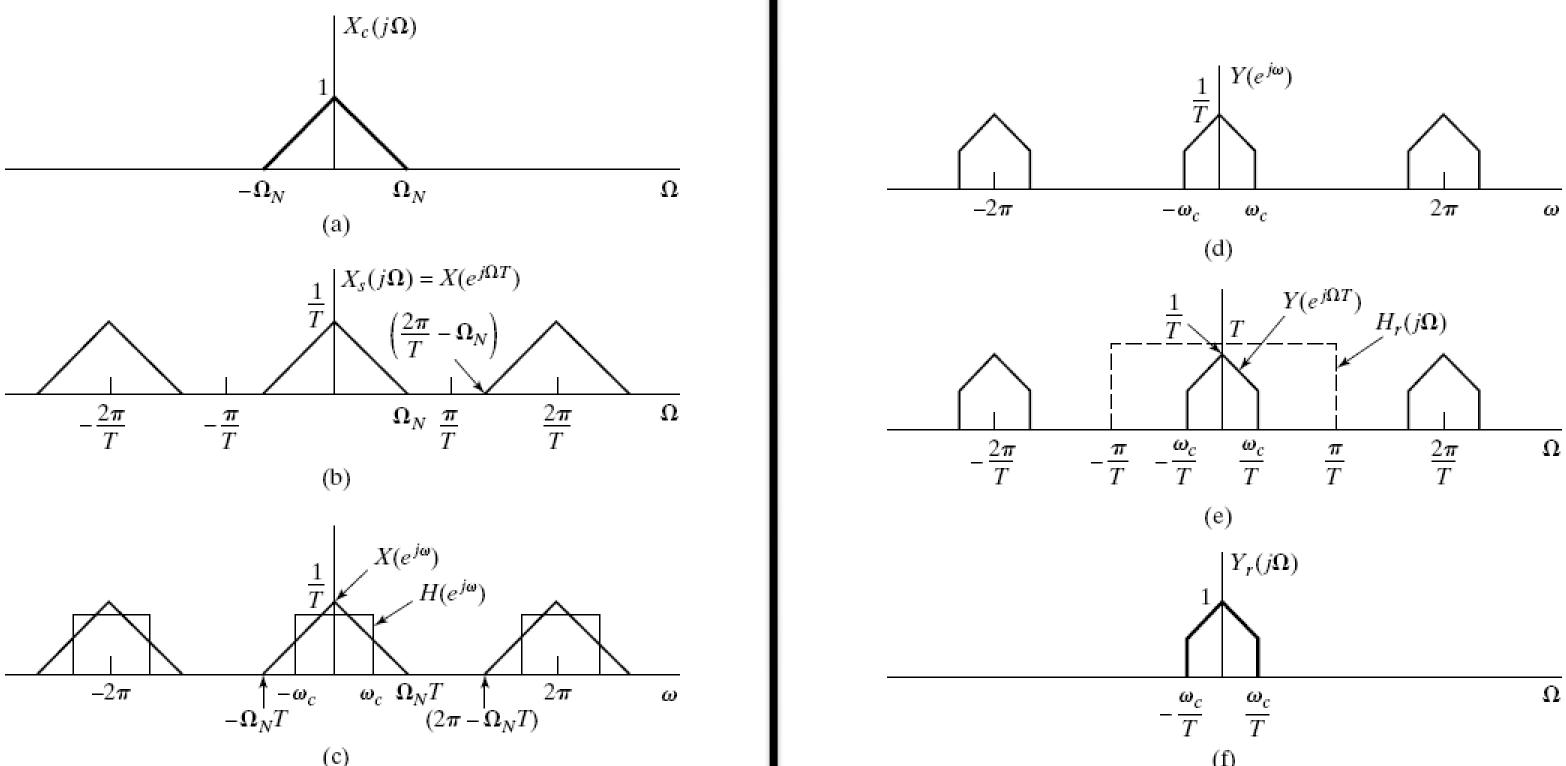
Reconstruction of a Bandlimited Signal from Its Samples

$$\left.\begin{array}{l} x_r(t) = \sum_{n=-\infty}^{+\infty} x[n] h_r(t - nT) \\ h_r(0) = 1 \\ h_r(nT) = 0 \quad n = \pm 1, \pm 2, \ldots \end{array}\right\rangle \Rightarrow x_r(mT) = x_c(mT)$$

For all integer values of m. independent from the sampling period T.

Therefore the resulting signal is an exact reconstruction of $x_c(t)$ at the sampling times. the fact that, if there is no aliasing, the low pass filter interpolates the correct reconstruction between the samples, and if there is aliasing, it can't interpolate them correctly.

Example: Ideal Continuous-Time Lowpass Filtering Using a Discrete-Time Lowpass Filter



Example: A discrete-time lowpass filter obtained by impulse invariance

- We want to obtain an ideal lowpass discrete-time filter with cutoff frequency $\omega_c < \pi$. we can do this by sampling a continuous-time ideal lowpass filter with cutoff frequency $\Omega_c = \omega_c / T < \pi / T$

$$H_c(j\Omega) = \begin{cases} 1, & |\Omega| < \Omega_c \\ 0, & |\Omega| \geq \Omega_c \end{cases}$$

$$h_c(t) = \frac{\sin(\Omega_c t)}{\pi t}$$

$$h[n] = T h_c(nT) = T \frac{\sin(\Omega_c nT)}{\pi nT} = \frac{\sin(\omega_c n)}{\pi n}$$