Infinite Impulse Response Filter – Bilinear Transformation Method

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Design of IIR filter using Bilinear Transformation Technique

The bilinear transformation is a conformal mapping that transforms the jΩ axis into the unit circle in the z – plane only once. Thus aliasing of frequency components is avoided.

Furthermore, all points in the LHP of 's' are mapped inside the unit circle in the z – plane and all points in the RHP of 's' are mapped into corresponding points outside the unit circle in the z – plane.

Let us consider the transfer function,

$$H(s) = \frac{b}{s+a} \qquad \text{...........(1)}$$

Or

$$\frac{Y(s)}{X(s)} = \frac{b}{s+a}$$

Or $\quad s\,Y(s) + a\,Y(s) = b\,X(s)$

This can be characterized by the differential equations,

$$\frac{dy(t)}{dt} + ay(t) = bx(t) \qquad \text{..........(2)}$$

y(t) can be approximated by the trapezoidal formula.

Thus,

$$y(t) = \int_{t_o}^{t} y'(\tau)d\tau + y(t_o)$$

y'(t) denotes the derivative of y(t)

$$\boxed{\int_a^b f(x)dx = \frac{(b-a)}{2}(f(b) + f(a))}$$

The approximation of the integral by the trapezoidal formula at t = nT and t_o = nT-T yields,

$$y(nT) = \frac{T}{2}[y'(nT) + y'(nT-T)] + y(nT-T) \qquad \text{............(3)}$$

from (2)

$$y'(nT) = -a\,y(nT) + b\,x(nT)$$

substituting the above expression in (3), rearranging and using y(nT) = y(n), we get,

$$\left[1+\frac{aT}{2}\right]y(n) - \left[1-\frac{aT}{2}\right]y(n-1) = \frac{bT}{2}\left[x(n)+x(n-1)\right]$$

applying Z – transform, we get,

$$\left[1+\frac{aT}{2}\right]Y(z)-\left[1-\frac{aT}{2}\right]z^{-1}Y(Z)=\frac{bT}{2}\left[X(z)+z^{-1}X(z)\right]$$

$$H(z)=\frac{Y(z)}{X(z)}=\frac{\frac{bT}{2}\left[1+z^{-1}\right]}{\left[1+\frac{aT}{2}\right]-\left[1-\frac{aT}{2}\right]z^{-1}}$$

$$=\frac{\frac{bT}{2}\left[1+z^{-1}\right]}{\left[1-z^{-1}\right]+\frac{aT}{2}\left[1+z^{-1}\right]}$$

Dividing by $(T/2)(1+z^{-1})$

$$H(z)=\frac{b}{\frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]+a}$$

Therefore

$$s=\frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]$$

This relationship between s and z is known as bilinear transformation.

Let $z = r\,e^{j\omega}$ and $s = \sigma + j\,\Omega$, then the relationship between s and z can be expressed as

$$s = \frac{2}{T}\left[\frac{z-1}{z+1}\right] = \frac{2}{T}\left[\frac{r\,e^{j\omega}-1}{r\,e^{j\omega}+1}\right]$$

$$\sigma + j\Omega = \frac{2}{T}\left[\frac{r\cos\omega + jr\sin\omega - 1}{r\cos\omega + jr\sin\omega + 1}\right]$$

$$= \frac{2}{T}\left[\frac{r\cos\omega - 1 + jr\sin\omega}{r\cos\omega + 1 + jr\sin\omega}\right] * \left[\frac{r\cos\omega + 1 - jr\sin\omega}{r\cos\omega + 1 - jr\sin\omega}\right]$$

$$= \frac{2}{T}\left[\frac{r^2 - 1 + 2jr\sin\omega}{r^2 + 1 + 2r\cos\omega}\right]$$

Equating the real and imaginary parts, we get,

$$\sigma = \frac{2}{T}\left[\frac{r^2-1}{r^2+1+2r\cos\omega}\right] \qquad \Omega = \frac{2}{T}\left[\frac{2r\sin\omega}{r^2+1+2r\cos\omega}\right]$$

If r ≤ 1then σ ≤ 0 and if r > 1then σ > 0. when r = 1 then σ = 0 and

$$\omega = 2\tan^{-1}\left(\frac{\Omega T}{2}\right) \quad \text{(Or)} \quad \Omega = \frac{2}{T}\tan\frac{\omega}{2}$$

Warping effect

We have,

$$\Omega = \frac{2}{T} \tan \frac{\omega}{2}$$

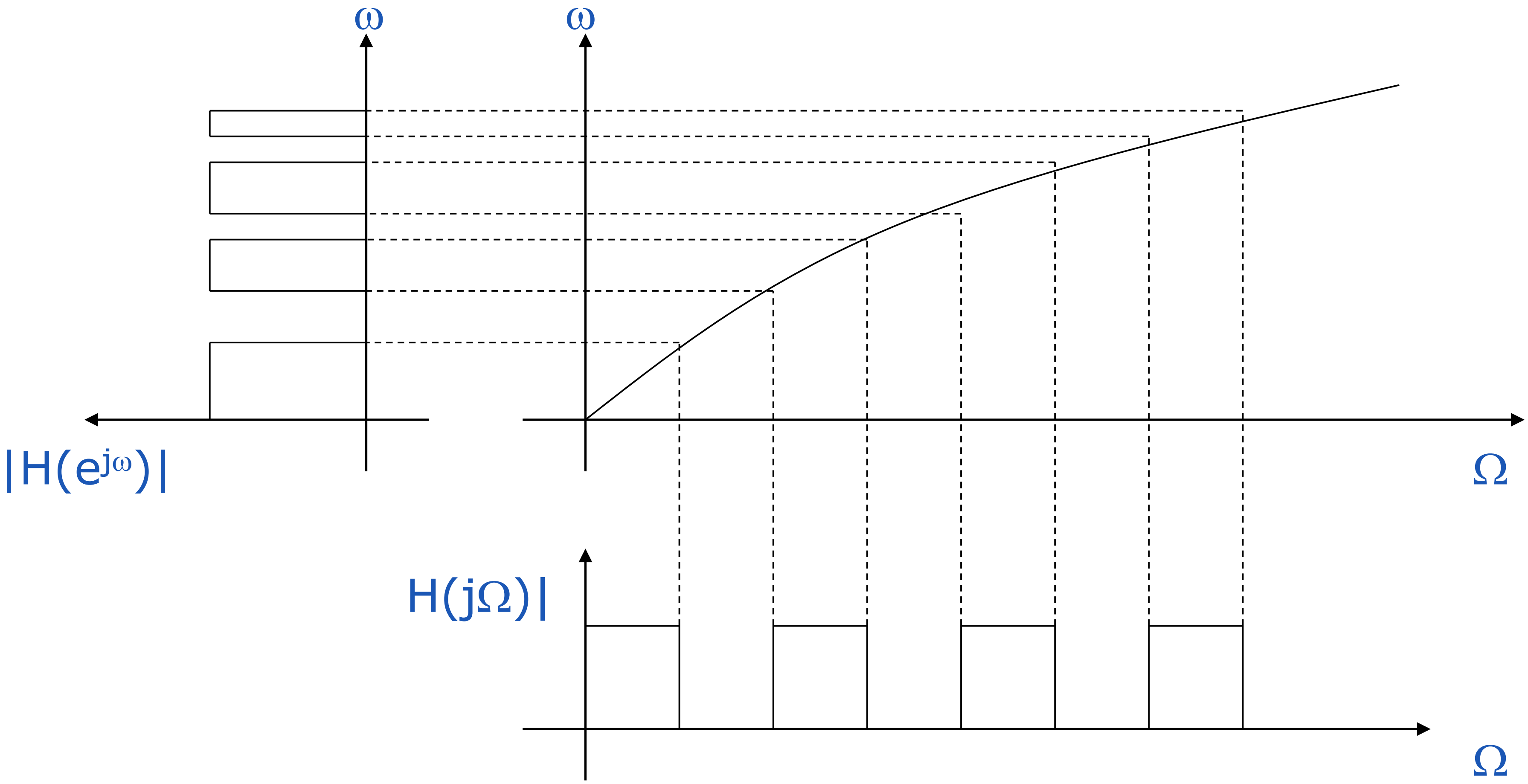
for small ω, $\tan(\omega/2) = \omega/2$, therefore $\Omega = \omega/T$ or $\omega = \Omega T$

For low frequencies, the relationship between Ω and ω are linear, hence the digital filter have the same amplitude response as the analog filter.

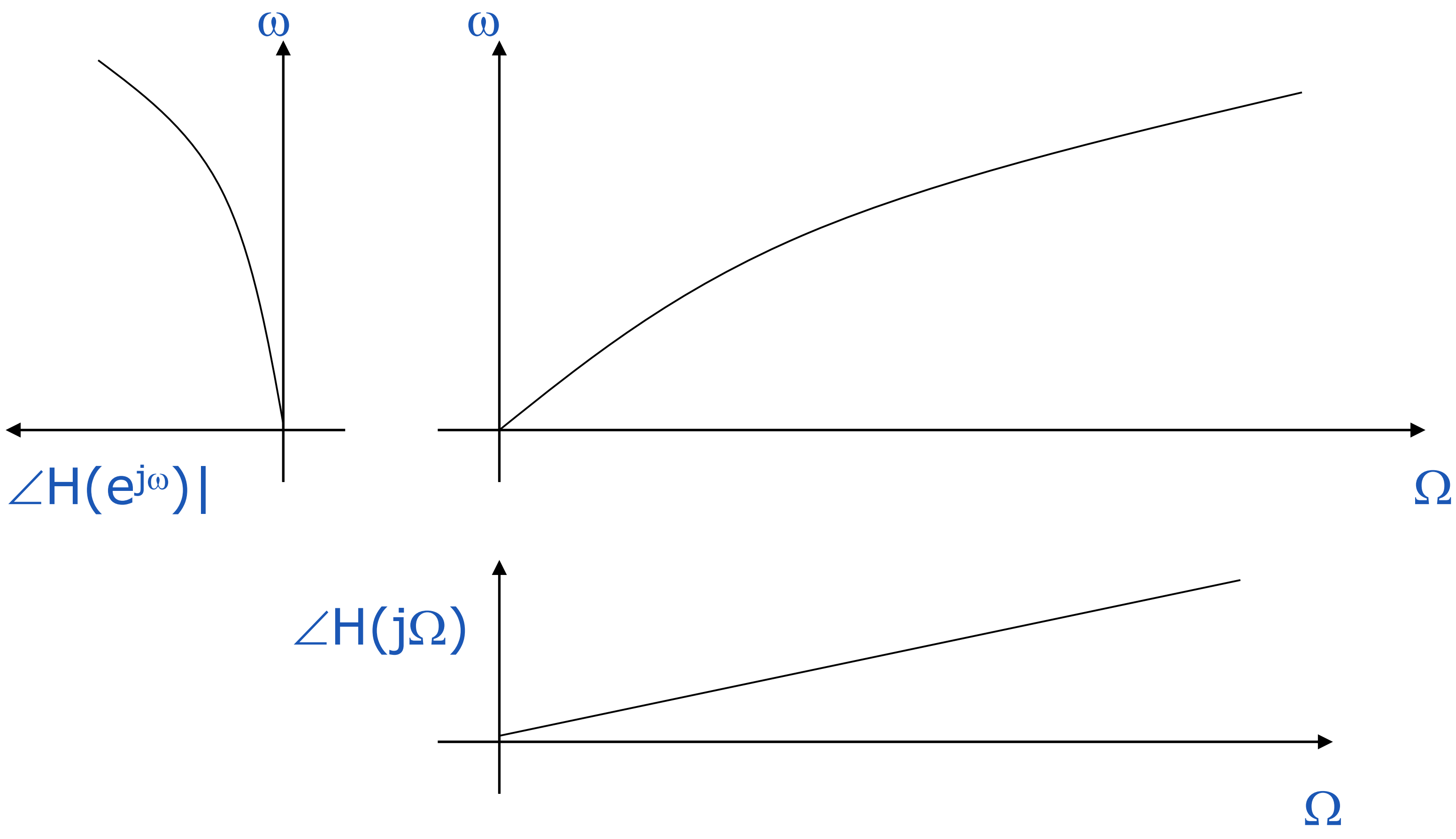
For high frequencies, the relationship becomes non linear and distortion is introduced in the frequency scale of the digital filter to that of the analog filter.

This is known as Warping effect.

Effect on magnitude response



Effect on phase response



Prewarping effect:

The warping effect can be eliminated by prewarping the analog filter. To find the prewarping analog frequencies we use the formula,

$$\Omega = \frac{2}{T}\tan\frac{\omega}{2}$$

and

$$\Omega_p = \frac{2}{T}\tan\frac{\omega_p}{2} \quad \& \quad \Omega_s = \frac{2}{T}\tan\frac{\omega_s}{2}$$