Z- Transform & its Properties

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Objective

- At the end of this session, students will understand the mathematical operation of Z transform for finite sequence and infinite sequence and its region of convergence

The Z-Transform

- Counterpart of the Laplace transform for discrete-time signals
- Generalization of the Fourier Transform

  Fourier Transform does not exist for all signals
- Definition:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

- Compare to DTFT definition:

$$X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

- z is a complex variable that can be represented as

  $z = r e^{j\omega}$
- Substituting $z = e^{j\omega}$ will reduce the z-transform to DTFT

The Z-Transform (Contd...)

$$Z\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = X(z)$$

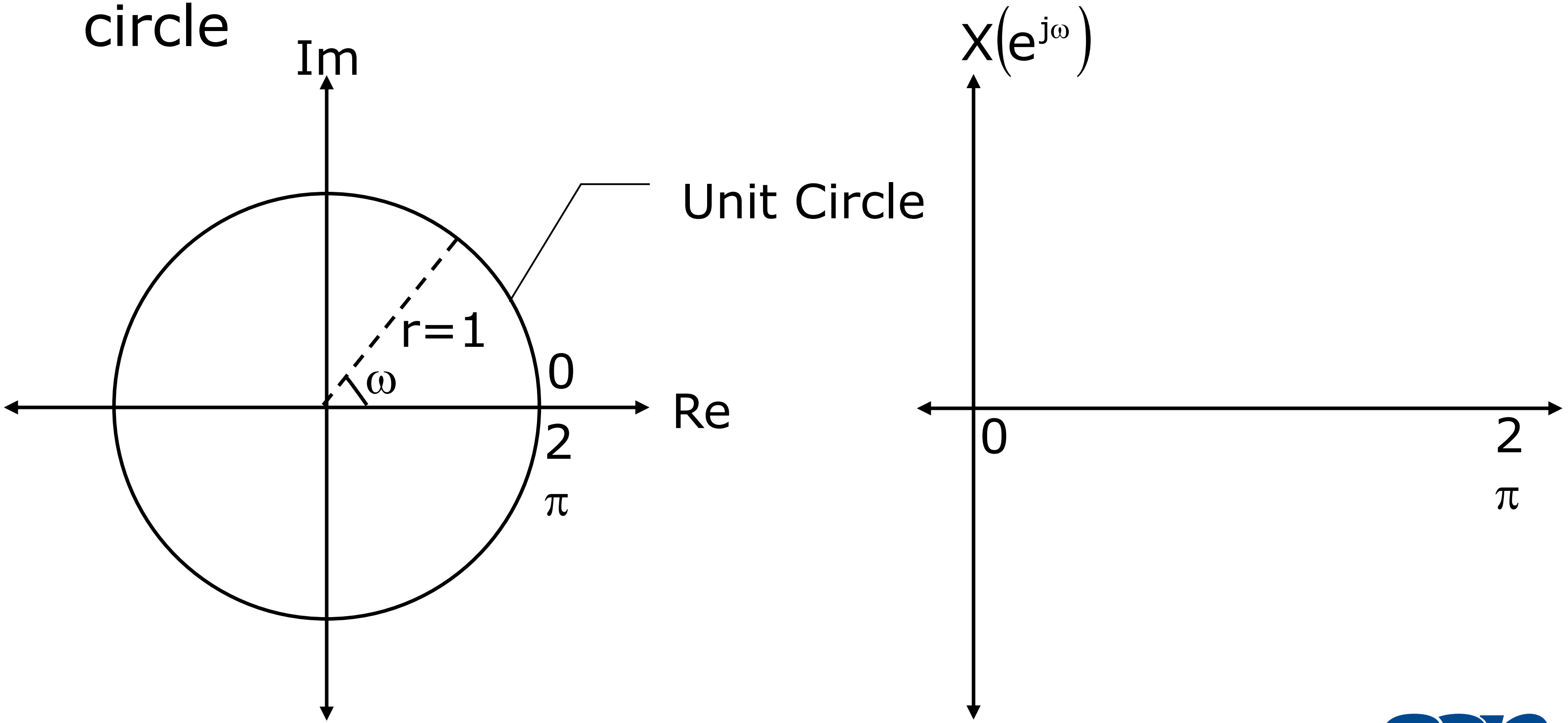
$$x[n] \xleftrightarrow{z} X(z)$$

$$X(z) = \sum_{n=0}^{\infty} x[n]z^{-n}$$

$$z = re^{j\omega}$$

The z-transform and the DTFT

- Convenient to describe on the complex z-plane
- If we plot $z=e^{j\omega}$ for $\omega=0$ to 2π we get the unit circle



Convergence of the z-Transform

- DTFT does not always converge $$X\left(e^{j\omega}\right)=\sum_{n=-\infty}^{\infty}x[n]e^{-j\omega n}$$
  Example: $x[n] = a^n u[n]$ for $|a|>1$ does not have a DTFT

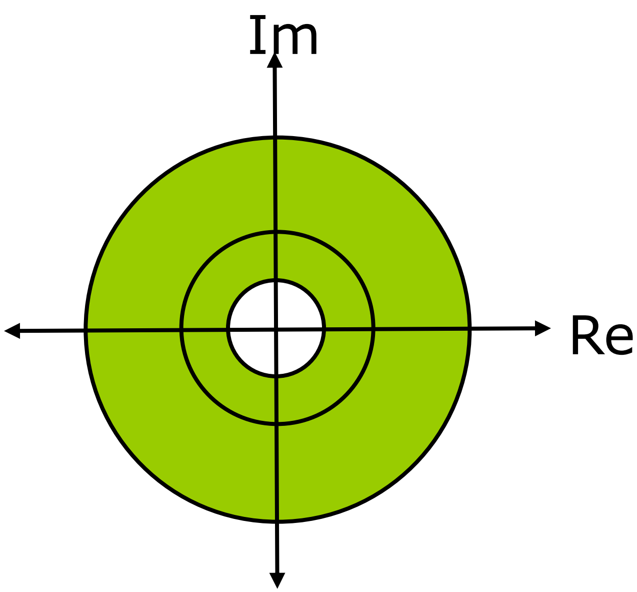
- Complex variable z can be written as $r\,e^{j\omega}$ so the z-transform $$X\left(re^{j\omega}\right)=\sum_{n=-\infty}^{\infty}x[n]\left(\mathrm{r}e^{-j\omega}\right)^{-n}=\sum_{n=-\infty}^{\infty}\left(x[n]\mathrm{r}^{-n}\right)e^{-j\omega n}$$

  convert to the DTFT of x[n] multiplied with exponential sequence r^{-n}

- For certain choices of r the sum $$\sum_{n=-\infty}^{\infty}\left|x[n]\mathrm{r}^{-n}\right|<\infty$$
  maybe made finite

Region of Convergence (ROC)

- **ROC:** The set of values of z for which the z-transform converges
- The region of convergence is made of circles



- Example: z-transform converges for values of $0.5<r<2$
  - *ROC is shown on the left*
  - *In this example the ROC includes the unit circle, so DTFT exists*

Region of Convergence (ROC)- (Contd…)

- Example:

Doesn't converge for any r.

DTFT exists.

It has finite energy.

DTFT converges in a mean square sense.

$$x[n] = \cos(\omega_o n)$$

- *Example:*

Doesn't converge for any r.

It doesn't have even finite energy.

But we define a useful DTFT with impulse function.

$$x[n] = \frac{\sin \omega_c n}{\pi n}$$

Example 1: Right-Sided Exponential Sequence

$$x[n] = a^n u[n] \quad \Rightarrow \quad X(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} \left(az^{-1}\right)^n$$

- For Convergence we require

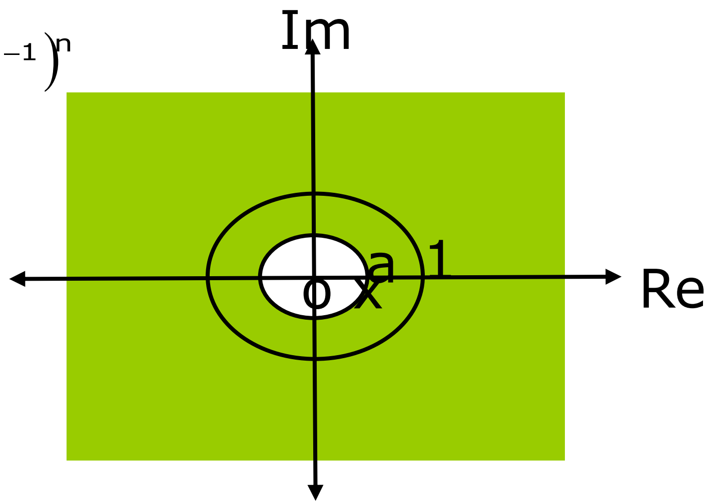
$$\sum_{n=0}^{\infty} \left|az^{-1}\right|^n < \infty$$

- Hence the ROC is defined as

$$\left|az^{-1}\right|^n < 1 \Rightarrow |z| > |a|$$

- Inside the ROC series converges to

$$X(z) = \sum_{n=0}^{\infty} \left(az^{-1}\right)^n = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}$$



- Region outside the circle of radius a is the ROC
- Right-sided sequence ROCs extend outside a circle

Example 2: Left-Sided Exponential Sequence

$$x[n] = -a^n u[-n-1]$$

$$X(z) = -\sum_{n=-\infty}^{\infty} a^n u[-n-1] z^{-n} = -\sum_{n=-\infty}^{-1} a^n z^{-n}$$

$$= -\sum_{n=1}^{\infty} \left(a^{-1}z\right)^n = 1 - \sum_{n=0}^{\infty} \left(a^{-1}z\right)^n$$

$ROC:$

$$\sum_{n=0}^{\infty} \left|a^{-1}z\right|^n < \infty \Rightarrow \left|a^{-1}z\right| < 1 \Rightarrow |z| < |a|$$

$$X(z) = 1 - \frac{1}{1-a^{-1}z} = \frac{1}{1-az^{-1}} = \frac{z}{z-a}$$

Example 3: Two-Sided Exponential Sequence

$$x[n] = \left(-\frac{1}{3}\right)^n u[n] - \left(\frac{1}{2}\right)^n u[-n-1]$$

$$\sum_{n=0}^{\infty}\left(-\frac{1}{3}z^{-1}\right)^n = \frac{\left(-\frac{1}{3}z^{-1}\right)^0 - \left(-\frac{1}{3}z^{-1}\right)^\infty}{1+\frac{1}{3}z^{-1}} = \frac{1}{1+\frac{1}{3}z^{-1}}$$

$$\sum_{n=-\infty}^{-1}\left(\frac{1}{2}z^{-1}\right)^n = \frac{\left(\frac{1}{2}z^{-1}\right)^{-\infty} - \left(\frac{1}{2}z^{-1}\right)^0}{1-\frac{1}{2}z^{-1}} = \frac{-1}{1-\frac{1}{2}z^{-1}}$$

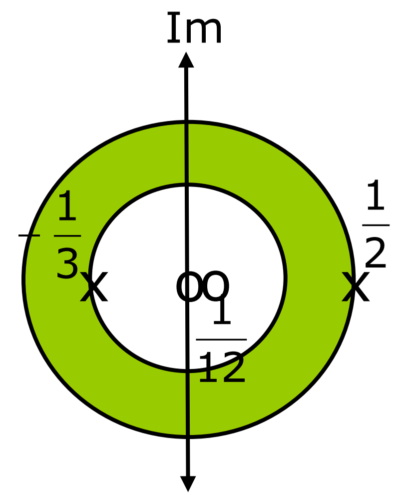
$$X(z) = \frac{1}{1+\frac{1}{3}z^{-1}} + \frac{1}{1-\frac{1}{2}z^{-1}} = \frac{2z\left(z-\frac{1}{12}\right)}{\left(z+\frac{1}{3}\right)\left(z-\frac{1}{2}\right)}$$

$$\text{ROC}: \left|-\frac{1}{3}z^{-1}\right| < 1$$

$$\frac{1}{3} < |z|$$

$$\text{ROC}: \left|\frac{1}{2}z^{-1}\right| > 1$$

$$\frac{1}{2} > |z|$$

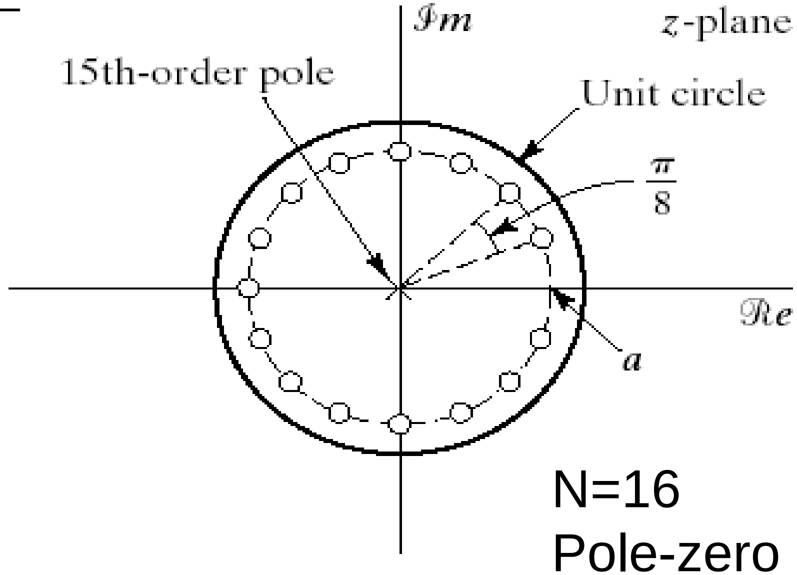


Example 4: Finite Length Sequence

$$x[n] = a^n \left(u[n] - u[n-N]\right) \qquad x[n] = \begin{cases} a^n & 0 \le n \le N-1 \\ 0 & \text{otherwise} \end{cases}$$

$$X(z) = \sum_{n=0}^{N-1} a^n z^{-n} = \sum_{n=0}^{N-1} \left(az^{-1}\right)^n = \frac{1-\left(az^{-1}\right)^N}{1-az^{-1}}$$

$$= \frac{1}{z^{N-1}} \cdot \frac{z^N - a^N}{z-a}$$



N=16
Pole-zero plot

$ROC:$

$$\sum_{n=0}^{N-1} \left|az^{-1}\right|^n < \infty \Rightarrow \left|az^{-1}\right| < \infty \Rightarrow z \neq 0$$

Some common Z-transform pairs

Sequence	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3. $-u[-n-1]$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
4. $\delta[n-m]$	z^{-m}	All z except 0 (if $m>0$) or ∞ (if $m<0$)
5. $a^n u[n]$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
6. $-a^n u[-n-1]$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
7. $na^n u[n]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
8. $-na^n u[-n-1]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
9. $[\cos\omega_0 n]u[n]$	$\dfrac{1-[\cos\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
10. $[\sin\omega_0 n]u[n]$	$\dfrac{[\sin\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
11. $[r^n\cos\omega_0 n]u[n]$	$\dfrac{1-[r\cos\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
12. $[r^n\sin\omega_0 n]u[n]$	$\dfrac{[r\sin\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
13. $\begin{cases} a^n, & 0\le n\le N-1, \\ 0, & \text{otherwise} \end{cases}$	$\dfrac{1-a^N z^{-N}}{1-az^{-1}}$	$\lvert z\rvert > 0$

Some common Z-transform pairs (Contd..)

SEQUENCE	TRANSFORM	ROC
$\delta[n]$	1	ALL z
$u[n]$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
$-u[-n-1]$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
$\delta[n-m]$	z^{-m}	All z except 0 (if $m > 0$) or ∞ (if $m < 0$)

Some common Z-transform pairs (Contd...)

$$a^n u[n] \xleftrightarrow{Z} \frac{1}{1-az^{-1}} \qquad ROC: \quad |z| > |a|$$

$$a^n u[-n-1] \xleftrightarrow{Z} \frac{1}{1-az^{-1}} \qquad ROC: \quad |z| < |a|$$

$$na^n u[n] \xleftrightarrow{Z} \frac{az^{-1}}{(1-az^{-1})^2} \qquad ROC: \quad |z| > |a|$$

$$-na^n u[-n-1] \xleftrightarrow{Z} \frac{az^{-1}}{(1-az^{-1})^2} \qquad ROC: \quad |z| < |a|$$

$$[\cos \omega_0 n] u[n] \xleftrightarrow{Z} \frac{1-[\cos \omega_0] z^{-1}}{1-[2\cos \omega_0] z^{-1} + z^{-2}} \qquad ROC: \quad |z| > 1$$

Some common Z-transform pairs (Contd…)

$$[\sin\omega_0 n]u[n] \overset{Z}{\longleftrightarrow} \frac{[\sin\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}} \qquad ROC: \quad |z|>1$$

$$[r^n\cos\omega_0 n]u[n] \overset{Z}{\longleftrightarrow} \frac{1-[r\cos\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}} \qquad ROC: \quad |z|>r$$

$$[r^n\sin\omega_0 n]u[n] \overset{Z}{\longleftrightarrow} \frac{[r\sin\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}} \qquad ROC: \quad |z|>r$$

$$\begin{cases} a^n & 0\le n\le N-1 \\ 0 & otherwise \end{cases} \overset{Z}{\longleftrightarrow} \frac{1-a^N z^{-N}}{1-az^{-1}} \qquad ROC: \quad |z|>0$$

Properties of The ROC of Z-Transform

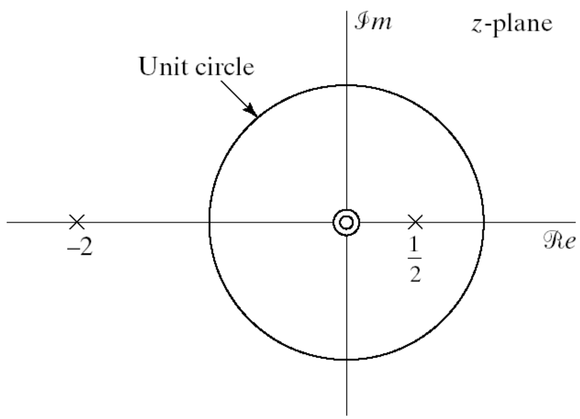
- The ROC is a ring or disk centered at the origin
- DTFT exists if and only if the ROC includes the unit circle
- The ROC cannot contain any poles
- The ROC for finite-length sequence is the entire z-plane

  except possibly $z=0$ and $z=\infty$
- The ROC for a right-handed sequence extends outward from the outermost pole possibly including $z=\infty$
- The ROC for a left-handed sequence extends inward from the innermost pole possibly including $z=0$
- The ROC of a two-sided sequence is a ring bounded by poles
- The ROC must be a connected region
- A z-transform does not uniquely determine a sequence without specifying the ROC

Stability, Causality, and the ROC

- Consider a system with impulse response h[n]
- The z-transform H(z) and the pole-zero plot shown below
- Without any other information h[n] is not uniquely determined

  $|z|>2$ or $|z|<½$ or $½<|z|<2$
- If system stable ROC must include unit-circle: $½<|z|<2$
- If system is sided: $|z|>2$



Z-Transform Properties: Linearity

- Notation

$$x[n] \overset{Z}{\longleftrightarrow} X(z) \qquad \text{ROC} = R_x$$

- Linearity

$$ax_1[n] + bx_2[n] \overset{Z}{\longleftrightarrow} aX_1(z) + bX_2(z) \qquad \text{ROC} = R_{x_1} \cap R_{x_2}$$

  – Note that the ROC of combined sequence may be larger than either ROC
  – This would happen if some pole/zero cancellation occurs
  – Example:

$$x[n] = a^n u[n] - a^n u[n - N]$$

    • Both sequences are right-sided
    • Both sequences have a pole z=a
    • Both have a ROC defined as |z|>|a|
    • In the combined sequence the pole at z=a cancels with a zero at z=a
    • The combined ROC is the entire z plane except z=0

Z-Transform Properties: Time Shifting

$$x[n-n_o] \xleftrightarrow{Z} z^{-n_o}X(z) \qquad \mathrm{ROC} = R_x$$

- Here n_o is an integer
  - *If positive the sequence is shifted right*
  - *If negative the sequence is shifted left*
- The ROC can change
  - The new term may add or remove poles at $z=0$ or $z=\infty$
- Example

$$X(z) = z^{-1}\left(\frac{1}{1-\frac{1}{4}z^{-1}}\right) \qquad |z| > \frac{1}{4}$$

$$x[n] = \left(\frac{1}{4}\right)^{n-1} u[n-1]$$

Z-Transform Properties: Multiplication by Exponential

$$z_o^n x[n] \stackrel{Z}{\longleftrightarrow} X(z/z_o) \qquad \mathrm{ROC} = |z_o| R_x$$

- ROC is scaled by $|z_o|$
- All pole/zero locations are scaled
- If z_o is a positive real number: z-plane shrinks or expands
- If z_o is a complex number with unit magnitude it rotates

$$u[n] \stackrel{Z}{\longleftrightarrow} \frac{1}{1 - z^{-1}} \qquad \mathrm{ROC}: |z| > 1$$

- Example: We know the z-transform pair

$$x[n] = r^n \cos(\omega_o n) u[n] = \frac{1}{2}\left(re^{j\omega_o}\right)^n u[n] + \frac{1}{2}\left(re^{-j\omega_o}\right)^n u[n]$$

- Let's find the z-transform of

$$X(z) = \frac{1/2}{1 - re^{j\omega_o} z^{-1}} + \frac{1/2}{1 - re^{-j\omega_o} z^{-1}} \qquad |z| > r$$

Z-Transform Properties: Differentiation

$$nx[n] \xleftrightarrow{\;z\;} -z\frac{dX(z)}{dz} \qquad \text{ROC} = R_x$$

- Example: We want the inverse z-transform of

$$X(z) = \log\left(1 + az^{-1}\right) \qquad |z| > |a|$$

- Let's differentiate to obtain rational expression

$$\frac{dX(z)}{dz} = \frac{-az^{-2}}{1 + az^{-1}} \Rightarrow -z\frac{dX(z)}{dz} = az^{-1}\frac{1}{1 + az^{-1}}$$

- Making use of z-transform properties and ROC

$$nx[n] = a(-a)^{n-1}u[n-1]$$

$$x[n] = (-1)^{n-1}\frac{a^n}{n}u[n-1]$$

Z-Transform Properties: Conjugation

$$x^*[n] \xleftrightarrow{Z} X^*(z^*) \qquad \text{ROC} = R_x$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$X^*(z) = \left(\sum_{n=-\infty}^{\infty} x[n] z^{-n} \right)^* = \sum_{n=-\infty}^{\infty} x^*[n] z^{n}$$

$$X^*(z^*) = \sum_{n=-\infty}^{\infty} x^*[n] \left(z^{n}\right)^* = \sum_{n=-\infty}^{\infty} x^*[n] z^{-n} = Z\left\{x^*[n]\right\}$$

Z-Transform Properties: Time Reversal

$$x[-n] \overset{z}{\longleftrightarrow} X(1/z) \qquad \text{ROC} = \frac{1}{R_x}$$

- ROC is inverted
- Example:

$$x[n] = a^{-n}u[-n]$$

- Time reversed version of $a^n u[n]$

$$X(z) = \frac{1}{1-az} = \frac{-a^{-1}z^{-1}}{1-a^{-1}z^{-1}} \qquad |z| < |a^{-1}|$$

Z-Transform Properties: Convolution

$$x_1[n] * x_2[n] \xleftrightarrow{z} X_1(z)X_2(z) \qquad \text{ROC}: R_{x1} \cap R_{x_2}$$

- Convolution in time domain is multiplication in z-domain
- Example: Let's calculate the convolution of

$$x_1[n] = a^n u[n] \text{ and } x_2[n] = u[n]$$

$$X_1(z) = \frac{1}{1 - az^{-1}} \quad \text{ROC}: |z| > |a| \qquad X_2(z) = \frac{1}{1 - z^{-1}} \quad \text{ROC}: |z| > 1$$

- Multiplications of z-transforms is

$$Y(z) = X_1(z)X_2(z) = \frac{1}{(1 - az^{-1})(1 - z^{-1})}$$

- ROC: if |a|<1 ROC is |z|>1 if |a|>1 ROC is |z|>|a|
- Partial fractional expansion of Y(z)

$$y[n] = \frac{1}{1 - a}\left(u[n] - a^{n+1}u[n]\right)$$

Some Z-transform properties

Section Reference	Sequence	Transform	ROC
	$x[n]$	$X(z)$	R_x
	$x_1[n]$	$X_1(z)$	R_{x_1}
	$x_2[n]$	$X_2(z)$	R_{x_2}
3.4.1	$ax_1[n]+bx_2[n]$	$aX_1(z)+bX_2(z)$	Contains $R_{x_1} \cap R_{x_2}$
3.4.2	$x[n-n_0]$	$z^{-n_0}X(z)$	R_x, except for the possible addition or deletion of the origin or ∞
3.4.3	$z_0^n x[n]$	$X(z/z_0)$	$\lvert z_0 \rvert R_x$
3.4.4	$nx[n]$	$-z\dfrac{dX(z)}{dz}$	R_x, except for the possible addition or deletion of the origin or ∞
3.4.5	$x^*[n]$	$X^*(z^*)$	R_x
	$\mathcal{R}e\{x[n]\}$	$\frac{1}{2}[X(z)+X^*(z^*)]$	Contains R_x
	$\mathcal{J}m\{x[n]\}$	$\frac{1}{2j}[X(z)-X^*(z^*)]$	Contains R_x
3.4.6	$x^*[-n]$	$X^*(1/z^*)$	$1/R_x$
3.4.7	$x_1[n]*x_2[n]$	$X_1(z)X_2(z)$	Contains $R_{x_1} \cap R_{x_2}$
3.4.8	Initial-value theorem:		
	$x[n]=0,\quad n<0$	$\lim_{z\to\infty} X(z)=x[0]$	

Summary

- Z transform, its ROC and its properties are discussed with some examples.