Design of FIR Filters using Frequency Sampling method

I.Nelson
SSN College of Engineering

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- Let h(n) be the filter coefficients of an FIR filter and H(k) is the DFT of h(n). Then we have,

$$h(n)=\frac{1}{N}\sum_{k=0}^{N-1}H(k)\,e^{j2\pi kn/N} \qquad ; \; n=0,1,\ldots\ldots\ldots,N-1 \qquad \ldots\ldots\ldots\ldots(1)$$

and

$$H(k)=\sum_{n=0}^{N-1}h(n)\,e^{-j2\pi kn/N} \qquad ; \; k=0,1,\ldots\ldots\ldots,N-1 \qquad \ldots\ldots\ldots\ldots(2)$$

- The DFT samples H(k) for an FIR sequence can be regarded as samples of the filter z- transform evaluated at N points equally spaced around the unit circle.

i.e., $$H(k) = H(z)\big|_{z=e^{j2\pi k/N}} \qquad \text{.............(3)}$$

- The transfer function H(z) of an FIR filter with impulse response is given by,

$$H(z) = \sum_{n=0}^{N-1} h(n)\, z^{-n} \qquad \text{................(4)}$$

➢ Substituting (1) in (4) we get,

$$H(z)=\sum_{n=0}^{N-1}\left[\frac{1}{N}\sum_{k=0}^{N-1}H(k)e^{j2\pi kn/N}\right]z^{-n}$$

$$=\sum_{k=0}^{N-1}\frac{H(k)}{N}\sum_{n=0}^{N-1}\left(e^{j2\pi k/N}z^{-1}\right)^{n}$$

$$=\sum_{k=0}^{N-1}\frac{H(k)}{N}\left(\frac{1-\left(e^{j2\pi k/N}z^{-1}\right)^{N}}{1-e^{j2\pi k/N}z^{-1}}\right)$$

$$=\frac{1-z^{-N}}{N}\sum_{k=0}^{N-1}\frac{H(k)}{1-e^{j2\pi k/N}z^{-1}} \qquad \text{..............(5)}$$

- We know,

$$H(e^{j\omega})\big|_{\omega=\frac{2\pi k}{N}} = H(e^{j2\pi k/N}) = H(k) \qquad \ldots\ldots\ldots\ldots(6)$$

i.e., H(k) is the k^{th} DFT component obtained by sampling the frequency response $H(e^{j\omega})$.

Frequency response of FIR filter:

- The frequency response of the FIR filter can be obtained by setting $z=e^{j\omega}$ in (5), we get,

$$H(e^{j\omega}) = \frac{1-e^{-j\omega N}}{N}\sum_{k=0}^{N-1}\frac{H(k)}{1-e^{j2\pi k/N}\,e^{-j\omega}}$$

$$= \frac{e^{-j\omega N/2}(e^{j\omega N/2}-e^{-j\omega N/2})}{N}\sum_{k=0}^{N-1}\frac{H(k)}{1-e^{-j(\omega-2\pi k/N)}}$$

$$= \frac{e^{-j\omega N/2}}{N}\sum_{k=0}^{N-1}\frac{H(k)(e^{j\omega N/2}-e^{-j\omega N/2})}{e^{-j(\omega-2\pi k/N)/2}(e^{j(\omega-2\pi k/N)/2}-e^{-j(\omega-2\pi k/N)/2})}$$

$$= \frac{e^{-j\omega N/2}}{N}\sum_{k=0}^{N-1}\frac{H(k)\sin(\omega N/2)}{e^{-j\omega/2}\,e^{j\pi k/N}\,\sin((\omega/2)-\pi k/N)}$$

$$= \frac{e^{-j\omega\frac{N-1}{2}}}{N}\sum_{k=0}^{N-1}\frac{H(k)e^{-j\pi k/N}\sin(\omega N/2)}{\sin((\omega/2)-\pi k/N)}$$

$$\sin ce, \quad \sin\left(\frac{\omega N}{2} - k\pi\right) = (-1)^k \sin\left(\frac{\omega N}{2}\right)$$

$$H(e^{j\omega}) = \frac{e^{-j\omega\frac{N-1}{2}}}{N} \sum_{k=0}^{N-1} \frac{H(k)(-1)^k e^{-j\pi k/N} \sin N\left(\frac{\omega}{2} - \frac{\pi k}{N}\right)}{\sin\left(\frac{\omega}{2} - \frac{\pi k}{N}\right)} \qquad \text{..........(7)}$$

Design:

- Based on the set of samples that we choose from the frequency response, there are two types of design.

Type 1design:

- The frequency samples of the desired response $H_d(e^{j\omega})$ are determined using the relation,

$$H(k)=H_d(e^{j\omega})\Big|_{\omega=\frac{2\pi}{N}k} \quad ; \quad k=0,1,\ldots\ldots\ldots,N-1 \qquad \ldots\ldots\ldots\ldots\ldots(8)$$

- The frequency samples can be expressed in the form,

$$H(k)=|H(k)|e^{j\theta(k)} \qquad \ldots\ldots\ldots\ldots\ldots(9)$$

- For linear phase,

$$\theta(k)=\begin{cases} -\alpha\omega\big|_{\omega=\frac{2\pi}{N}k} & ; \quad k=0,1,\ldots\ldots\ldots N-1 \\ -\left(\frac{N-1}{2}\right)\frac{2\pi}{N}k & \\ -\left(\frac{N-1}{N}\right)\pi k & ; \quad k=0,1,\ldots\ldots\ldots N-1 \end{cases} \qquad \ldots\ldots\ldots\ldots(10)$$

- The filter coefficients h(n) can be obtained by finding IDFT of H(k), i.e.,

$$h(n)=\frac{1}{N}\sum_{k=0}^{N-1}H(k)e^{j2\pi kn/N} \qquad ; \quad n=0,1,\ldots\ldots\ldots N-1 \qquad \ldots\ldots(11)$$

- If h(n), the impulse response of the filter is to be a real valued signal, the frequency samples H(k) must satisfy the symmetry requirement.

  for N odd or even,

  $H(N-k) = H^*(k)$; $k=0,1,\ldots\ldots..N-1$ …….(12)

  and also for N even, $H(N/2)=0$

- With the frequency response H(k), the magnitude response is an even function,

  $|H(k)| = |H(N-k)|$ $k=0,1,\ldots\ldots\ldots N-1$ …….(13)

  and the phase is an odd function

  $\theta(k) = -\theta(N-k)$ $k=0,1,\ldots\ldots\ldots N-1$ …….(14)

- Replacing k by (N-k) in (10), we get

$$\theta(N-k) = -\left(\frac{N-1}{N}\right)\pi(N-k)$$

$$= -(N-1)\pi + \left(\frac{N-1}{N}\right)\pi k$$

- To satisfy the requirements of (14), θ(k) for N odd is given by

$$\theta(k) = \begin{cases} -\left(\frac{N-1}{N}\right)\pi k & ; \quad k = 0,1,\ldots\ldots,\frac{N-1}{2} \\ (N-1)\pi - \left(\frac{N-1}{N}\right)\pi k & ; \quad k = \frac{N+1}{2},\ldots\ldots\ldots\ldots,N-1 \end{cases} \qquad \ldots\ldots\ldots(15)$$

- Similarly for N even,

$$\theta(k)=\begin{cases} -\left(\dfrac{N-1}{N}\right)\pi k & ; \quad k=0,1,\ldots\ldots,\dfrac{N}{2}-1 \\ (N-1)\pi-\left(\dfrac{N-1}{N}\right)\pi k & ; \quad k=\dfrac{N}{2}+1,\ldots\ldots\ldots\ldots,N-1 \\ 0 & ; \quad k=\dfrac{N}{2} \end{cases} \quad \ldots\ldots(16)$$

- Substituting (15) in (9), we get for N odd,

$$H(k)=\begin{cases} |H(k)|e^{-j(N-1)\pi k/N} & ; \quad k=0,1,\ldots\ldots\ldots\ldots,\dfrac{N-1}{2} \\ |H(k)|e^{j\left[(N-1)\pi-\left(\frac{N-1}{N}\right)\pi k\right]} & ; \quad k=\dfrac{N+1}{2},\ldots\ldots\ldots\ldots,N-1 \end{cases} \quad \ldots\ldots(17)$$

- Substituting (16) in (9), we get for N even,

$$H(k)=\begin{cases} |H(k)|e^{-j(N-1)\pi k/N} & ; \quad k=0,1,\ldots\ldots,\dfrac{N}{2}-1 \\ |H(k)|e^{j\left[(N-1)\pi-\left(\frac{N-1}{N}\right)\pi k\right]} & ; \quad k=\dfrac{N}{2}+1,\ldots\ldots,N-1 \\ 0 & ; \quad k=\dfrac{N}{2} \end{cases} \qquad \ldots\ldots(18)$$

- If the filter is to be linear phase, then h(n) must satisfy the symmetry condition,

  h(n) = h(N-1-n) …………..(19)

- Using this symmetry condition and symmetry condition of H(k) the filter coefficients can be written as,

$$for\ N\ odd$$

$$h(n)=\frac{1}{N}\left\{H(0)+2\sum_{k=1}^{\frac{N-1}{2}}\operatorname{Re}\left[H(k)\,e^{j2\pi kn/N}\right]\right\} \qquad .....(20)$$

$$for\ N\ even$$

$$h(n)=\frac{1}{N}\left\{H(0)+2\sum_{k=1}^{\frac{N}{2}-1}\operatorname{Re}\left[H(k)\,e^{j2\pi kn/N}\right]\right\} \qquad .....(21)$$

The system function of the filter is given by,

$$H(z)=\sum_{n=0}^{N-1}h(n)\,z^{-n} \qquad .....(22)$$

Type 2 design

- The frequency samples of the desired response $H_d(e^{j\omega})$ are determined using the relation,

$$H(k) = H_d(e^{j\omega})\Big|_{\omega=\frac{2\pi}{N}\left(k+\frac{1}{2}\right)} \quad ; \quad k = 0,1,\dots\dots\dots, N-1 \qquad \dots\dots\dots(23)$$

- The filter coefficients h(n) can be obtained by finding IDFT of H(k), i.e.,

$$h(n) = \frac{1}{N}\sum_{k=0}^{N-1} H(k)e^{j2\pi kn/N} \qquad ; \qquad n = 0,1,\dots\dots\dots N-1 \qquad (24)$$

- If h(n), the impulse response of the filter is to be a real valued signal, the frequency samples H(k) must satisfy the symmetry requirement.

$$for\ N\ odd,$$

$$H(N-k-1)=H^{*}(k) \qquad k=0,1,\ldots\ldots,\frac{N-1}{2}-1$$

$$H\left(\frac{N-1}{2}\right)=0$$

$$for\ N\ even$$

$$H(N-k-1)=H^{*}(k) \qquad k=0,1,\ldots\ldots,\frac{N}{2}-1$$

- Using this symmetry condition and symmetry condition of H(k) the filter coefficients can be written as,

$$for\ N\ odd$$

$$h(n)=\frac{2}{N}\sum_{k=1}^{\frac{N-3}{2}}\mathrm{Re}\left[H(k)e^{j\pi n(2k+1)/N}\right]$$

$$for\ N\ even$$

$$h(n)=\frac{2}{N}\sum_{k=1}^{\frac{N-2}{2}}\mathrm{Re}\left[H(k)e^{j\pi n(2k+1)/N}\right]$$