Design of FIR Filters using Windowing Techniques

I.Nelson
SSN College of Engineering

Design of FIR filters using windows:

- The desired frequency response $H_d(e^{j\omega})$ of a filter is periodic in frequency and can be expanded using Fourier series. The resultant series is

$$H_d(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h_d(n) e^{-j\omega n}$$

where

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\omega})$$

are known as Fourier coefficients having infinite length.

- One possible way of obtaining FIR filter is to truncate the infinite Fourier series at n = ± (N-1)/2, where N is the length of the desired sequence.

- But abrupt truncation results in oscillation in the passband and stopband. These oscillations are due to slow convergence of the Fourier series and this effect is known as Gibbs phenomenon.

- To reduce these oscillations, the Fourier coefficients of the filter are modified by multiplying the infinite impulse response with a finite weighing sequence w(n) called as window.

- The window sequence is given by,

$$
\begin{aligned}
w(n) &= w(-n) \neq 0 && \text{for} \quad |n| \leq (N-1)/2 \\
&= 0 && \text{for} \quad |n| > (N-1)/2
\end{aligned}
$$

- After multiplying window sequence w(n) and $h_d(n)$, we get a finite duration sequence h(n) that satisfies the desired magnitude response

$$
\begin{aligned}
h(n) &= h_d(n)\, w(n) \quad ; && |n| \leq (N-1)/2 \\
&= 0 \quad ; && \text{otherwise}
\end{aligned}
$$

- The frequency response $H(e^{j\omega})$ of the filter can be obtained by convolution of $H_d(e^{j\omega})$ and $W(e^{j\omega})$ given by

$H(e^{j\omega}) = H_d(e^{j\omega}) * W(e^{j\omega})$

$$H(e^{j\omega}) = \frac{1}{2\pi} \int H_d(e^{j\theta}) W(e^{j(\omega-\theta)}) d\theta$$

Note: We find that the frequency response of the filter $H(e^{j\omega})$ depends on the frequency response of window $W(e^{j\omega})$. Therefore, the window, chosen for truncating the infinite impulse response should have some desirable characteristics. They are

1. The central lobe of the frequency response of the window should contain most of the energy and should be narrow.
2. The highest side lobe level of the frequency response should be small.
3. The side lobes of the frequency response should decrease in energy rapidly as ω tends to π.

Rectangular window:

- The rectangular window sequence is given by,

$$w_R(n) = 1 \quad ; \quad -(N-1)/2 \le n \le (N-1)/2$$
$$= 0 \quad ; \quad \text{otherwise}$$

- The spectrum of the rectangular window is given by

$$W_R(e^{j\omega}) = \sum_{n=-\frac{N-1}{2}}^{\frac{N-1}{2}} e^{-j\omega n} \quad = e^{j\omega\frac{N-1}{2}} + \ldots\ldots\ldots\ldots + e^{j\omega} + 1 + e^{-j\omega} + \ldots\ldots\ldots\ldots + e^{-j\omega\frac{N-1}{2}}$$

$$= e^{j\omega\frac{N-1}{2}}\left[1 + e^{-j\omega} + \ldots\ldots\ldots\ldots + e^{-j\omega(N-1)}\right] \quad = e^{j\omega\frac{N-1}{2}}\left[\frac{1-e^{-j\omega N}}{1-e^{-j\omega}}\right]$$

$$= \frac{e^{j\omega\frac{N}{2}}(1-e^{-j\omega N})}{e^{j\omega\frac{1}{2}}(1-e^{-j\omega})} \quad = \frac{\sin\frac{\omega N}{2}}{\sin\frac{\omega}{2}}$$

- To get a finite impulse response filter we multiply $h_d(n)$ with a rectangular window, i.e.,

$$h(n) = h_d(n)w_R(n)$$

- The frequency response of the truncated filter can be obtained by periodic convolution,

$$H(e^{j\omega}) = \frac{1}{2\pi}\int H_d(e^{j\theta})W_R(e^{j(\omega-\theta)})d\theta$$

- Note: The Gibbs phenomenon can be reduced by using a less abrupt truncation of filter coefficients. This can be achieved using a window function that tapers smoothly towards at both ends, eg: Triangular window.

Triangular or Bartlett window:

- The N – point triangular window is given by,

$$w_T(n) = 1 - \frac{2|n|}{N-1} \quad ; \quad -\frac{(N-1)}{2} \le n \le \frac{(N-1)}{2}$$

- The Fourier transform of the triangular window is

$$W_T(e^{j\omega}) = \left(\frac{\sin\left(\frac{N-1}{4}\right)\omega}{\sin\frac{\omega}{2}} \right)^2$$

- To get a finite impulse response filter we multiply $h_d(n)$ with a triangular window, i.e.,

$$h(n) = h_d(n)\, w_T(n)$$

- The frequency response of the truncated filter can be obtained by periodic convolution,

$$H(e^{j\omega}) = \frac{1}{2\pi} \int H_d(e^{j\theta}) W_T(e^{j(\omega-\theta)}) d\theta$$

- Due to the following reasons, the triangular window is not usually a good choice.
  1. Transition region is more
  2. Stopband attenuation is very less

Raised Cosine Window:

- The raised cosine window multiplies the central Fourier coefficients by approximately unity and smoothly truncate the Fourier coefficients toward the ends of the filter.

- The window sequence is of form,

$$w_\alpha(n) = \alpha + (1-\alpha)\cos\left(2\pi n/(N-1)\right) \quad ; \quad -((N-1)/2) \le n \le ((N-1)/2)$$

$$= 0 \quad ; \text{otherwise}$$

- The frequency response $w_\alpha(n)$ is given by,

$$W_\alpha(e^{j\omega})=\sum_{n=-\frac{N-1}{2}}^{\frac{N-1}{2}}\left[\alpha+(1-\alpha)\cos\frac{2\pi n}{N-1}\right]e^{-j\omega n}$$

$$=\alpha\frac{\sin\frac{\omega N}{2}}{\sin\frac{\omega}{2}}+\frac{(1-\alpha)\sin\left(\frac{\omega N}{2}-\frac{\pi N}{N-1}\right)}{2\sin\left(\frac{\omega}{2}-\frac{\pi}{N-1}\right)}+\frac{(1-\alpha)\sin\left(\frac{\omega N}{2}+\frac{\pi N}{N-1}\right)}{2\sin\left(\frac{\omega}{2}+\frac{\pi}{N-1}\right)}$$

Hanning window:

- The Hanning window sequence can be obtained by substituting $\alpha = 0.5$ in the Raised Cosine function.

$W_{Hn}(n) = 0.5 + 0.5 \cos(2\pi n/(N-1))$; $-((N-1)/2) \le n \le ((N-1)/2)$

$= 0$; otherwise

- The frequency response $w_\alpha(n)$ is given by,

$$W_{Hn}(e^{j\omega}) = 0.5\frac{\sin\frac{\omega N}{2}}{\sin\frac{\omega}{2}} + 0.25\frac{\sin\left(\frac{\omega N}{2} - \frac{\pi N}{N-1}\right)}{\sin\left(\frac{\omega}{2} - \frac{\pi}{N-1}\right)} + 0.25\frac{\sin\left(\frac{\omega N}{2} + \frac{\pi N}{N-1}\right)}{\sin\left(\frac{\omega}{2} + \frac{\pi}{N-1}\right)}$$

- To get a finite impulse response filter we multiply $h_d(n)$ with a rectangular window, i.e.,

$$h(n) = h_d(n)w_{Hn}(n)$$

- The frequency response of the truncated filter can be obtained by periodic convolution,

$$H(e^{j\omega}) = \frac{1}{2\pi}\int H_d(e^{j\theta})W_{Hn}(e^{j(\omega-\theta)})d\theta$$

- **Note:** At higher frequencies, the stopband attenuation is ever greater.

Hamming Window

- The equation for Hamming window is obtained by substituting $\alpha = 0.54$ in the Raised Cosine function.

$$w_H(n) = 0.54 + 0.46 \cos(2\pi n/(N-1)) \quad ; \quad -((N-1)/2) \le n \le ((N-1)/2)$$

$$= 0 \quad ; \quad \text{otherwise}$$

- The frequency response $w_\alpha(n)$ is given by,

$$W_H(e^{j\omega}) = 0.54 \frac{\sin \frac{\omega N}{2}}{\sin \frac{\omega}{2}} + 0.23 \frac{\sin\left(\frac{\omega N}{2} - \frac{\pi N}{N-1}\right)}{\sin\left(\frac{\omega}{2} - \frac{\pi}{N-1}\right)} + 0.23 \frac{\sin\left(\frac{\omega N}{2} + \frac{\pi N}{N-1}\right)}{\sin\left(\frac{\omega}{2} + \frac{\pi}{N-1}\right)}$$

- To get a finite impulse response filter we multiply $h_d(n)$ with a rectangular window, i.e.,

$$h(n) = h_d(n)w_H(n)$$

- The frequency response of the truncated filter can be obtained by periodic convolution,

$$H(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\theta})W_H(e^{j(\omega-\theta)})d\theta$$

- **Note:** At higher frequencies, the stopband attenuation is ever greater.

Blackman Window:

The Blackman window sequence is given by,

$$w_B(n) = 0.42 + 0.5\cos(2\pi n/(N-1)) + 0.08\cos(4\pi n/(N-1)) \quad ; -((N-1)/2) \le n \le ((N-1)/2)$$

$$= 0 \quad ;\text{otherwise}$$

Note: Additional cosine terms reduces the side lobes but main lobe width is increased.

- To get a finite impulse response filter we multiply $h_d(n)$ with a rectangular window, i.e.,

$$h(n) = h_d(n)w_B(n)$$

- The frequency response of the truncated filter can be obtained by periodic convolution,

$$H(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\theta})W_B(e^{j(\omega-\theta)})d\theta$$

Comparison of different types of windows:

Window type	Main lobe width	Highest side lobe level	Minimum stopband attenuation
Rectangular	$4\pi/N$	-13 dB	-21 dB
Bartlett or Triangular	$8\pi/N$	-25 dB	-25 dB
Hanning	$8\pi/N$	-32 dB	-44 dB
Hamming	$8\pi/N$	-43 dB	-53 dB
Blackman	$12\pi/N$	-58 dB	-74 dB