Unit-3

Graph Theory

Definition: A graph G is an ordered triple (V(G),E(G),ψ) consisiting of

i. a nonempty finite set V(G).

ii. A finite set E(G), which is disjoint from V(G), and

iii. an incidence function ψ that associates with each element of E(G) an unordered pair of elements of V(G).

The elements of V(G) are called vertices of G, and the elements of E(G) are called edges of G.

Ex.1 G=(V(G),E(G),ψ) where $V(G) = \{v_1, v_2, \dots, v_6\}, E(G) = \{e_1, e_2, \dots, e_{10}\}$ and ψ is defined by

$$\psi(e_1) = v_1v_3, \psi(e_2) = v_1v_5, \psi(e_3) = v_1v_5, \psi(e_4) = v_2v_2, \psi(e_5) = v_2v_4, \psi(e_6) = v_2v_5, \psi(e_7) = v_3v_4, \psi(e_8) = v_4v_5, \psi(e_9) = v_5v_5, \psi(e_{10}) = v_5v_6.$$

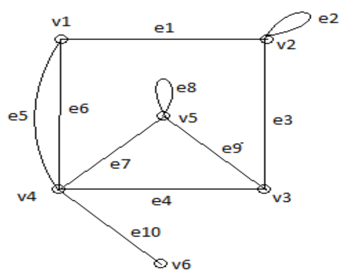


Figure 1.

Definition 2. If e is an edge in a graph G such that $\psi(e) = (u, u)$ for some vertex $u \in V(G)$, then e is said to be a loop in G, In figure 1, e_2 is a loop. If e_1, e_2 are edges in G such that $\psi(e_1) = \psi(e_2)$, then $e_1 and\ e_2$ are said to be parallel edges joining same end vertices, in figure 1, $e_5 and\ e_6$ are parallel edges..

Definition 3. A graph which has no loop and no parallel edge is said to be a simple graph. If G is a simple graph and $\psi(e) = (u, v)$ for and edge e, then e is denoted by uv.

Note: A graph is finite if both V(G) and E(G) are finite sets.

Throughout this chapter we deal with finite and simple graphs.

Terminology:

- If $e = (u, v)\ or\ simply\ e = uv$ is and edge then, we say that e is incident with u and v. Also u and v are said to incident with e.
- Two vertices u & v are said to be adjacent if there exists $e \in E(G)$ such that $e = uv$.
- For example in Figure 1, vertex v_1 is **adjacent** to $v_4 and\ v_2$. The edge e_1 is incident with $v_1 and\ v_2$, the vertex v_2 is incident with e_1, e_2 and e_3.

Definition: Let v be a vertex in a graph G. Then the degree $d_G(v)$ of the vertex v in G is the number of edges of G that are incident with v, each **loop is counted twice**. The $d_G(v)$ also denoted by $\deg_G(v)$.

In Fig. 1, $\deg(v_1) = \deg(v_3) = 3, \deg(v_2) = \deg(v_5) = 4, \deg(v_4) = 5,\ and \deg(v_6) = 1.$

Theorem 1 (Handshaking Lemma or Handshaking Theorem)

Let G be a graph with vertex set $\{v_1, v_2, \ldots, v_n\}$. Then $\sum_{i=1}^{n} d(v_i) = 2\epsilon, where$ ϵ is the number of edges in G.

Proof. Let G be a graph with ϵ edges and n vertices $v_1, v_2, \ldots, v_n$. Since each edge contributes two degrees to the sum of the degrees of all vertices in G, we have $\sum_{i=1}^{n} d(v_i) = 2\epsilon$. □

Note: Check the validity of the above theorem with Fig. 1.

Theorem 2. In any graph, the number of vertices of odd degree is even.

Proof. Let $V_1 and\ V_2$ be the sets of vertices of odd and even degrees in G. Then $V = V_1 \cup V_2\ and\ V_1 \cap V_2 = \phi$ and $\sum_{v \in V(G)} d(v) = \sum_{v \in V_1} d(v) + \sum_{v \in V_2} d(v)$ ------------------------------------ (1)

By Theorem 1, $\sum_{v \in V} d(v) = 2\epsilon$ and hence is even. Since d(v) is even for $v \in V_2$, the sum $\sum_{v \in V_2} d(v)$ is even. From (1), it follows that $\sum_{v \in V_1} d(v)$ is an even number. Since each term d(v) is odd, the total number of terms must be even to make the sum an even number. Thus $|V_1|$ is even.□

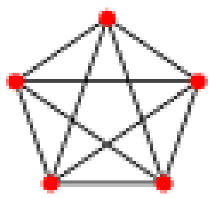
Notation: Let G be a graph. We denote $\delta(G) and\ \Delta(G)$ the minimum and maximum degrees, resply, of vertices of G.

That is, $\delta(G) = min\{d(v): v \in V(G)\}\ and\ \Delta(G) = max\{d(v): v \in V(G)\}$. Clearly, $\delta(G) \leq \Delta(G)$.

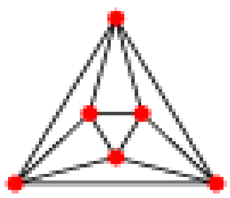
In Fig. 1. $\delta(G) = 1\ and\ \Delta(G) = 7$.

Definition: Regular Graph

If all the vertices of G are of equal degree, then G is said to be regular. If all the vertices have degree k, then G is said to be k-regular. The following graph are 4-regular.



K_5



Octahedral Graph

Figure 2. Four Regular Graphs

Definition. Let v be a vertex in G. Then (i). v is said to be isolated vertex in G, if deg(v)=0, (ii). v is called a pendant vertex in G, if deg(v)=1. A graph is said to be a null graph, if every vertex of G is an isolated vertex.

Note: For a simple graph G, with |V(G)|=n, we have $0 \leq \delta(G) \leq \deg(v) \leq \Delta(G) \leq n - 1$, for all vertices of G.

Example 1. For a graph G, show that $\delta(G) \leq \frac{2\epsilon}{n} \leq \Delta(G)$.

Solution: For a graph G, we have $\delta(G) \leq \deg(v) \leq \Delta(G), \forall v \in V(G)$.

Hence $\sum \delta(G) \leq \sum_{v \in V(G)} \deg(v) \leq \sum \Delta(G)$. If |V(G)=n, then we have

$$n\delta(G) \leq \sum_{v \in V(G)} \deg(v) \leq n\Delta(G)$$

and

$$\delta(G) \leq \frac{1}{n} \sum_{v \in V(G)} \deg(v) \leq \Delta(G)$$

that is,
$\delta(G) \leq average\ degree \leq \Delta(G)$, as $\sum_{v \in V(G)} \deg(v) = 2\epsilon$, we have $\delta(G) \leq \frac{2\epsilon}{n} \leq \Delta(G)$. $\square$

Definition. Walk

A walk in G is a finite non-null sequence $W = v_0 e_1 v_1 e_2 v_2 e_3 v_3 \ldots e_k v_k$, where terms are alternately vertices and edges, such tat for $1 \leq i \leq k$, the vertices v_{i-1} and v_i are the ends of the edge e_i.

- We call the above walk as (v_0, v_k)-walk of length k.
- If $v_0 = v_k$ then W is a closed walk
- If the edges of W are distinct then W is called a trail, and if the vertices are distinct then W is said to be a path or (v_0, v_k) −path or simply a path of length k.
- If the vertices are distinct and if $v_0 = v_k$, then W is said to be a cycle or k-cycle. If k is odd then W is odd cycle otherwise W is an even cycle
- A 3-cycle is said to be a triangle.

Problem. Show that if there is a (v_0, v_k)-walk then there is a (v_0, v_k)-path. (refer class notes for the proof.)

Definition. If there is a u-v path in G, then a u-v path of least length is called a **shortest path from u to v**. The **length** of a shortest path between u and v denoted by **d(u,v).** If there is no u-v path in G, the its length is defined to be ∞.

- d(u,u)=0, that is distance from the vertex to itself is zero.
- d(u,v)=d(v,u), that is the distance from u to v and the distance from v to u are same.

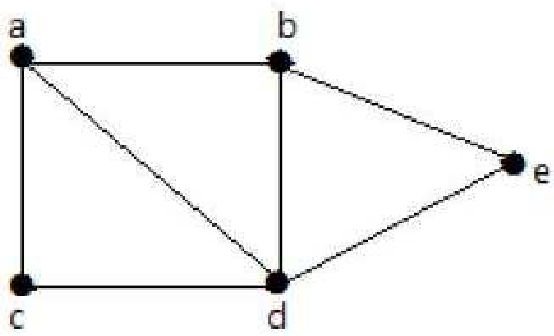
Definition. Let G be a graph. Two vertices u and v of G are said to be connected if either u=v or there is a (u,v)-path in G.

- In V(G), we define a relation ~ by u~v iff either u=v or there is a u-v path in G.
- Clearly, the relation ~ is
  - reflexive, that is u~u
  - symmetric, that is, u~v implies v~u
  - transitive, that is, if u~v and v~w then u~w.

Since ~ is an equivalence relation, it partitions the graph into connected subgraphs called connected components. That is, if $V_1, V_2, \ldots V_n$ is the partition of V with respect to the relation ~, then $G[V_1], G[V_2], \ldots, G[V_n]$ are called connected components of G. If G has exactly one component then G is said to be connected. If G is not connected then we say that G is disconnected.

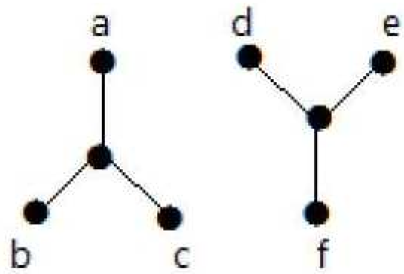
Example.

In the following graph, it is possible to travel from one vertex to any other vertex. For example, one can traverse from vertex 'a' to vertex 'e' using the path 'a-b-e'.



Example.

In the following example, traversing from vertex 'a' to vertex 'f' is not possible because there is no path between them directly or indirectly. Hence it is a disconnected graph.



Theorem. A graph G is disconnected if and only if its vertex set V can be partitioned into two nonempty subsets V_1 and V_2 such that there exists no edge in G whose one end in V_1 and other end in V_2.

Proof. Assume that G be disconnected. Consider a vertex $u \in V$. Let V_1={$v \in V|$ $there\ is\ a\ u-v\ path\ in\ G$}. As G is disconnected $V_1 \neq V$. Let $V_2 = V - V_1$. Then $V_2 = \phi$. We claim that there is no edge with one end in V_1 and other end in V_2. If it is not so, let e=ab be an edge in G such that $a \in V_1$ and $b \in V_2$. As $a \in V_1$, either a=u or there i a u-a path in G. If a=u, then e=ab=ub and $b \in V_1$, which is a contradiction. So $a \neq u$. Let $ux_1x_2 \ldots x_m a$ be a u-a path in G. Clearly as $x_1x_2 \ldots x_m$ are in $V_1, b \neq x_i$ for any i and hence $ux_1x_2 \ldots x_m ab$ is a u-b path in G, which is a contradiction as $b \notin V_1$. Thus there is no edge e in G with one end of e in V_1 and other end in V_2.

Conversely, assume that the vertex set V can be partitioned into two disjoint, nonempty subsets $V_1 and\ V_2$ such that there exists no edge in G whose one end is in V_1 and other end in V_2. Consider a vertex $a \in V_1, b \in V_2$. We claim that there is no a-b path in G. Suppose there is a a-b path $x_0x_1 \ldots x_m x_{m+1}$, where $x_m = a\ and\ x_{m+1} = b$, in G. Let i be the least positive integer such that $x_i \in V_2$ (such an i exists since $x_{m+1} \in V_2$). Clearly, $i \geq 1, since x_0 = a \notin V_2$. Also $x_{i-1} \in V_1$. Thus there is an edge with x_{i-1} and x_i as end vertices and $x_{i-1} \in V_1$ and $x_i \in V_2$. This is a contradiction. Thus there is no a-b path in G. Hence G is not connected. □

Definition: Subgraph

A graph $H = (V(H), E(H), \psi_H)$ is said to be a **subgraph** of a graph G= $(V(G), E(G), \psi_G)$ if $V(H) \subseteq V(G), E(H) \subseteq E(G)$and the map ψ_H is the restriction of ψ_G to E(H). If H is a subgraph of G, we denote by $\boldsymbol{H} \subseteq \boldsymbol{G}$. If H contains all the vertices of G, then we say that H is a **spanning subgraph** of G.

Examples.

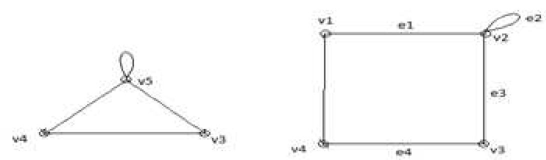
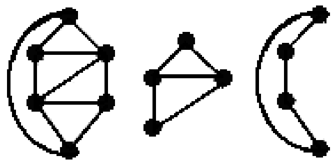


Fig. 3. Examples of Subgraph of Figure 1.

Definition: Vertex induced subgraph

Let V' be a nonempty subset of V(G). The subgraph of G whose vertex set is V' and whose edge set is the set of those edges of G that have both the ends in V' is called the **induced subgraph of G** induced by V' and is denoted by $G[V']$.

Example.



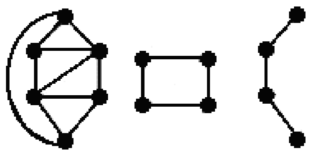
G G_1 G_2

Figure 4. G_1 and G_2 are vertex induced subgraphs of G

Definition: Edge induced subgraph

If E' is the nonempty subset of E(G), the subgraph of G, whose vertex set = $\{v \in V(G) | \, v \; is \; an \; end \; of \; some \; edge \; e \in E'\}$ and whose edge set is E', is called the **edge induced subgraph of G** induced by E' and is denoted by $G[E']$.

Example.



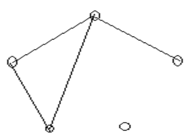
G G_1 G_2

Figure 5. G_1 and G_2 are edge induced subgraphs of G

The sequence $(d_1, d_2, .., d_n)$ in a graph G with n vertices is said to be a **degree sequence**, if each d_i is a degree of some vertex of G. The sequence $(d_1, d_2, .., d_n)$ is said to be **graphic** if there is a graph G with degree sequence as above.

Example 1: Is it possible to construct a graph with degrees (i) (2,1,3,3,4). (ii). (1,1,2,3), (iii). (2,2,4,6). (iv). (5,4,3,2,1), (v) (3,2,2,1,0).

Soln. (i) & (ii) are not graphic, since there are odd number of odd degree vertices. (iii) is not graphic as there are 4 vertices and one vertex has degree more than number of vertices. (iv) is also not graphic as there are 5 vertices and the maximum degree in a graph will be at most n-1. (v) is graphic and the graph is



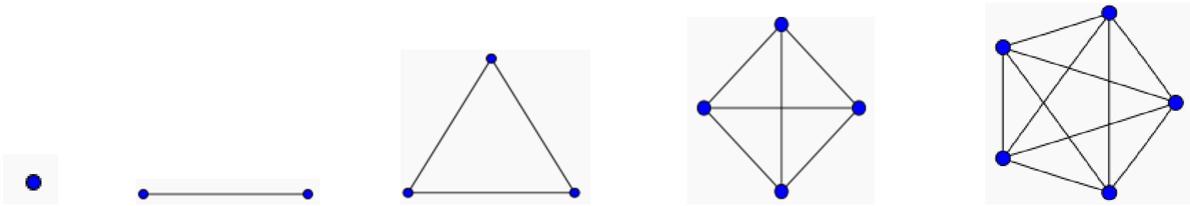
Example 2: Show that maximum no. of edges in a simple graph with n vertices $\binom{n}{2} = \frac{n(n-1)}{2}$.

Since G is simple, $0 \leq \deg(v_i) \leq n - 1$. ---------(1)

By handshaking theorem, $\sum_{i=1}^{n} \deg(v_i) = 2\epsilon$. Summing overall i, in (1) we get, $2\epsilon = \sum_{i=1}^{n} \deg(v_i) \leq n(n-1)$ and hence $\epsilon \leq \frac{n(n-1)}{2}$. □

Definition. (Complete Graph). If each vertex v is adjacent to every other vertex in a graph, then the graph is said to be a complete graph and is denoted by K_n, where n is the number of vertices.

Example.



K_1 K_2 K_3 K_4 K_5

Figure 6. Complete Graphs

Definition. (Wheel Graph) A Wheel graph is a **graph** formed by connecting a single vertex to all vertices of a cycle of length n-1. A **wheel graph** with n vertices is denoted by W_n. K_4 is a wheel graph on 4 vertices.

Definition. (n-cubes or hypercube Q_n) The n-dimensional hypercube, or n-cube Q_n is the graph that has vertices as 2^n bit strings of length n. Two vertices are adjacent if and only if the bit strings that they represent differ in exactly one bit position.

Example.

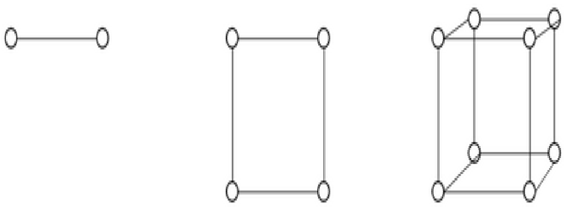


Figure 7. Hypercubes $Q_1, Q_2 and\ Q_3$

Note: 1. Q_n has 2^n vertices and $n2^{n-1}$ edges.

Problem 1. Does there exist a 4-regular graph on 6 vertices. If so construct a graph.

Problem 2. For which values of n are these graphs regular?

(a). K_n (b). C_n (cycle on n vertices) (c). W_n (d). Q_n

Definition (Bipartite graph). An undirected graph G is said to be bipartite if its vertex set V(G) can be partitioned into subsets V_1 and V_2 such that every edge f G has one end in V_1 and other end in V_2. Such a partition (V_1, V_2) is called a bipartition G.

Example.

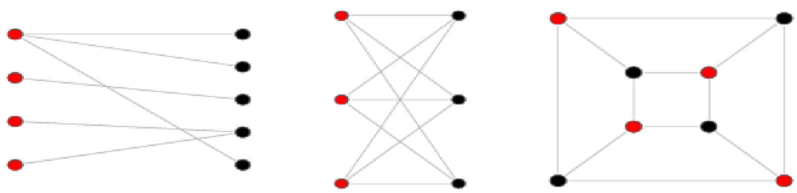
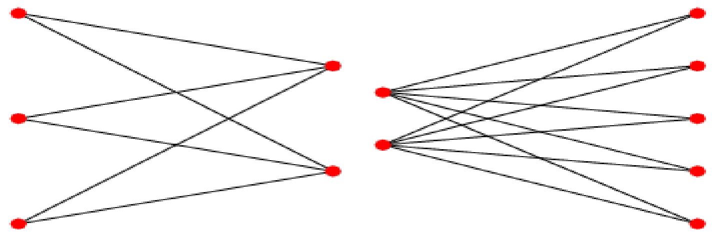


Figure 8. Bipartite Graphs

Definition (Complete bipartite Graph). A simple bipartite graph G with bipartition (V_1, V_2) is said to be a complete bipartite graph if every vertex of V_1 is adjacent to every vertex of V_2 and if $|V_1| = m \; and \; |V_2| = n$, then we denote this graph by $K_{m,n}$.

Example.



$\boldsymbol{K_{3,2}}$ $\qquad\qquad$ $\boldsymbol{K_{2,5}}$

Figure 9. Complete bipartite graphs.

Note. 1. As an example every even cycle is a bipartite graph.

2. No. of edges of $K_{m,n}$ is mn.

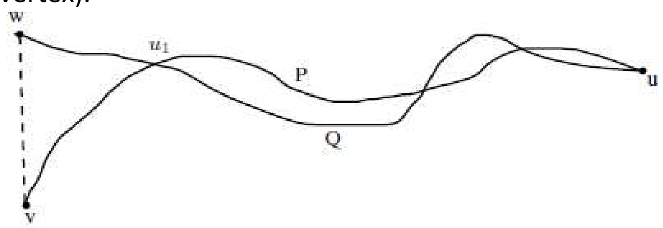
Theorem.(A characterization for bipartite graphs)

An undirected graph G is bipartite if and only if G contains no odd cycles.

Proof.

Let G be a bipartite graph with bipartition (X,Y). Let $C = v_1 v_2 v_3 \dots v_k v_1$ be a cycle of length k. We claim that k is even. Without loss of generality we may assume that $v_1 \in X$, then as $v_1 v_2 \in E(G), we \; have \; v_2 \in Y$. Similarly as $v_2 v_3$ is an edge, we have $v_3 \in X \; etc.$, Proceeding like this, we have $v_{2i} \in Y$ and $v_{2i+1} \in X$. As $v_1 \in X$ and $v_1 v_k$ is an edge, we have $v_k \in Y$ and hence k is even. Thus G contains no odd cycle.

Conversely, assume G contains no odd cycles. Let u be a vertex of G. Let $X = \{v \in V | d(u,v) is \; even\}$ and $Y = \{v \in V | d(u,v) \; is \; odd\}$. We claim that (X,Y) is a bipartition of G and G is a bipartite graph. Let $v, w \in V(G)$. Let P be a shortest u-v path and Q be a shortest u-w path. Let u_1 be a last vertex common to both P and Q (since u is already a common vertex to P and Q, there must be such vertex).



Since P and Q are shortest paths, the (u, u_1)-sections of both P and Q have same length. As length of P and Q are even, the (u_1, v)-section P_1 of P and (u_1, w)-section Q_1 of Q have same parity. If vw were an edge of G, then $P_1 vw Q_1^{-1}$ is an odd cycle, a contradiction. Hence vw is not an edge and no two vertices of X are adjacent. Similarly no two vertices of Y are adjacent. Hence (X,Y) is bipartition and G is a bipartite graph. □

Theorem. A simple graph with n vertices and k components can have at most $\frac{(n-k)(n-k+1)}{2}$ edges.

Proof. refer class work note.

Definition: Adjacency matrix.

Let G be a graph with n vertices and no parallel edges. Let $V = \{v_1, v_2, \dots, v_n\}$ be the vertex set of G. The adjacency matrix $A = A(G) = (a_{ij})_{n \times n}$ of G is an $n \times n$ matrix defined by

$$a_{ij} = \begin{cases} 1, if \; there \; is \; edge \; joining \; v_i and \; v_j \\ 0, \;\; otherwise \end{cases}$$

Note:

(i). A(G) is symmetric and is a 0,1- matrix.

(ii). $a_{ii} = 0$

(iii). i'th row sum = deg(v_i).

Problem. Let r be a positive integer. Let A be the adjacency matrix of a simple graph G. Then the ij-th entry in A^r is the number of different walks of length r between the vertices v_i *and* v_j.
Soution:

We prove this by induction on r. LEt r=1. Then $A^r = A$. The ij-th entry in A is 1 if $v_i v_j$ is an edge in G, otherwise it is 0. There is a $v_i - v_j$ walk in G if and only if $v_i v_j$ is an edge in G. In this case there is only one $v_i - v_j$ walk of length 1 in G. Thus if the ij-th entry of A = the no. of walks of length r between the vertices v_i*and* v_j.
Assume that the result is true for r=k. We show that the result is true for r=k+1.

Now ij-th entry of A^{r+1} = dot product of i-th row of A^r and j-th column of A
= $\sum_1^n ik$-th entry of A^r . kj-th entry of A

Note that if $v_k v_j$ is an edge in G, then every v_i - v_k walk $v_i v_1 v_2 \dots v_{r-1} v_k$ of length r can be extended to a v_i-v_j walk of $v_i v_1 v_2 \dots v_{r-1} v_k v_j$ of length r+1 and vice versa. So, the number of v_i - v_j walks of length r+1
= $\sum$(the number of v_i - v_k walks of length r)) .(kj-th entry in A)
=$\sum_{k=1}^n (ik$-entry in A^r). (kj-th entry in A) =ij-th entry of A^{r+1}.

Hence the result is true for any r by mathematical induction principle. □

Definition: Incidence matrix

Let G be a graph with n vertices and has no loops. Let $V = \{v_1, v_2, \dots, v_n\}$ and E(G)= $\{e_1, e_2, \dots, e_m\}$. Then incidence matrix M(G) is defined as follows:

$$m_{ij} = \begin{cases} 1, if\ v_i is\ incident\ with\ e_j \\ 0, \quad otherwise \end{cases}$$

Note:
(i). Row sum is degree of that vertex
(ii). Columns sum is two.

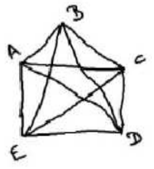
Definition: Isomorphism
Two Graphs G and H are isomorphic if there exists a function $f: V(G) \to V(H)$ such that (i). f is one-to-one, (ii). f is onto, (iii). if $uv \in E(G)$ then $f(u)f(v) \in E(H)$ and if $uv \notin E(G)$ then $f(u)f(v) \notin E(H)$.
Example 1. Are the simple graphs with the following adjacency matrices isomorphic?

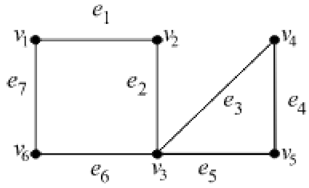
$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Problems...
Defnition: Complement of a graph
The complement $\bar{G}$ of a graph G is defined as a simple graph with the same vertex set as G and two vertices u and v are adjacent if and only if u and v are not adjacent in G.
Example: If G is a simple graph with n vertices and ϵ edges then $\epsilon(\bar{G}) = \frac{n(n-1)}{2} - \epsilon$.
Example: $\epsilon(G) + \epsilon(\bar{G}) = \frac{n(n-1)}{2}$
Definition: Self complementary graph
A graph G is said to be self complementary, if G is isomorphic to its complement. That is, $G \equiv \bar{G}$
Problem 1. If G is self complementary graph then $n \equiv 0\ or\ 1\ (mod\ 4)$.
Solution. Let G be self complementary with n vertices. Then G contains exactly half of the total possible edges. That is, the number of edges $\epsilon(G) = \frac{\frac{n(n-1)}{2}}{2} = \frac{n(n-1)}{4}$. Now $\epsilon(G)$ is an integer if and only if $n \equiv 0\ or\ 1\ (mod\ 4)$.
Definition: Eulerian graph.

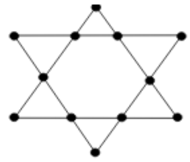
A closed trail containing all the edges of G exactly once is called an eulerian trail. A closed eulerian trail is called an eulerian tour. A graph having an eulerian tour is called an eulerian graph.

Examples: for eulerian graph



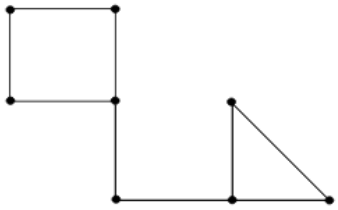


(a)



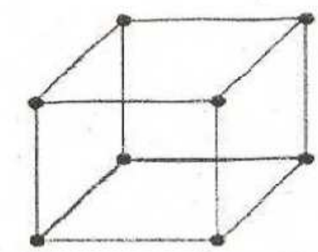
Two Euler graphs (b)

Examples: for non-eulerian graph



(b)

Non-Eulerian Graph



Problem 1. A graph with $\delta(G) \geq 2$ contains a cycle.
Solution. Let P=$u_0u_1u_2 \dots u_i \dots u_k$ be a longest path in a graph G of length k starting at the vertex u_0 and ends at u_k. Since $\delta(G) \geq 2$, we have $\deg(u_i) \geq 2$ for every vertex of G. In particular, $\deg(u_0) \geq 2$ and $\deg(u_k) \geq 2$. As $\deg(u_0) \geq 2$, it must be adjacent to some more vertex other than $u_1, say\ u_t$. If u_t does not coincide with any of the vertices in a path P, then $u_tu_0u_1u_2 \dots u_k$ will be a longest path in G, a contradiction to the choice of P. Hence u_t must coincide with any of the vertex in P, say u_i. Then $u_0u_1u_2 \dots u_iu_0$ is the required cycle in G.
Problem 2. If G is a graph with $\delta(G) \geq 2$, then G contains a cycle of length at least $\delta + 1$.
Solution. P=$u_0u_1u_2 \dots u_i \dots u_k$ be a longest path in a graph G of length k starting at the vertex u_0 and ends at u_k. Proceeding as in the previous problem, we must have $\delta(G)$ distinct vertices in the path P which are adjacent to u_0. Let u_t be the last vertex in the path P which is adjacent to u_0. Clearly $u_0u_1u_2 \dots u_tu_0$ is cycle of length at least $\delta + 1$.
Theorem: Characterization for Eulerian Graphs (Important)
A nonempty connected graph is eulerian if and only if it has no odd degree vertices.
Proof. <u>Necessity</u>

Let G be an eulerian graph and let C be an Euler tour of G with origin and terminus u. Each time a vertex v occurs as an internal vertex of C, two of the edges incident with v accounted for. Since an Euler tour contains every edge of G, d(v) is even for all $v \neq u$. Similarly, since C starts and ends at u, d(u) is also even. Thus G has no odd vertices.

Conversely, assume G has no odd degree vertices. To prove G is eulerian.

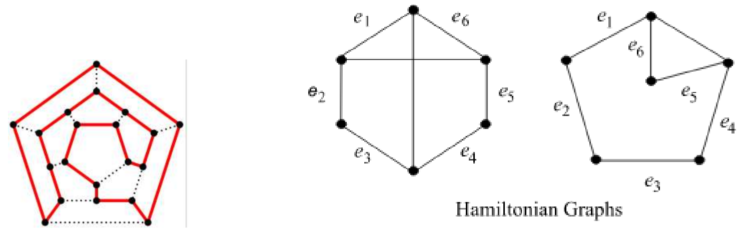
Among all graphs without odd vertices, let G be a noneulerian connected graph with least no. of edges. Since each vertex has even degree, it contains a closed trail (by previous problem 1 as $\delta(G) \geq 2$). Let C be a closed trail of maximum length in G. By assumption C is not an Euler tour and hence $G - E(C)$ has some component G' with $\epsilon(G') > 0$.
Since C is itself eulerian, C contains no odd vertices; then the connected component G' has no vertex of odd degree.

By the choice of G, the Graph G' (since no. of edges in G' is lesser that no. of edges in G) contains an Euler tour and let it be C'. As the graph G is connected, $V(G) \cap V(G')$ is nonempty. Without loss of generality, assume $v \in V(G) \cap V(G')$. Let v be the origin and terminus of both C and C'. Then the trail CC' (the concatenation of the trails C and C') is the closed trail in G with more no. of edges then C, contradiction the choice of C. Hence G is eulerian.
Hence Prove.
Corollary 1: A connected graph has an Euler trail if and only if it has at most two vertices of odd degree.
Proof. If G has an Euler trail then, as in the proof of the previous theorem, each vertex other than the origin and terminus of this trail has even degree. That is we have exactly two odd degree vertices.

Conversely, suppose that G is a non-trivial connected graph with at most two odd degree vertices. If G has no two odd vertices, then we are done (by previous theorem). Otherwise, G has exactly two odd degree vertices u and v. In this case join u and v by a new edge, say e=uv. Clearly each vertex of G+e has even degree, by previous theorem, we have an Euler tour in G started from the vertex $ue_1v_1e_2v_2e_3 \dots e_{\epsilon+1}v_{\epsilon+1}$ where $e_1 = e$ and $v_1 = v$. Then the required eulerian trail is $v_1e_2v_2e_3 \dots e_{\epsilon+1}v_{\epsilon+1}$ in G.
Corollary 2: In a connected graph with exactly 2k odd vertices, there exist k edge-disjoint subgraphs such that they together contain all the edges of G and that each is a Euler trail.
Proof: is similar to Corollary 1.
Definition: Hamilton Path and Hamilton Cycle.

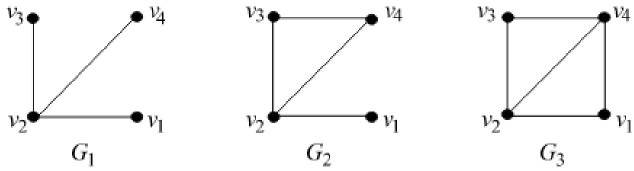
A path in a graph G that passes through every vertex of G exactly once is called a Hamilton path.

A cycle in G that passes through every vertex of G exactly once is called a Hamilton cycle.
Notation: A graph G is said to be hamiltonian if it contains a Hamilton cycle.
Example: Hamilton cycle in a graph

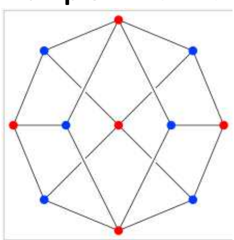


Hamiltonian Graphs

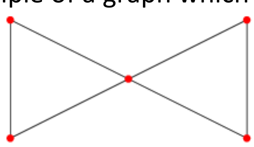
In the first graph, the Hamilton cycle is denoted by red lines. In second and third graphs $e_1e_2e_3e_4e_5e_6$ constitute a Hamilton cycle.
Example: for hamiltonian path



In G_1, there is no Hamilton path. In G_2 and G_3, $v_1v_2v_3v_4$ and $v_1v_2v_4v_3$ are Hamilton paths.
Example: A non-hamiltonian graph



This graph does not possess a Hamilton cycle and hence is not a hamiltonian graph.
Problem 1: Give an example of a graph which has an Euler tour but no Hamilton cycle.



Problem 2: Show that a bipartite graph with odd no.of vertices does not contain a Hamilton cycle.
Since a Hamilton cycle contains all the vertices of G, it contains an odd cycle. As a graph is bipartite, there will be no odd cycle in G. Hence no bipartite graph with odd no. of vertices has a Hamilton cycle.
Theorem 1: If G is a simple undirected graph with $n \geq 3$ vertices. Let u and v be two nonadjacent vertices in G such that deg(u)=deg(v)≥ n in G. Then G is hamiltonian if and only if G+uv is hamiltonian.
Proof. If G is hamiltonian then, clearly a G contains a Hamilton cycle, say, C. This cycle C is contained in G+uv too. Hence H+uv is hamiltonian.
Conversely, assume G+uv is hamiltonian and let C be the Hamilton cycle. Clearly, every Hamilton cycle contains the edge uv, in particular, C contains the edge uv (otherwise this cycle C also contained in G itself, a contradiction). Clearly $C - uv$ is a Hamilton path from u to v in G. Let it be $v_1v_2v_3 \dots v_n$, where $u = v_1$ and $v = v_n$.

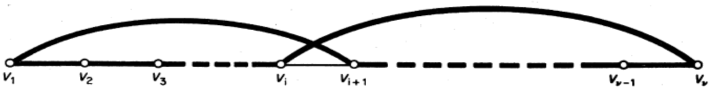
Let $S = \{v_i |\, uv_{i+1} \in V(G), 1 \leq i \leq n-1\}$ and $T = \{v_i | v_iv \in E(G)\}$.

Clearly, $v = v_n \notin S \cup T$ (otherwise, there is a Hamilton cycle in G)

$\Longrightarrow |S \cup T| \leq n - 1.$

that is, |S|+|T|=deg(u)+deg(v)$\geq n$. Hence $|S \cup T| + |S \cap T| = |S| + |T| \geq n.$

Hence $|S \cup T| \geq n - (n-1) = 1$. Let $v_i \in S \cap T$. Then uu_{i+1} and $u_i v$ are edges in G, see the following Figure.



Thus, $v_1 v_{i+1} v_{i+2} v_{i+3} \dots v_{n-1}\, v_n v_i v_{i-1} v_{i-2} \dots v_2 v_1$ is the Hamilton cycle in G.

Theorem: If G is simple undirected graph with $n \geq 3$ vertices and the minimum degree $\delta(G) \geq \frac{n}{2}$, then G is hamiltonian.

Proof. If G is K_n, then it is hamiltonian. So assume G is not complete. Let $(u_1, v_1), (u_2, v_2), \dots (u_k, v_k)$ be the pairs of non-adjacent vertices. As $\delta(G) \geq \frac{n}{2}, we\ have\ \deg(u_i) + deg(v_i) \geq n$ for all $1 \leq i \leq k$. Let $G_m = G + \{e_1, e_2, \dots, e_k\}$, where $e_i = u_i v_i, 1 \leq i \leq k$. As these are the only non-adjacent vertices in G, G_k is a complete graph K_n and is hamiltonian as $n \geq 3$. By previous theorem, G_k is hamiltoninan $\xLeftrightarrow[if\ and\ only\ if]{}$ G_{k-1} is hamiltonian $\Longleftrightarrow$ G_{k-2} is hamiltonian $\Longleftrightarrow$ G_{k-3} is hamiltonian $\Longleftrightarrow$ .... $\Longleftrightarrow$ G_2 is hamiltonian $\Longleftrightarrow$ $G_1 = G + u_1 v_1$ is hamiltonian$\Longleftrightarrow$ G is hamiltonian.

Hence the proof.