Peano Curves

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Space Filling Curves

- An N-dimensional space-filling curve is a continuous, surjective (onto) function from the unit interval $[0,1]$ to the N-dimensional unit hypercube $[0,1]^N$.
- In particular, a 2-dimensional space-filling curve is a continuous curve that passes through every point of the unit square $[0,1]^2$.

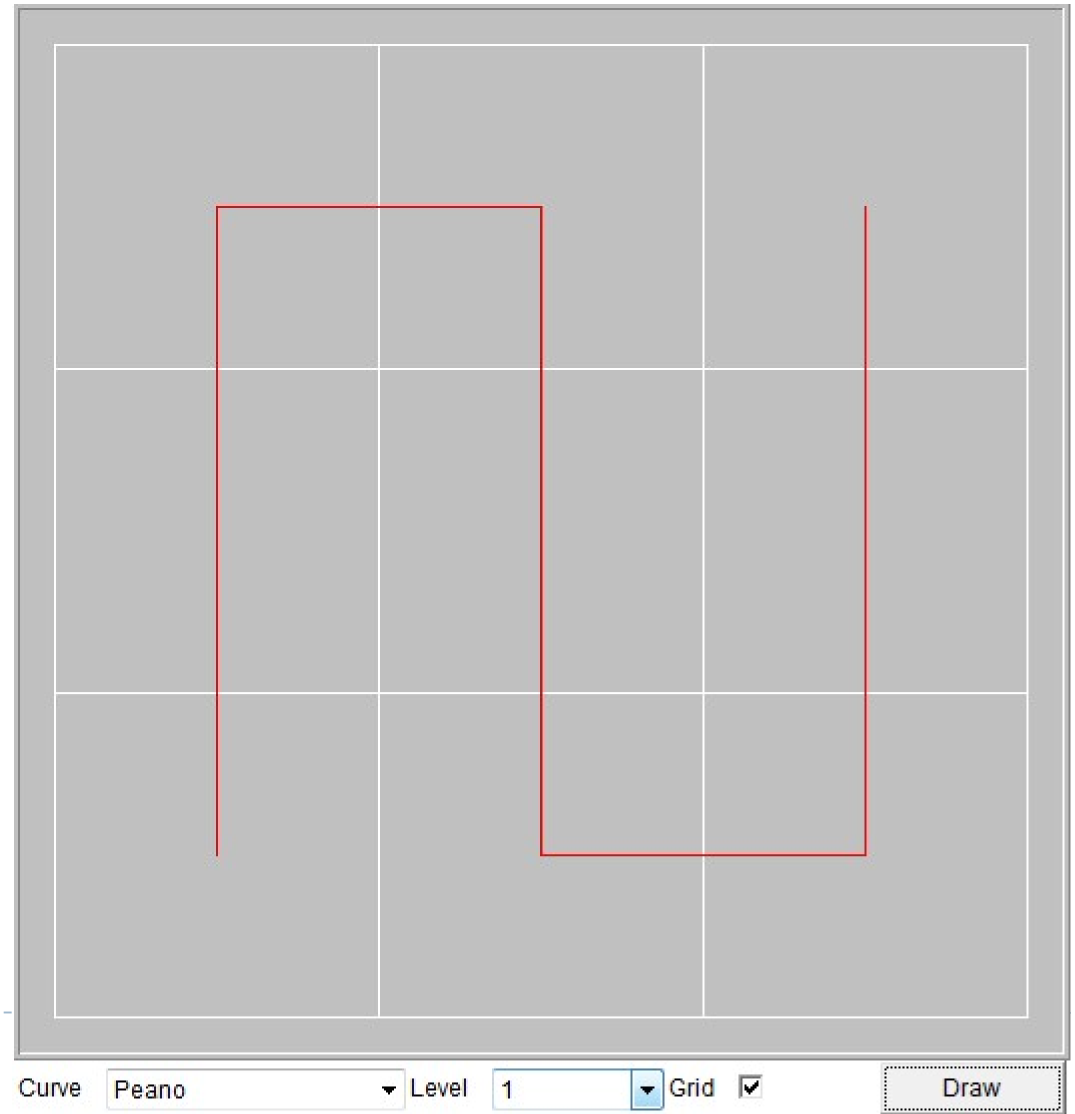
Peano Curves

- First described by Italian mathematician Guiseppe Peano (1858–1932).
- Space-filling curves in the 2-dimensional plane are commonly called *Peano curves*.

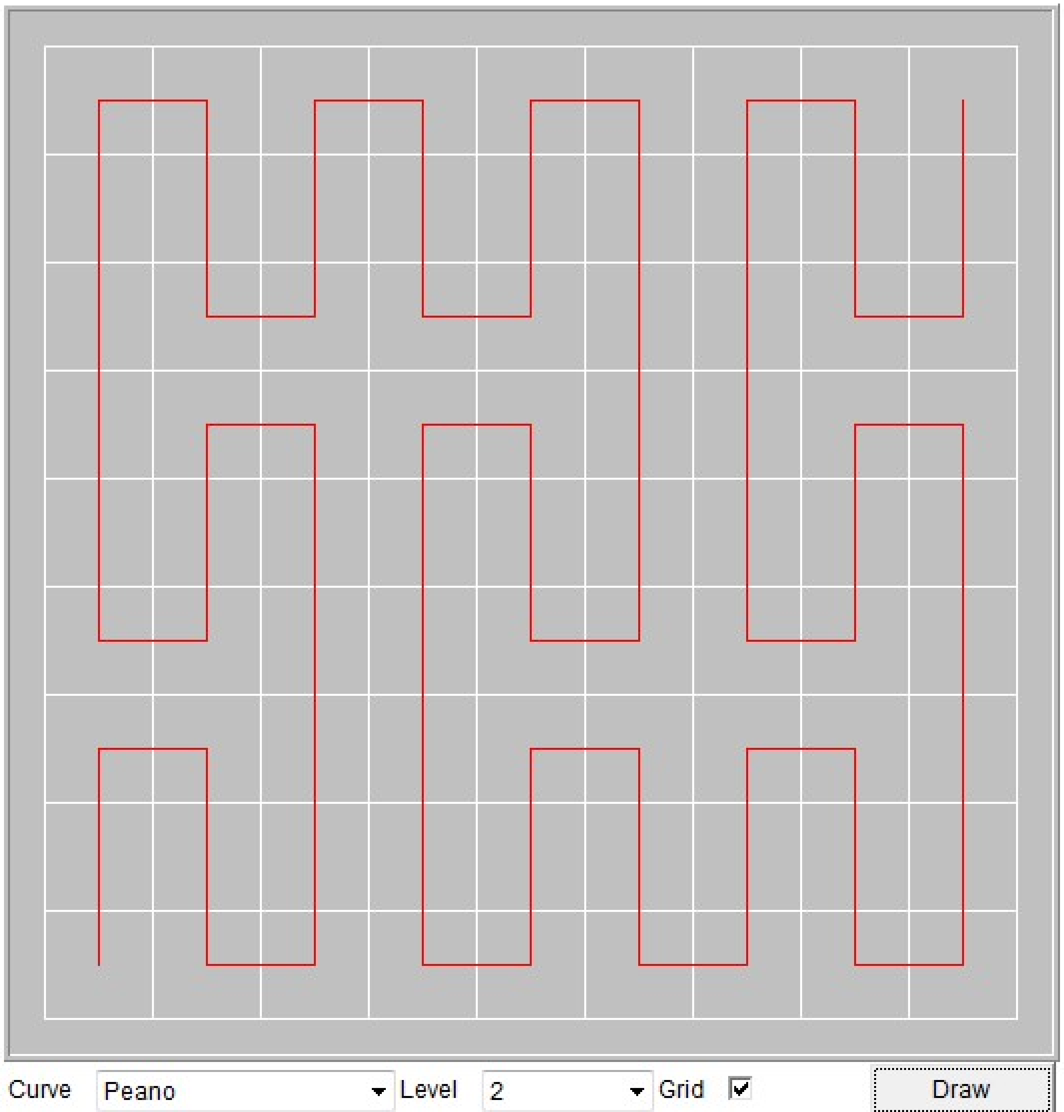
How to draw a Peano curve?

- Drawing a curve may be viewed as a two step process:
  1) Subdividing the unit square (the drawing area) into a number of cells;
  2) Traversing the cells in a characteristic order, subject to the following rules:
     - Each cell may be visited only once;
     - Cells visited in succession must be neighboring cells.
- There are some curves such as the Z-curve that do not follow the second rule.

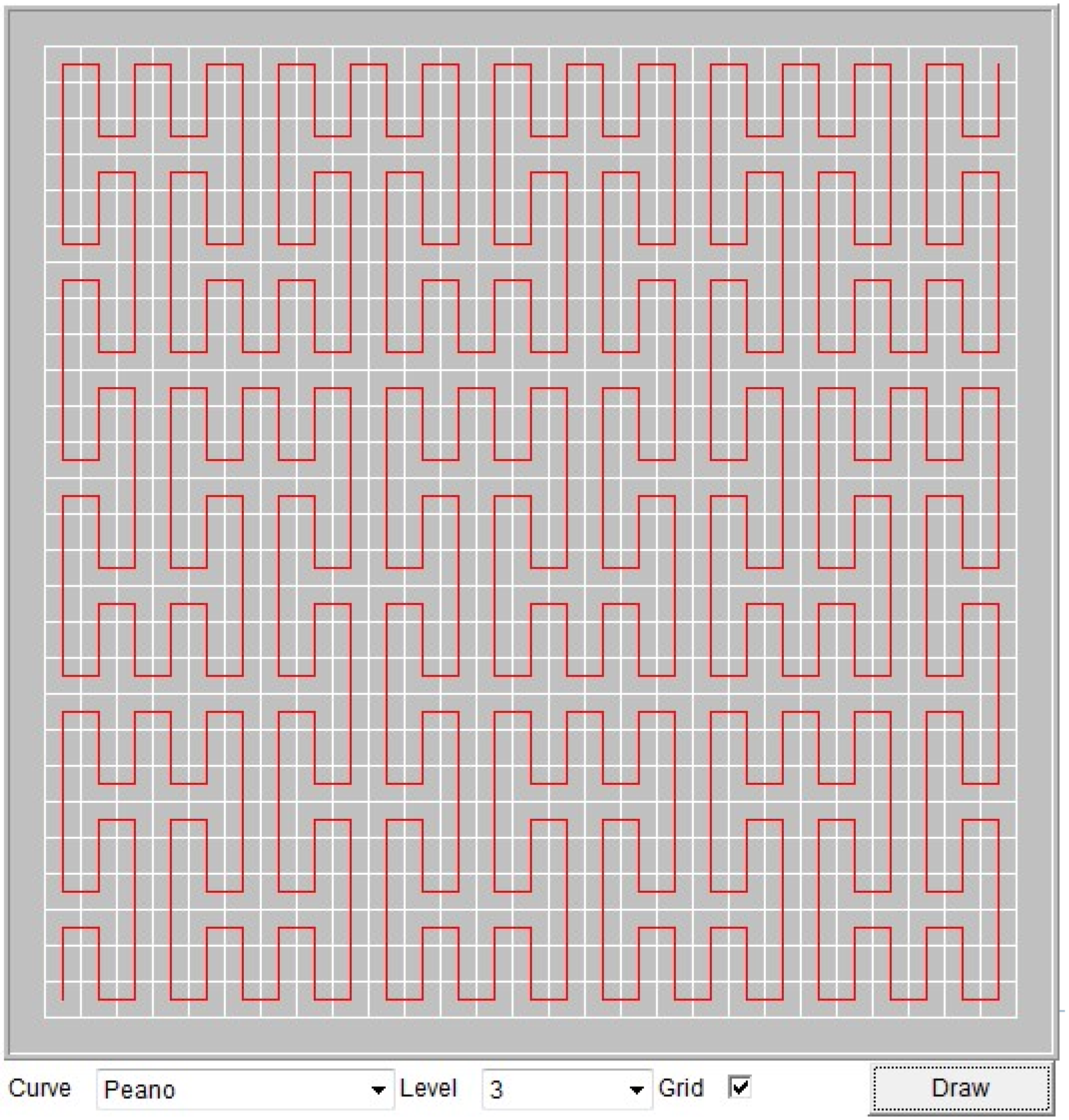
Drawing Peano curve – Level 1



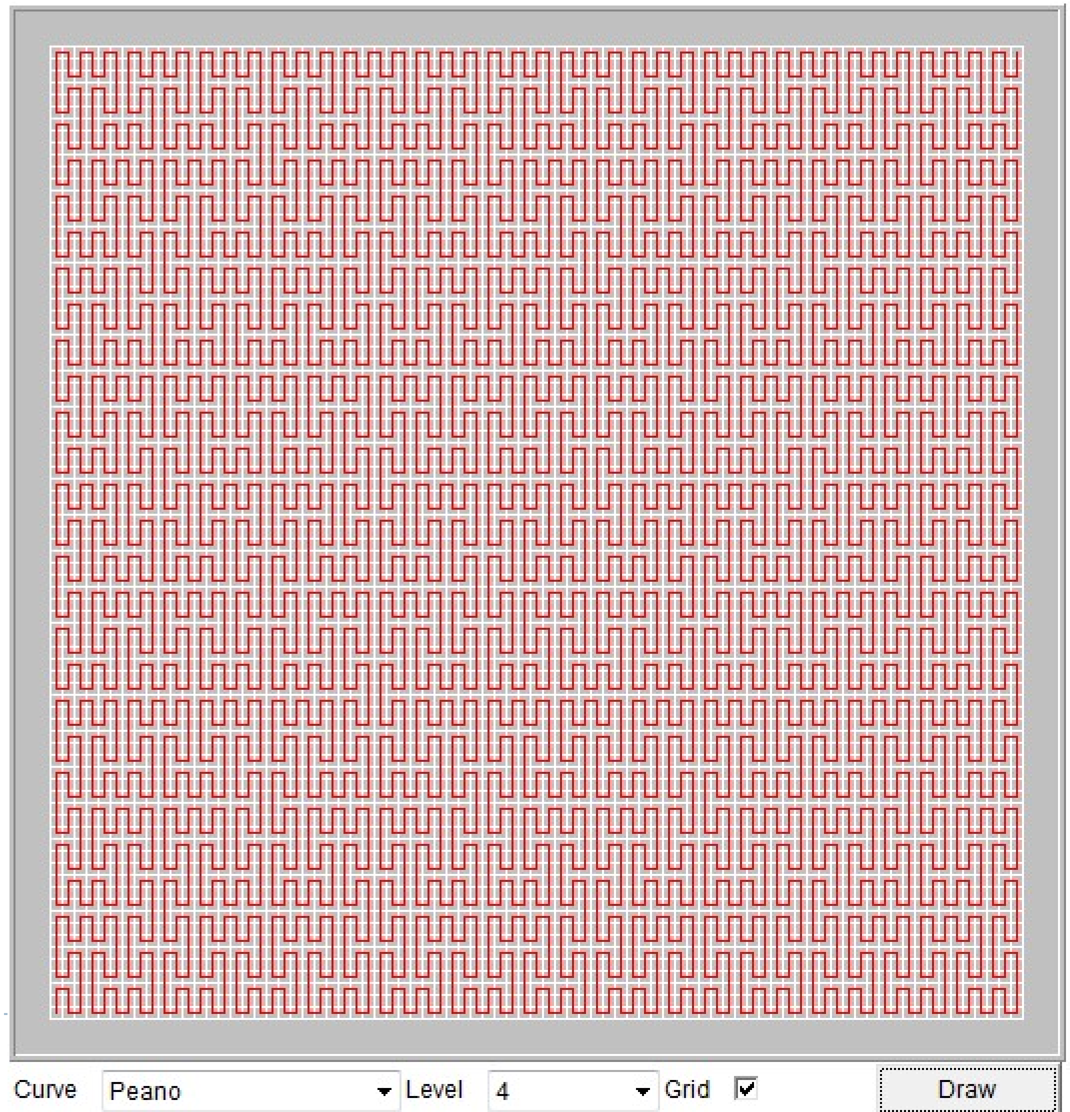
Drawing Peano curve – Level 2



Drawing Peano curve – Level 3



Drawing Peano curve – Level 4



Peano curve

- The Peano curves use square cells
- Each cell is subdivided into 9 cells(as opposed to 4 in Hilbert curve) at the next recursive level.
- The increased choice in traversing 9 cells gives rise to the variety in the Peano curves.
  - The labels Peano_S, Peano_R and Peano_M stand, respectively, for the serpentine, reflected and meandering variations of the Peano curve.

Properties of a peano curve

- Surjective(not injective)
- Self-intersecting

Self-intersecting - Proof

- We start with one curve
  - f_0: $[0, 1] \rightarrow [0, 1] \times [0, 1]$

    defined on the interval $[0, 1]$ with values in the square $[0, 1] \times [0, 1]$
- Divide the square into four smaller squares I_{00}, I_{01}, I_{10}, and I_{11}
- Consider the first fourth of our interval (currently it's $[0, 1]$).

  f_0 maps it into I_{00}
- Split I_{00} into smaller squares and shifting points of $[0, 1/4]$ until the first fourth of this interval maps into the first small square, its second fourth maps into the second square and so forth.

Note

The proof explains self-intersection property using hilbert curve.

Proof - contd

- Use the same procedure for I_{01}, I_{10} and I_{11}.

- So we define a function $f_1:[0, 1]\rightarrow[0, 1]\times[0, 1]$ such that when the square is split into 16 intervals, f_1 maps [0, 1/16] into the first square,[1/16, 2/16] into the second and so on.

- We obtain a sequence of functions f_0, f_1, f_2, ... each mapping the unit interval into the unit square

- For every point t, (t € [0,1])
  - the distance between its values f_n and f_{n+1} does not exceed the diagonal of the square obtained on the nth step - $\sqrt{2} \cdot 1/2^n$ or $2^{1/2-n}$

Proof - Contd

- We have,
  - $|f_{n+m}(t) - f_n(t)| < |f_{n+1}(t)-f_n(t)| + \dots + |f_{n+m}(t) - f_{n+m-1}(t)|$
  - $|f_{n+m}(t) - f_n(t)| < 2^{1/2-n} + 2^{1/2-(n+1)} + \dots + 2^{1/2-(n+m-1)}$
- Summing up the geometric series, we can transform this into
  - $|f_{n+m}(t) - f_n(t)| < 2^{1/2-n}(1 + 2^{-1} + \dots + 2^{-(m-1)}) < 2^{3/2-n}$
  - This means that for every $t \in [0,1]$ we have a Cauchy sequence $f_0(t), f_1(t), \dots$

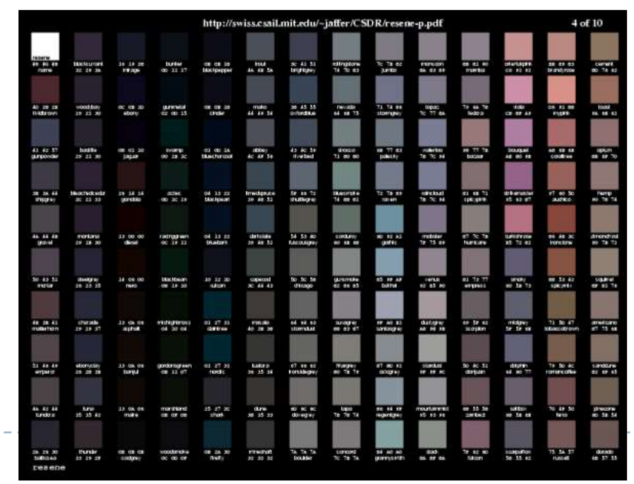
A sequence $x_0, x_1, \dots$ of elements of a metric space is said to be a *Cauchy sequence* if differences $|x_{n+m} - x_n|$ are uniformly small in m (i.e. do not depend on m) and tend to 0 as n grows.

Proof - Contd

- The plane is known to be a complete space implying that the sequence converges to a point in the unit square which is denoted by f(t).
- By construction, the curve passes arbitrarily close to any point in the square.
- Thus for any point (x, y) in the square it's possible to select a sequence of the function f values that converge to that point.
- This values are taken on by a sequence t_n of points in [0,1].
- Out of this sequence it's possible to extract a subsequence convergent to a point, say, t.
  - Then f(t) = (x, y).
  - It follows from a theorem by L. Brouwer that not only f maps a line interval onto a square it actually is self-intersecting.

Applications of Space filling curves

- In addition to their mathematical importance, space-filling curves have applications to
  1) Dimension reduction – a concept in statistics to reduce the number of random variables.
  2) Mathematical programming
  3) Sparse multi-dimensional database indexing
  4) Radio-frequency electronics
  5) Biology .



References

- http://www.cs.utexas.edu/~vbb/misc/sfc/0index.html

http://www.cut-the-knot.org/do_you_know/hilbert.shtml