Properties of Discrete Fourier Transform

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1. Periodicity:

If $x(n)$ and $X(k)$ are an N-point DFT pair, then

$x(n+N) = x(n)$ for all n

$X(k+N) = X(k)$ for all k

2. Linearity:

If $x_1(n) \xleftrightarrow{DFT} X_1(k)$

and $x_2(n) \xleftrightarrow{DFT} X_2(k)$

then for any real valued or complex – valued constants a_1 and a_2,

$$a_1 x_1(n) + a_2 x_2(n) \xleftrightarrow{DFT} a_1 X_1(k) + a_2 X_2(k)$$

3. Circular Symmetries of a Sequence:

The N – point DFT of a finite duration sequence, x(n) of length L $\leq$ N is equivalent to the N – point DFT of a periodic sequence $x_p(n)$, of period N, which is obtained by periodically extending x(n), that is,

$$x_p(n) = \sum_{l=-\infty}^{\infty} x(n - lN)$$

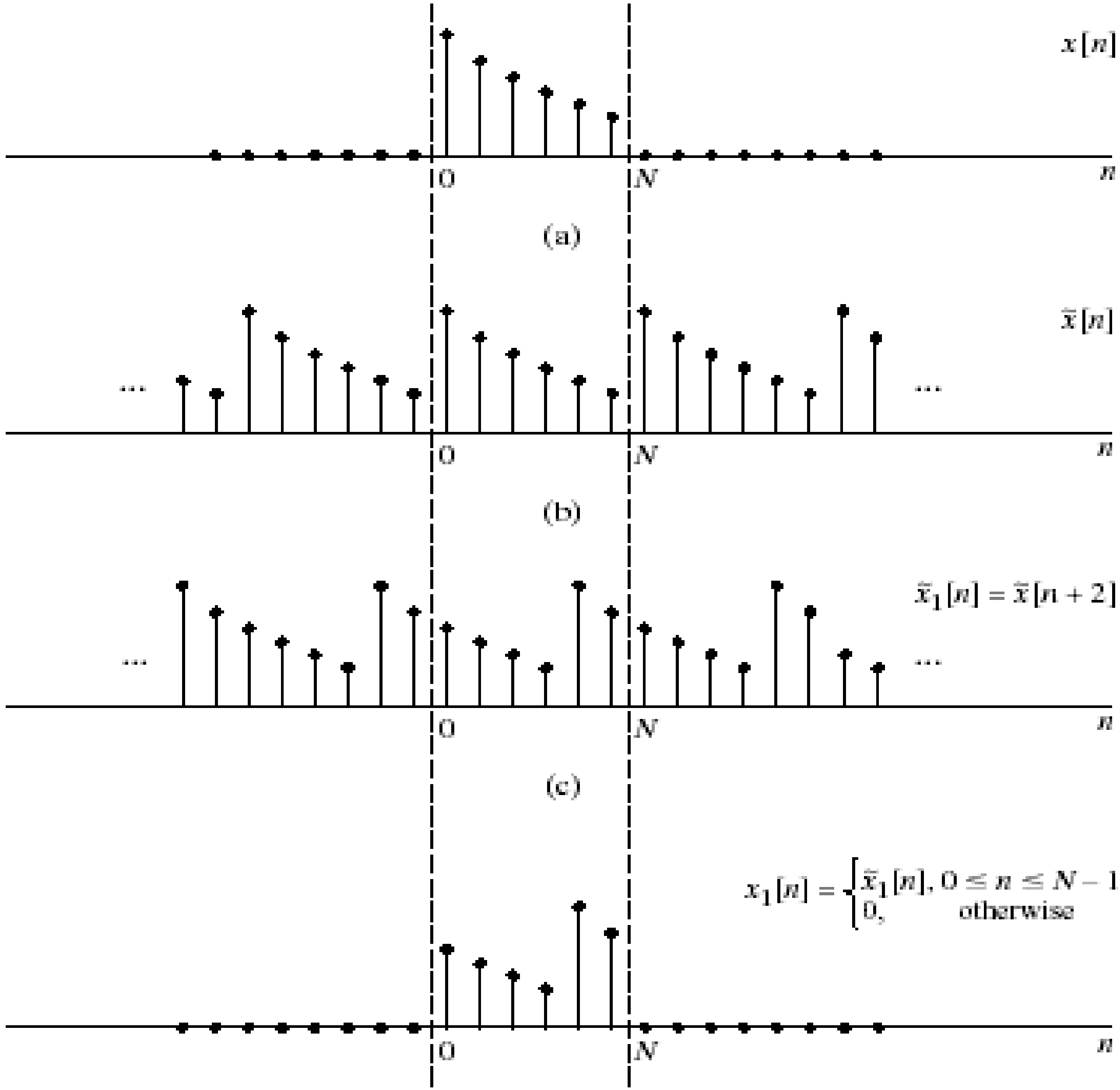
Now suppose that we shift the periodic sequence $x_p(n)$ by k values to the right. Thus we obtain another periodic sequence

$$x'_p(n) = x_p(n-k) = \sum_{l=-\infty}^{\infty} x(n-k-lN)$$

The finite duration sequence

$$x'(n) = \begin{cases} x_p^{'}(n), & 0 \le n \le N-1 \\ 0, & otherwise \end{cases}$$

is related to the original sequence x(n) by a circular shift. This relationship is illustrated in the following figure for N=4.



x [n]
0
N
n
(a)
x̃ [n]
...
(b)
x̃1[n] = x̃[n + 2]
(c)
x1[n] = x̃1[n], 0 ≤ n ≤ N − 1
0, otherwise

In general, the circular shift of the sequence can be represented as the index modulo N. Thus we can write,

$x(n) = x(n-k, \text{ modulo } N) = x((n-k))_N$

For eg., if k=2 and N=4, we have

$x'(n) = x((n-2))_4$

which implies that

$x'(0) = x((-2))_4 = x(2)$
$x'(1) = x((-1))_4 = x(3)$
$x'(2) = x((0))_4 = x(0)$
$x'(3) = x((1))_4 = x(1)$

Hence x’(n) is simply x(n) shifted circularly by two units in time, where the counterclockwise direction has been arbitrarily selected as the positive direction. Thus we conclude that a circular shift of an N – point sequence is equivalent to a linear shift of its periodic extension and vice versa.

Note:

1. An N – point sequence is called circularly even, if it is symmetric about the point zero on the circle. i.e.,

$$x(N-n) = x(n) \qquad 0 \le n \le N-1$$

2. An N – point sequence is called circularly odd, if it is antisymmetric about the point zero on the circle. i.e.,

$$x(N-n) = -x(n) \qquad 0 \le n \le N-1$$

3. The time reversal of an N – point sequence is attained by reversing its samples about the point zero on the circle. Thus the sequence $x((-n))_N$ is simply given as

$$x((-n))_N = x(N-n) \qquad 0 \le n \le N-1$$

4. This time reversal is equivalent to plotting x(n) in a clockwise direction on a circle.

4. Symmetry properties of the DFT:

Let us assume that the N – point sequence x(n) and its DFT are both complex valued. Then the sequences can be expressed as

$$x(n) = x_R(n) + j\,x_I(n) \qquad 0 \le n \le N-1 \qquad \ldots\ldots(1)$$

$$X(k) = X_R(k) + j\,X_I(k) \qquad 0 \le k \le N-1 \qquad \ldots\ldots.(2)$$

By substituting the expression (1) into DFT expression, we obtain

$$X_R(k) = \sum_{n=0}^{N-1}\left[x_R(n)\cos\left(\frac{2\pi kn}{N}\right) + x_I(n)\sin\left(\frac{2\pi kn}{N}\right)\right]$$

$$X_I(k) = -\sum_{n=0}^{N-1}\left[x_R(n)\sin\left(\frac{2\pi kn}{N}\right) - x_I(n)\cos\left(\frac{2\pi kn}{N}\right)\right]$$

Similarly, by substituting the expression (2) into IDFT expression, we obtain

$$x_R(n) = \frac{1}{N}\sum_{k=0}^{N-1}\left[X_R(k)\cos\left(\frac{2\pi kn}{N}\right) - X_I(k)\sin\left(\frac{2\pi kn}{N}\right)\right]$$

$$x_I(n) = \frac{1}{N}\sum_{k=0}^{N-1}\left[X_R(k)\sin\left(\frac{2\pi kn}{N}\right) + X_I(k)\cos\left(\frac{2\pi kn}{N}\right)\right]$$

Real and even sequence:

If the sequence x(n) is real and even, that is,

$$x(N-n) = x(n) \qquad 0 \le n \le N-1$$

and $x_I(n) = 0$

then $X_I(k) = 0$. Hence the DFT reduces to

$$X(k) = X_R(k) = \sum_{n=0}^{N-1} x(n) \cos\left(\frac{2\pi kn}{N}\right) \qquad 0 \le k \le N-1$$

which is real – valued and even.

Furthermore, the IDFT reduces to

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) \cos\left(\frac{2\pi kn}{N}\right) \qquad 0 \le n \le N-1$$

Real and odd sequence:

If the sequence x(n) is real and odd, that is,

$x(N-n) = -x(n) \qquad 0 \le n \le N-1$

and $x_I(n) = 0$

then $X_R(k) = 0$. Hence the DFT reduces to

$$X(k) = X_I(k) = -j\sum_{n=0}^{N-1} x(n)\sin\left(\frac{2\pi kn}{N}\right) \qquad 0 \le k \le N-1$$

which is purely imaginary and odd. Furthermore, the IDFT reduces to

$$x(n) = j\frac{1}{N}\sum_{k=0}^{N-1} X(k)\sin\left(\frac{2\pi kn}{N}\right) \qquad 0 \le n \le N-1$$

Purely imaginary sequences:

If the sequence $x(n) = jx_I(n)$, then

$$X_R(k) = \sum_{n=0}^{N-1} x_I(n)\sin\left(\frac{2\pi kn}{N}\right)$$

$$X_I(k) = \sum_{n=0}^{N-1} x_I(n)\cos\left(\frac{2\pi kn}{N}\right)$$

we observe that $X_R(k)$ is odd and $X_I(k)$ is even.

If $x_I(n)$ is odd, then $X_I(k) = 0$ and hence $X(k)$ is purely real. On the other hand, if $x_I(n)$ is even, then $X_R(k) = 0$ and hence $X(k)$ is purely imaginary.

5. Multiplication of two DFTs and Circular convolution:

Suppose that we have two finite – duration sequences of length N, $x_1(n)$ and $x_2(n)$. Their respective N – point DFTs are

$$X_1(k)=\sum_{n=0}^{N-1}x_1(n)e^{-j\frac{2\pi}{N}kn} \qquad ; \quad k=0,1,\dots\dots,N-1$$

$$X_2(k)=\sum_{n=0}^{N-1}x_2(n)e^{-j\frac{2\pi}{N}kn} \qquad ; \quad k=0,1,\dots\dots,N-1$$

If we multiply the two DFTs together, the result is a DFT, $X_3(k)$, of a sequence $x_3(n)$ of length N.

Let us determine the relationship between $x_3(n)$ and the sequence $x_1(n)$ and $x_2(n)$.

We have

$$X_3(k) = X_1(k) \, . \, X_2(k) \qquad\qquad k = 0, 1, 2, \ldots\ldots. N\text{-}1$$

The IDFT of $X_3(k)$ is

$$x_3(m) = \frac{1}{N}\sum_{k=0}^{N-1} X_3(k)\, e^{j\frac{2\pi}{N}km} \qquad ;\ m = 0,1,\ldots\ldots, N-1$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X_1(k) X_2(k)\, e^{j\frac{2\pi}{N}km} \qquad ;\ m = 0,1,\ldots\ldots, N-1$$

Substituting for $X_1(k)$ and $X_2(k)$ using the DFTs, we obtain,

$$x_3(m) = \frac{1}{N}\sum_{k=0}^{N-1}\left[\sum_{n=0}^{N-1} x_1(n) e^{-j\frac{2\pi}{N}kn}\right]\left[\sum_{l=0}^{N-1} x_2(l) e^{-j\frac{2\pi}{N}kl}\right] e^{j\frac{2\pi}{N}km}$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} x_1(n) \sum_{l=0}^{N-1} x_2(l)\left[\sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}k(m-n-l)}\right]$$

The sum in the brackets reduced to

$$\sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}k(m-n-l)} = \begin{cases} N, & l = m-n+pN = ((m-n))_N \\ 0, & otherwise \end{cases}$$

Therefore we obtain the desired expression for $x_3(n)$ in the form,

$$x_3(m) = \sum_{n=0}^{N-1} x_1(n)\, x_2((m-n))_N \qquad ; \qquad m = 0,1,\ldots\ldots\ldots N-1$$

The above expression has the form in convolution sum. But it is not the ordinary linear convolution. Instead, the convolution sum involves the index $((m-n))_N$ and is called Circular convolution.

Thus we conclude that multiplication of the DFTs of two sequences is equivalent to the circular convolution of the two sequences in the time domain.

The property can be summarized as follows

If $x_1(n) \xleftrightarrow{DFT} X_1(k)$ and $x_2(n) \xleftrightarrow{DFT} X_2(k)$ then

$$x_1(n) \otimes x_2(n) \xleftrightarrow{DFT} X_1(k) X_2(k)$$

where $x_1(n)$ N $x_2(n)$ (or) $x_1(n)$ $*$ $x_2(n)$ denotes the circular convolution of the sequence $x_1(n)$and $x_2(n)$.

Additional properties of DFT

6. Time reversal of a sequence:

If $x(n) \xleftrightarrow{DFT} X(k)$ then

$$x((-n))_N = x(N-n) \xleftrightarrow{DFT} X((-k))_N = X(N-k)$$

7. Circular time shift of a sequence:

If $x(n) \xleftrightarrow{DFT} X(k)$ then

$$x((n-l))_N \xleftrightarrow{DFT} X(k)e^{-j\frac{2\pi}{N}kl}$$

8. Circular frequency shift of a sequence:

If $x(n) \xleftrightarrow{DFT} X(k)$ then

$$x(n)e^{j\frac{2\pi}{N}\ln} \xleftrightarrow{DFT} X((k-l))_N = X(N+k-l)$$

9. Complex conjugate property:

If $x(n) \xleftrightarrow{DFT} X(k)$ then

$$x^*(n) \xleftrightarrow{DFT} X^*((-k))_N = X^*(N-k)$$

10. Circular correlation property:

If $x(n) \xleftrightarrow{DFT} X(k)$ and $y(n) \xleftrightarrow{DFT} Y(k)$, then

$$x(n) \otimes y^*(-n) \xleftrightarrow{DFT} X(k)Y^*(k)$$

If y(n) = x(n), then

$$x(n) \otimes x^*(-n) \xleftrightarrow{DFT} |X(k)|^2 = X(k)\,X^*(k)$$

11. Multiplication of two sequences:

If $x_1(n) \xleftrightarrow{DFT} X_1(k)$ and $x_2(n) \xleftrightarrow{DFT} X_2(k)$, then

$$x_1(n)\, x_2(n) \xleftrightarrow{DFT} \frac{1}{N} X_1(k) \otimes X_2(k)$$

12. Parseval's theorem:

If $x(n) \xleftrightarrow{DFT} X(k)$ and $y(n) \xleftrightarrow{DFT} Y(k)$, then

$$\sum_{n=0}^{N-1} x(n) y^*(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) Y^*(k)$$

If y(n) = x(n), then

$$\sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X(k)|^2$$