1 Fast Fourier Transform

Fast (Discrete) Fourier Transform (DFT): An algorithm for quickly computing change of coordinates to or from Fourier basis.

$$\left\{ \begin{pmatrix}1\\1\\1\\1\\1\\1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\\omega\\\omega^2\\\omega^3\\\omega^4\\\omega^5\\\omega^6\\\omega^7\end{pmatrix}, \begin{pmatrix}1\\\omega^2\\\omega^4\\\omega^6\\1\\\omega^2\\\omega^4\\\omega^6\end{pmatrix}, \begin{pmatrix}1\\\omega^3\\\omega^6\\\omega\\\omega^4\\\omega^7\\\omega^2\\\omega^5\end{pmatrix}, \begin{pmatrix}1\\\omega^4\\1\\\omega^4\\1\\\omega^4\\1\\\omega^4\end{pmatrix}, \begin{pmatrix}1\\\omega^5\\\omega^2\\\omega^7\\\omega^4\\\omega\\\omega^6\\\omega^3\end{pmatrix}, \begin{pmatrix}1\\\omega^6\\\omega^4\\\omega^2\\1\\\omega^6\\\omega^4\\\omega^2\end{pmatrix}, \begin{pmatrix}1\\\omega^7\\\omega^6\\\omega^5\\\omega^4\\\omega^3\\\omega^2\\\omega\end{pmatrix} \right\}$$

$$F_8 = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & \omega_8 & \omega_8^2 & \omega_8^3 & \omega_8^4 & \omega_8^5 & \omega_8^6 & \omega_8^7 \\
1 & \omega_8^2 & \omega_8^4 & \omega_8^6 & \omega_8^8 & \omega_8^{10} & \omega_8^{12} & \omega_8^{14} \\
1 & \omega_8^3 & \omega_8^6 & \omega_8^9 & \omega_8^{12} & \omega_8^{15} & \omega_8^{18} & \omega_8^{21} \\
1 & \omega_8^4 & \omega_8^8 & \omega_8^{12} & \omega_8^{16} & \omega_8^{20} & \omega_8^{24} & \omega_8^{28} \\
1 & \omega_8^5 & \omega_8^{10} & \omega_8^{15} & \omega_8^{20} & \omega_8^{25} & \omega_8^{30} & \omega_8^{35} \\
1 & \omega_8^6 & \omega_8^{12} & \omega_8^{18} & \omega_8^{24} & \omega_8^{30} & \omega_8^{36} & \omega_8^{42} \\
1 & \omega_8^7 & \omega_8^{14} & \omega_8^{21} & \omega_8^{28} & \omega_8^{35} & \omega_8^{42} & \omega_8^{49}
\end{pmatrix}$$

$$= \begin{pmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & \omega_8 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & \omega_8^2 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & \omega_8^3 \\
1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & -\omega_8 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & -\omega_8^2 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & -\omega_8^3
\end{pmatrix}
\begin{pmatrix}
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & \omega_4 & \omega_4^2 & \omega_4^3 & 0 & 0 & 0 & 0 \\
1 & \omega_4^2 & \omega_4^4 & \omega_4^6 & 0 & 0 & 0 & 0 \\
1 & \omega_4^3 & \omega_4^6 & \omega_4^9 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & \omega_4 & \omega_4^2 & \omega_4^3 \\
0 & 0 & 0 & 0 & 1 & \omega_4^2 & \omega_4^4 & \omega_4^6 \\
0 & 0 & 0 & 0 & 1 & \omega_4^3 & \omega_4^6 & \omega_4^9
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

2 Image compression

3 Abstract vector spaces

A vector space V over the field $\mathbb{F}$ is a set of objects which can be

- added in such a way that the sum of two elements of V is again an element of V, and
- multiplied by elements of $\mathbb{F}$, in such a way that the product of an element of V by an element of $\mathbb{F}$ is an element of V,

and the following properties are satisfied:

VS 1 Given elements $\mathbf{u}, \mathbf{v}, \mathbf{w}$ of V, we have

$$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w}).$$

VS 2 There is an element of V, denoted by $\mathbf{0}$, such that

$$\mathbf{0} + \mathbf{u} = \mathbf{u} + \mathbf{0} = \mathbf{u}$$

for all elements $\mathbf{u}$ of V.

VS 3 Given an element $\mathbf{u}$ of V, there exists an element $-\mathbf{u}$ in V such that

$$\mathbf{u} + (-\mathbf{u}) = \mathbf{0}.$$

VS 4 For all elements $\mathbf{u}, \mathbf{v}$ of V, we have

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}.$$

VS 5 If c is a number, then $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$.

VS 6 If a, b are two numbers, then $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$.

VS 7 If a, b are two numbers, then $(ab)\mathbf{v} = a(b\mathbf{v})$.

VS 8 For all elements $\mathbf{u}$ of V, we have $1\mathbf{u} = \mathbf{u}$ (1 here is the number one).

4 Vector spaces of functions

Example: Let $C([-\pi,\pi])$ be the vector space of continuous real-valued functions on $[-\pi,\pi]$.

Inner product:

- linear in first coordinate:

  * $<cf,g> \;=\; c<f,g>$
  * $<f+g,h> \;=\; <f,h>+<g,h>$

- conjugate symmetry:

  * $<f,g> \;=\; \overline{<g,f>}$

- positive definite:

  * $<f,f> \;\geq\; 0$
  * $<f,f> \;=\; 0 \iff f=0$

Orthogonal functions:

Distance between functions:

Convergence for sequences of functions:

5 Fourier series

The set of functions

$$\{\sin nx : n = 1, 2, 3, \ldots\} \cup \{\cos nx : n = 1, 2, 3, \ldots\} \cup \left\{\frac{1}{2}\right\}$$

E.g. Fourier series for step function:

$$f = \frac{1}{2} + \sum_{m=1}^{\infty} \frac{1 + (-1)^m}{m\pi} \sin mx$$

Other Fourier series

- Sawtooth:

$$f(x) = 2\sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{n} \sin(nx)$$

- Parabolic pulse:

$$f(x) = \frac{\pi^2}{12} - \sum_{m=1}^{\infty} \frac{2}{(2m)^2} \cos(2mx) + \sum_{m=1}^{\infty} \frac{4}{\pi(2m-1)^3} \sin((2m-1)x)$$

- Triangle pulse:

$$f(x) = \frac{\pi}{8} - \frac{4}{\pi}\sum_{m=1}^{\infty} \frac{1}{\pi(4m-2)^2} \cos((4m-2)x) + \frac{2}{\pi}\sum_{m=1}^{\infty} \frac{(-1)^{n-1}}{(2m-1)^2} \sin((2m-1)x)$$

- Triangle wave:

$$f(x) = 2\sum_{m=0}^{\infty} \frac{2}{\pi(4n+1)^2} \sin((4n+1)x) - 2\sum_{m=0}^{\infty} \frac{2}{\pi(4n+3)^2} \cos((4n+3)x)$$