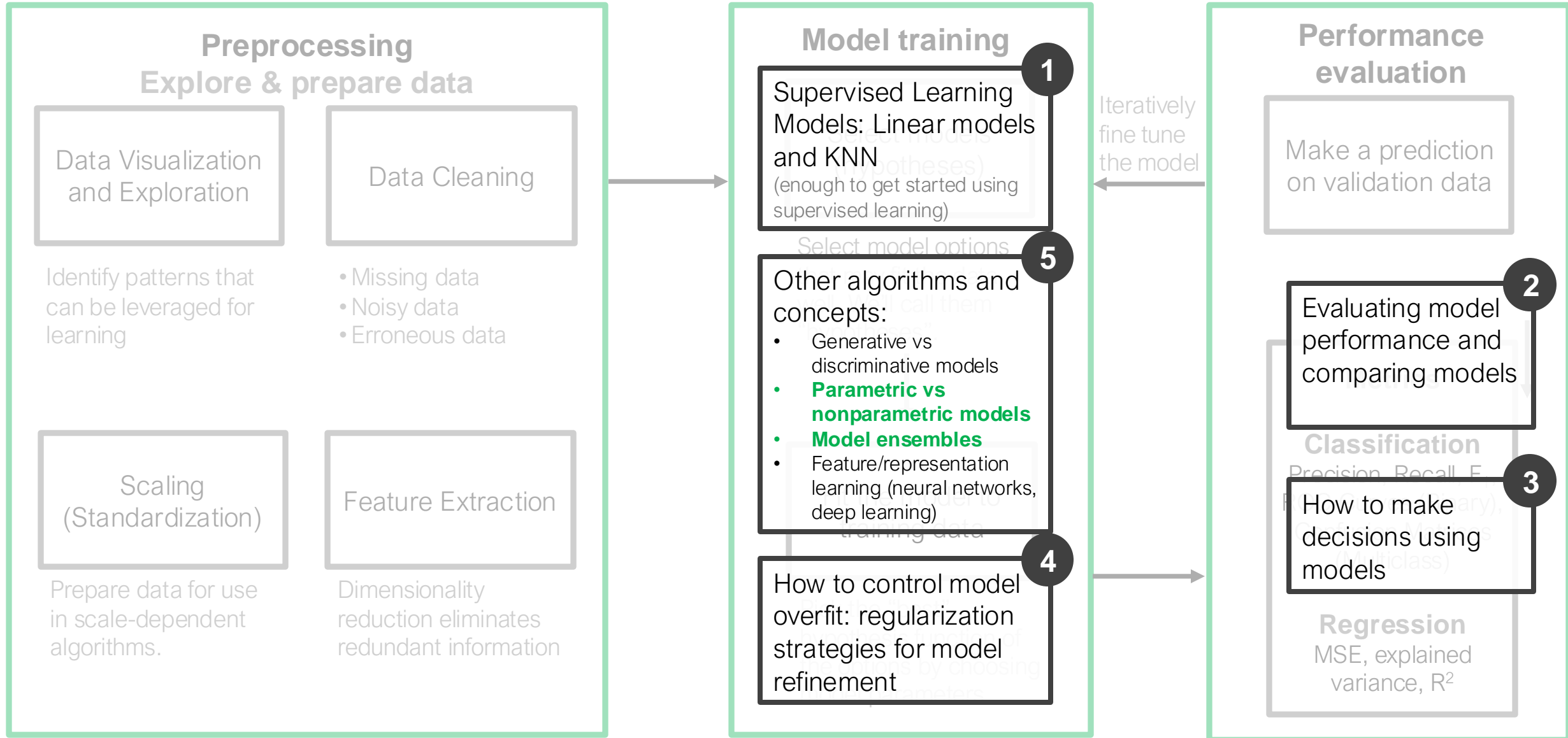


# Tree-based Models and Ensembles

# Supervised learning in practice



# Supervised Learning Techniques

Covered so far

Linear Regression

K-Nearest Neighbors

Perceptron

Logistic Regression

Linear Discriminant Analysis

Quadratic Discriminant Analysis

Naïve Bayes

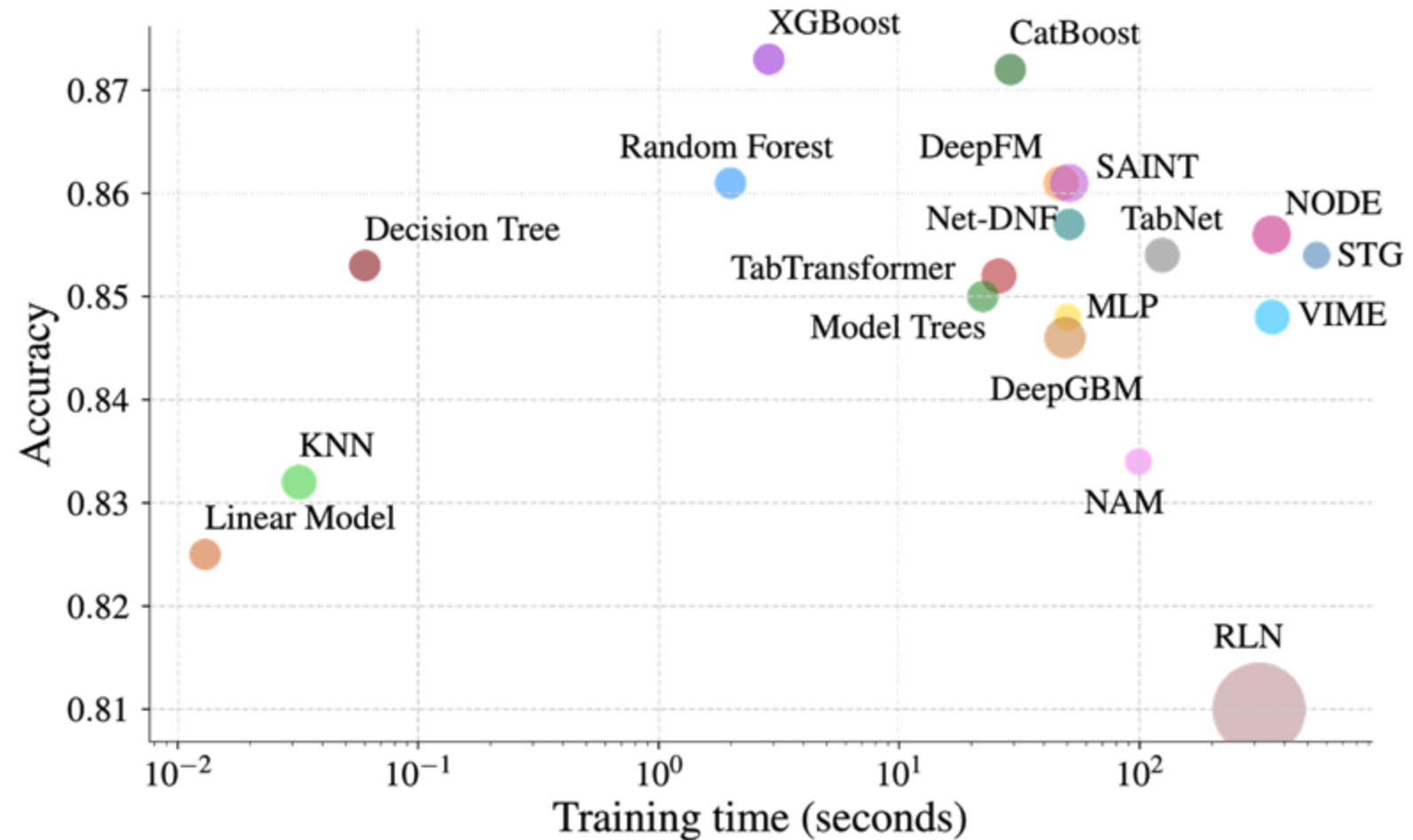
Decision Trees and Random Forests

Ensemble methods (bagging, boosting, stacking)

# Decision Tree Ensembles

"**gradient-boosted tree ensembles** still mostly outperform deep learning models on supervised learning tasks [on **heterogeneous tabular data**]"

a data set with a fixed number of features that are either continuous or categorical



Results on Adult Income dataset from UCI repository. Task: predict whether income exceeds \$50K/yr based on census data

Gradient-boosted decision tree ensembles	Benchmark datasets: HELOC <b>bold ones are the best ones by comparison</b> Adult HIGGS Covertypes Cal. Housing									
		Acc ↑	AUC ↑	Acc ↑	AUC ↑	Acc ↑	AUC ↑	Acc ↑	AUC ↑	MSE ↓
	Linear Model	73.0±0.0	80.1±0.1	82.5±0.2	85.4±0.2	64.1±0.0	68.4±0.0	72.4±0.0	92.8±0.0	0.528±0.008
	KNN [65]	72.2±0.0	79.0±0.1	83.2±0.2	87.5±0.2	62.3±0.1	67.1±0.0	70.2±0.1	90.1±0.2	0.421±0.009
	Decision Tree [197]	80.3±0.0	89.3±0.1	85.3±0.2	89.8±0.1	71.3±0.0	78.7±0.0	79.1±0.0	95.0±0.0	0.404±0.007
	Random Forest [198]	82.1±0.2	90.0±0.2	86.1±0.2	91.7±0.2	71.9±0.0	79.7±0.0	78.1±0.1	96.1±0.0	0.272±0.006
	XGBoost [53]	<u>83.5±0.2</u>	92.2±0.0	<u>87.3±0.2</u>	<u>92.8±0.1</u>	<u>77.6±0.0</u>	<u>85.9±0.0</u>	<b>97.3±0.0</b>	<b>99.9±0.0</b>	0.206±0.005
	LightGBM [78]	<u>83.5±0.1</u>	<u>92.3±0.0</u>	<b>87.4±0.2</b>	<b>92.9±0.1</b>	77.1±0.0	85.5±0.0	93.5±0.0	99.7±0.0	<b>0.195±0.005</b>
	CatBoost [79]	<b>83.6±0.3</b>	<b>92.4±0.1</b>	87.2±0.2	<u>92.8±0.1</u>	77.5±0.0	85.8±0.0	<u>96.4±0.0</u>	<u>99.8±0.0</u>	<u>0.196±0.004</u>
	Model Trees [199]	82.6±0.2	91.5±0.0	85.0±0.2	90.4±0.1	69.8±0.0	76.7±0.0	-	-	0.385±0.019
Deep Learning	MLP [200]	73.2±0.3	80.3±0.1	84.8±0.1	90.3±0.2	77.1±0.0	85.6±0.0	91.0±0.4	76.1±3.0	0.263±0.008
	DeepFM [15]	73.6±0.2	80.4±0.1	86.1±0.2	91.7±0.1	76.9±0.0	83.4±0.0	-	-	0.260±0.006
	DeepGBM [70]	78.0±0.4	84.1±0.1	84.6±0.3	90.8±0.1	74.5±0.0	83.0±0.0	-	-	0.856±0.065
	RLN [72]	73.2±0.4	80.1±0.4	81.0±1.6	75.9±8.2	71.8±0.2	79.4±0.2	77.2±1.5	92.0±0.9	0.348±0.013
	TabNet [5]	81.0±0.1	90.0±0.1	85.4±0.2	91.1±0.1	76.5±1.3	84.9±1.4	93.1±0.2	99.4±0.0	0.346±0.007
	VIME [88]	72.7±0.0	79.2±0.0	84.8±0.2	90.5±0.2	76.9±0.2	85.5±0.1	90.9±0.1	82.9±0.7	0.275±0.007
	TabTransformer [98]	73.3±0.1	80.1±0.2	85.2±0.2	90.6±0.2	73.8±0.0	81.9±0.0	76.5±0.3	72.9±2.3	0.451±0.014
	NODE [6]	79.8±0.2	87.5±0.2	85.6±0.3	91.1±0.2	76.9±0.1	85.4±0.1	89.9±0.1	98.7±0.0	0.276±0.005
	Net-DNF [57]	82.6±0.4	91.5±0.2	85.7±0.2	91.3±0.1	76.6±0.1	85.1±0.1	94.2±0.1	99.1±0.0	-
	STG [201]	73.1±0.1	80.0±0.1	85.4±0.1	90.9±0.1	73.9±0.1	81.9±0.1	81.8±0.3	96.2±0.0	0.285±0.006
	NAM [202]	73.3±0.1	80.7±0.3	83.4±0.1	86.6±0.1	53.9±0.6	55.0±1.2	-	-	0.725±0.022
	SAINT [9]	82.1±0.3	90.7±0.2	86.1±0.3	91.6±0.2	<b>79.8±0.0</b>	<b>88.3±0.0</b>	96.3±0.1	<u>99.8±0.0</u>	0.226±0.004

# Parametric vs Nonparametric techniques

## Non-parametric Models

Complexity of the model grows with the size of the training data

How is KNN more complex based on data size?

- K-Nearest Neighbors
- Decision Trees

## Parametric Models

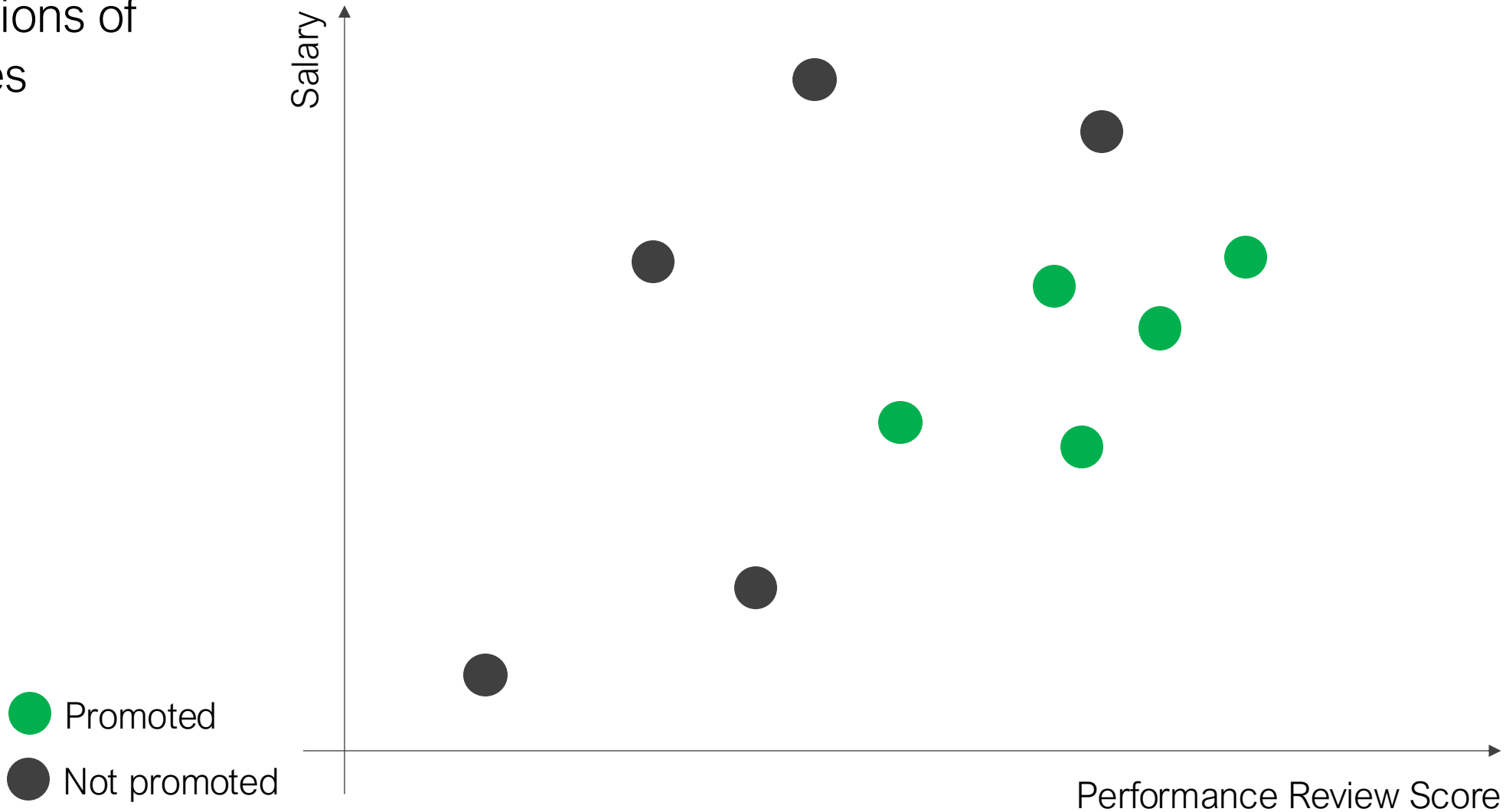
Fixed number of parameters (i.e. a fixed structure)

- Linear regression
- Logistic regression
- LDA, QDA
- Naïve Bayes with Gaussian likelihoods

# Classification and Regression Trees (CART)

Classification trees = decision trees

Predicting promotions of  
salaried employees



# Classification and Regression Trees (CART)

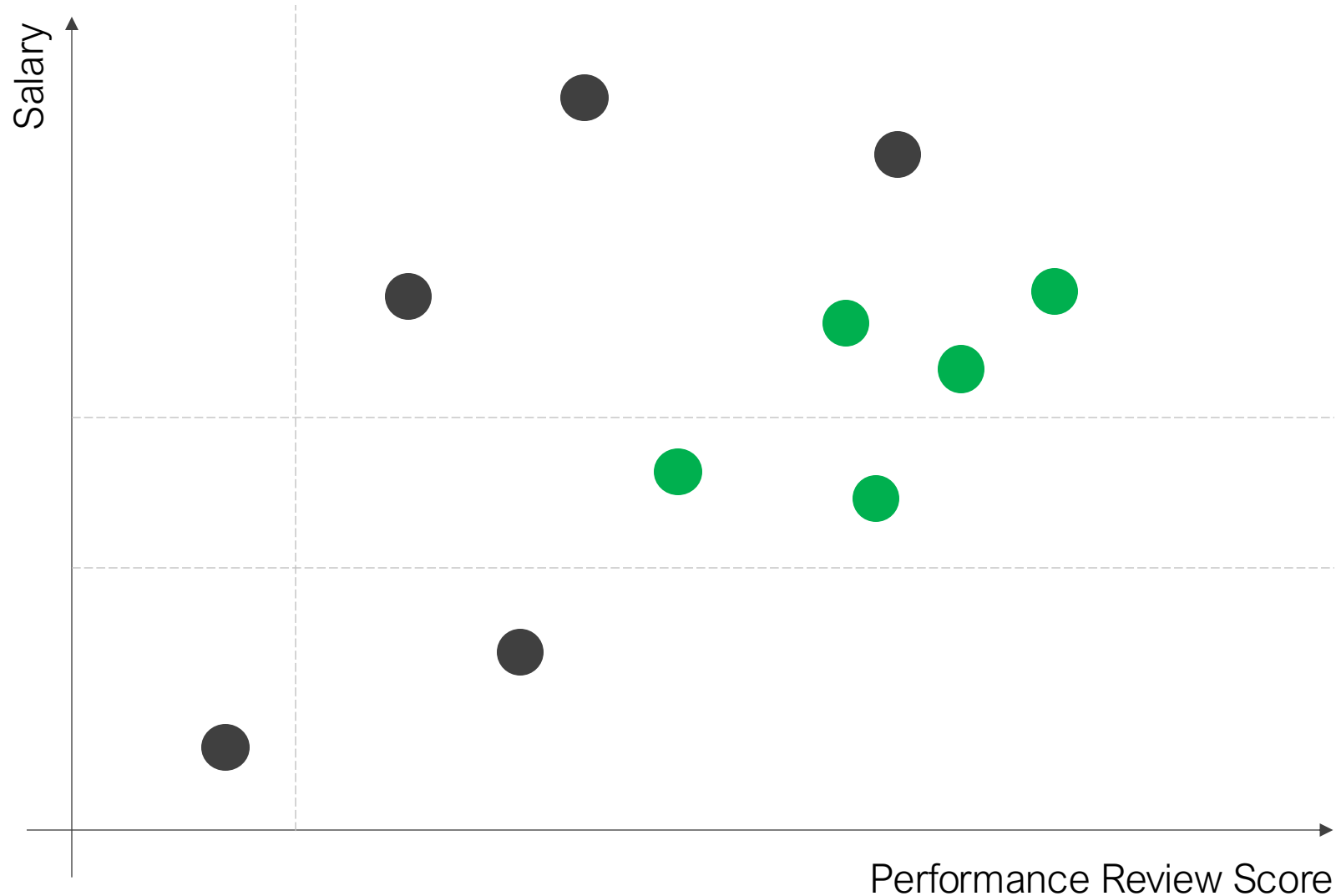
Predicting promotions of  
salaried employees

1

Find the best “split” in any one  
feature (that best classifies the  
data) that divides the region in  
two

● Promoted

● Not promoted





# Classification and Regression Trees (CART)

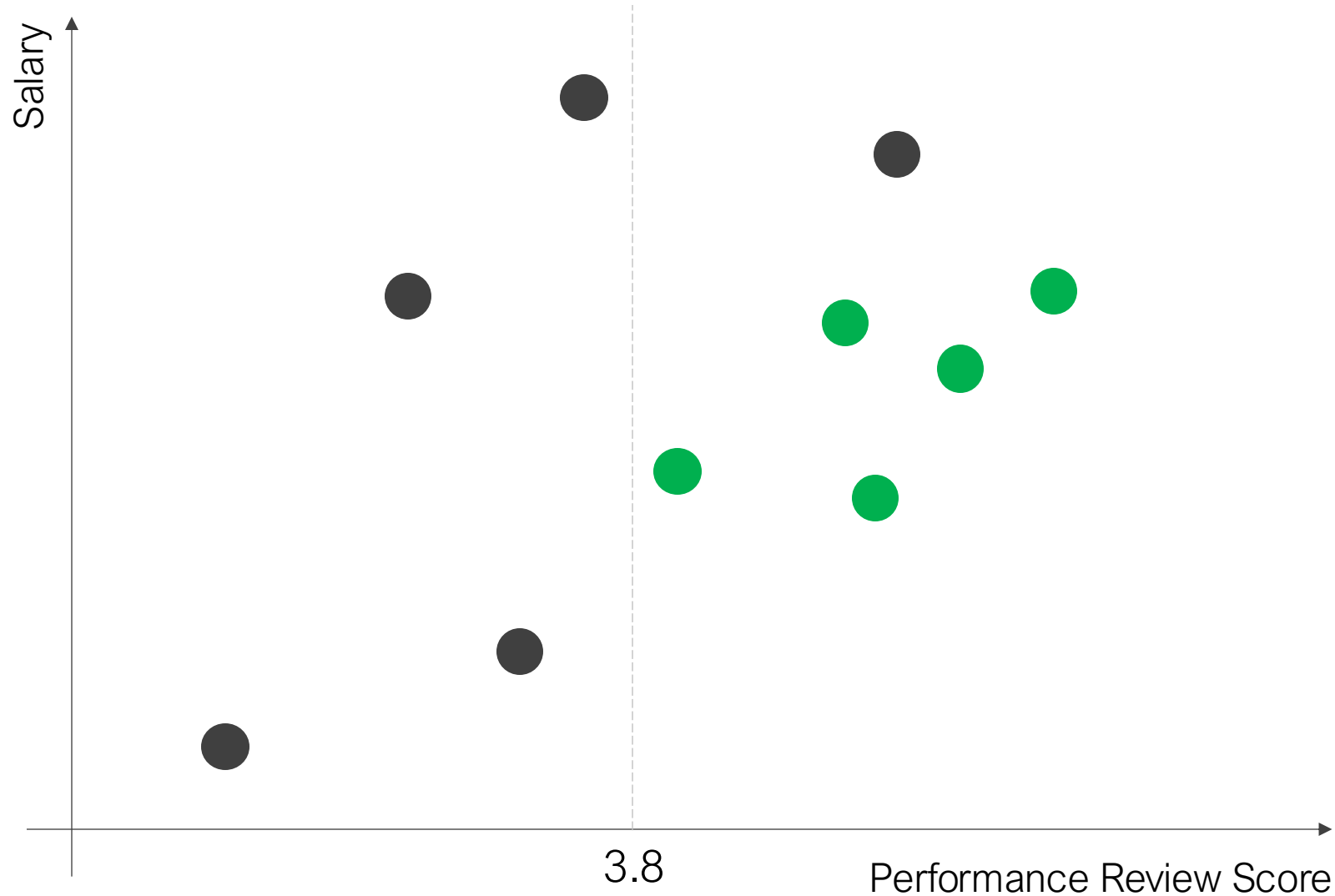
Predicting promotions of  
salaried employees

1

Find the best “split” in any one  
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data) that divides the region in  
two

● Promoted

● Not promoted



# Classification and Regression Trees (CART)

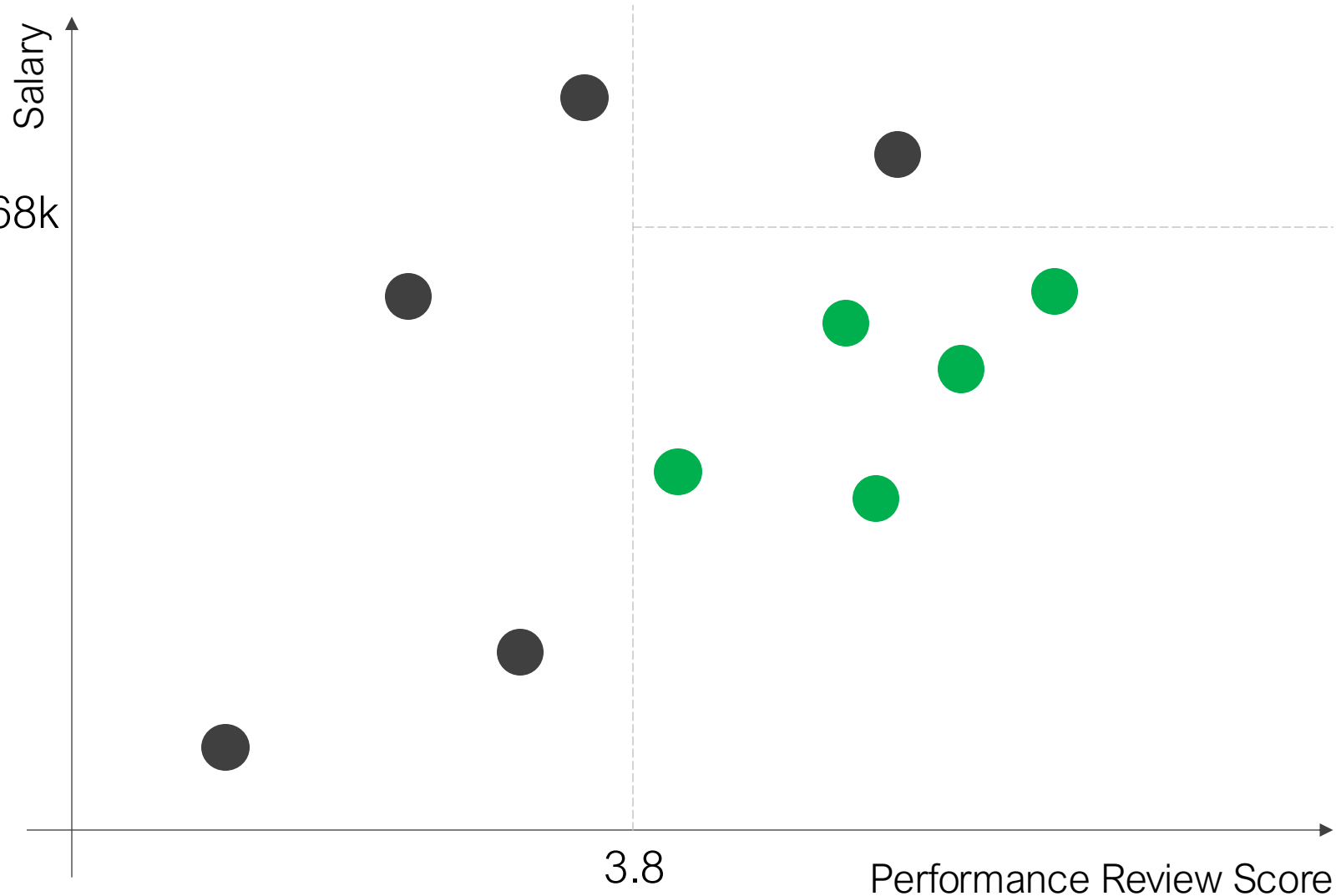
Predicting promotions of salaried employees

**1** Find the best “split” in any one feature (that best classifies the data) that divides the region in two

**2** Continue splitting regions (1 feature at a time) until a stopping criterion is reached (e.g. there are at most  $N$  samples in any region)

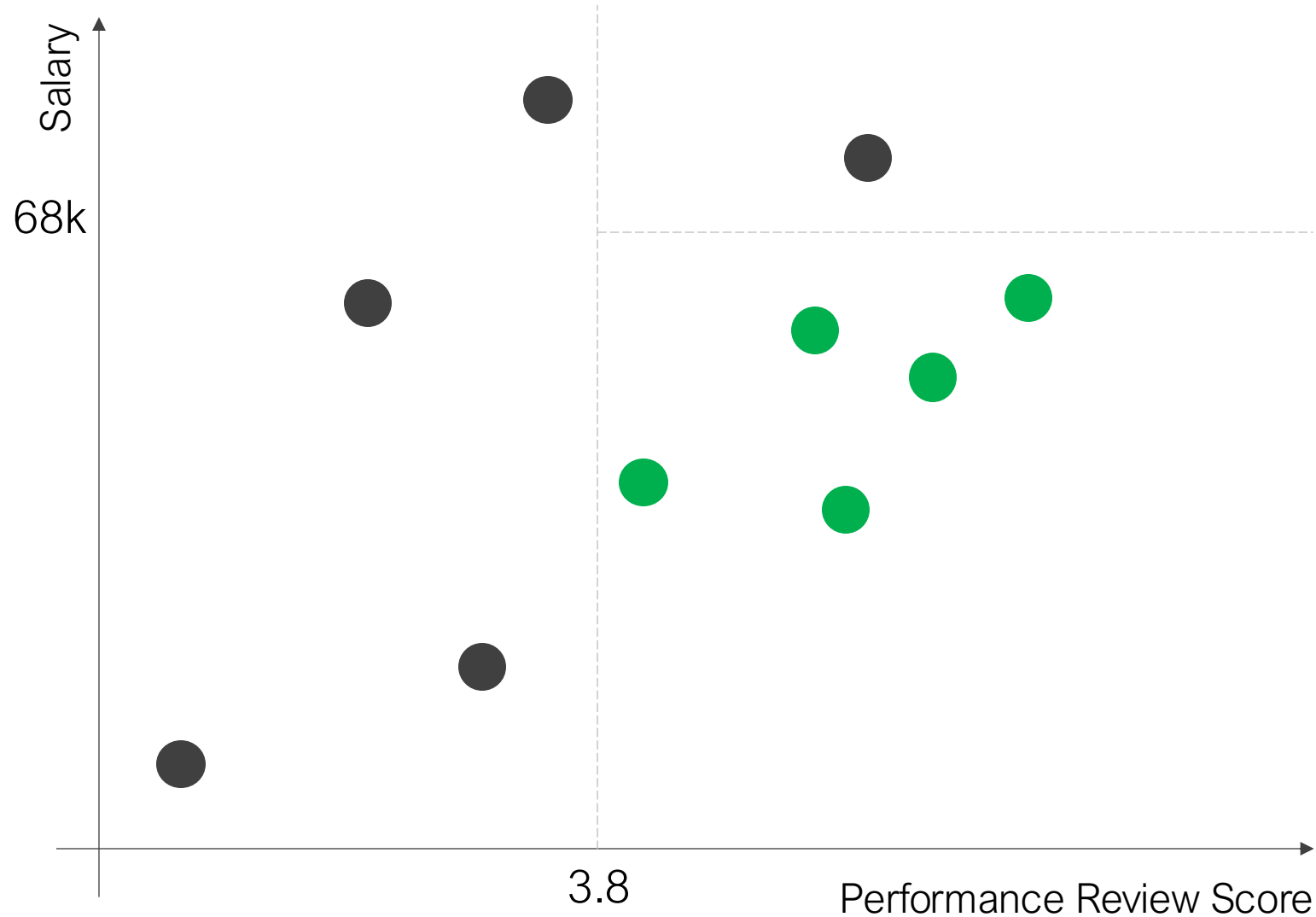
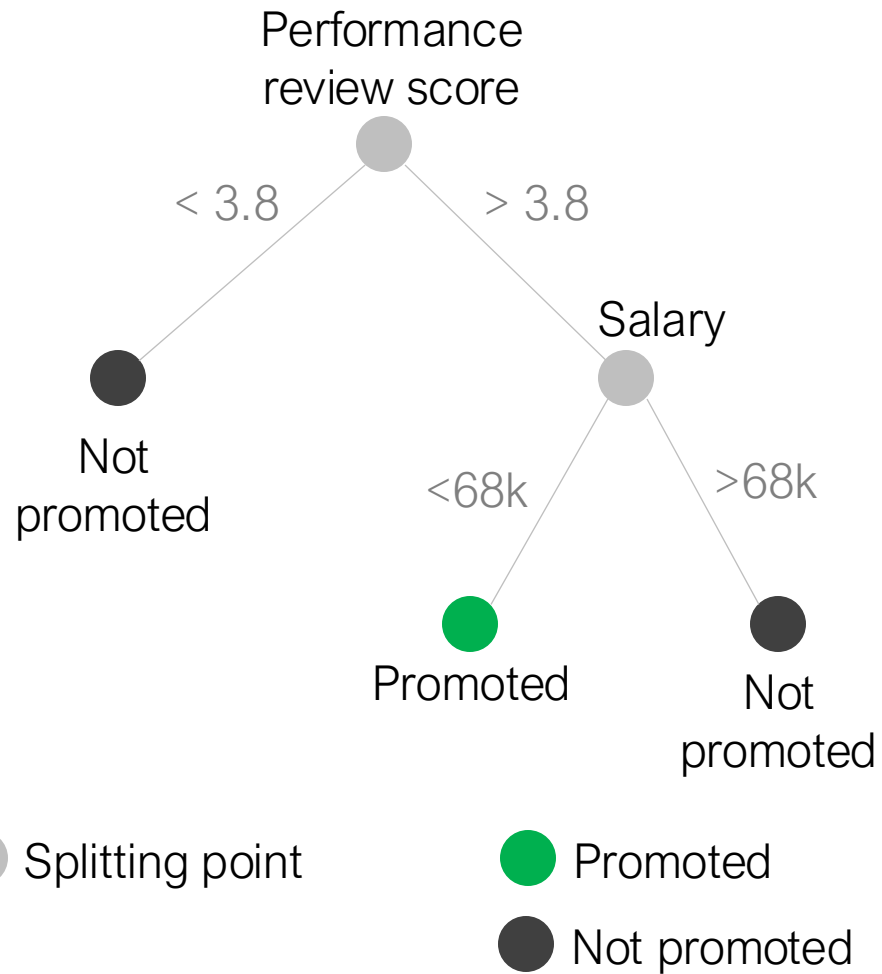
**Greedy, recursive  
binary tree**

● Promoted  
● Not promoted



# Classification and Regression Trees (CART)

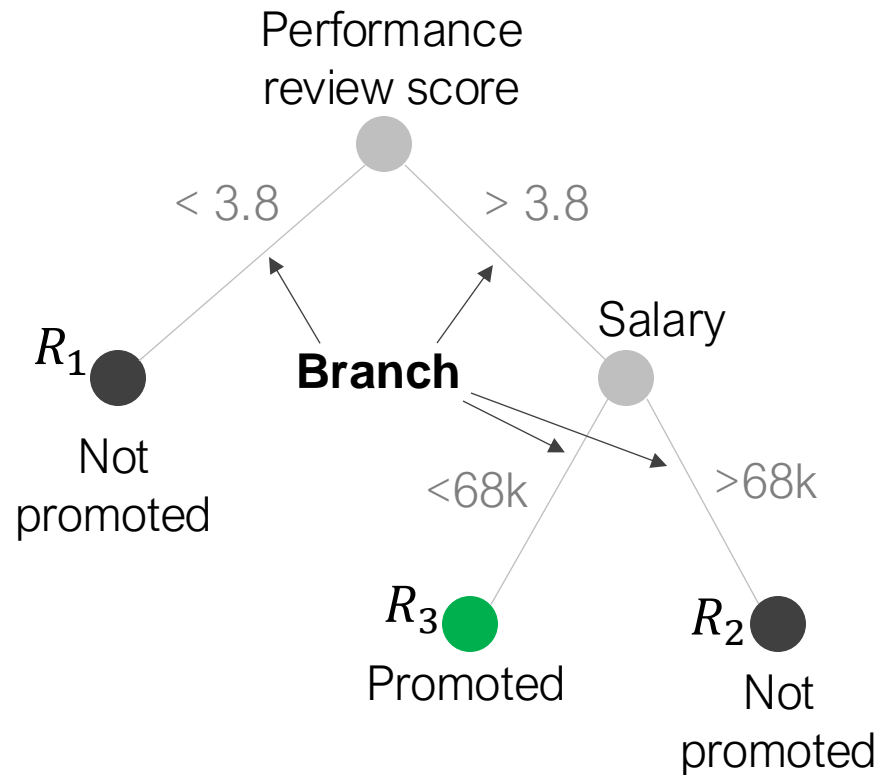
Tree representation:



# Classification and Regression Trees (CART)

Tree representation:

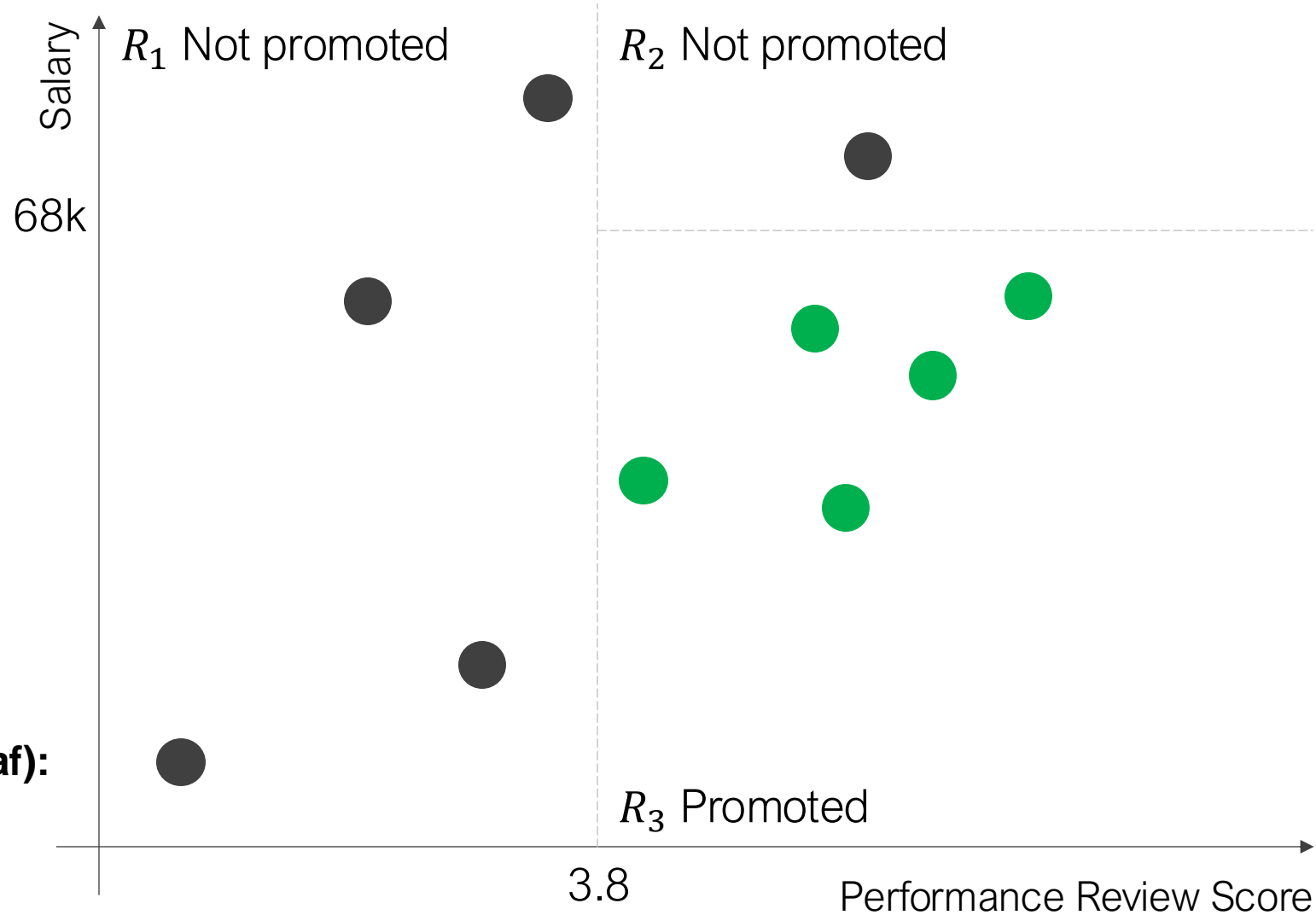
continue splitting until each region only has one class



**Internal node:**  
● Splitting point

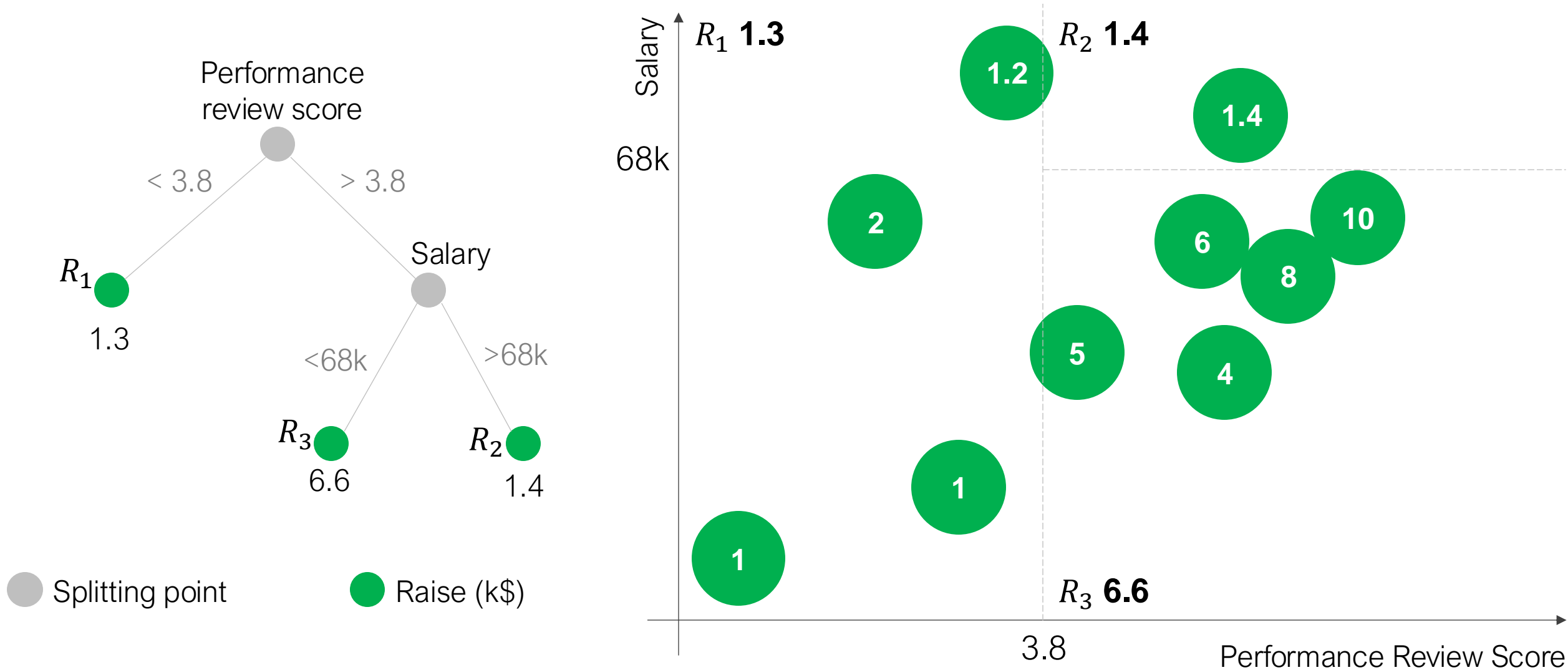
**Terminal node (leaf):**  
● Promoted

● Not promoted



# The Regression Setting

In this case, each region is represented by an average of the values it contains



# How do we determine which split to make?

Pick the split that reduces the error/cost criterion most after the split

## Splitting criterion

$$C = \sum_{r=1}^{R_{tot}} Q(r)$$

*(Note:  $R_{tot}$  is labeled "total" in the original image)*

## Regression

Mean square error

$$Q_{MSE}(r) = \sum_{i \in R_r} (y_i - \hat{y}_{R_r})^2$$

$y_i$  = training data response  $i$   
 $\hat{y}_{R_r}$  = mean value in region  $r$ , (where  $R_r$  is the set of samples in region  $r$ )

## Classification

Misclassification rate

$$Q_{Misclass}(r) = 1 - \max_k (\hat{p}_{rk})$$

Gini impurity

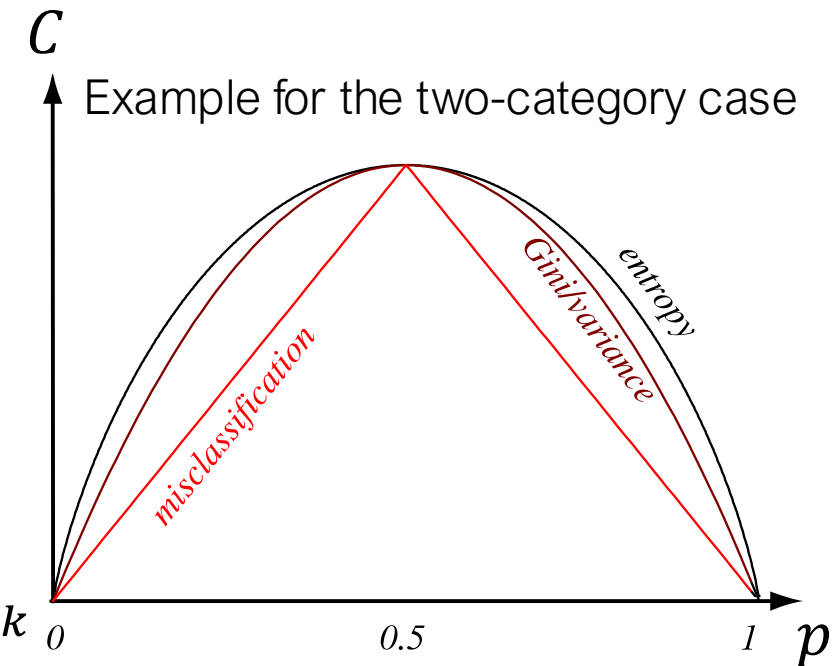
Measure of how often a randomly chosen element from the set would be incorrectly labeled if it was randomly labeled

$$Q_{Gini}(r) = \sum_{k=1}^K \hat{p}_{rk}(1 - \hat{p}_{rk})$$

Cross-entropy

$$Q_{entropy}(r) = - \sum_{k=1}^K \hat{p}_{rk} \log \hat{p}_{rk}$$

$\hat{p}_{rk}$  = proportion of training observations in the  $r^{\text{th}}$  region from the  $k^{\text{th}}$  class



Duda, Hart, and Stork., Pattern Classification

# How to measure quality of split for classification?

$\hat{p}_{rk}$  = proportion of training observations in the  $r^{\text{th}}$  region from the  $k^{\text{th}}$  class

Class 1 ●  
Class 2 ●

**For each region:**

Misclassification rate

$$Q_{\text{Misclass}}(r) = 1 - \max_k (\hat{p}_{rk})$$

	1	2
	0.333	0.167

Gini impurity

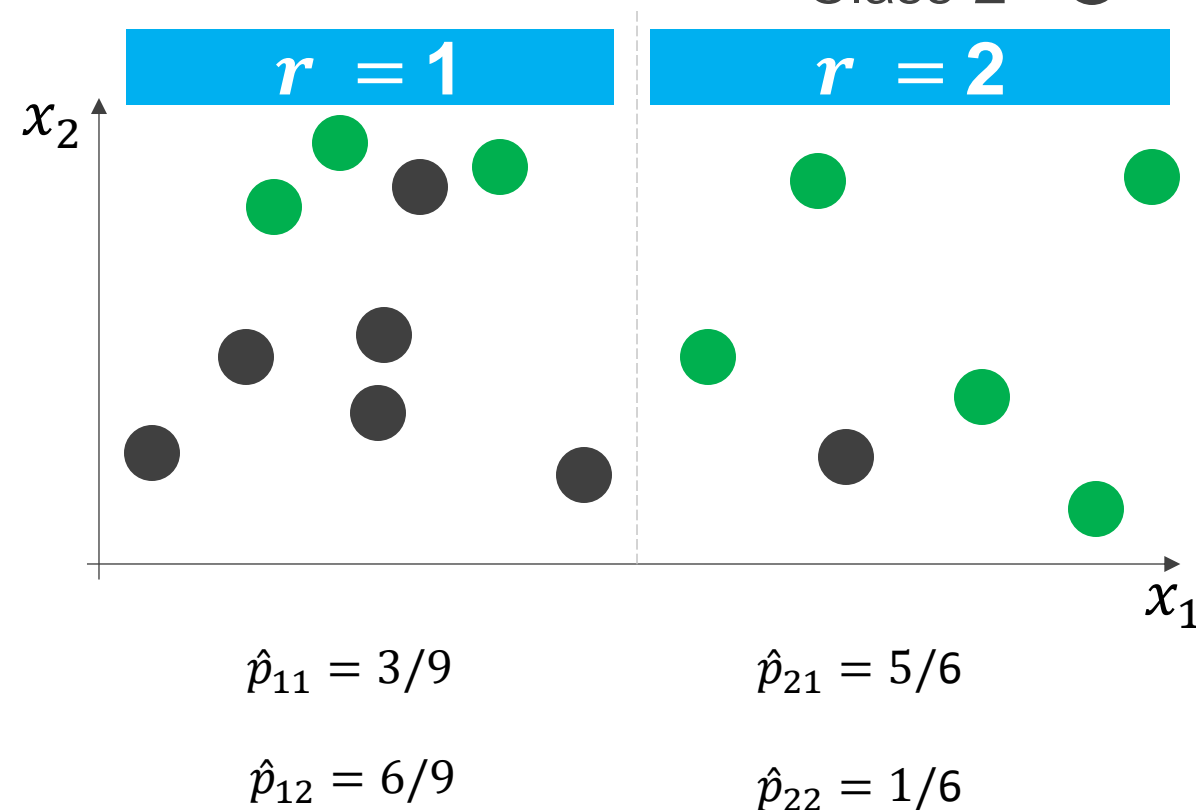
$$Q_{\text{Gini}}(r) = \sum_{k=1}^K \hat{p}_{rk}(1 - \hat{p}_{rk})$$

	0.444	0.278
--	-------	-------

Cross-entropy

$$Q_{\text{entropy}}(r) = - \sum_{k=1}^K \hat{p}_{rk} \log \hat{p}_{rk}$$

	0.637	0.450
--	-------	-------



# Tree Pruning

Trees have the tendency to overfit the data

Consider the stopping rule: stop splitting once there is only 1 class of observations in each region (leads to complete overfit)

**Pruning** the tree reduces this overfit (removing splits after the tree is formed)

Pruning can be optimized through a penalty on the number of terminal nodes:

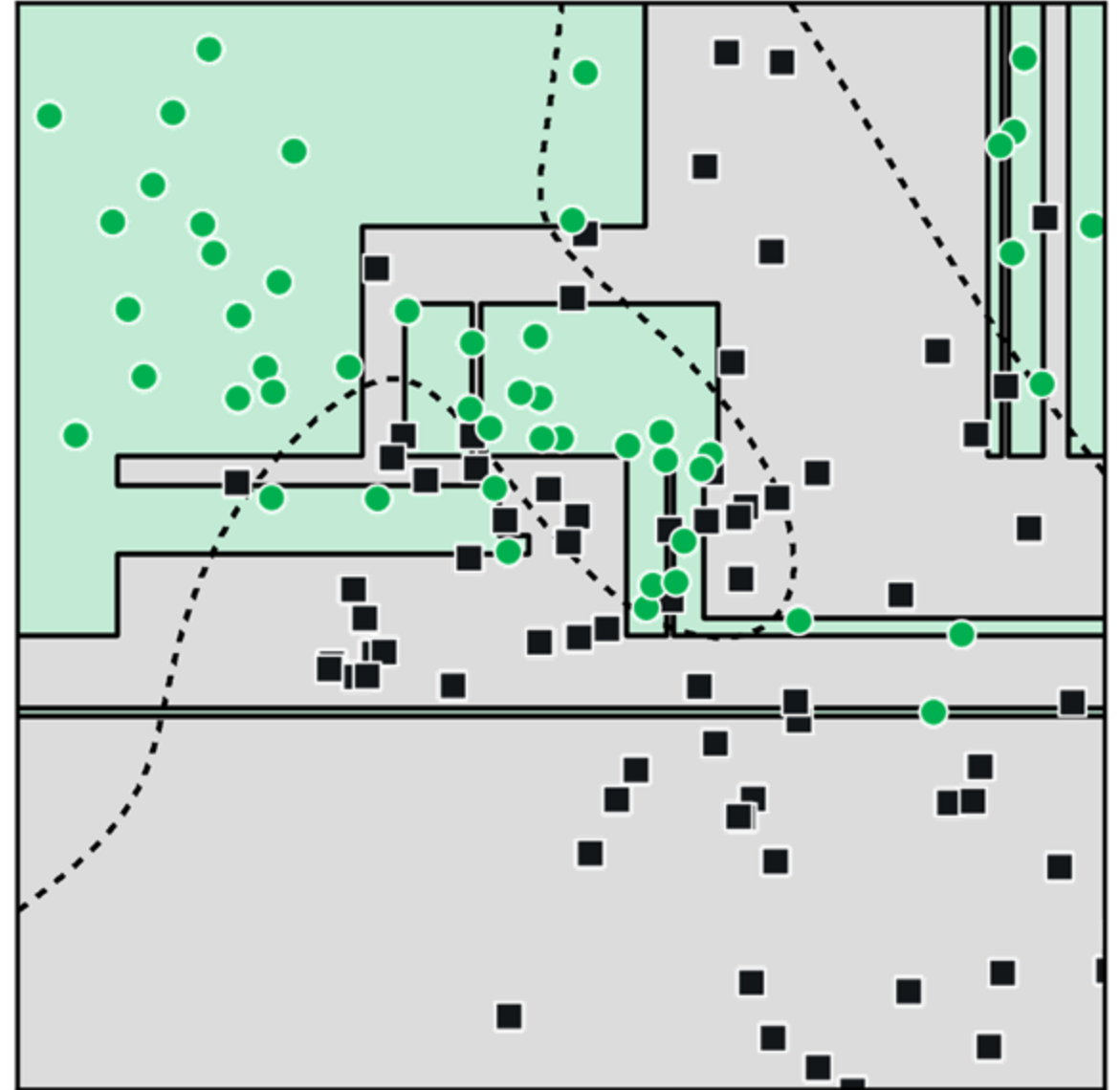
$$C_{Prune} = \sum_{j=1}^T \sum_{i \in R_j} (y_i - \hat{y}_{R_j})^2 + \alpha T$$

not used for complex problem

penalty on number of terminal nodes

number of terminal nodes

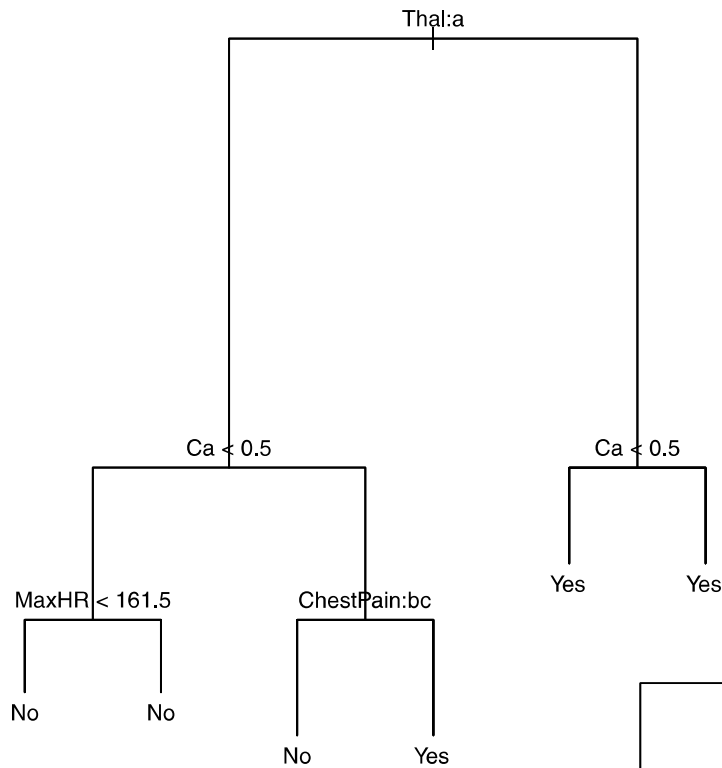
Decision Tree





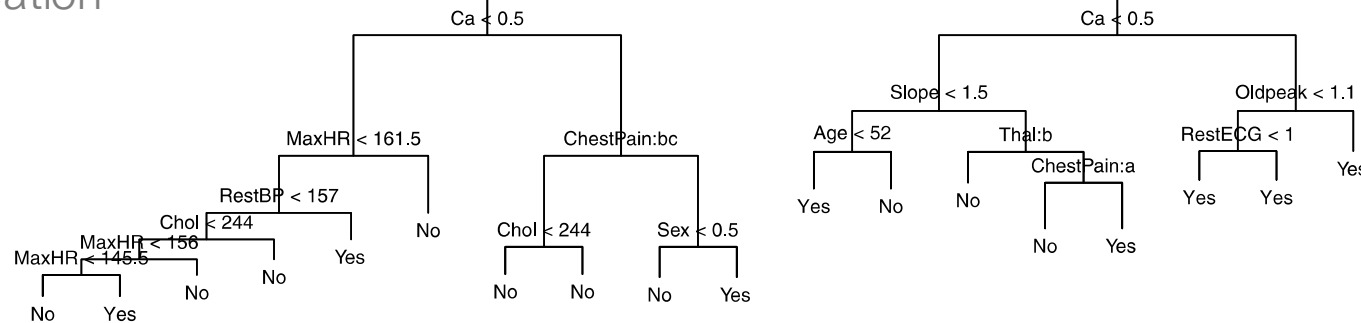
# Pruning example

Pruned Tree

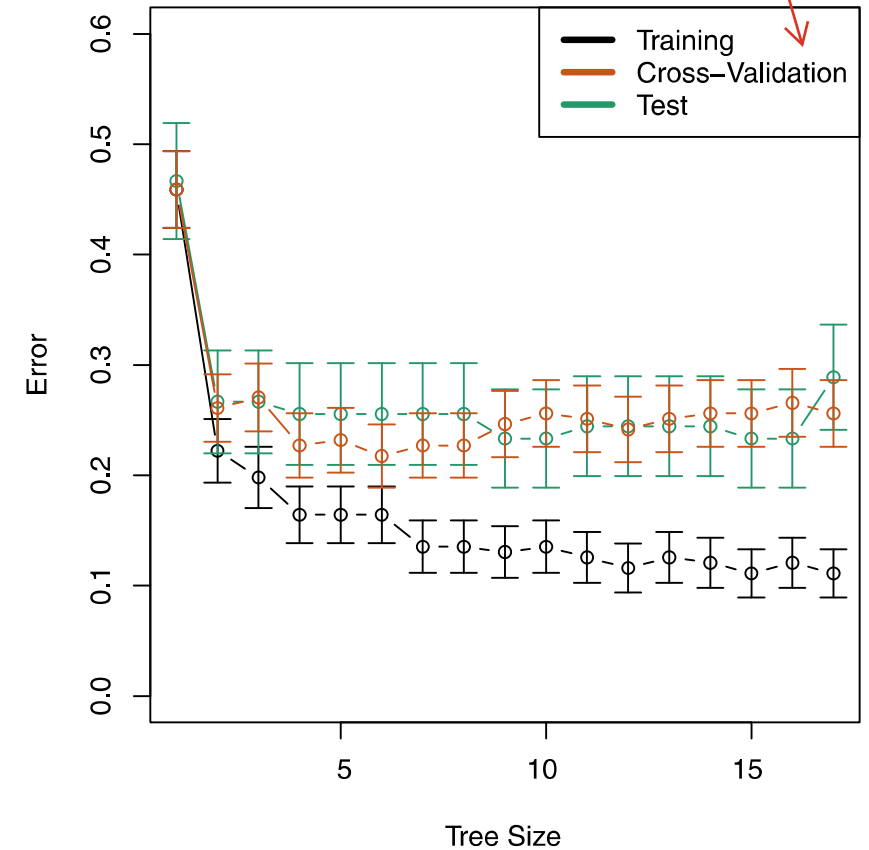


Original Tree

Example: heart disease classification

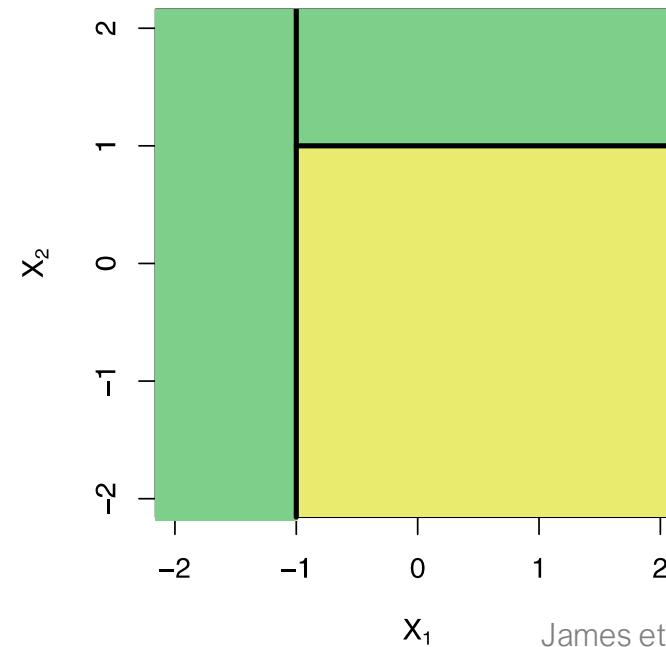
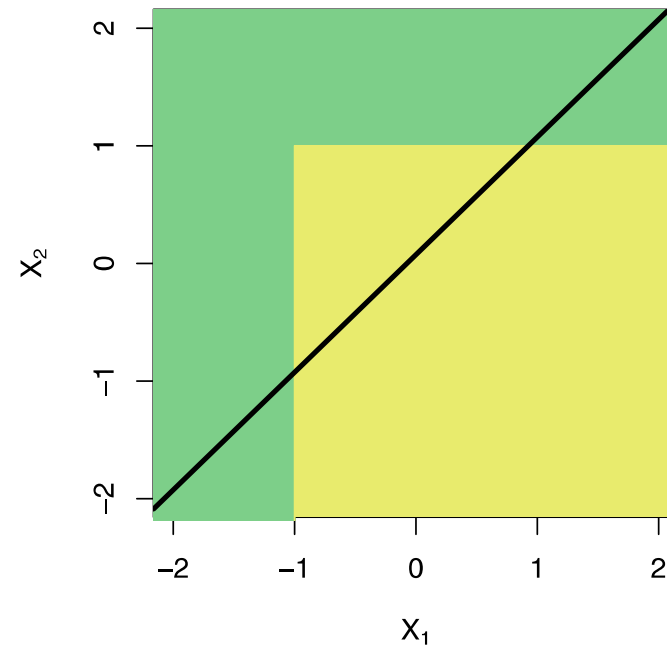
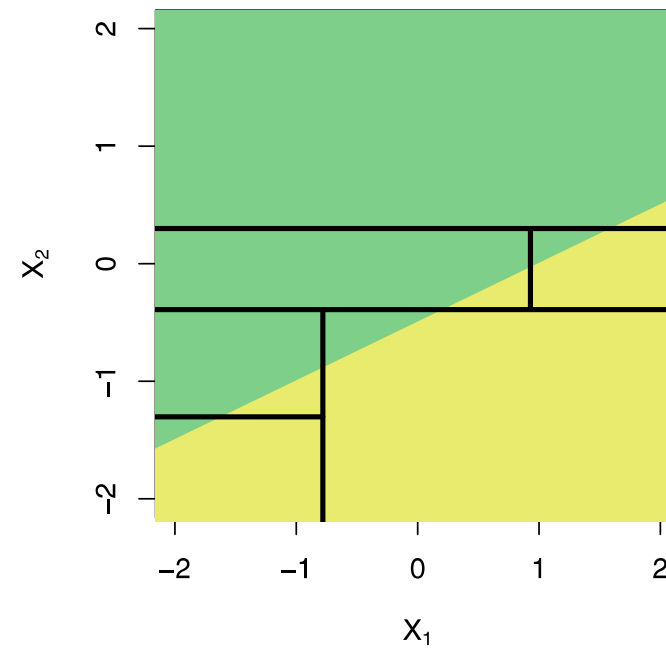
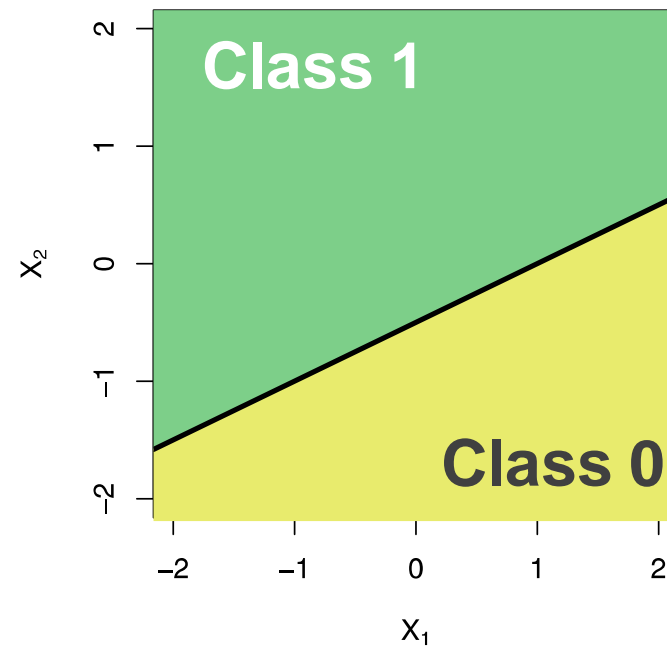


Performance



James et al., An Introduction to Statistical Learning

# Linear model



# Classification Tree

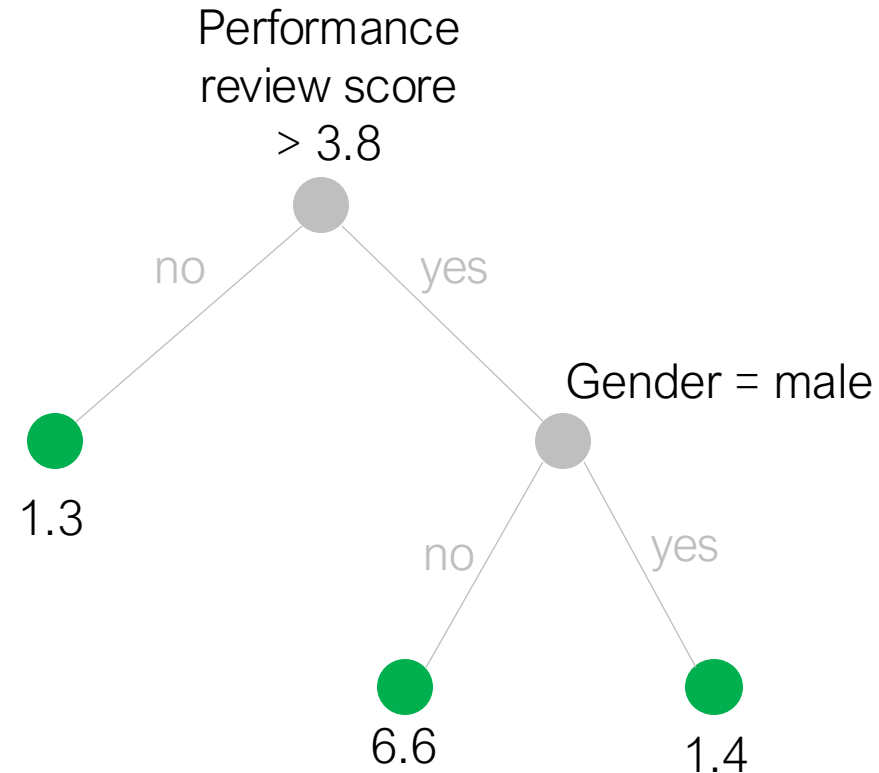
Struggle when the boundary is not parallel to an axis

...nonlinear feature transforms could help...

# Pros/Cons

**Numerical** data

**Categorical** data



## Pros:

Trees easily handle multiple types of data

Trees are easy to interpret

## Cons:

Trees do not typically have the same level of predictive accuracy of other methods

Tend to overfit  
(have high variance)

# Ensemble learning

Combining models to improve performance beyond any individual model alone

Bagging (bootstrap aggregation)

Random forests (tree-specific modification of bagging)

Gradient boosting

# Reducing Variance or Bias through ensembles

## Bagging

Models in ensemble:

high variance, low bias  
(i.e. overfit models)

Effect of aggregating:

Reduce variance through averaging output

## Boosting

high bias, low variance  
(i.e. underfit models, “weak learners”)

Reduce bias through sequentially fitting models to previous model errors

# Bagging

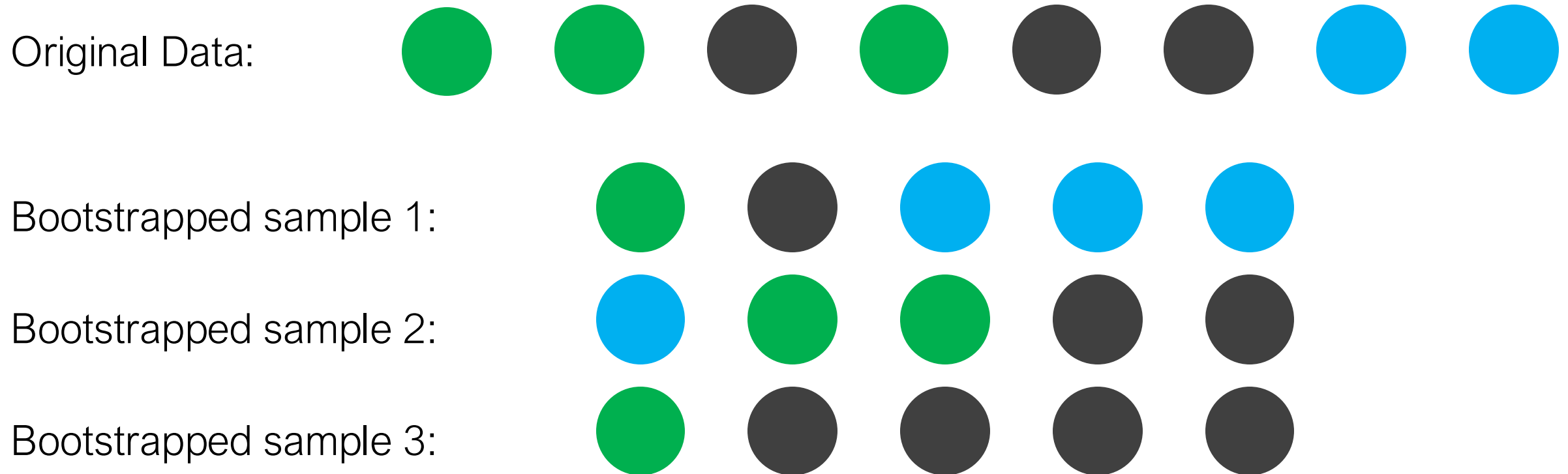
= Bootstrap aggregation

we cannot have so many copies of the model. we could only have one

model is overfit but different

Trees **overfit** (have high variance). Averaging over observations **reduces variance**

Recall bootstrap sampling (sampling with replacement):



# Bagging

Decision trees are a popular  
base model

Bootstrap aggregation

- 1 Create a random bootstrap sample from the training data
- 2 Train a model on that bootstrap sample and call it  $\hat{f}_b(\mathbf{x})$
- 3 Repeat 1 and 2 until we have  $B$  models trained on different bootstrap samples
- 4 Take the average of the output for our new model estimate:

$$\hat{f}_{bag}(\mathbf{x}) = \frac{1}{B} \sum_{b=1}^B \hat{f}_b(\mathbf{x})$$

(for classification models we can get the average class confidence or take a majority vote)

# Bagging

Tree Number:

1

2

3

4

Observations  
Included:  
(out of 1-9)

[1,2,3,3,8]

[1,2,4,7,7]

[1,5,6,8,9]

[2,2,2,4,9]

Features list:

[A, B, C, D]

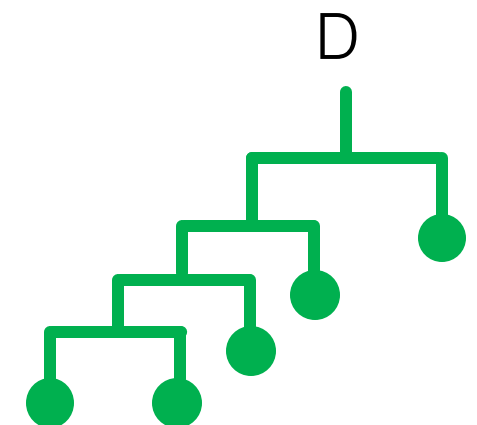
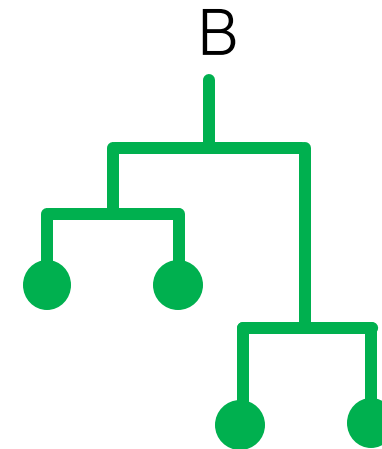
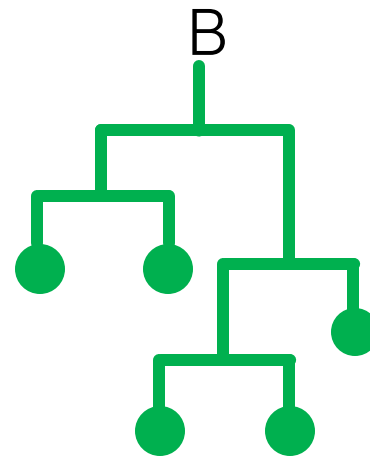
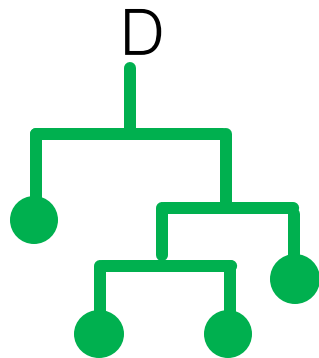
[A, B, C, D]

[A, B, C, D]

[A, B, C, D]

First split:

Trees:





# Variable Importance

Decision trees are very interpretable, but this is lost with bagging

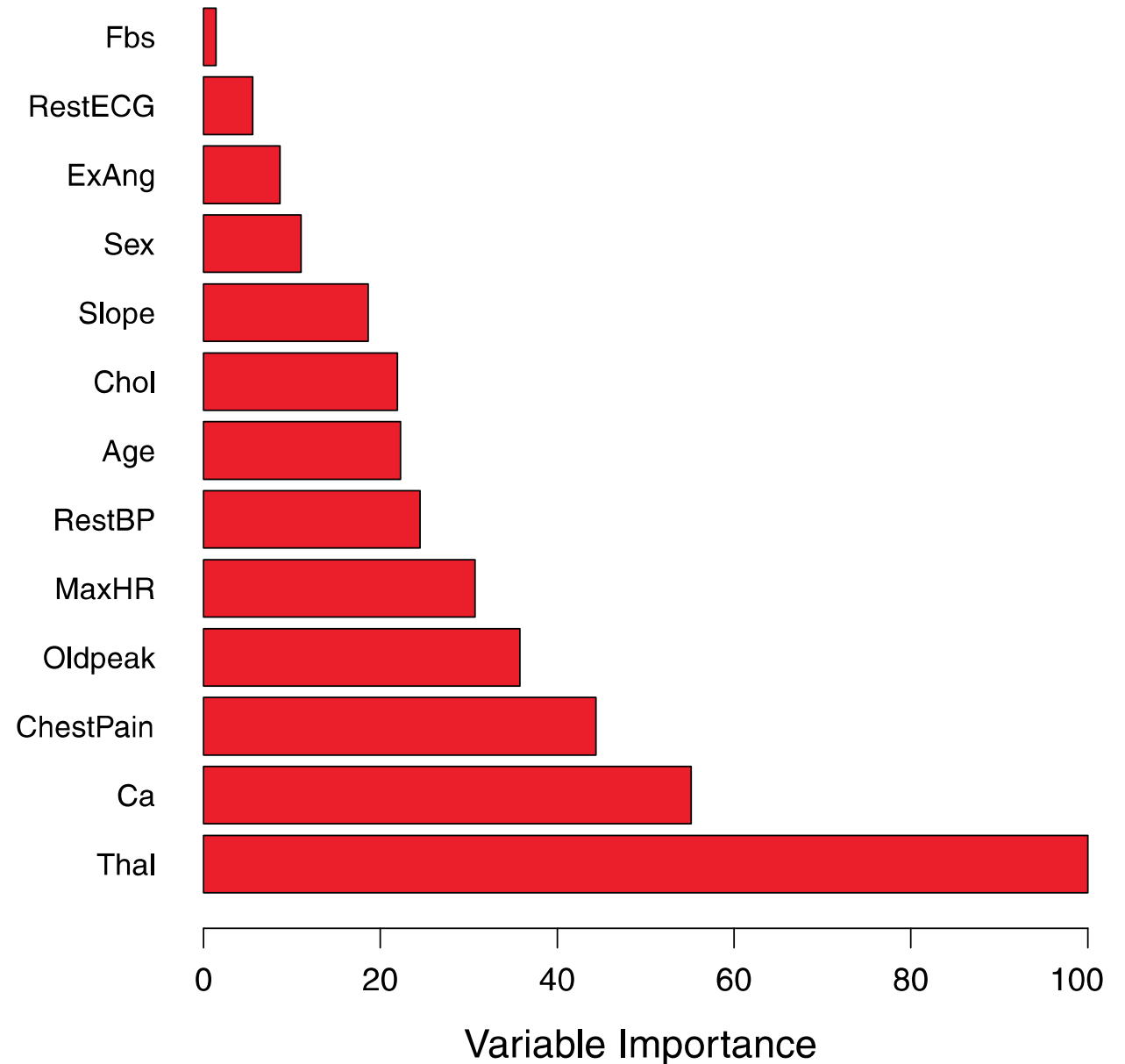
We can construct another measure called “variable importance” to **compare feature contributions**

1

Calculate the total amount the error (or impurity) decreased by splitting on each feature.

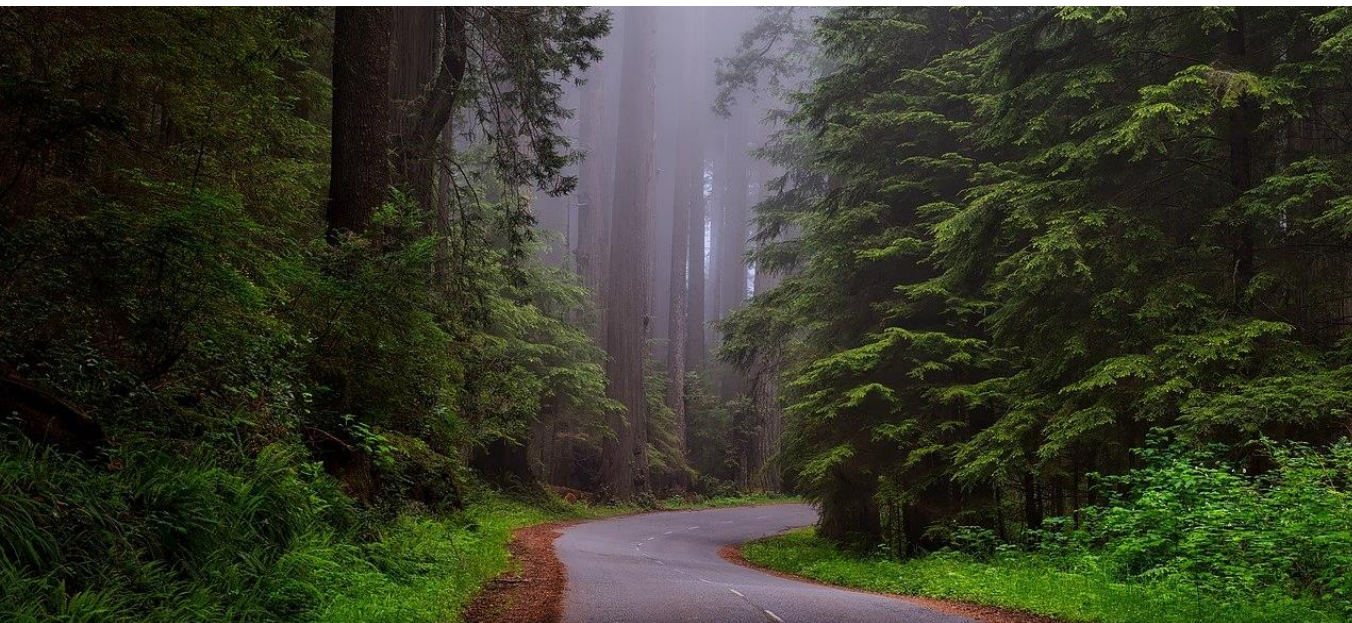
2

Average over all the trees resulting from bagging





# Random Forests





# Random Forests

A **small tweak on bagging**

Random forests  
**decorrelate**  
the bagged trees

Decision trees are constructed greedily

This can lead to highly correlated trees

“Strong” features will typically be split before moderately strong predictors.

Each time a split is considered, a **random subset of  $m$  features** is selected as candidates from the full set of  $p$  features

Typically chose:  $m = \sqrt{p}$

(If  $m = p$ , then we would be back to the bagging approach)

# Bagging

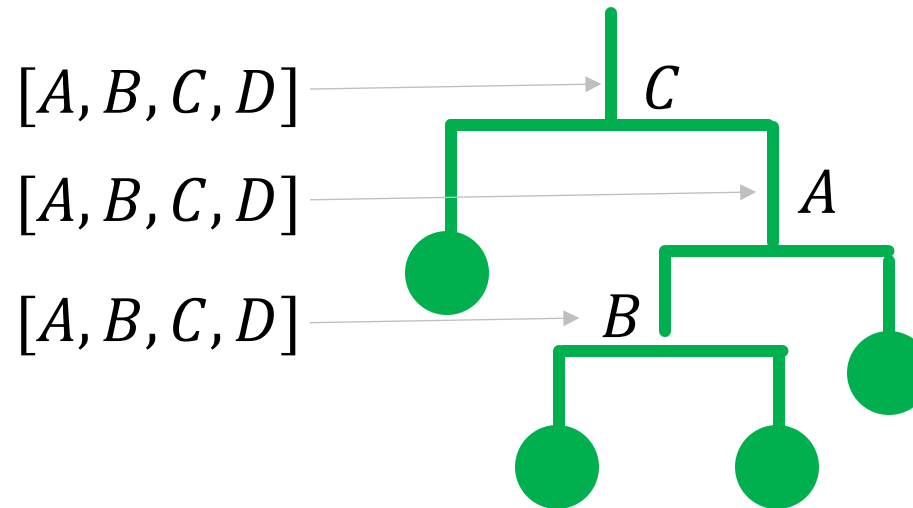
Observations  
Included:  
(out of 1-9)

Features list:

Feature options for  
each split:

$[1, 2, 3, 3, 8]$

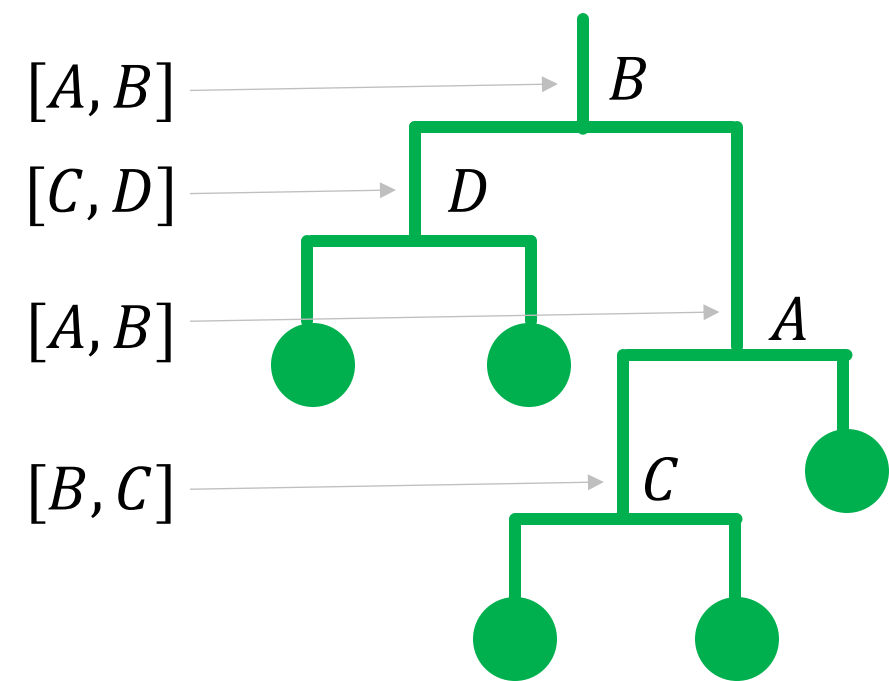
$[A, B, C, D]$



# Random forests

$[1, 2, 3, 3, 8]$

$[A, B, C, D]$

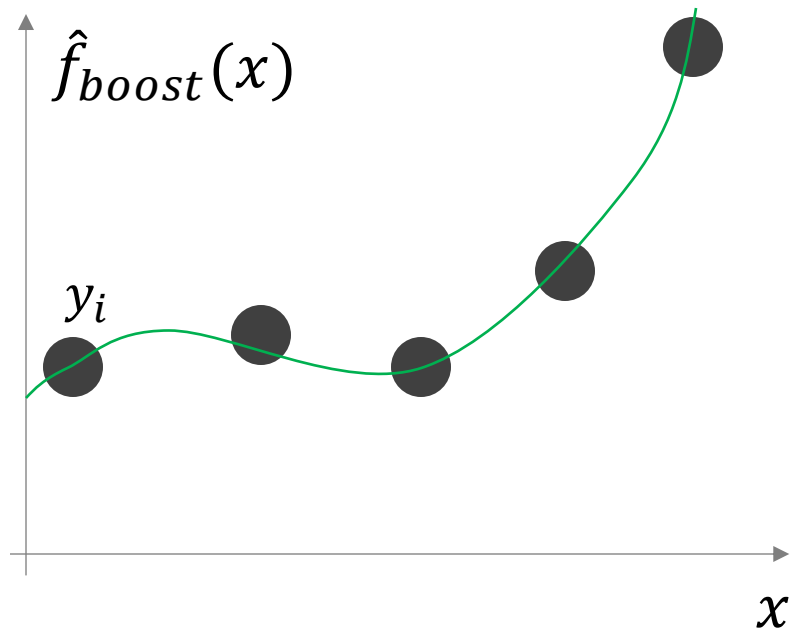
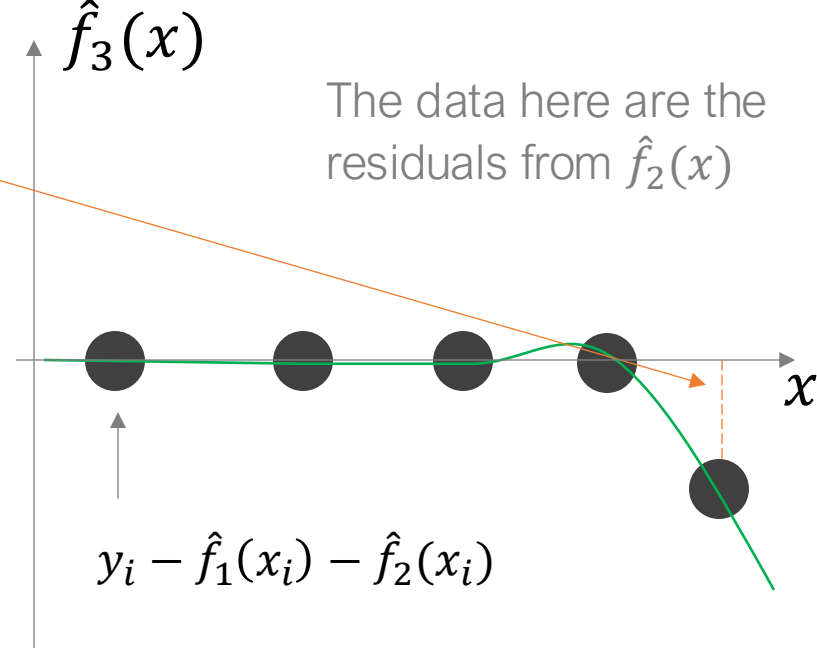
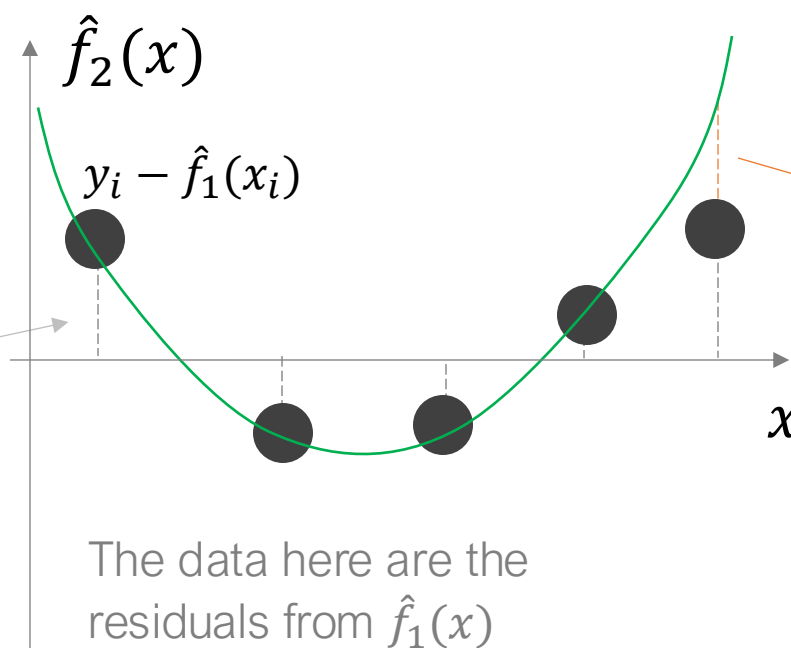
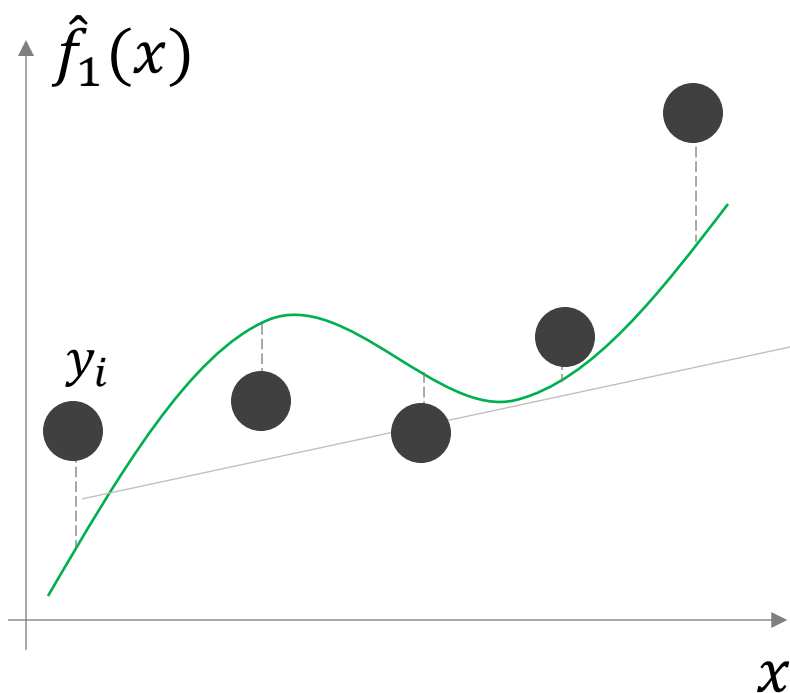


# Boosting

Decision trees are a popular  
base model

**Bagging** created trees that were designed to be as independent as possible

**Boosting** involves building trees **sequentially**, each building on the errors of the last



We build 连续的 consecutive models, each fit to the residuals of the last model

We sum models output to get the boosted prediction  

$$\hat{f}_{boost}(x) = \hat{f}_1(x) + \hat{f}_2(x) + \hat{f}_3(x)$$

# Boosting

# Boosting for regression trees

- 1 Select the number of models to train,  $B$ , and learning rate  $\lambda$
- 2 Set  $\hat{f}(\mathbf{x}) = 0$  and  $r_i = y_i$  for all the training data
- 3 Fit a tree,  $\hat{f}_b(\mathbf{x})$  to the residuals,  $r_i$  (with  $d$  splits)
- 4 Update  $\hat{f}(\mathbf{x}) \leftarrow \hat{f}(\mathbf{x}) + \lambda \hat{f}_b(\mathbf{x})$
- 5 Update the residuals  $r_i \leftarrow r_i - \lambda \hat{f}_b(\mathbf{x}_i)$
- 6 Output the boosted model: 
$$\hat{f}(\mathbf{x}) = \sum_{b=1}^B \lambda \hat{f}_b(\mathbf{x})$$

$\lambda$  slows down the learning process to avoid overfitting

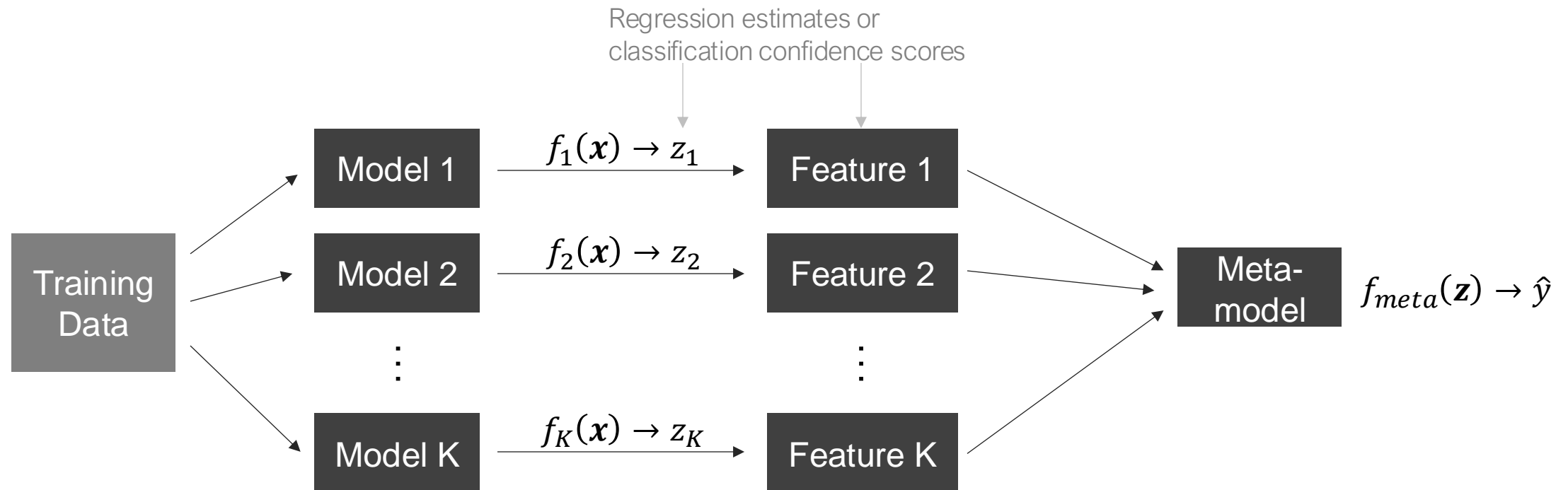
Often this is just a small number of splits (a stump)

Repeat  $B$  times

# Model Stacking

Train multiple supervised learning techniques (could be different models)

THEN Train a supervised learning technique that includes the **outputs** of the other models as **features**





# Supervised Learning Techniques

Covered so far

- Linear Regression
- K-Nearest Neighbors
- Perceptron
- Logistic Regression
- Linear Discriminant Analysis
- Quadratic Discriminant Analysis
- Naïve Bayes
- Decision Trees and Random Forests
- Ensemble methods (bagging, boosting, stacking)

Can be used with numerous machine learning techniques, often CART

Appropriate for:  
● Classification  
● Regression