Forecasting Methods and Applications
Applied Data Analysis School

Lecture 3

Exponential Smoothing Methods

November 2021

Cristina Amado
University of Minho

Outline of the lecture:

- Smoothing methods
- Averaging methods
- Simple exponential smoothing
- Holt's linear trend method
- Holt-Winters seasonal method
- Applications

References:

- Diebold, F. X. (2007), Elements of Forecasting, 4th edition, South-Western College Publishing.
- Montgomery, D. C., Jennings, C. L. and Kulahci, M. (2015), Introduction to Time Series Analysis and Forecasting, 2nd edition, Wiley.
- Brockwell, P. J. and Davis, R. A. (2002), Introduction to Time Series and Forecasting, 2nd edition, Springer-Verlag, New York.

Smoothing methods

- In this lecture we introduce models applicable to time series data with seasonal, trend, or both seasonal and trend component and stationary data.
- Data can often be seen as consisting of two distinct components: signal and noise. We may wish to smooth the data in order to see a trend in the signal or the general pattern of the data.
- Smoothing is the technique to separate the signal and the noise as much as possible and it acts as a filter to obtain an estimate for the signal.

Smoothing methods

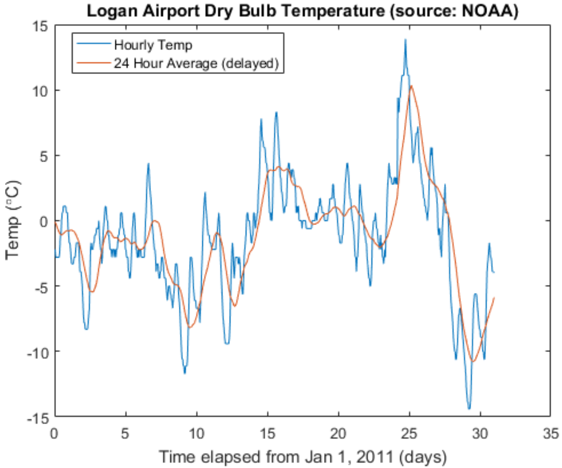

Figure 1: Smoothing the hourly temperature (in Celsius) at Logan Airport for January, 2011.

Averaging methods

Averaging methods are suitable for stationary time series data where the series is in equilibrium around a constant value with a constant variance over time.

Average method:

- All future forecasts are equal to a simple average of the observed data

$$\widehat{y}_{T+1|T} = \frac{1}{T}\sum_{t=1}^{T} y_t$$

  for $h = 1, 2, \dots$

- Hence, the **average method** assumes that all observations are of equal importance and they are given equal weight when computing forecasts.
- This method is appropriate when there is no trend or seasonality.

Averaging methods

Moving average:

- The moving average method is based on an average of observations where equal weights are assigned to each observation to smooth out short-term irregularity in the series.
- The moving average method is unable to capture peaks and troughs of the series, and it obviously fails to deal with non-stationary data, trends and seasonality.
- In the moving average of order k, MA(k), the one-step ahead forecast is

$$\widehat{y}_{T+1|T} = \frac{y_T + y_{T-1} + \ldots + y_{T-k+1}}{k} = \frac{1}{k} \sum_{t=T-k+1}^{T} y_t$$

Averaging methods

Moving average:

- The moving average for time period t is the mean of the k most recent observations, where the constant k is determined by the user.
- When the series is moving down persistently, the moving average forecast tends to over-predict, while when the series is going up, the moving average forecast tends to under-predict the actual value.

Example: For quarterly data, a four-quarter moving average, MA(4), eliminates or averages out seasonal effects.

Note:

- The smaller k, the more weight is given to recent periods; The greater k, the less weight is given to more recent periods.
- A large k is desirable when there are wide fluctuations in the series; A small k is desirable when there are sudden shifts in the level of series.

Averaging methods

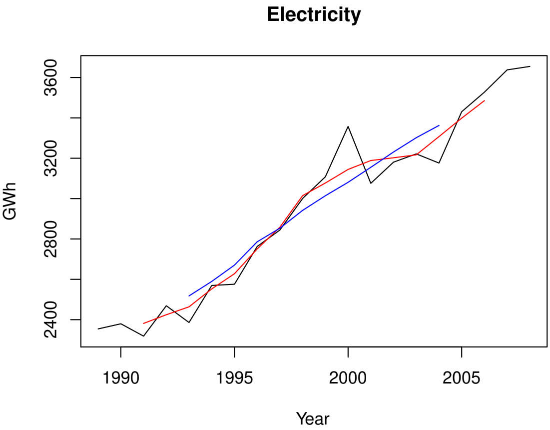

Figure 2: Residential electricity sales (black) along with the MA(5) and MA(9) estimate of the trend-cycle (red).

Simple exponential smoothing

Several methods:

- Simple exponential smoothing
- Holt's linear trend method
- Holt-Winters seasonal method

Simple exponential smoothing

- The simple moving average method assigns equal weights $(1/k)$ to all k data points.
- However, recent observations provide more relevant information than past observations. Thus, we need a weighting scheme that assigns decreasing weights to more distant observations.
- Forecasts produced are weighted averages of past observations, with the weights decaying exponentially as the observations become more distant.

Simple exponential smoothing

- Appropriate for data with no trend or seasonal pattern, but the mean of the time series is slowly changing over time.

- Formally, the exponential smoothing equation is

$$\widehat{y}_{t+1|t} = \alpha y_t + (1 - \alpha)\widehat{y}_{t|t-1}$$

  for $t = 1, ..., T$, where $0 < \alpha < 1$ is the smoothing parameter.

- Smaller α leads to less adjustment that takes place in the forecast in the direction of the previous data point, so the one-step within-sample forecasts are smoother than for larger α.

- The process has to start at some first forecast of y_1, denoted by $\widehat{y}_0$. Commonly, set $\widehat{y}_0 = y_1$ or $\widehat{y}_0 = \bar{y}$.

Simple exponential smoothing

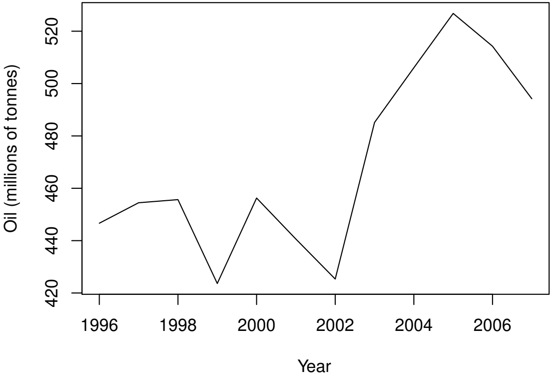

Figure 3: Oil production in Saudi Arabia from 1996 to 2007.

Simple exponential smoothing

- Then,

$$
\begin{aligned}
\widehat{y}_{2|1} &= \alpha y_1 + (1-\alpha)\widehat{y}_0 \\
\widehat{y}_{3|2} &= \alpha y_2 + (1-\alpha)\widehat{y}_{2|1} \\
\widehat{y}_{4|3} &= \alpha y_3 + (1-\alpha)\widehat{y}_{3|2} \\
& \ldots \\
\widehat{y}_{T+1|T} &= \alpha y_T + (1-\alpha)\widehat{y}_{T|T-1}
\end{aligned}
$$

- By substitution:

$$\widehat{y}_{T+1|T} = \alpha y_T + \alpha(1-\alpha) y_{T-1} + \alpha(1-\alpha)^2 y_{T-2} + \ldots + (1-\alpha)^T \widehat{y}_0$$

where $0 < \alpha < 1$.

Simple exponential smoothing

- Note that forecasts are calculated using weighted averages where smallest weights are associated with the oldest observations.
- The component form of simple exponential smoothing is given by:

$$\begin{aligned} \text{Forecast equation} \quad &: \quad \widehat{y}_{t+1|t} = \ell_t \\ \text{Smoothing equation} \quad &: \quad \ell_t = \alpha y_t + (1-\alpha)\ell_{t-1} \end{aligned}$$

  where is the level (or the smoothest value) of the series at time t, and $0 < \alpha < 1$. This implies

$$\widehat{y}_{T+1|T} = \ell_t,$$

  i.e., the most recent level.
- Simple exponential smoothing has a flat forecast function, so for longer forecast horizons

$$\widehat{y}_{T+h|T} = \widehat{y}_{T+1|T} = \ell_t, \; h = 2, 3, \ldots$$

Simple exponential smoothing

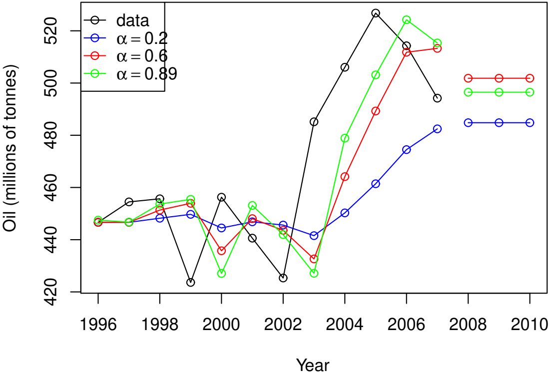

Figure 4: Simple exponential smoothing applied to oil production in Saudi Arabia.

Holt's linear trend method

- The Holt's linear method is an extension of the simple exponential smoothing to allow forecasting data with a trend.

- If a time series is slowly changing over time approximately at a fixed rate, then it may be described by the linear trend model

$$y_t = \beta_0 + \beta_1 T + \varepsilon_t, \;\; T = 1, 2, \ldots$$

- This method involves a forecast equation and two smoothing equations (one for the level and one for the trend):

$$\begin{aligned}
\text{Forecast equation} \quad &: \quad \widehat{y}_{t+h|t} = \ell_t + hT_t \\
\text{Level equation} \quad &: \quad \ell_t = \alpha y_t + (1-\alpha)(\ell_{t-1} + T_{t-1}) \\
\text{Trend equation} \quad &: \quad T_t = \beta(\ell_t - \ell_{t-1}) + (1-\beta) T_{t-1}
\end{aligned}$$

where $0 < \alpha < 1$ and $0 < \beta < 1$.

Holt's linear trend method

where ℓ_t is an estimate of the level of the series at time t, T_t is the estimate of the trend at time t, α is the smoothing parameter for the level, and β is the smoothing parameter for the trend.

- Large weights result in more rapid changes in the component. Small weights result in less rapid changes.
- The h-step-ahead forecast is equal to the last estimated level plus h times the last estimated trend value. Thus, the forecasts are no longer flat but trending and a linear function of h.

Holt's linear trend method

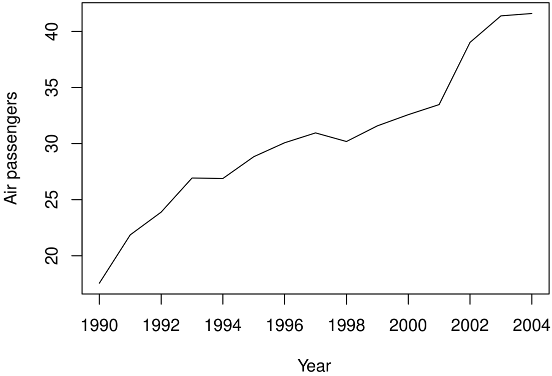

Figure 5: Air Passengers in an Australia airline (thousands of passengers).

Holt's linear trend method

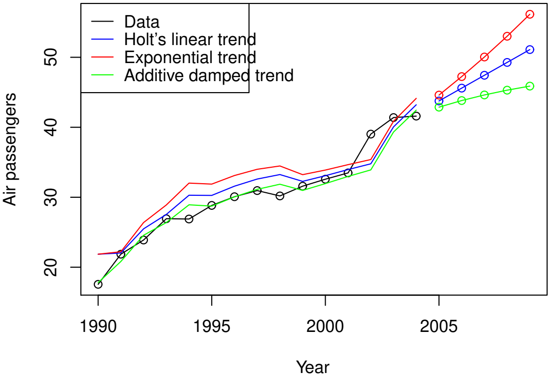

Figure 6: Forecasting air passengers in an Australia airline.

Holt-Winters seasonal method

- **Holt-Winter seasonal method** is the second extension of the simple exponential smoothing model.
- It is useful for data that exhibit both trend and seasonality.
- Two Holt-Winters methods:
  - **Additive Holt-Winters method:** for time series with roughly constant seasonal variations.
  - **Multiplicative Holt-Winters method:** for time series when the seasonal variations with changing seasonal variations.

Holt-Winters seasonal method

Additive Holt-Winters method:

- This method useful to forecasting a time series that can be described by the equation:

$$y_t = \beta_0 + \beta_1 T + S_t + \varepsilon_t, \;\; T = 1, 2, \ldots$$

  where $S_t =$ seasonal pattern and $\varepsilon_t =$ irregular component.

- It is appropriate when a time series has a linear trend with a constant seasonal pattern and the seasonal pattern may be slowly changing over time.

Holt-Winters seasonal method

Additive Holt-Winters method:

The component form for the additive method is:

$$\begin{aligned}
\text{Forecast equation} \quad &: \quad \widehat{y}_{t+h|t} = \ell_t + hT_t + S_{t+h-s} \\
\text{Level equation} \quad &: \quad \ell_t = \alpha(y_t - S_{t-s}) + (1-\alpha)(\ell_{t-1} + T_{t-1}) \\
\text{Trend equation} \quad &: \quad T_t = \beta(\ell_t - \ell_{t-1}) + (1-\beta)T_{t-1} \\
\text{Seasonality equation} \quad &: \quad S_t = \gamma(y_t - \ell_t) + (1-\gamma)S_{t-s}
\end{aligned}$$

where $0 < \alpha < 1$, $0 < \beta < 1$, and $0 < \gamma < 1$, and s is the number of

seasons in a year ($s = 12$ for monthly data, and $s = 4$ for quarterly data).

To initialize the seasonal indices we use

$$S_1 = y_1 - \ell_s,\ S_2 = y_2 - \ell_s, \ldots, S_s = y_s - \ell_s$$

Holt-Winters seasonal method

Multiplicative Holt-Winters method:

- This method useful to forecasting a time series that can be described by the equation:

$$y_t = \beta_0 + \beta_1 T \times S_t \times \varepsilon_t, \;\; T = 1, 2, \ldots$$

  where $S_t =$ seasonal pattern and $\varepsilon_t =$ irregular component.

- This method is appropriate when a time series has a linear trend with a multiplicative seasonal pattern, and the seasonal pattern may be slowly changing over time.

Holt-Winters seasonal method

Multiplicative Holt-Winters method:

The component form for the multiplicative method is:

$$
\begin{aligned}
\text{Forecast equation} &: \quad \widehat{y}_{t+h|t} = (\ell_t + hT_t)S_{t+h-s} \\
\text{Exponentially smooth series} &: \quad \ell_t = \alpha \frac{y_t}{S_{t-s}} + (1-\alpha)(\ell_{t-1} + T_{t-1}) \\
\text{Trend equation} &: \quad T_t = \beta(\ell_t - \ell_{t-1}) + (1-\beta)\,T_{t-1} \\
\text{Seasonality equation} &: \quad S_t = \gamma \frac{y_t}{\ell_t} + (1-\gamma)S_{t-s}
\end{aligned}
$$

where $h =$ number of periods in the forecast lead period, $\ell_t =$ level of the series, $T_t =$ trend estimate, $S_t =$ estimate of the seasonal component, $s =$ number of periods in the season.

Holt-Winters seasonal method

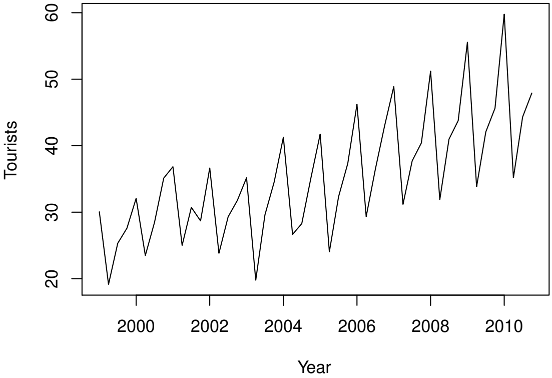

Figure 7: International visitor nights in Australia for 2005Q1–2010Q4.

Holt-Winters seasonal method

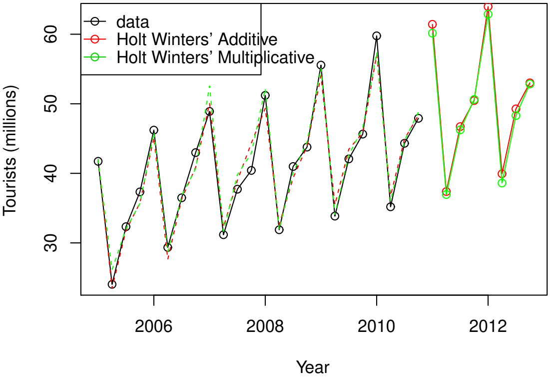

Figure 8: Forecasting international visitor nights in Australia.