Dartmouth COSC 89.18/189.02 Computational Methods for Physical Systems, Winter 2019

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Course Notes 3: Particle Fluid (SPH)

In particle fluid simulation, the fluid motion is described by a set of particles. Each particle carries a set of physical properties, including position $\vec{x}$, velocity $\vec{u}$, mass m, density ρ, and pressure p. For a single particle i, we conduct a smooth approximation of its physical property A_i by considering the weighed sum of all its neighboring particles j (including itself) within a kernel radius h:

$$A_i^s = \sum_j A_j V_j W(|\vec{x}_i - \vec{x}_j|, h)$$

with $V_j = m_j/\rho_j$.

The gradient of A can be further approximated by the weighted sum of the kernel gradient:

$$\nabla A_i^s = \sum A_j V_j \nabla W(|\vec{x}_i - \vec{x}_j|, h)$$

And similarly, the Laplacian is expressed as the weighted sum of the kernel Laplacian:

$$\nabla^2 A_i^s = \sum A_j V_j \nabla^2 W(|\vec{x}_i - \vec{x}_j|, h)$$

1. Neighbor Search

A spatial hashing algorithm is used to search the neighbors with a kernel radius h. A background grid is built conceptually to discretize the domain. The origin of the grid can be arbitrary (e.g., zero). The size of the grid cell is comparable to the kernel radius h to ensure that a one-ring search (3^2 cells in 2D and 3^3 cells in 3D) can get all the neighbors within h.

A hash table is initialized to store the occupied grid cells and the list of particle indices inside each cell in memory. The table key is the cell coordinate (Vector2i in 2D and Vector3i in 3D), and the table item is the list of indices). Every time when a point is inserted to the hash table, the coordinate of its occupied cell is calculated first. If the coordinate exists in the table already, we append the particle index to the end of its list; otherwise we add the coordinate as a new key and insert the particle index as the first element in its list.

We refresh the hash table (by removing and re-inserting all the particles) at the beginning of each time step.

Then, the neighbors for each particle within h can be found by searching the 9-(or 27) neighboring cells. We precompute the neighbors and store them in memory at the beginning of each time step.

2. Fluid Force Calculation

The momentum equation for the particle fluid is:

$$\rho \frac{D\vec{u}}{Dt} = -\nabla p + \mu \nabla \cdot \nabla \vec{u} + \rho \vec{g}$$

We first approximate the density of each particle by:

$$\rho_i = \sum_j m_j W_{ij}$$

Here we use W_{ij} as the abbreviation of $W(\vec{x}_i - \vec{x}_j, h)$ (see Section 3 below).

Then the pressure for each particle is calculated as a linear function of its density (to roughly approximate the incompressibility):

$$p_i = k(\rho_i - \rho_0)$$

The smoothed pressure force is calculated as:

$$\vec{f}_i^p = -\sum_j \frac{(p_i + p_j)}{2} \frac{m_j}{\rho_j} \nabla W_{ij}$$

The smoothed viscosity force is:

$$\vec{f}_i^v = \mu \sum_j (\vec{u}_j - \vec{u}_i) \frac{m_j}{\rho_j} \nabla^2 W_{ij}$$

And the body force is:

$$\vec{f}_i^g = \rho \vec{g}$$

We also consider a boundary penalty force if a particle is intersecting with some implicit geometry ϕ (e.g., the container):

$$\vec{f}_i^b = k_s(\phi(\vec{c}_i) - r_i)(-\nabla\phi)$$

The total force applied on a fluid particle can be expressed as:

$$\vec{f}_i = \vec{f}_i^p + \vec{f}_i^v + \vec{f}_i^g + \vec{f}_i^b$$

This total force can be used to update the velocity and position of each particle using the explicit Euler time integration (remember that the force is expressed in a volume weighted fashion so we need to divide f by density instead of mass when calculating the acceleration).

3. Kernel function

We can use different kernel functions to approximate different terms. In particular, we use the Spiky kernel to approximate every term in the momentum equation except the viscosity:

$$W_{spiky}(\vec{r}, h) = \frac{15}{\pi h^6} \begin{cases} (h - |\vec{r}|)^3 & 0 \le r \le h \\ 0 & otherwise \end{cases}$$

And, we use the viscosity kernel to approximate the viscosity force:

$$W_{vis}(\vec{r}, h) = \frac{15}{2\pi h^3} \begin{cases} -\frac{|\vec{r}|^3}{2h^3} + \frac{|\vec{r}|^2}{h^2} + \frac{h}{2|\vec{r}|} - 1 & 0 \le r \le h \\ 0 & otherwise \end{cases}$$

For particle i and its neighbor j, we define $W_{ij} = W(\vec{x}_i - \vec{x}_j, h)$, with the variable $\vec{r}$ defined as $\vec{x}_i - \vec{x}_j$.

The gradient(or Laplacian) of W is defined as the gradient (or Laplacian) of W w.r.t. $\vec{x}_i$ (remember that the gradient is a vector and the Laplacian is a scalar).