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A Practical Method for Decomposition of the Essential Matrix

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Abstract

One well-known approach for reconstruction of a 3D object from two images obtained by calibrated cameras is based on the essential matrix computation. The extrinsic camera parameters can be found by decomposing the essential matrix into skew-symmetric and rotation parts, and then the reconstruction problem can be solved by triangulation. In this paper we present a direct way for such a decomposition based only on the operations with the vector-rows of the essential matrix. The obtained new identities for scalar products of these vector-rows are naturally integrated in a 3D reconstruction scheme.

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1 Introduction

Reconstruction of a 3D object from two perspective images is a significant task in Photogrammetry, Computer Vision, Robotics and related ields. This

3D reconstruction problem can be solved using triangulation, if the perspective images are obtained by cameras with known intrinsic and extrinsic parameters. The connection between the corresponding points on the two images can be expressed in terms of homogeneous coordinates. In case of known intrinsic parameters and unknown extrinsic parameters, this connection is represented by the essential matrix. The estimation of the essential matrix is possible, if at least five pairs of corresponding points are given. The extrinsic parameters can be found by decomposing the essential matrix, and then, the reconstruction problem is solved by triangulation. The key role in this scheme plays the decomposition of the essential matrix into the product of a skew-symmetric matrix and a rotation matrix. A detailed treatment of the 3D reconstruction problem can be found in many books (see for example [1] and [4]).

There are two main approaches to decompose a given essential matrix into skew-symmetric and rotation factors. The first approach is based on singular value decomposition (SVD) of the essential matrix. An excellent description is given in [4] and [10]. The second approach avoids SVD and uses specific relations of the essential matrix with its skew-symmetric and rotation factors. This point of view has been introduced by Horn in [5]. Recent applications are given in [8]. The aim of this paper is to give a direct way for a mentioned decomposition of the essential matrix. We extract the information for the two camera frames from the essential matrix. The rigid motion between two camera frames is a composition of a translation and a rotation which correspond to the skew-symmetric matrix and the rotation matrix, respectively. Using this we can express the coordinates of the points in both images with respect to one and the same camera frame. Then we may solve 3D reconstruction problem. The paper is organized as follows. The next section contains some basic definitions, properties and facts from perspective projection and epipolar geometry. In Section 3., main relations between the vector-rows of the essential matrix and the elements of its skew-symmetric and rotation parts are formulated and proved. Section 4 is devoted to a novel algorithm for a factorization of the essential matrix. The paper concludes with final remarks.

2 Two View Geometry

In this section we recall basic definitions and notations concerning the essential matrix. More details and historical remarks can be found in [4, Ch. 9] .

2.1 Perspective Projection

Let us consider the perspective (or central) projection in the Euclidean 3-space $\mathbb{E}^3$ determined by the projection center $\overline{S} \in \mathbb{E}^3$ and the image (projection) plane $\omega \subset \mathbb{E}^3$ ($\overline{S} \notin \omega$). In other words, the central projection $\overline{S}, \omega$ maps an arbitrary point $\overline{M} \in \mathbb{E}^3 \setminus \{\overline{S}\}$ to the intersecting point $M = \overline{S}\,\overline{M} \cap \omega$. It is clear that M is a point at infinity, if and only if, the line $\overline{S}\,\overline{M}$ is parallel to the plane ω. Suppose that homogeneous coordinates of the point $\overline{M}$ with respect to a fixed Euclidean coordinate system in the space $\mathbb{E}^3$ are presented by the vector-column $\overline{\mathbf{M}} = (\overline{X}, \overline{Y}, \overline{Z}, 1)^T \in \mathbb{R}^4$, and the homogeneous coordinates of the point M with respect to a fixed Euclidean coordinate system in the plane ω are presented by the vector-column $\mathbf{m} = (m_1, m_2, m_3)^T \in \mathbb{R}^3 \setminus \{(0,0,0)\}$ (m_3 is equal to either 1 or 0). Then there exists a 3×4 matrix $\mathbf{P}$ with a non-singular left 3×3-submatrix such that

$$\sigma \mathbf{M} = \mathbf{P}\,\overline{\mathbf{M}}\,, \tag{1}$$

where σ is a non-zero factor. The matrix $\mathbf{P}$ is called the projection matrix of the central projection. There is a Cartesian coordinate system $\overline{O}\overline{X}\,\overline{Y}\,\overline{Z}$ in the Euclidean 3-space, so-called the camera frame, which is related to the considered central projection. Its origin $\overline{O}$ is placed at the center $\overline{S}$ and the $\overline{Z}$-axis coincides with the principle ray of the camera, i.e. it is perpendicular to the plane ω and its orientation is from $\overline{S}$ to ω. The pedal point H of $\overline{S}$ with respect to ω is called a principal point, and the distance $\left|\overline{S}H\right| = d$ between $\overline{S}$ and ω is the focal length. In the image plane, there is an associated Cartesian coordinate system Oxy to the camera frame. Its origin O is placed at the point H and the axes x and y are parallel to $\overline{X}$ and $\overline{Y}$, respectively. Under above assumptions, the projection matrix possesses the simplest form, i.e.

$$\mathbf{P} = \begin{bmatrix} d & 0 & 0 & 0 \\ 0 & d & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Let us consider a more general case. Assume that: (i) the coordinate vector $\overline{\mathbf{M}} = (\overline{X}, \overline{Y}, \overline{Z}, 1)^T$ is relative to an arbitrary world coordinate system,
(ii) the coordinate vector $\mathbf{M} = (m_1, m_2, m_3)^T$ is relative to an arbitrary plane coordinate system in ω. Then, the projection matrix can be decomposed as

$$\mathbf{P} = \begin{bmatrix} d\kappa_x & s & x_0 \\ 0 & d\kappa_y & y_0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} r_{11} & r_{12} & r_{13} & t_1 \\ r_{21} & r_{22} & r_{23} & t_2 \\ r_{31} & r_{32} & r_{33} & t_3 \end{bmatrix} \tag{2}$$

where: κ_x, κ_y are scale factors in the x and y direction, respectively; s is a skew parameter; x_0 , y_0 are the coordinates of the principal point; $r_{i.j}$ form a rotation matrix and t_1, t_2, t_3 are coordinates of a vector. The upper triangular matrix

$$\mathbf{K} = \begin{bmatrix} d\kappa_x & s & x_0 \\ 0 & d\kappa_y & y_0 \\ 0 & 0 & 1 \end{bmatrix}$$

is known as a calibration matrix and $d\kappa_x$, $d\kappa_y$, s, x_0, y_0 are called intrinsic camera parameters. The rotation matrix $\mathbf{R} = [r_{i.j}]$ and the translation vector $\mathbf{t} = (t_1, t_2, t_3)^T$ in (2) are the extrinsic parameters expressing the relationship between the world coordinate system and the camera frame. Then, the equation (1) can be rewritten in the form $\sigma\mathbf{m} = \mathbf{K}\,[\mathbf{R}|\mathbf{t}]\overline{\mathbf{M}}$. Since the calibration matrix is non-singular, it is determined the non-zero vector $\mathbf{m}^{\mathcal{N}} = \mathbf{K}^{-1}\mathbf{m} = [\mathbf{R}|\mathbf{t}]\overline{\mathbf{M}}$ representing the image point in normalized coordinates. The 3×4-matrix $\mathbf{K}^{-1}\mathbf{P} = [\mathbf{R}|\mathbf{t}]$ is called a normalized camera matrix.

2.2 Epipolar Geometry

Let us consider two perspective projections, or equivalently two camera systems, determined by the pairs $(\overline{S}, \omega)$ and $(\overline{S'}, \omega')$ (see Figure 1). Assume that their normalized camera matrices $\mathbf{P}$ and $\mathbf{P'}$ are related to the first camera frame. Then, the first matrix is $\mathbf{P} = [\mathbf{I}\big|\mathbf{o}]$, where $\mathbf{I}$ is the 3×3 identity matrix and $\mathbf{o} = (0,0,0)^T$ is three-dimensional zero vector. The second matrix expresses the position of the second camera with respect to the first one, and therefore it can be written in the form $\mathbf{P'} = [\mathbf{R}\big|\mathbf{t}]$, where $\mathbf{R}$ is the 3×3 rotation matrix and $\mathbf{t} = \overrightarrow{\overline{S}\,\overline{S'}}$ is a three-dimensional translation vector. If we suppose

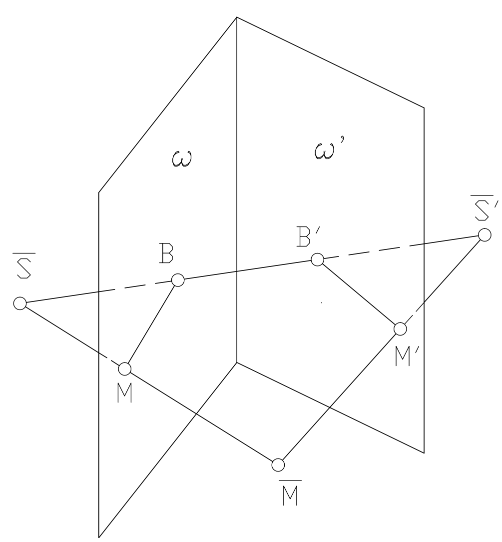

Figure 1: Epipolar Geometry

that $\overline{M} \in \mathbb{E}^3$ is an arbitrary point, different from $\overline{S}$ and $\overline{S'}$, then we get the intersection points $M = \overline{S}\,\overline{M} \bigcap \omega$, $M' = \overline{S'}\,\overline{M} \bigcap \omega'$ which are its perspective projections in the image planes ω and ω'. Thus, a pair of corresponding points

$(M,\, M')$ is associated to the point $\overline{M} \in \mathbb{E}^3 \setminus \{\overline{S}, \overline{S'}\}$. Let the vectors

$$\mathbf{m} = (m_1, m_2, m_3)^T \quad \texttt{and} \quad \mathbf{m}' = (m_1', m_2', m_3')^T \tag{3}$$

represent normalized coordinates of M and M'. Then, there exists a 3×3 matrix $\mathbf{E}$ such that $\texttt{rank}(\mathbf{E}) = 2$ and

$$(\mathbf{m}')^T \mathbf{E} \mathbf{m} = 0. \tag{4}$$

This matrix is called an essential matrix and possesses a decomposition

$$\mathbf{E} = \mathbf{S}\,\mathbf{R}, \tag{5}$$

where $\mathbf{S}$ is a 3×3 skew-symmetric matrix ($\mathbf{S}^T = -\mathbf{S}$) and $\mathbf{R}$ is a 3×3 rotation matrix ($\mathbf{R}^T = \mathbf{R}^{-1}$, $\texttt{det}(\mathbf{R}) = 1$). The essential matrix was introduced by Longuet-Higgins in [7].

If the coordinate vectors (3) are related to uncalibrated cameras, i.e. cameras with unknown intrinsic parameters, there exists another 3×3 matrix $\mathbf{F}$ such that $\texttt{rank}(\mathbf{F}) = 2$ and $(\mathbf{m}')^T \mathbf{F} \mathbf{m} = 0$. The matrix $\mathbf{F}$ is called a fundamental matrix. It is related to $\mathbf{E}$ by $\mathbf{E} = (\mathbf{K}')^T \mathbf{F} \mathbf{K}$, where $\mathbf{K}$ and $\mathbf{K}'$ are the calibration matrices of the camera systems $(\overline{S}, \omega)$ and $(\overline{S'}, \omega')$, respectively.

The line passing through the centers $\overline{S}$ and $\overline{S'}$ is called a baseline and the intersection points $B = \overline{S}\,\overline{S'} \bigcap \omega$ and $B' = \overline{S}\,\overline{S'} \bigcap \omega'$ are known as epipolar points. An epipolar plane is any plane containing the baseline. The points $\overline{S}$, $\overline{S'}$, M and M' lie on one and the same epipolar plane. Epipolar lines are the intersection lines of an epipolar plane with projection planes. A detailed information concerning epipolar geometry can be found in [1], [4] and [10]. See also [3].

3 Identities for the vector-rows of the essential matrix

In the rest of the paper, we will use the assumptions and notation from Subsection 2.2. Let $(x_{\overline{S'}}, y_{\overline{S'}}, z_{\overline{S'}})$ be the Cartesian coordinates of the second projection center $\overline{S'}$ with respect to the first camera frame. Then, the position vector $\overrightarrow{\overline{S}\,\overline{S'}} = (x_{\overline{S'}}, y_{\overline{S'}}, z_{\overline{S'}})^T$ of $\overline{S'}$ coincides with the translation vector $\mathbf{t}$ and the skew-symmetric matrix $\mathbf{S}$ in the equation (5) is given by $\mathbf{S} = \mathbf{t}_\times = \begin{bmatrix} 0 & -z_{\overline{S'}} & y_{\overline{S'}} \\ z_{\overline{S'}} & 0 & -x_{\overline{S'}} \\ -y_{\overline{S'}} & x_{\overline{S'}} & 0 \end{bmatrix}$. The rotation matrix $\mathbf{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$ in the same equation has vector-rows $\mathbf{r}_i = (r_{i1}, r_{i2}, r_{i3}), \quad i = 1, 2, 3$. Similarly, if $\mathbf{E} = \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix}$ is the essential matrix determined by (4), the vector-rows of $\mathbf{E}$ are denoted by $\mathbf{e}_i = (e_{i1}, e_{i2}, e_{i3})\,, \quad i = 1, 2, 3.$

Using the matrix equation $\mathbf{E} = \mathbf{SR}$, we obtain the basic relations between vector-rows of the essential matrix and the position vector of the second projection center. From $\texttt{rank}(\mathbf{E}) = 2$ it follows that $\texttt{rank}(\mathbf{S}) = 2$, i.e. $(x_{\overline{S'}}, y_{\overline{S'}}, z_{\overline{S'}}) \neq (0, 0, 0)$.

Lemma 3.1. *The Cartesian coordinates of the second projection center $\overline{S'}$ and the vector-rows of the essential matrix $\mathbf{E}$ are related by*

$$\mathbf{e}_1 \cdot \mathbf{e}_1 = z^2_{\overline{S'}} + y^2_{\overline{S'}} \qquad \mathbf{e}_2 \cdot \mathbf{e}_2 = z^2_{\overline{S'}} + x^2_{\overline{S'}} \qquad \mathbf{e}_3 \cdot \mathbf{e}_3 = y^2_{\overline{S'}} + x^2_{\overline{S'}} \tag{6}$$

and

$$\mathbf{e}_1 \cdot \mathbf{e}_2 = -x_{\overline{S'}} y_{\overline{S'}} \qquad \mathbf{e}_1 \cdot \mathbf{e}_3 = -x_{\overline{S'}} z_{\overline{S'}} \qquad \mathbf{e}_2 \cdot \mathbf{e}_3 = -z_{\overline{S'}} y_{\overline{S'}}. \tag{7}$$

Depending on the position of $\overline{S'}$, there are three possibilities for the scalar products $\mathbf{e}_1 \cdot \mathbf{e}_2$, $\mathbf{e}_1 \cdot \mathbf{e}_3$ and $\mathbf{e}_2 \cdot \mathbf{e}_3$, (1) each of them is a non-zero number and $(\mathbf{e}_1 \cdot \mathbf{e}_2)(\mathbf{e}_1 \cdot \mathbf{e}_3)(\mathbf{e}_2 \cdot \mathbf{e}_3) < 0$, if and only if $\overline{S'}$ does not lie on the coordinate planes, (2) exactly one of them is a non-zero number if and only if $\overline{S'}$ lies on a coordinate plane, (3)each of them is equal to zero if and only if $\overline{S'}$ lies on a coordinate axes.

Proof. From $\mathbf{E} = \mathbf{SR}$ it follows that $\mathbf{E}^T = \mathbf{R}^T\mathbf{S}^T$. Applying $\mathbf{R}^T = \mathbf{R}^{-1}$ and $\mathbf{S}^T = -\mathbf{S}$ we get $\mathbf{E}\mathbf{E}^T = \mathbf{SRR}^T\mathbf{S}^T = -\mathbf{S}\,\mathbf{S}$. This yields (6) and (7). For the second statement we observe that $(\mathbf{e}_1 \cdot \mathbf{e}_2)(\mathbf{e}_1 \cdot \mathbf{e}_3)(\mathbf{e}_2 \cdot \mathbf{e}_3) = -\,x^2_{\overline{S'}}\, y^2_{\overline{S'}}\, z^2_{\overline{S'}} \leq 0$. This means that the three scalar products are nonzero numbers whenever all coordinates of $\overline{S'}$ are different from 0. By (7) we conclude that if a one scalar product is zero, then at least two scalar products are zeros. In fact, there is an exactly one non-zero coordinate of $\overline{S'}$ whenever exactly two scalar products are zeros, and two coordinates of $\overline{S'}$ are zero whenever all scalar products are zeros. $\square$

Now, we examine separately the above three possibilities.

Theorem 3.2. *Let $\mathbf{E}$ be a given essential matrix with*

$$(\mathbf{e}_1 \cdot \mathbf{e}_2)(\mathbf{e}_1 \cdot \mathbf{e}_3)(\mathbf{e}_2 \cdot \mathbf{e}_3) < 0.$$

Then, the Cartesian coordinates of the second projection center can be com-

puted by

$$\begin{aligned} x_{\overline{S'}} &= -\varepsilon\, \mathit{sgn}(\mathbf{e}_1\cdot\mathbf{e}_3)\sqrt{-\frac{(\mathbf{e}_1\cdot\mathbf{e}_2)\,(\mathbf{e}_1\cdot\mathbf{e}_3)}{(\mathbf{e}_2\cdot\mathbf{e}_3)}} \\ y_{\overline{S'}} &= -\varepsilon\, \mathit{sgn}(\mathbf{e}_2\cdot\mathbf{e}_3)\sqrt{-\frac{(\mathbf{e}_1\cdot\mathbf{e}_2)\,(\mathbf{e}_2\cdot\mathbf{e}_3)}{(\mathbf{e}_1\cdot\mathbf{e}_3)}} \\ z_{\overline{S'}} &= \varepsilon\sqrt{-\frac{(\mathbf{e}_1\cdot\mathbf{e}_3)\,(\mathbf{e}_2\cdot\mathbf{e}_3)}{(\mathbf{e}_1\cdot\mathbf{e}_2)}}\,, \end{aligned} \tag{8}$$

where $\varepsilon \in \{+1, -1\}$ *and* $\mathit{sgn}(.)$ *is the sign function. Moreover, any scalar product* $\mathbf{e}_k\cdot\mathbf{e}_k$ *(*$k = 1,2,3$*) is a positive number, and the following equalities are satisfied*

$$\begin{aligned} (\mathbf{e}_1\cdot\mathbf{e}_1)\,(\mathbf{e}_1\cdot\mathbf{e}_2)\,(\mathbf{e}_1\cdot\mathbf{e}_3) &= -(\mathbf{e}_2\cdot\mathbf{e}_3)\left\{(\mathbf{e}_1\cdot\mathbf{e}_2)^2+(\mathbf{e}_1\cdot\mathbf{e}_3)^2\right\} \\ (\mathbf{e}_1\cdot\mathbf{e}_2)\,(\mathbf{e}_2\cdot\mathbf{e}_2)\,(\mathbf{e}_2\cdot\mathbf{e}_3) &= -(\mathbf{e}_1\cdot\mathbf{e}_3)\left\{(\mathbf{e}_1\cdot\mathbf{e}_2)^2+(\mathbf{e}_2\cdot\mathbf{e}_3)^2\right\} \\ (\mathbf{e}_1\cdot\mathbf{e}_3)\,(\mathbf{e}_2\cdot\mathbf{e}_3)\,(\mathbf{e}_3\cdot\mathbf{e}_3) &= -(\mathbf{e}_1\cdot\mathbf{e}_2)\left\{(\mathbf{e}_1\cdot\mathbf{e}_3)^2+(\mathbf{e}_2\cdot\mathbf{e}_3)^2\right\} \end{aligned} \tag{9}$$

Proof. Solving the nonlinear system (7) with respect to $x_{\overline{S'}}$, $y_{\overline{S'}}$ and $z_{\overline{S'}}$, we get (8). As consequence, we have $x_{\overline{S'}} \neq 0$, $y_{\overline{S'}} \neq 0$ and $z_{\overline{S'}} \neq 0$. By (6) all $\mathbf{e}_k\cdot\mathbf{e}_k$ are positive. Substituting (8) in the right-hand sides of the equalities (6) we obtain (9). □

Theorem 3.3. *Suppose that* $\mathbf{E}$ *is a given essential matrix whose vector-rows satisfy the conditions* $\mathbf{e}_k\cdot\mathbf{e}_i = 0$, $\mathbf{e}_k\cdot\mathbf{e}_j = 0$ *and* $\mathbf{e}_i\cdot\mathbf{e}_j \neq 0$ *for a certain subset* $\{i,j\} \subset \{1,2,3\}$ *and* $k = \{1,2,3\}\setminus\{i,j\}$*. Then, the scalar product* $(\mathbf{e}_l\cdot\mathbf{e}_l)$ *is non-zero for* $l = 1,2,3$*, the Cartesian coordinates of the second projection center are*

$$\begin{aligned} &x_{\overline{S'}} = 0,\ y_{\overline{S'}} = \varepsilon\,\mathit{sgn}(\mathbf{e}_2\cdot\mathbf{e}_3)\sqrt{(\mathbf{e}_3\cdot\mathbf{e}_3)},\ z_{\overline{S'}} = -\varepsilon\sqrt{(\mathbf{e}_2\cdot\mathbf{e}_2)},\ (i,j,k) = (2,3,1) \\ &x_{\overline{S'}} = \varepsilon\,\mathit{sgn}(\mathbf{e}_1\cdot\mathbf{e}_2)\sqrt{(\mathbf{e}_2\cdot\mathbf{e}_2)},\ y_{\overline{S'}} = -\varepsilon\sqrt{(\mathbf{e}_1\cdot\mathbf{e}_1)},\ z_{\overline{S'}} = 0,\ (i,j,k) = (1,2,3) \\ &x_{\overline{S'}} = -\varepsilon\sqrt{(\mathbf{e}_3\cdot\mathbf{e}_3)},\ y_{\overline{S'}} = 0,\ z_{\overline{S'}} = \varepsilon\,\mathit{sgn}(\mathbf{e}_1\cdot\mathbf{e}_3)\sqrt{(\mathbf{e}_1\cdot\mathbf{e}_1)},\ (i,j,k) = (1,3,2) \end{aligned} \tag{10}$$

$(\varepsilon = \pm 1)$ *and the following equalities hold*

$$\mathbf{e}_k\cdot\mathbf{e}_k = \mathbf{e}_i\cdot\mathbf{e}_i + \mathbf{e}_j\cdot\mathbf{e}_j \qquad (\mathbf{e}_i\cdot\mathbf{e}_j)^2 = (\mathbf{e}_i\cdot\mathbf{e}_i)\,(\mathbf{e}_j\cdot\mathbf{e}_j)\,. \tag{11}$$

Proof. Consider the case $(i, j.k) = (2,3,1)$. From (7) it may be concluded that $x_{\overline{S'}} = 0$, $y_{\overline{S'}} \neq 0$ and $z_{\overline{S'}} \neq 0$. Hence, the right-hand sides in (6) are positive, i.e. $(\mathbf{e}_l\cdot\mathbf{e}_l) > 0$. Combining $\mathbf{e}_2\cdot\mathbf{e}_2 = z^2_{\overline{S'}}$ and $\mathbf{e}_3\cdot\mathbf{e}_3 = y^2_{\overline{S'}}$ with $\mathbf{e}_1\cdot\mathbf{e}_1 = y^2_{\overline{S'}} + z^2_{\overline{S'}}$ and $\mathbf{e}_2\cdot\mathbf{e}_3 = -z_{\overline{S'}}y_{\overline{S'}}$, we get (10) and (11). In the remaining two cases the proof is the same. □

Theorem 3.4. *If* $\mathbf{E}$ *is a given essential matrix and if* $\mathbf{e}_i \cdot \mathbf{e}_i > 0$, $\mathbf{e}_j \cdot \mathbf{e}_j > 0$, $\mathbf{e}_i \cdot \mathbf{e}_j = \mathbf{e}_i \cdot \mathbf{e}_k = \mathbf{e}_j \cdot \mathbf{e}_k = \mathbf{e}_k \cdot \mathbf{e}_k = 0$ *for a certain subset* $\{i, j\} \subset \{1, 2, 3\}$ *and* $k = \{1, 2, 3\} \setminus \{i, j\}$, *then the following condition is fulfilled*

$$\mathbf{e}_i \cdot \mathbf{e}_i = \mathbf{e}_j \cdot \mathbf{e}_j \tag{12}$$

and the cartesian coordinates of S' *are*

$$\begin{array}{lllll} x_{\overline{S'}} = 0 & y_{\overline{S'}} = 0 & z_{\overline{S'}} = \varepsilon\sqrt{(\mathbf{e}_1 \cdot \mathbf{e}_1)} & (\varepsilon = \pm 1) & (i, j.k) = (1, 2, 3) \\ x_{\overline{S'}} = 0 & y_{\overline{S'}} = \varepsilon\sqrt{(\mathbf{e}_3 \cdot \mathbf{e}_3)} & z_{\overline{S'}} = 0 & (\varepsilon = \pm 1) & (i, j.k) = (1, 3, 2) \\ x_{\overline{S'}} = \varepsilon\sqrt{(\mathbf{e}_2 \cdot \mathbf{e}_2)} & y_{\overline{S'}} = 0 & z_{\overline{S'}} = 0 & (\varepsilon = \pm 1) & (i, j.k) = (2, 3, 1) \end{array} \tag{13}$$

Proof. Consider the case $(i, j.k) = (1, 2, 3)$. From (6) and $\mathbf{e}_3 \cdot \mathbf{e}_3 = 0$, it can be concluded that $x_{\overline{S'}} = 0$, $y_{\overline{S'}} = 0$, $\mathbf{e}_1 \cdot \mathbf{e}_1 = \mathbf{e}_2 \cdot \mathbf{e}_2$ and $z_{\overline{S'}} = \varepsilon\sqrt{(\mathbf{e}_1 \cdot \mathbf{e}_1)}$. The other two cases follow in the same way. □

The condition $\texttt{rank}(\mathbf{E}) = 2$ means that three vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ are linearly dependent, but there are two of them which are linearly independent.

Lemma 3.5. *Let* a, b, c *be three real constants, not all zero, such that*

$$a\,\mathbf{e}_1 + b\,\mathbf{e}_2 + c\,\mathbf{e}_3 = \mathbf{o} = (0, 0, 0). \tag{14}$$

Then, the following equalities are fulfilled:

$$b\,x_{\overline{S'}} = a\,y_{\overline{S'}}, \qquad b\,z_{\overline{S'}} = c\,y_{\overline{S'}}, \qquad c\,x_{\overline{S'}} = a\,z_{\overline{S'}}. \tag{15}$$

Proof. Since $\mathbf{E}\mathbf{R}^T = \mathbf{S}$, the scalar product $\mathbf{e}_i \cdot \mathbf{r}_j$ is equal to the element s_{ij} of the matrix $\mathbf{S}$. Then, the equalities (15) follow from the equalities (14) and $a\,(\mathbf{e}_1 \cdot \mathbf{r}_j) + b\,(\mathbf{e}_2 \cdot \mathbf{r}_j) + c\,(\mathbf{e}_3 \cdot \mathbf{r}_j) = 0$, $j = 1, 2, 3$. □

Theorem 3.6. *Assume that the vector-rows* $\mathbf{e}_1$ *and* $\mathbf{e}_2$ *of the essential matrix* $\mathbf{E}$ *are linearly independent, or equivalently, the third coordinate* $z_{\overline{S'}}$ *of the second projection center* $\overline{S'}$ *is non-zero. Then the vector-rows of the rotation matrix* $\mathbf{R}$ *can be computed by the formulas*

$$\mathbf{r}_k = a_k\mathbf{e}_1 + b_k\mathbf{e}_2 + c_k(\mathbf{e}_1 \times \mathbf{e}_2), \qquad k = 1, 2, 3, \tag{16}$$

where

$$a_k = -\frac{(\mathbf{e}_2 \cdot \mathbf{e}_k)}{\|\mathbf{e}_1 \times \mathbf{e}_2\|^2}\, z_{\overline{S'}}, \quad b_k = \frac{(\mathbf{e}_1 \cdot \mathbf{e}_k)}{\|\mathbf{e}_1 \times \mathbf{e}_2\|^2}\, z_{\overline{S'}} \quad k = 1, 2, 3 \tag{17}$$

and

$$c_1 = -\frac{(\mathbf{e}_1 \cdot \mathbf{e}_3)}{\|\mathbf{e}_1 \times \mathbf{e}_2\|^2},\ c_2 = -\frac{(\mathbf{e}_2 \cdot \mathbf{e}_3)}{\|\mathbf{e}_1 \times \mathbf{e}_2\|^2},\ c_3 = a_1\,b_2 - b_1\,a_2 = \frac{z_{\overline{S'}}^2}{\|\mathbf{e}_1 \times \mathbf{e}_2\|^2}. \tag{18}$$

Proof. First, we will show the equivalence of the assumptions. From the linear independence of the vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ we see that the coefficient c in (14) is nonzero. Applying Proposition 3.5 we get $z_{\overline{S'}} \neq 0$. Conversely, if $z_{\overline{S'}}$ is nonzero, then by (15) c is also non-zero. Thus, the condition $\mathtt{rank}(\mathbf{E}) = 2$ implies that $\mathbf{e}_1$ and $\mathbf{e}_2$ are linearly independent.

Without loss of generality we may suppose that $c = -1$. Then, the vector equality (14) can be rewritten in the form $\mathbf{e}_3 = a\,\mathbf{e}_1 + b\,\mathbf{e}_2$, and the second equality in (15) becomes $-y_{\overline{S'}} = b\,z_{\overline{S'}}$.

From the matrix equation $\mathbf{E}\mathbf{R}^{\mathbf{T}} = \mathbf{S}$ it follows that

$$\mathbf{e}_1 \cdot \mathbf{r}_1 = 0 \quad \mathbf{e}_2 \cdot \mathbf{r}_1 = z_{\overline{S'}} \quad \mathbf{e}_3 \cdot \mathbf{r}_1 = -y_{\overline{S'}}$$

The third scalar product is a consequence of the first two scalar products. In fact, we have $\mathbf{e}_3 \cdot \mathbf{r}_1 = (a.\mathbf{e}_1 + b.\mathbf{e}_2) \cdot \mathbf{r}_1 = a.0 + b.z_{\overline{S'}} = -y_{\overline{S'}}$. Since the vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ are linearly independent, the vectors $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_1 \times \mathbf{e}_2$ form a basis in $\mathbb{R}^3$. Therefore the vector $\mathbf{r}_1$ can be expressed as

$$\mathbf{r}_1 = a_1\mathbf{e}_1 + b_1\mathbf{e}_2 + c_1(\mathbf{e}_1 \times \mathbf{e}_2).$$

Taking the scalar products of both sides of this vector equality with the vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ we get the system of two equations

$$\left| \begin{array}{l} a_1.(\mathbf{e}_1 \cdot \mathbf{e}_1) + b_1.(\mathbf{e}_1 \cdot \mathbf{e}_2) = 0 \\ a_1.(\mathbf{e}_1 \cdot \mathbf{e}_2) + b_1.(\mathbf{e}_2 \cdot \mathbf{e}_2) = z_{\overline{S'}} \end{array} \right.$$

Solving the system by Cramer's rule we have $a_1 = -\dfrac{(\mathbf{e}_2 \cdot \mathbf{e}_1)}{\|\mathbf{e}_1 \times \mathbf{e}_2\|^2}.z_{\overline{S'}}$ and $b_1 = \dfrac{((\mathbf{e}_1 \cdot \mathbf{e}_1))}{\|\mathbf{e}_1 \times \mathbf{e}_2\|^2}.z_{\overline{S'}}$. Similarly, the coefficients a_2, b_2, a_3 and b_3 can be calculated by (17) for $k = 2, 3$.

The matrix $\mathbf{R}$ is orthogonal and $\det(\mathbf{R}) = 1$. As consequence we have $\mathbf{r}_1 \times \mathbf{r}_2 = \mathbf{r}_3$. On the other hand, the vectors $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$ are linear combinations (16) of the basic vectors $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_1 \times \mathbf{e}_2$. Thus, we can compute the vector cross product

$$\begin{aligned}
\mathbf{r}_1 \times \mathbf{r}_2 &= (a_1\,b_2 - b_1\,a_2)(\mathbf{e}_1 \times \mathbf{e}_2) \\
&\quad +(a_1\,c_2 - c_1\,a_2)\{\mathbf{e}_1 \times (\mathbf{e}_1 \times \mathbf{e}_2)\} + (b_1\,c_2 - c_1\,b_2)\{\mathbf{e}_2 \times (\mathbf{e}_1 \times \mathbf{e}_2)\} \\
&= (a_1\,b_2 - b_1\,a_2)(\mathbf{e}_1 \times \mathbf{e}_2) \\
&\quad +(a_1\,c_2 - c_1\,a_2)\{(\mathbf{e}_1 \cdot \mathbf{e}_2)\mathbf{e}_1 - (\mathbf{e}_1 \cdot \mathbf{e}_1)\mathbf{e}_2\} \\
&\quad +(b_1\,c_2 - c_1\,b_2)\{(\mathbf{e}_2 \cdot \mathbf{e}_2)\mathbf{e}_1 - (\mathbf{e}_1 \cdot \mathbf{e}_2)\mathbf{e}_2\} \\
&= \{(a_1\,c_2 - c_1\,a_2)(\mathbf{e}_1 \cdot \mathbf{e}_2) + (b_1\,c_2 - c_1\,b_2)(\mathbf{e}_2 \cdot \mathbf{e}_2)\}\mathbf{e}_1 \\
&\quad -\{(a_1\,c_2 - c_1\,a_2)(\mathbf{e}_1 \cdot \mathbf{e}_1) + (b_1\,c_2 - c_1\,b_2)(\mathbf{e}_1 \cdot \mathbf{e}_2)\}\mathbf{e}_2 \\
&= +(a_1\,b_2 - b_1\,a_2)(\mathbf{e}_1 \times \mathbf{e}_2)
\end{aligned}$$

Comparing this representation of $\mathbf{r}_3 = \mathbf{r}_1 \times \mathbf{r}_2$ and $\mathbf{r}_3 = a_3\mathbf{e}_1 + b_3\mathbf{e}_2 + c_3(\mathbf{e}_1 \times \mathbf{e}_2)$ we obtain $a_1b_2 - b_1a_2 = c_3$ and

$$\left| \begin{array}{rcl} \big(-a_2(\mathbf{e}_1\cdot\mathbf{e}_2) - b_2(\mathbf{e}_2\cdot\mathbf{e}_2)\big)c_1 + \big(a_1(\mathbf{e}_1\cdot\mathbf{e}_2) + b_1(\mathbf{e}_2\cdot\mathbf{e}_2)\big)c_2 & = & a_3 \\ \big(a_2(\mathbf{e}_1\cdot\mathbf{e}_1) - b_2(\mathbf{e}_1\cdot\mathbf{e}_2)\big)c_1 - \big(a_1(\mathbf{e}_1\cdot\mathbf{e}_1) + b_1(\mathbf{e}_1\cdot\mathbf{e}_2)\big)c_2 & = & b_3 \end{array} \right. .$$

Then, the equalities (18) are an immediate consequence of (17).

The matrix $\mathbf{R}$ is rotational if and only if its vector-rows satisfy $\mathbf{r}_1\cdot\mathbf{r}_1 = 1$, $\mathbf{r}_2\cdot\mathbf{r}_2 = 1$, $\mathbf{r}_1\cdot\mathbf{r}_2 = 0$ and $\mathbf{r}_1\times\mathbf{r}_2 = \mathbf{r}_3$. Since the last condition is fulfilled we have to prove the other three conditions. From (16), (17) and (18) we can express $\mathbf{r}_1\cdot\mathbf{r}_1$ as follows

$$\begin{aligned} \mathbf{r}_1\cdot\mathbf{r}_1 &= \big(a_1\mathbf{e}_1 + b_1\mathbf{e}_2 + c_1(\mathbf{e}_1\times\mathbf{e}_2)\big)\cdot\big(a_1\mathbf{e}_1 + b_1\mathbf{e}_2 + c_1(\mathbf{e}_1\times\mathbf{e}_2)\big) \\ &= a_1^2(\mathbf{e}_1\cdot\mathbf{e}_1) + b_1^2(\mathbf{e}_2\cdot\mathbf{e}_2) + 2a_1b_1(\mathbf{e}_1\cdot\mathbf{e}_2) + c_1^2\|\mathbf{e}_1\times\mathbf{e}_2\|^2 \\ &= \frac{(\mathbf{e}_1\cdot\mathbf{e}_2)^2(\mathbf{e}_1\cdot\mathbf{e}_1) + (\mathbf{e}_1\cdot\mathbf{e}_1)^2(\mathbf{e}_2\cdot\mathbf{e}_2) - 2(\mathbf{e}_1\cdot\mathbf{e}_2)^2(\mathbf{e}_1\cdot\mathbf{e}_1)}{\|\mathbf{e}_1\times\mathbf{e}_2\|^4}\, z_{S'}^2 \\ &\quad + \frac{(\mathbf{e}_1\cdot\mathbf{e}_3)^2}{\|\mathbf{e}_1\times\mathbf{e}_2\|^4}\|\mathbf{e}_1\times\mathbf{e}_2\|^2 \\ &= \frac{(\mathbf{e}_1\cdot\mathbf{e}_1)\, z_{S'}^2 + (\mathbf{e}_1\cdot\mathbf{e}_3)^2}{\|\mathbf{e}_1\times\mathbf{e}_2\|^2} \end{aligned} \tag{19}$$

Suppose that the conditions of Theorem 3.2 are satisfied. From (8) and the first equality in (9) it follows that $z_{S'}^2 = -\dfrac{(\mathbf{e}_1\cdot\mathbf{e}_3)\,(\mathbf{e}_2\cdot\mathbf{e}_3)}{(\mathbf{e}_1\cdot\mathbf{e}_2)} > 0$ and $(\mathbf{e}_1\cdot\mathbf{e}_3)^2 = -\dfrac{(\mathbf{e}_1\cdot\mathbf{e}_1)\,(\mathbf{e}_1\cdot\mathbf{e}_3)\,(\mathbf{e}_1\cdot\mathbf{e}_3)}{(\mathbf{e}_2\cdot\mathbf{e}_3)} - (\mathbf{e}_1\cdot\mathbf{e}_2)^2$. Using the above expression (19) for $\mathbf{r}_1\cdot\mathbf{r}_1$ and the second equality in (9) we obtain

$$\begin{aligned} \mathbf{r}_1\cdot\mathbf{r}_1 &= \frac{-\dfrac{(\mathbf{e}_1\cdot\mathbf{e}_1)\,(\mathbf{e}_1\cdot\mathbf{e}_3)\,(\mathbf{e}_2\cdot\mathbf{e}_3)}{(\mathbf{e}_1\cdot\mathbf{e}_2)} - \dfrac{(\mathbf{e}_1\cdot\mathbf{e}_1)\,(\mathbf{e}_1\cdot\mathbf{e}_2)\,(\mathbf{e}_1\cdot\mathbf{e}_3)}{(\mathbf{e}_2\cdot\mathbf{e}_3)} - (\mathbf{e}_1\cdot\mathbf{e}_2)^2}{\|\mathbf{e}_1\times\mathbf{e}_2\|^2} \\ &= \frac{(\mathbf{e}_1\cdot\mathbf{e}_1)\,(\mathbf{e}_1\cdot\mathbf{e}_3)\left(-\dfrac{(\mathbf{e}_2\cdot\mathbf{e}_3)^2 + (\mathbf{e}_1\cdot\mathbf{e}_2)^2}{(\mathbf{e}_1\cdot\mathbf{e}_2)\,(\mathbf{e}_2\cdot\mathbf{e}_3)}\right) - (\mathbf{e}_1\cdot\mathbf{e}_2)^2}{\|\mathbf{e}_1\times\mathbf{e}_2\|^2} \\ &= \frac{(\mathbf{e}_1\cdot\mathbf{e}_1)\,(\mathbf{e}_1\cdot\mathbf{e}_3)\left(\dfrac{(\mathbf{e}_1\cdot\mathbf{e}_2)\,(\mathbf{e}_2\cdot\mathbf{e}_2)\,(\mathbf{e}_2\cdot\mathbf{e}_3)}{(\mathbf{e}_1\cdot\mathbf{e}_2)\,(\mathbf{e}_2\cdot\mathbf{e}_3)\,(\mathbf{e}_1\cdot\mathbf{e}_3)}\right) - (\mathbf{e}_1\cdot\mathbf{e}_2)^2}{\|\mathbf{e}_1\times\mathbf{e}_2\|^2} \\ &= \frac{(\mathbf{e}_1\cdot\mathbf{e}_1)\,(\mathbf{e}_2\cdot\mathbf{e}_2) - (\mathbf{e}_1\cdot\mathbf{e}_2)^2}{\|\mathbf{e}_1\times\mathbf{e}_2\|^2} = 1 \end{aligned}$$

If the conditions of Theorem 3.3 (or Theorem 3.4) are satisfied and $z_{S'}^2 > 0$, then using (19), (10) and the identities (11) (or (19), (13) and the identities (12)) we get $\mathbf{r}_1\cdot\mathbf{r}_1 = 1$. The conditions $\mathbf{r}_2\cdot\mathbf{r}_2 = 1$ and $\mathbf{r}_1\cdot\mathbf{r}_2 = 0$ can be proved in the same way. $\square$

In case the coordinate $x_{\overline{S'}}$ (respectively $y_{\overline{S'}}$) is non-zero, we see that Theorem 3.6 remains true changing $z_{\overline{S'}}$ with $x_{\overline{S'}}$ (respectively $y_{\overline{S'}}$) and making a cyclic replacement of indices $(123) \to (231)$ (respectively $(123) \to (312)$).

4 The essential matrix and 3D reconstruction

One well-known approach for reconstruction of a 3D object from two images obtained by calibrated cameras is based on the essential matrix computation. This approach can be divided into three steps: (i) estimating the essential matrix up to a scale factor, (ii) decomposing the essential matrix into a skew-symmetric matrix and a rotation matrix, (iii) determining the 3D object by triangulation. Different ways for the estimation of the essential matrix are given in [2], [6], [9] and [10]. In the previous section we examine several properties of the essential matrix which can be used for both checking and decomposition of the essential matrix. The following algorithm summarizes these properties.

Algorithm 4.1. *A factorization of a given essential matrix with vector-rows* $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ *into a product of a skew-symmetric matrix and a rotation matrix.*

1. *Calculate the six scalar products* $\mathbf{e}_1 \cdot \mathbf{e}_2 = q_{12}$, $\mathbf{e}_1 \cdot \mathbf{e}_3 = q_{13}$, $\mathbf{e}_2 \cdot \mathbf{e}_3 = q_{23}$, $\mathbf{e}_1 \cdot \mathbf{e}_1 = q_{11}$, $\mathbf{e}_2 \cdot \mathbf{e}_2 = q_{22}$ *and* $\mathbf{e}_3 \cdot \mathbf{e}_3 = q_{33}$.

2. *If* $q_{12}q_{13}q_{23} < 0$, *test the conditions (9), compute the coordinates (8) of the second projection center and go to step 5,*

3. *If* $q_{ij} = q_{ik} = 0$ *and* $q_{jk} \neq 0$ *for a certain permutation* (ijk) *of* $\{1, 2, 3\}$, *verify the conditions (11), compute the coordinates (10) of the second projection center and go to step 5,*

4. *In case of* $q_{ij} = q_{ik} = q_{jk} = q_{kk} = 0$, $q_{ii} \neq 0$, $q_{jj} \neq 0$ *for a certain permutation* (ijk) *of* $\{1, 2, 3\}$, *verify the conditions (12) and compute the coordinates (13) of the second projection center.*

5. *Construct the skew-symmetric matrix* $\mathbf{S}$,

6. *Choose a non-zero coordinate of the second projection center and apply Theorem 3.6 for obtaining the vector-rows* $\mathbf{r}_i$ *of the rotation matrix. Use a suitable cyclic replacement of the indices if necessary.*

7. *Construct the rotation matrix.*

Recall that the Cartesian coordinates of the second projection center $\overline{S'}$ form the translation vector $\mathbf{t} = (x_{\overline{S'}}, y_{\overline{S'}}, z_{\overline{S'}})^T$ and $\mathbf{t}_\times = \mathbf{S}$. It is determined a linear map of $\mathbf{R}^4$ into itself by the nonsingular 4×4 matrix

$$\begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{o} & 1 \end{bmatrix},$$

where $\mathbf{o} = (0, 0, 0)$. If a 4×1 vector-column $\mathbf{m}'$ represents homogeneous coordinates of a point $M' \in \omega'$ with respect to the second camera frame, then the vector $\mathbf{m}'_1 = \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{o} & 1 \end{bmatrix} \mathbf{m}'$ represents homogeneous coordinates of the same point M' with respect to the first camera frame. Thus the point $\overline{M} \in \mathbb{E}^3$ can be determined by its projections $M \in \omega$ and $M' \in \omega'$ related to the first camera frame as the intersection $\overline{M} = \overline{S}\, M \bigcap \overline{S'}\, M'$. The point $\overline{M}$ us called the reconstruction point.

Example 4.1. *Facade reconstruction of an old Bulgarian country house. Let us consider two pictures of an old country house. We choose ten pairs of corresponding points in these pictures (See Figure 2 and Figure 3). Then we determine homogeneous coordinates of the selected points. The results are written in the Table 1. Following the procedure described in [10] we obtain the*

Table 1: Homogeneous coordinates of the selected points

N:	Points from the first picture			N:	Points from the second picture		
1	-0,0658	0,1476	1	1	-0,1466	0,0981	1
2	0,0137	0,1498	1	2	-0,0755	0,0982	1
3	-0,0652	0,092	1	3	-0,1462	0,0425	1
4	-0,182	0,1462	1	4	-0,289	0,0984	1
5	0,0785	0,0496	1	5	-0,0199	-0,0036	1
6	0,1689	0,0632	1	6	0,0559	0,0099	1
7	0,2043	0,0685	1	7	0,827	0,0147	1
8	0,2222	0,0712	1	8	0,0982	0,0175	1
9	-0,0285	-0,0621	1	9	-0,1065	-0,1238	1
10	0,1509	-0,0593	1	10	0,0476	-0,1127	1

essential matrix

$$\mathbf{E} = \begin{bmatrix} 22,5273 & -54,1562 & 9,337 \\ -54,8582 & -23,7347 & -0,0369 \\ 8,5515 & -5,7703 & 1,3872 \end{bmatrix}$$

Since the conditions of Theorem 3.2 are satisfied, the coordinates of second projection center are expressed by (8), i.e.
$(x_{\overline{S'}}, y_{\overline{S'}}, z_{\overline{S'}}) = (8.7624, -5.6187, 59.1269)$, *or equivalently*

$$\mathbf{S} = \begin{bmatrix} 0 & 59.1269 & -5.6187 \\ -59.1269 & 0 & -8.7624 \\ 5.6187 & 8.7624 & 0 \end{bmatrix}$$

Applying Theorem 3.6 we get the rotational matrix

$$\mathbf{R} = \begin{pmatrix} 0,9224 & 0,3593 & -0,1414 \\ 0,3844 & -0,8889 & 0,249 \\ 0,0362 & 0,284 & 0,9581 \end{pmatrix}.$$

Then, we compute homogeneous coordinates of the points from the second picture with respect to the first camera frame and determine the reconstruction points corresponding to the selected pairs. Finally, the reconstruction points from the facade are related to the world coordinate system whose X-axes and Y-axes are the lines 12 *and* 13 *, respectively. The result of the reconstruction of the facade is given in Figure 4.*

5 Conclusion

The proposed method for decomposition of the essential matrix has two main advantages. First, the elements of the skew-symmetric matrix and the rotation matrix are expressed explicitly by scalar products of the vector-rows of the essential matrix. Second, the obtained new identities for these scalar products are naturally integrated in a 3D reconstruction scheme.

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Figure 2: An old country house - first picture.

Figure 3: An old country house - second picture.

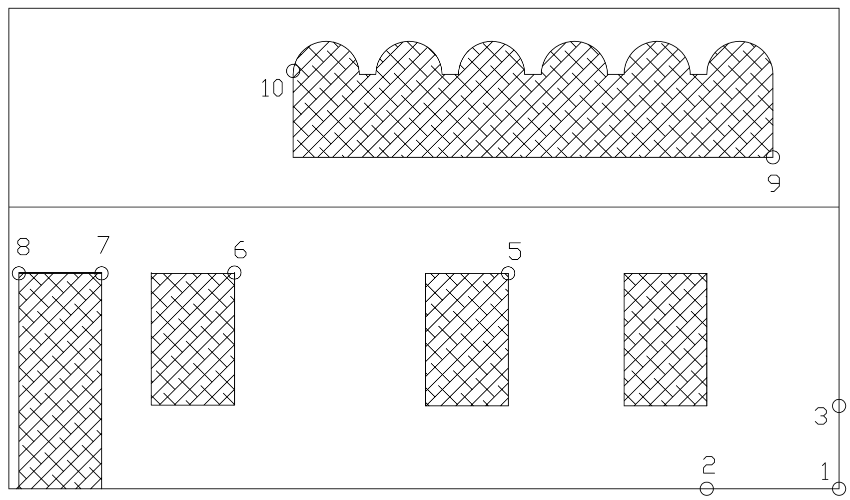

Figure 4: Reconstruction of the facade.

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