Homework 1

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October 8, 2015

1. **Relations**

   (a) To show a relation to be an equivalence relation, we must show it to be reflexive, symmetric, and transitive.

   Reflexive

   Given an element $x \in A$, we must evaluate the pair *(x, x)*. The sum of digits in x equal the sum of digits in y, since $x = y$.

   Symmetric

   Given elements *(x, y)* $\in A$, we must prove the implication:

   $$x \sim y \rightarrow y \sim x \tag{0.1}$$

   For brevity, we define *d(x)* to be a function returning the digits of x. We apply the relation to the LHS of the implication.

   $$\sum d(x) = \sum d(y) \tag{0.2}$$

   Trivially, we can flip the equation to match the RHS.

   $$\sum d(y) = \sum d(x) \tag{0.3}$$

   Transitive

   Given elements *x,y,z* $\in A$, we must prove the implication:

   $$(x \sim y) \wedge (y \sim z) \rightarrow (x \sim z) \tag{0.4}$$

   We expand the LHS of the implication as follows:

   $$\sum d(x) = \sum d(y) = \sum d(z) \tag{0.5}$$

   Trivially, we can see the following

$$\sum d(x) = \sum d(z) \tag{0.6}$$

(b) The equivalence classes of A under R:

$$\begin{aligned}
[10]_R &= \{10\} \\
[24]_R &= \{24\} \\
[11]_R &= [20]_R = \{11, 20\} \\
[12]_R &= [21]_R = \{12, 21\} \\
[13]_R &= [22]_R = \{13, 22\} \\
[14]_R &= [23]_R = \{14, 23\}
\end{aligned} \tag{0.7}$$

2. **Functions**

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ n-1 & \text{if } n \text{ is odd} \end{cases} \tag{0.8}$$

(a) For a given $a,\ b \in Z$, we examine the implication:

$$(f(a) = f(b)) \to (a = b) \tag{0.9}$$

Assume *f(a)* = *f(b)*, one of three cases is true. Either a and b are both even, a and b are both odd, or a is even and b is odd. We will show a contradiction in the final case.

$$\frac{a}{2} = b - 1 \tag{0.10}$$

Multiply both sides by 2.

$$a = 2 * b - 1 \tag{0.11}$$

The above contradictions (0.9), therefore *f(x)* is *not injective.*

(b) To prove f is surjective, we must show:

$$(\forall b \in Z)(\exists a \in Z)[f(a) = b] \tag{0.12}$$

For a given $b \in Z$, we can construct an a such that *f(a)* = b by simply setting $a = 2 * b$. As such, a will always be even and therefore result in the first branch of the piecewise function.

3. **Sets and Counterexamples**

   (a) $(A \cap B) \supset (A \cap B \cap C)$

   $$A = B = C = \{1\} \tag{0.13}$$

   $\{1\}$ is not a superset of $\{1\}$.

   (b) $C \subseteq (A \cup B) \rightarrow (A \cap B) \subseteq C$

   $$A = \{1, 2\}, B = \{2\}, C = \{1\} \tag{0.14}$$

   C is a subset of $\{1, 2\}$, however, $\{2\}$ is not a subset of C.

   (c) $(A \cap B) \subseteq C \rightarrow C \subseteq (A \cup B)$

   $$A = \{1\}, B = \{1, 2\}, C = \{1, 4\} \tag{0.15}$$

   $\{1\}$ is a subset of C, however, C is not a subset of $\{1, 2\}$.

4. **Proof by Contradiction**

   Assume the graph G is disconnected and has at least two components. We can choose two nodes (u, v) from two separate components. These nodes should share no common neighbors. We define the function *N(v)* to be the set of neighbors of node v. No common neighbors is represented as, $|N(u) \cap N(v)| = 0$.

   By the Inclusion-Exclusion principle

   $$|N(u) \cap N(v)| = |N(u)| + |N(v)| - |N(u) \cup N(v)| \tag{0.16}$$

   $$|N(u) \cap N(v)| = \frac{n}{2} + \frac{n}{2} - |N(u) \cup N(v)| \tag{0.17}$$

   The intersection of both sets of neighbors cannot include $\{u, v\}$ since these two vertices are in separate components.

   $$|N(u) \cup N(v)| \leq n - 2 \tag{0.18}$$

   By substitution into (0.17), we arrive at the following:

   $$|N(u) \cap N(v)| \geq 2 \tag{0.19}$$

This contradicts the requirement that u and v should share no common neighbors, therefore, a graph in which all nodes have degree $\frac{n}{2}$ is a connected graph.

5. **Inductive Proof**

   (a) Binary tree with $n = 1$ degree 2 nodes and $n+1 = 2$ leaves.

   By definition, a single degree 2 node contains 2 leaves.

   (b) Assuming a binary tree with n degree 2 nodes has $n+1$ leaves, then a tree with $n+1$ degree 2 nodes has $n+2$ leaves.

   In order to add a node to our binary tree of size n, we must add a child two one of the leaves. This decrements the leaf count to n. However, our new child node have degree two, so we add two leaves to our child node. This increases the leaf count by 2, resulting in $n+2$ leaves.

6. **Stable Marriage**

   In every instance of the Stable Marriage problem, there is a stable matching containing a pair (m, w) such that m is ranked first on the preference list of w and w is ranked first on the preference list of m.

   Imagine a population with the following preferences:

   - m_1 prefers w_1
   - m_2 prefers w_2
   - w_1 prefers m_2
   - w_2 prefers m_1

   The four valid pairs are: (m_1, w_1), (m_1, w_2), (m_2, w_1), (m_2, w_2). As there are no possible pairs, the problem description is false.