

Homework 1

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September 15, 2015

1. Relations

- (a) To show a relation to be an equivalence relation, we must show it to be reflexive, symmetric, and transitive.

Reflexive

Given an element $x \in A$, we must evaluate the pair (x, x) . The sum of digits in x equal the sum of digits in y , since $x = y$.

Symmetric

Given elements $(x, y) \in A$, we must prove the implication:

$$x \sim y \rightarrow y \sim x \quad (0.1)$$

For brevity, we define $d(x)$ to be a function returning the digits of x . We apply the relation to the LHS of the implication.

$$\sum d(x) = \sum d(y) \quad (0.2)$$

Trivially, we can flip the equation to match the RHS.

$$\sum d(y) = \sum d(x) \quad (0.3)$$

Transitive

Given elements $x, y, z \in A$, we must prove the implication:

$$(x \sim y) \wedge (y \sim z) \rightarrow (x \sim z) \quad (0.4)$$

We expand the LHS of the implication as follows:

$$\sum d(x) = \sum d(y) = \sum d(z) \quad (0.5)$$

Trivially, we can see the following

$$\sum d(x) = \sum d(z) \quad (0.6)$$

(b) The equivalence classes of A under R:

$$\begin{aligned} [10]_R &= \{10\} \\ [24]_R &= \{24\} \\ [11]_R &= [20]_R = \{11, 20\} \\ [12]_R &= [21]_R = \{12, 21\} \\ [13]_R &= [22]_R = \{13, 22\} \\ [14]_R &= [23]_R = \{14, 23\} \end{aligned} \quad (0.7)$$

2. Functions

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ n - 1 & \text{if } n \text{ is odd} \end{cases} \quad (0.8)$$

(a) For a given $a, b \in Z$, we examine the implication:

$$(f(a) = f(b)) \rightarrow (a = b) \quad (0.9)$$

Assume $f(a) = f(b)$, one of three cases is true. Either a and b are both even, a and b are both odd, or a is even and b is odd. We will show a contradiction in the final case.

$$\frac{a}{2} = b - 1 \quad (0.10)$$

Multiply both sides by 2.

$$a = 2 * b - 1 \quad (0.11)$$

The above contradictions (0.9), therefore $f(x)$ is *not injective*.

(b) To prove f is surjective, we must show:

$$(\forall b \in Z)(\exists a \in Z)[f(a) = b] \quad (0.12)$$

For a given $b \in Z$, we can construct an a such that $f(a) = b$ by simply setting $a = 2 * b$. As such, a will always be even and therefore result in the first branch of the piecewise function.

3. Sets and Counterexamples

$$(a) (A \cap B) \supset (A \cap B \cap C)$$

$$A = B = C = \{1\} \quad (0.13)$$

$\{1\}$ is not a superset of $\{1\}$.

$$(b) C \subseteq (A \cup B) \rightarrow (A \cap B) \subseteq C$$

$$A = \{1, 2\}, B = \{2\}, C = \{1\} \quad (0.14)$$

C is a subset of $\{1, 2\}$, however, $\{2\}$ is not a subset of C .

$$(c) (A \cap B) \subseteq C \rightarrow C \subseteq (A \cup B)$$

$$A = \{1\}, B = \{1, 2\}, C = \{1, 4\} \quad (0.15)$$

$\{1\}$ is a subset of C , however, C is not a subset of $\{1, 2\}$.

4. Proof by Contradiction

Assume the graph G is disconnected and has at least two components. We can choose two nodes (u, v) from two separate components. These nodes should share no common neighbors. We define the function $N(v)$ to be the set of neighbors of node v . No common neighbors is represented as, $|N(u) \cap N(v)| = 0$.

By the Inclusion-Exclusion principle

$$|N(u) \cap N(v)| = |N(u)| + |N(v)| - |N(u) \cup N(v)| \quad (0.16)$$

$$|N(u) \cap N(v)| = \frac{n}{2} + \frac{n}{2} - |N(u) \cup N(v)| \quad (0.17)$$

The intersection of both sets of neighbors cannot include $\{u, v\}$ since these two vertices are in separate components.

$$|N(u) \cup N(v)| \leq n - 2 \quad (0.18)$$

By substitution into (0.17), we arrive at the following:

$$|N(u) \cap N(v)| \geq 2 \quad (0.19)$$

This contradicts the requirement that u and v should share no common neighbors, therefore, a graph in which all nodes have degree $\frac{n}{2}$ is a connected graph.

5. Inductive Proof

- (a) Binary tree with $n = 1$ degree 2 nodes and $n+1 = 2$ leaves.

By definition, a single degree 2 node contains 2 leaves.

- (b) Assuming a binary tree with n degree 2 nodes has $n+1$ leaves, then a tree with $n+1$ degree 2 nodes has $n+2$ leaves.

In order to add a node to our binary tree of size n , we must add a child to one of the leaves. This decrements the leaf count to n . However, our new child node has degree two, so we add two leaves to our child node. This increases the leaf count by 2, resulting in $n+2$ leaves.

6. Stable Marriage

In every instance of the Stable Marriage problem, there is a stable matching containing a pair (m, w) such that m is ranked first on the preference list of w and w is ranked first on the preference list of m .

Imagine a population with the following preferences:

- m_1 prefers w_1
- m_2 prefers w_2
- w_1 prefers m_2
- w_2 prefers m_1

The four valid pairs are: (m_1, w_1) , (m_1, w_2) , (m_2, w_1) , (m_2, w_2) . As there are no possible pairs, the problem description is false.